Math 3032: Abstract Algebra

Final exam - solutions

24 April 2023

Your name:

University academic honour statement:

Dalhousie University has adopted the following statement, based on "The Fundamental Values of Academic Integrity" developed by the International Center for Academic Integrity (ICAI):

> Academic integrity is a commitment to the values of learning in an academic environment. These values include honesty, trust, fairness, responsibility, and respect. All members of the Dalhousie community must acknowledge that academic integrity is fundamental to the value and credibility of academic work and inquiry. We must seek to uphold academic integrity through our actions and behaviours in all our learning environments, our research, and our service.

Please **sign here** to confirm that you will uphold these values, and that the work you submit on this exam will be your own.

Exam structure

There are six questions, each worth ten points.

Question 1.

Suppose that R is a unital ring. State the definition of *idempotent* ("self-powerful") in R. Prove that if $p \in R$ is idempotent, then so is $1-p$. Prove that the ideal $\langle p \rangle$ consists exactly of the elements $r \in R$ that solve $rp = r$.

$p \in R$ is *idempotent* if $p^2 = p$. In this case, $(1-p)^2 = 1 - 2p + p^2 = 1 - 2p + p = 1 - p$, so $1-p$ is also idempotent.

For the second claim, first note that an element of $\langle p \rangle$ is always of the form $r = sp$ for some $s \in R$. But $sp^2 = sp$ since $p^2 = p$, so such an element solves $rp = r$. On the other hand, suppose that r solves $rp = r$. Then obviously r is a multiple of p.

Question 2.

Give an example of a ring homomorphism $f : R \to S$ such that both R and S are unital, but f is not unital.

One example: the map $\mathbb{Z}_3 \to \mathbb{Z}_6$ given by $[x \mod 3] \mapsto [2x \mod 6]$.

Another example: let S be any unital ring and consider the inclusion $\{0\} \hookrightarrow S$.

Question 3.

Suppose that R is a unital commutative ring. When is an ideal $J \subset R$ called *prime*? When is it called *maximal*? Give an example of a prime ideal in $\mathbb{Z}$ that is not maximal.

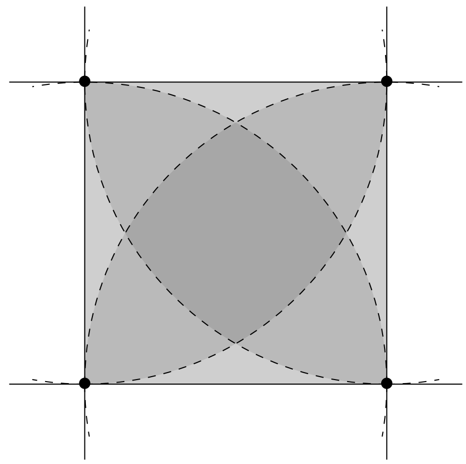
One answer: An ideal $J \subset R$ is *prime* if R/J is an integral domain, and *maximal* if R/J is a field. Another answer: an ideal $J \subset R$ is *prime* if whenever $ab \in J$, at least one of a or b is in J, and *maximal* if $J \neq R$ and there are no ideals I with $J \subsetneq I \subsetneq R$.

The ideal $\{0\} \subset \mathbb{Z}$ is prime but not maximal.

Question 4.

Draw a picture to illustrate why $\mathbb{Z}[i]$ is a Euclidean domain.

The point is that a square is covered by four quarter-circles:



Question 5.

Let R be a commutative ring, with ideals $I, J \subset R$. Recall that the *product* of I and J is the set of sums of products of an element in I with an element in J:

$$I \cdot J = \{\sum_{k=1}^{n} a_k b_k : n \in \mathbb{N}, a_k \in I, b_k \in J\}.$$

Prove that $I \cdot J$ is an ideal in R.

We must show that $I \cdot J$ is closed under addition, and absorbing for multiplication.

The closure under addition is obvious, since already $I \cdot J$ consists of sums of arbitrary length.

For the absorption statement, suppose that $a_k \in I$ and $b_k \in J$, $k = 1, \ldots, n$, and that $r \in R$. Then

$$r \sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} r a_k b_k$$

which is in $I \cdot J$ because $r a_k \in I$ for each a_k (because I is absorbing).

Question 6.

Run Buchberger's algorithm to find a Gröbner basis for the ideal $\langle x^2y + xy^2, xy - x\rangle \subset \mathbb{R}[x, y]$ with respect to the ordering $x \gg y$.

We run long division, replacing

$$x^2y + xy^2 \rightsquigarrow (x^2y + xy^2) - x(xy - x) = xy^2 + x^2 = x^2 + xy^2.$$

$\boxed{\text{The basis } \langle x^2 + xy^2, xy - x\rangle \text{ is already Gröbner.}}$ This can be seen by Buchberger's criterion. Indeed, the s-element is

$$s = S(x^2 + xy^2, xy - x) = y(x^2 + xy^2) - x(xy - x) = xy^3 + x^2 = x^2 + xy^3,$$

which transforms to 0 under long division by our basis:

$$s \rightsquigarrow (x^2 + xy^3) - (x^2 + xy^2) = xy^3 - xy^2 \rightsquigarrow (xy^3 - xy^2) - y^2(xy - x) = 0.$$

One can alternately perform further subleading long divisions, replacing

$$x^2 + xy^2 \rightsquigarrow (x^2 + xy^2) - y(xy - x) = x^2 + xy \rightsquigarrow (x^2 + xy) - (xy - x) = x^2 + x$$

to produce the Gröbner basis $\boxed{\langle x^2 + x, xy - x\rangle}$. Buchberger's criterion easily checks that this is Gröbner: $s = y(x^2 + x) - x(xy - x) = x^2 + xy$, which is the sum of the two basis elements.