MATH 5055: Abstract Algebra

Assignment 2

due 6 February 2025, end of day

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please title the file in a useful way, for example `Math5055_HW#_Name.pdf`.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You are expected to attempt every problem on every assignment, but you are not expected to solve every problem on every assignment. The purpose of homework assignments is to learn.

1. Show that the functorial image of an isomorphism is an isomorphism. Give an example of a functor which takes a non-isomorphism to an isomoprhism.

2. Given categories $\mathcal{C}, \mathcal{D}$, show that there is a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are the functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are the natural transformations. In particular, show that the composition of natural transformations is again a natural transformation.

3. (a) Let $\mathcal{C}, \mathcal{D}$ be a category, with opposite categories $\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}$. Given a functor $\Phi : \mathcal{C} \to \mathcal{D}$, show that the same assignment on objects and morphisms is also a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. This same assignment, thought of as a functor on the opposite categories, is denoted Φ^{op}
   (b) Given $\Phi : \mathcal{C} \to \mathcal{D}$ and $\Psi : \mathcal{D} \to \mathcal{E}$, how are Φ^{op}, Ψ^{op}, and $(\Psi\Phi)^{\mathrm{op}}$ related?
   (c) Show that the assignment $\Phi \mapsto \Phi^{\mathrm{op}}$ extends naturally to natural transformations, where provides a contravariant equivalence between $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})$.

4. Write $(\mathbb{N}, \leq)$ for the poset of natural numbers, made into a category in the usual way. What are all the categories $\mathcal{C}$ with a surjective (on both objects and morphisms) functor $(\mathbb{N}, \leq) \to \mathcal{C}$?

5. (a) Let $f, g : (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$ be monotonic maps, thought of as functors. When is there a natural transformation $f \Rightarrow g$?
   (b) Show that the category of functors and natural transformations from $(\mathbb{R}, \leq)$ to itself is a poset.

6. Suppose that $\Phi : \mathcal{D} \to \mathcal{C}$ and $\Psi : \mathcal{E} \to \mathcal{C}$ are functors. Prove that there is a category, called the *comma category* and denoted $\Phi \downarrow \Psi$, with:
   - objects are triples $(D \in \mathcal{D}, E \in \mathcal{E}, c : \Phi D \to \Psi E)$
   - morphisms $(D, E, f) \to (D', E', f')$ are pairs $(d : D \to D', e : E \to E')$ such that $c \circ \Phi d = \Psi e \circ c$.

7. Fix a set A. Show that the following operations **Set** $\to$ **Set** are naturally functors:
   (a) The assignment $X \mapsto X \sqcup A$.

(b) The assignment $X \mapsto X \times A$.

(c) The assignment $X \mapsto X^A$.

(d) Show that $X \mapsto A^X$ is a contravariant functor.

8. Show that the contravariant endofunctor $X \mapsto A^X$ of **Set** is self-adjoint on the right.

9. Let **Mnd** denote the category of monoids and monoid homomorphisms. Let $\Upsilon : \mathbf{Mnd} \to \mathbf{Set}$ denote the functor that forgets the monoid structure, and $\Phi : \mathbf{Set} \to \mathbf{Mnd}$ the functor that takes a set to the monoid of all finite-length lists (aka words) in that set. Show that there is an adjunction $\Phi \dashv \Upsilon$.

10. Consider the assignment $V \mapsto \mathrm{End}(V), \mathbf{Vec} \to \mathbf{Ring}$, that takes a vector space to its ring of endomorphisms. Show that this assignment is not a functor.