MATH 5055: Abstract Algebra

Assignment 1

due 23 January 2025, end of day

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please title the file in a useful way, for example `Math5055_HW#_Name.pdf`.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You are expected to attempt every problem on every assignment, but you are not expected to solve every problem on every assignment. The purpose of homework assignments is to learn.

1. Let F be a finite field. Prove that $|F| = p^n$ for some prime p and some integer n.

2. Let θ be a root of $x^3 + 9x + 6$. Show that $\mathbb{Q}(\theta)$ is a degree-3 extension of $\mathbb{Q}$ (hint: show that $x^3 + 9x + 6$ is irreducible over $\mathbb{Q}$) and compute $(1 + \theta)^{-1}$.

3. Show that $x^3 + x + 1$ is irreducible over $\mathbb{F}_2$. Let θ be a root, and compute the powers of θ in $\mathbb{F}_2(\theta)$.

4. For which n is $x^3 - nx + 2$ irreducible over $\mathbb{Q}$?

5. Prove that if $[F(\alpha) : F]$ is odd, then $F(\alpha) = F(\alpha^2)$.

6. Prove that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Conclude that $\sqrt{2} + \sqrt{3}$ is the root of a quartic polynomial over $\mathbb{Z}$. Find that polynomial.

7. Prove directly that $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is an automorphism of $\mathbb{Q}(\sqrt{2})$.

8. Prove that the only unital ring endomorphism of $\mathbb{R}$ is the identity. Hint: $x \leq y$ iff $\sqrt{y - x} \in \mathbb{R}$.

9. (a) Let $\sqrt{3 + 4\mathbf{i}}$ denote the square root of $3 + 4i$ that lies in the first quadrant and let $\sqrt{3 - 4\mathbf{i}}$ denote the square root of $3 - 4\mathbf{i}$ that lies in the fourth quadrant. Show that $[\mathbb{Q}(\sqrt{3 + 4\mathbf{i}} + \sqrt{3 - 4\mathbf{i}}) : \mathbb{Q}] = 1$.

   (b) Compute $[\mathbb{Q}(\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}) : \mathbb{Q}]$.

10. Let $f(x)$ be an irreducible polynomial of degree n over some field F, and let $g(x)$ be arbitrary. Prove that every irreducible factor of $(f \circ g)(x) \in F[x]$ has degree divisible by n.

11. A field F is *formally real* if -1 is not a sum of squares in F. Suppose that F is formally real and that $f(x) \in F[x]$ is irreducible of odd degree. Pick a root α of f. Prove that $F(\alpha)$ is formally real. Hint: Consider a counterexample of minimal degree. Show that there exists $g(x)$ of odd degree $< \deg(f)$ such that $-1 + f(x)g(x)$ is a sum of squares in $F[x]$. Show that g would give a new counterexample, violating minimality of f.