Math 4055/5055: Advanced Algebra II

Assignment 3

due 25 February 2025, end of day

Noncommutative separability

1. Let F be a field and A an associative and unital, but possibly noncommutative, F-algebra. Show that the multiplication map $m : A \otimes A \to A$ is always a homomorphism of A-bimodules. Show that $m : A \otimes A \to A$ is a homomorphism of algebras if and only if A is commutative.

2. A is called *separable* if there exists an A-bilinear splitting $\Delta : A \to A \otimes A$ of the multiplication map m. Such a Δ is called a *separation* of A. Show that every separation $\Delta : A \to A \otimes A$ is *coassociative*:
$$(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta : A \to A \otimes A \otimes A.$$
The name is because the dual map $\Delta^* : A^* \otimes A^* \to (A \otimes A)^* \to A$ is associative. Hint: Precompose both sides with $\mathrm{id}_A = m \circ \Delta$.

3. Show that a separation Δ is determined by the element $u = \Delta(1) = \sum_i u_i^{(1)} \otimes u_i^{(2)} \in A \otimes A$, and that u must satisfy the equations
$$\sum_i a u_i^{(1)} \otimes u_i^{(2)} = \sum_i u_i^{(1)} \otimes u_i^{(2)} a \in A \otimes A, \qquad \sum_i u_i^{(1)} u_i^{(2)} = 1 \in A,$$
and that these are the only conditions.

4. Let $A = K = F[\alpha]$ be a field extension, such that the minimal polynomial q of α is separable. Then in $K[x]$, we can factor $q(x) = (x - \alpha) \sum_n b_n x^n$, and by separability, $q'(\alpha) \neq 0$. Show that
$$u = \sum \alpha^n \otimes \frac{b_n}{q'(\alpha)} \in K \otimes K$$
provides a separation for K. In other words, K is separable as in the noncommutative sense.

5. Compute (the dimension of) the space of separations for the quaternion algebra $A = \mathbb{H}$ thought of as an algebra over the real field $F = \mathbb{R}$. Hint: $\mathbb{H}^e \cong \mathrm{Mat}_4(\mathbb{R})$. This algebra is semisimple with a unique simple left module. Who are $A \otimes A$ and A in terms of that unique simple left module? Hint: The category of left $\mathrm{Mat}_4(\mathbb{R})$-modules is equivalent to $\mathrm{Vec}_{\mathbb{R}}$ via the functor that sends the unique simple left module to $\mathbb{R}^1$.

6. Recall that a finite dimensional algebra A is *semisimple* if all of its finite-dimensional left modules are projective. Show that if A is finite-dimensional and separable, then it is semisimple. Hint: Let M be a left A-module. Use the isomorphism $A \otimes_A M \cong M$. Now use Δ to write A as a direct summand of $A \otimes A$.

7. Prove that the tensor product of separable algebras is again separable.

8. Suppose that A is finite-dimensional, and let A^{op} denote its opposite algebra. The *enveloping algebra* of A is $A^e := A \otimes A^{\mathrm{op}}$. Prove that A is separable if and only if A^e is semisimple. Hint: A^e-modules are the same as A-bimodules. In one direction, use semisimplicity of A^e to find a separation of A. In the other direction, use the previous two questions.

9. Suppose now that $A = K = F[\alpha]$ such that the minimal polyonial q of α is inseparable. Show that $K^e = K \otimes K$ contains a nilpotent element. Explain why this shows that the converse of question 6 fails.

   More generally, a field extension $F \subset E$ is *inseparable* if E contains some nonzero element α whose minimal polynomial is inseparable. Show again that E^e contains a nilpotent element and hence is not semisimple.

   Conclude that a finite-degree field extension is separable-in-the-field-extension-sense — every element has separable minimal polyonimal — if and only if it is separable-in-the-algebra-sense.