MATH 3032: Abstract Algebra

Assignment 2

due 6 February 2025, end of day

Homework should be submitted as a single PDF attachment to `denisalja@dal.ca`. Please title the file in a useful way, for example `Math3032_HW#_Name.pdf`.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You are expected to attempt every problem on every assignment, but you are not expected to solve every problem on every assignment. The purpose of homework assignments is to learn.

1. Decide which of the following are ideals in $\mathbb{Z}\times\mathbb{Z}$. Include a brief justification for your answers.
   (a) $\{(a,a) : a \in \mathbb{Z}\}$
   (b) $\{(2a,2b) : a,b \in \mathbb{Z}\}$
   (c) $\{(2a,0) : a \in \mathbb{Z}\}$
   (d) $\{(a,-a) : a \in \mathbb{Z}\}$.

2. Decide which of the following are ideals in $\mathbb{Z}[x]$. Include a brief justification for your answers.
   (a) The set of polynomials whose constant term is 3.
   (b) The set of polynomials whose constant term is a multiple of 3.
   (c) The set of polynomials whose linear term is a multiple of 3.
   (d) The set of polynomials whose constant and linear terms are both multiples of 3.
   (e) $\mathbb{Z}[x^2]$, i.e. the set of polynomials in which only even powers of x appear.

3. Show that the ideal $(3,x) \subset \mathbb{Z}[x]$ is not principal.

4. Suppose that A is an abelian group, written additively. Given an integer n and an element $a \in A$, define

$$a \cdot n := \begin{cases} \underbrace{a + a + \cdots + a}_{n}, & n > 0, \\ 0, & n = 0, \\ \underbrace{(-a) + (-a) + \cdots + (-a)}_{-n}, & n < 0. \end{cases}$$

   For $n > 0$, a unital ring R is said to have *characteristic* n if n is the smallest positive integer such that $n \cdot 1 = 0$. If no such integer exists, then R is said to have *characteristic* 0.
   (a) Prove that the function $n \mapsto n \cdot 1, \mathbb{Z} \to R$ is a ring homomorphism.
   (b) Prove that for $n \geq 0$, the kernel of this homomorphism is $n\mathbb{Z}$ if and only if R has characteristic n. (This is why we say "characteristic zero" and not "characteristic infinity.")

(c) Determine the characteristic of the rings $\mathbb{Q}$, $\mathbb{Z}[x]$, and $(\mathbb{Z}/n\mathbb{Z})[x]$.

(d) Prove that if p is prime and R is a ring of characteristic p, then

$$(a+b)^p = a^p + b^p$$

for all $a, b \in R$. This is called the *Freshman's Dream.*

5. Prove directly that a commutative ring is a field if and only if $\{0\}$ is a maximal ideal.

6. Let $\phi : R \to S$ a ring homomorphism, and $J \subset S$ an ideal. Recall that its *preimage*, defined by

$$\phi^{-1}(J) := \{r \in R : \phi(r) \in J\},$$

is an ideal in R. (If you do not recall this, then prove it again!)

   (a) Prove that the preimage of a prime ideal is prime.

   (b) Prove that the preimage of a maximal ideal is maximal.

   (c) Give an example showing that the preimage of a principal ideal might not be principal. Hint: consider the "mod 3" map $\mathbb{Z}[x] \to (\mathbb{Z}/3\mathbb{Z})[x]$.

7. Suppose that R and S are unital rings, with units denoted $1_R \in R$ and $1_S \in S$, and that $\phi : R \to S$ is a nonunital homomorphism.

   (a) Show that $\phi(1_R) \in S$ is *idempotent*: it solves the equation $x^2 = x$.

   (b) Show that $\phi(1_R)$ is a zero divisor.

   (c) Show that the principal ideal generated by $1_S - \phi(1_R)$ is unital when thought of as a ring, even though it is not a unital subring of S.

   (d) Show that every element of S can be written uniquely as a sum of an element of the principle ideal $(\phi(1_R))$ and an element of the principle ideal $(1_S - \phi(1_R))$.

8. Prove that the intersection of a collection of ideals is an ideal.

9. Let R be a commutative unital ring. Prove that the principal ideal $(x) \subset R[x]$ is prime if and only if R is an integral domain.

10. Let R be an integral domain. Prove that if $(a) = (b)$, then there exists an invertible $u \in R$ such that $a = ub$.

11. Consider the ring $R = (\mathbb{Z}/2\mathbb{Z})[x]$, and the principle ideal $I = (x^2 + x + 1)$. Given $r \in R$, write $\overline{r} := r \mod I$ for its image in R/I. Show that $R/I = \{\overline{0}, \overline{1}, \overline{x}, \overline{1+x}\}$. Work out the addition and multiplication tables on R/I. Show that R/I is a field.

12. Recall that a ring R is *Boolean* if every element $a \in R$ satisfies $a^2 = a$.

   (a) Prove that if R is Boolean, then every finitely generated ideal is principal.

   (b) Prove that if R is Boolean, then every prime ideal is maximal.