Math 3032: Abstract Algebra

Final exam

24 April 2023

Your name:

University academic honour statement:

Dalhousie University has adopted the following statement, based on "The Fundamental Values of Academic Integrity" developed by the International Center for Academic Integrity (ICAI):

> Academic integrity is a commitment to the values of learning in an academic environment. These values include honesty, trust, fairness, responsibility, and respect. All members of the Dalhousie community must acknowledge that academic integrity is fundamental to the value and credibility of academic work and inquiry. We must seek to uphold academic integrity through our actions and behaviours in all our learning environments, our research, and our service.

Please **sign here** to confirm that you will uphold these values, and that the work you submit on this exam will be your own.

Exam structure

There are six questions, each worth ten points.

Question 1.

Suppose that R is a unital ring. State the definition of *idempotent* ("self-powerful") in R. Prove that if $p \in R$ is idempotent, then so is $1-p$. Prove that the ideal $\langle p \rangle$ consists exactly of the elements $r \in R$ that solve $rp = r$.

Question 2.

Give an example of a ring homomorphism $f : R \to S$ such that both R and S are unital, but f is not unital.

Question 3.

Suppose that R is a unital commutative ring. When is an ideal $J \subset R$ called *prime*? When is it called *maximal*? Give an example of a prime ideal in $\mathbb{Z}$ that is not maximal.

Question 4.

Draw a picture to illustrate why $\mathbb{Z}[i]$ is a Euclidean domain.

Question 5.

Let R be a commutative ring, with ideals $I, J \subset R$. Recall that the *product* of I and J is the set of sums of products of an element in I with an element in J:

$$I \cdot J = \{\sum_{k=1}^{n} a_k b_k : n \in \mathbb{N}, a_k \in I, b_k \in J\}.$$

Prove that $I \cdot J$ is an ideal in R.

Question 6.

Run Buchberger's algorithm to find a Gröbner basis for the ideal $\langle x^2y + xy^2, xy - x \rangle \subset \mathbb{R}[x, y]$ with respect to the ordering $x \gg y$.