SPT phases and generalized cohomology

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Algebraic Structures in Quantum Computation IV
UBC (online), 23 June 2020

Based on arxiv:1712.07950, joint with Davide Gaiotto.

These slides are available at categorified.net/ASQC4.pdf.

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Goals for the talk:

- The spaces $\mathcal{I}^n$ of n-dimensional invertible gapped topological systems compile into a spectrum.
- For any group G, the set of n-dimensional G-SPTs is the relative generalized cohomology $\mathcal{I}^n(\mathrm{B}G, \mathrm{pt}) = \mathcal{I}^n(\mathrm{B}G, \mathrm{pt})$.
- Spectra are rigid and highly constrained. Topologists know lots of methods for computing with them.

Conventions for the talk:

- system = gapped topological quantum matter system.
- n-dimensional = spacetime dimension ($n = n{-}1 + 1$).

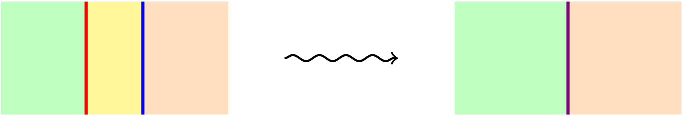
There is no complete definition of "gapped topological system." The problem is to make sense of "gapped" in a fully local way: what should be done about boundary conditions?

Physical assumptions:

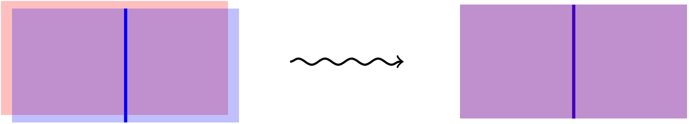
- There is a well-defined topological space $\{n\text{-dim'l systems}\}$. Paths in this space are continuous paths that do not close the gap. **Defn:** An n-dim'l phase is a connected component of this space.
- Given a system, modifications (operator insertions, defects, interfaces, ...) in one region of spacetime do not affect the choices of modifications in macroscopically-separated regions. Collections of possible modifications are again topological spaces, and inserting is a continuous operation.

(modification = gapped topological modification)

Under these assumptions, *n*-dim'l systems are the objects of an *n*-category. 1-morphisms = interfaces. 2-morphisms = junctions between interfaces. Composition = fusion: place interfaces next to each other, and then zoom out.



Systems themselves may be fused, aka stacked. A system (or interface) X is invertible if there exists some other system Y such that $X \otimes Y$ is in the trivial phase. Interfaces and defects may also be stacked.

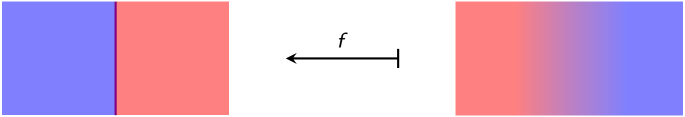


Fundamental result (Kitaev, JF–Gaiotto):

For n-dim'l systems X and Y, there is a homotopy equivalence:

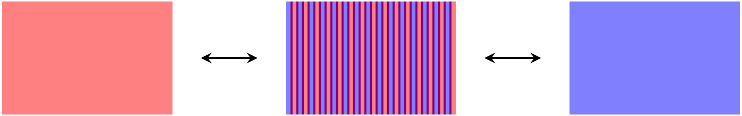
$$\{\text{invertible defects } X \mid Y\} \xrightarrow{\sim} \{\text{cont's paths } X \rightsquigarrow Y\}.$$

Proof: Run the path adiabatically in space, and then zoom out.

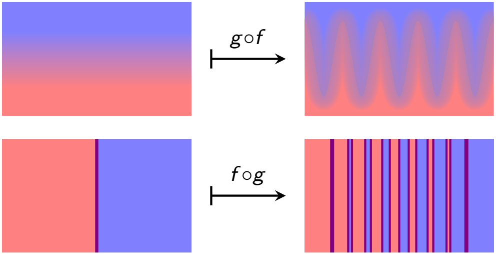


To show that f is a homotopy equivalence, it suffices to describe a map g in the other direction and check that $f \circ g \simeq \mathrm{id}$ and $g \circ f \simeq \mathrm{id}$. We do not need to check any "higher" homotopies.

Proof (cont'd): To build g, consider a system which at a mesoscopic scale alternates X and Y slabs. Invertibility of the interface ⇒ this can be cont'sly deformed to both X and Y.



Now compute:



Fundamental result repeated:

$$\{\text{invertible defects } X \mid Y\} \xrightarrow{\sim} \{\text{cont's paths } X \rightsquigarrow Y\}.$$

Corollary: $\mathcal{I}^\bullet := \{\text{invertible } \bullet\text{-dim systems}\}$ is a spectrum:

$$\mathcal{I}^n \xrightarrow{\sim} \Omega\mathcal{I}^{n+1}.$$

For topological space $\mathcal{T}$ with a basepoint $1 \in \mathcal{T}$, $\Omega\mathcal{T} = \{\text{paths } 1 \rightsquigarrow 1\}$. Basepoint in $\mathcal{I}^n$ is the trivial n-dimensional system "1_n."

Proof: $\{n\text{-dim systems}\} = \{\text{interfaces } 1_{n+1} \rightsquigarrow 1_{n+1}\}$.

Definition: A prespectrum (aka Σ-spectrum) is a sequence $\mathcal{T}^0, \mathcal{T}^1, \mathcal{T}^2, \ldots$ of topological spaces with basepoints, together with maps

$$\mathcal{T}^n \longrightarrow \Omega\mathcal{T}^{n+1}.$$

A prespectrum is a spectrum (aka Ω-spectrum) if these maps are homotopy equivalences. Any prespectrum $\mathcal{T}^\bullet$ may be completed to a spectrum

$$[\mathcal{T}]^\bullet = \varinjlim_{k\to\infty} \Omega^k \mathcal{T}^{\bullet+k}$$

Information in $[\mathcal{T}]$ is stable. Info in $\mathcal{T}$ that is lost in $[\mathcal{T}]$ is unstable.

Warning: There are lots of models of the homotopy theory of spectra. Different models $\Rightarrow$ inequivalent, but homotopy equivalent, definitions. **Examples:** (1) Use category whose objects are prespectra, but $\hom(\mathcal{S}^\bullet, \mathcal{T}^\bullet) = \hom([\mathcal{S}]^\bullet, [\mathcal{T}]^\bullet)$. (2) Require that $\mathcal{T}^n \to \Omega\mathcal{T}^{n+1}$ is a homeomorphism.

Let G be a group. An n-dim'l G-SET is a system X enhanced with G-symmetry. It is a G-SPT if it is in the trivial phase ($X \simeq 1$) if you forget G.

By coupling to background G-connections, can encode a G-SET in the data: for each $g \in G$, a path $\rho(g) : X \rightsquigarrow X$; for each $g, h \in G$, a homotopy of paths $\rho(gh) \simeq \rho(g) \circ \rho(h)$; for each triple, a homotopy of homotopies; etc.

Assuming $X \in \mathcal{I}^n$, these data define a continuous map

$$(X, \rho) : \mathrm{B}G \to \mathcal{I}^n,$$

where $\mathrm{B}G$ is the classifying space of G.

Conversely, a map $\mathrm{B}G \to \mathcal{I}^n$ determines a G-SET: concentrate the paths onto defects; use a mesoscopic lattice to build a system with on-site symmetry.

Generalized cohomology:

Let $\mathcal{T}^\bullet$ be a spectrum (e.g. $\mathcal{I}^\bullet$), and $\mathcal{G}$ a top'l space (e.g. $\mathrm{B}G$). The $\mathcal{T}$-cohomology of $\mathcal{G}$ is

$$\mathcal{T}^\bullet(\mathcal{G}) = \pi_0 \,\mathsf{maps}(\mathcal{G}, \mathcal{T}^\bullet).$$

Examples:

- The homotopy groups of $\mathcal{I}$ are $\mathcal{I}^n(\mathrm{pt}) = \pi_0 \mathcal{I}^n =: I^n$.
- $\mathcal{I}^\bullet(\mathrm{B}G) =$ invertible G-SETs up to phase.
- Can stack any invertible G-SET with its inverse equipped with trivial G-symmetry. This gives an isomorphism

$$\mathcal{I}^\bullet(\mathrm{B}G) \cong \mathcal{I}^\bullet(\mathrm{pt}) \oplus \mathcal{I}^\bullet(\mathrm{B}G, \mathrm{pt}).$$

$\mathcal{I}^\bullet(\mathrm{B}G, \mathrm{pt}) =$ relative $\mathcal{I}$-cohomology of $\mathrm{B}G = G$-SPTs.

How to evaluate $\mathcal{I}^\bullet(\mathrm{B}G)$?

An n-dim'l G-SPT ρ determines an $(n-1)$d phase $\rho_1(g) \in I^{n-1}$. Group law: $\rho_1(gh) = \rho_1(g) \circ \rho_1(h)$. I.e. $\rho_1(-)$ is a 1-cocycle on G with values in I^{n-1}.

Once and for all, arbitrarily choose representative $\widehat{x} \in \mathcal{I}^n$ of each phase $x \in I^n = \pi_0\mathcal{I}^n$. The SPT ρ includes information about the homotopy $\widehat{\rho_1}(gh) \simeq \widehat{\rho_1}(g)\widehat{\rho_1}(h)$. Temporarily pretend $\widehat{xy} = \widehat{x}\widehat{y}$ for all x, y. Then this information is a new phase $\rho_2(g,h) \in I^{n-2}$.

Pretend Lemma: As paths $\widehat{\rho_1}(ghk) \rightsquigarrow \widehat{\rho_1}(g)\widehat{\rho_1}(h)\widehat{\rho_1}(k)$, have $\rho_2(h,k) + \rho_2(g,hk) \simeq \rho_2(g,h) + \rho_2(gh,k)$. I.e. ρ_2 is a 2-cocycle.

However, $\widehat{xy} \neq \widehat{x}\widehat{y}$. Once and for all, choose a path $\widehat{xy} \rightsquigarrow \widehat{x}\widehat{y}$. ρ will select some other path. Difference $= \rho_2(g,h) \in I^{n-2}$. Lemma is false, because the arbitrary choice $\widehat{xyz} \rightsquigarrow \widehat{x}\widehat{yz} \rightsquigarrow \widehat{x}\widehat{y}\widehat{z}$ is probably different from $\widehat{xyz} \rightsquigarrow \widehat{xy}\widehat{z} \rightsquigarrow \widehat{x}\widehat{y}\widehat{z}$.

$\widehat{xyz} \rightsquigarrow \widehat{x}\widehat{yz} \rightsquigarrow \widehat{x}\widehat{\widehat{y}\widehat{z}}$ is probably different from $\widehat{xyz} \rightsquigarrow \widehat{xy}\widehat{z} \rightsquigarrow \widehat{\widehat{x}\widehat{y}}\widehat{z}$.

Difference, applied to $(x, y, z) = (\rho_1(g), \rho_1(h), \rho_1(k))$, is some 3-cocyle $\mathfrak{k}^n_{12}(\rho_1)(g, h, k)$.

$$\mathfrak{k}^n_{1,2} : \{\text{1-cocycles in } I^{n-1}\} \to \{\text{3-cocycles in } I^{n-2}\}$$

is a universal operation, indep of G. Cochain ρ_2 will solve

$$\mathrm{d}\rho_2 + \mathfrak{k}^n_{1,2}(\rho_1) = 0.$$

Now repeat: compare some once-and-for-all choice of equivalences of phases with the choice determined by ρ to build a 3-cochain ρ_3. It will solve

$$\mathrm{d}\rho_3 + \mathfrak{k}^n_{2,3}(\rho_2) + \mathfrak{k}^n_{1,3}(\rho_1) = 0$$

for some universal operations $\mathfrak{k}^n_{2,3}$ and $\mathfrak{k}^n_{13}$.

Theorem: For each i, j, n, there is some universal operation $\mathfrak{k}^n_{i,j} : \{i\text{-cochains in } I^{n-i}\} \to \{(j+1)\text{-cochains in } I^{n-j}\}$.

An n-dim G-SPT $\rho \in \mathcal{I}^n(\mathrm{B}G, \mathrm{pt})$ is equivalent to a sequence of cochains $\rho_i \in \mathrm{C}^i(\mathrm{B}G; I^{n-i})$ solving

$$\begin{pmatrix} \mathrm{d} & \mathfrak{k}^n_{n-1,n} & \cdots & \mathfrak{k}^n_{1,n} \\ & \mathrm{d} & & \vdots \\ & & \ddots & \mathfrak{k}^n_{1,2} \\ & & & \mathrm{d} \end{pmatrix} \begin{pmatrix} \rho_n \\ \vdots \\ \rho_2 \\ \rho_1 \end{pmatrix} = 0$$

The maps $\mathfrak{k}^n_{i,j}$ are the Postnikov k-invariants of $\mathcal{I}^\bullet$.

Warning: $\mathfrak{k}^n_{i,j}$ are never linear on cochains. $\mathfrak{k}^n_{i,i+1}$ are cohomology operations: they take cocycles to cocycles. Rest are more complicated.

Because $\mathcal{I}^\bullet$ is a spectrum, the operations $\mathfrak{k}^n_{i,j}$ are stable of degree $j-i+1$. This means that there is a universal object

$$\mathfrak{k}_{n-i,n-j} : \mathrm{H}^\bullet(-; I^{n-i}) \to \mathrm{H}^{\bullet+(j-i)+1}(-; I^{n-j})$$

determining all $\mathfrak{k}^n_{i,j}$ in a systematic way.

There aren't very many stable operations.

Convention: $\mathrm{d}_1 = \mathrm{d}$. $\mathrm{d}_2 := \mathfrak{k}_{\bullet,\bullet+1}$. $\mathrm{d}_3 := \mathfrak{k}_{\bullet,\bullet+2}$. ...

Example: Fermionic phases without time-reversal symmetry.
Ingredient: $\mathcal{I}^0 = \mathrm{U}(1)$, i.e.

$$I^0 = \pi_0 \mathrm{U}(1) = 0, \quad I^{-1} = \pi_0 \Omega \mathrm{U}(1) = \mathbb{Z}, \quad I^{<-1} = 0.$$

We also know $I^1 = \{\text{bosonic line}, \text{fermionic line}\} = \mathbb{Z}_2$. There are only two stable operations $\mathbb{Z}_2 \to \mathbb{Z}$ of degree 3:

$$0 \quad \text{and} \quad \square \mathrm{Sq}^2.$$

Here Sq^2 is the second Steenrod square, and $\square$ is the integral Bockstein (basically, $(-1)^x : \mathbb{Z}_2 \to \mathrm{U}(1)$).

Ingredient: $\{\text{3d SPTs for } G = \mathbb{Z}_2 \text{ with } \rho_1 = 0\} = \mathbb{Z}_4 \neq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Corollary: k-invariant is nontrivial, hence must be $\square \mathrm{Sq}^2$.

Organizing the calculation: Recall: to find an n-dim SPT ρ, we need to solve the cocycle equation

$$\begin{pmatrix} \mathrm{d}_1 & \mathrm{d}_2 & \dots & \mathrm{d}_{n-1} \\ & \mathrm{d}_1 & & \vdots \\ & & \ddots & \mathrm{d}_2 \\ & & & \mathrm{d}_1 \end{pmatrix} \begin{pmatrix} \rho_n \\ \vdots \\ \rho_2 \\ \rho_1 \end{pmatrix} = 0$$

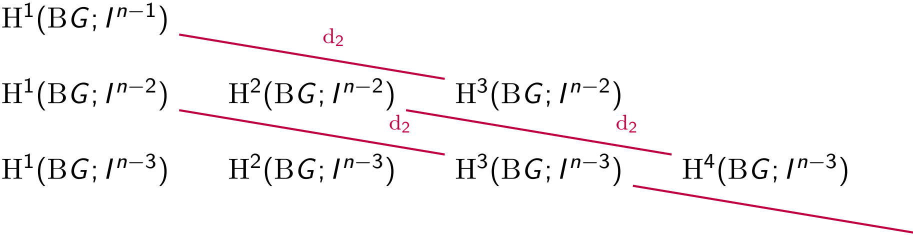
where $\mathrm{d}_1 = \mathrm{d}$ and $\mathrm{d}_r = \mathfrak{t}^n_{n-\bullet, n+1-r-\bullet}$ for $r > 1$.

Start with $\rho_1 \in \mathrm{H}^1(\mathrm{B}G; I^{n-1})$. If ρ_2 exists, then

$$\rho_1 \in \ker\left(\mathrm{H}^1(\mathrm{B}G; I^{n-1}) \xrightarrow{\mathrm{d}_2} \mathrm{H}^3(\mathrm{B}G; I^{n-2})\right).$$

Choice of ρ_2: If ρ_2', ρ_2'' are two possible choices of ρ_2, then $\rho_2' - \rho_2''$ is a cocycle, which will solve

$$\rho_2' - \rho_2'' \in \ker\left(\mathrm{H}^2(\mathrm{B}G; I^{n-2}) \xrightarrow{\mathrm{d}_2} \mathrm{H}^4(\mathrm{B}G; I^{n-3})\right).$$



$\mathrm{H}^1(\mathrm{B}G; I^{n-1}) \supset \ker(\mathfrak{k}) \ni \rho_1$.

$\mathrm{H}^2(\mathrm{B}G; I^{n-2}) \supset \ker(\mathfrak{k}) \ni$ ambiguity in ρ_2.

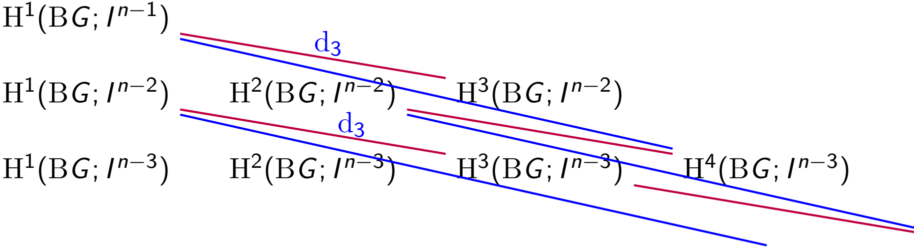
$\mathrm{H}^3(\mathrm{B}G; I^{n-1}) \supset \ker(\mathfrak{k}) \ni$ ambiguity in ρ_3? Not quite. Modify $\rho_2 \mapsto \rho_2 + \mathrm{d}\alpha$ for some $\alpha \in \mathrm{C}^1(\mathrm{B}G; I^{n-2})$. Then $\mathrm{d}_2(\rho_2)$ will change by some function of α. Absorb this into ρ_3.

Lemma: Ambiguity in ρ_3 lives in $\dfrac{\mathrm{H}^3(\mathrm{B}G; I^{n-3})}{\mathrm{d}_2(\mathrm{H}^1(\mathrm{B}G; I^{n-2}))}$.

$$\rho_1 \in \ker\left(\mathrm{H}^1(\mathrm{B}G; I^{n-1}) \xrightarrow{\mathrm{d}_2} \mathrm{H}^3(\mathrm{B}G; I^{n-2})\right)$$

is not the only relation that ρ_1 must solve. Existence of ρ_3 forces $\mathrm{d}_3(\rho_1) = \mathrm{d}_2(\text{something}) + \mathrm{d}(\text{something})$, i.e.

$$\mathrm{d}_3(\rho_1) = [0] \in \frac{\mathrm{H}^4(\mathrm{B}G; I^{n-3})}{\mathrm{d}_2(\mathrm{H}^2(\mathrm{B}G; I^{n-2}))}.$$



We saw already that d_2 is a (stable) cohomology operation: it takes d-cocycles to d-cocycles, and d-coboundaries to d-coboundaries. d_3 is not a d-cohomology operation. But it is a d_2-cohomology operation: it is well-defined as a map

$$\mathrm{d}_3 : \frac{\ker\big(\mathrm{d}_2 : \mathrm{H}^i(\mathrm{B}G; I^j) \to \mathrm{H}^{i+2}(\mathrm{B}G; I^{j-1})\big)}{\mathrm{im}\big(\mathrm{d}_2 : \mathrm{H}^{i-2}(\mathrm{B}G; I^{j+1}) \to \mathrm{H}^i(\mathrm{B}G; I^j)\big)} \longrightarrow \frac{\ker\big(\mathrm{d}_2 : \mathrm{H}^{i+3}(\mathrm{B}G; I^{j-2}) \to \mathrm{H}^{i+5}(\mathrm{B}G; I^{j-3})\big)}{\mathrm{im}\big(\mathrm{d}_2 : \mathrm{H}^{i+1}(\mathrm{B}G; I^{j-1}) \to \mathrm{H}^{i+3}(\mathrm{B}G; I^{j-2})\big)}.$$

Summary: $\{G\text{-SPTs}\}$ is built from the "d_∞-cohomology."

(d_∞: for fixed dimension n, $\mathrm{d}_{\gg n} = 0$. built from: the d_∞ cohomology gives the ambiguity in choosing ρ_i, not ρ_i itself.)

Vocabulary: The sequence d_* forms the Atiyah–Hirzebruch spectral sequence (AHSS) $\mathrm{H}^\bullet(\mathrm{B}G, \mathrm{pt}; I^\bullet) \Rightarrow \mathcal{I}^\bullet(\mathrm{B}G, \mathrm{pt})$.

Generalization: You can also consider SPTs for groups G that act nontrivially on $\mathcal{I}^\bullet$. For example, supposed G includes time-reversing symmetries, encoded by $G \xrightarrow{\tau} \mathbb{Z}_2^T \to \mathrm{Aut}(\mathcal{I}^\bullet)$. The set of G-SPTs is the τ-twisted relative cohomology $\mathcal{I}_\tau^\bullet(\mathrm{B}G, \mathrm{pt})$.

There is an AHSS $\mathrm{H}_\tau^\bullet(G; I^\bullet) \Rightarrow \mathcal{I}_\tau^\bullet(G)$.

Warning: Due to the twisting, $\mathcal{I}_\tau^\bullet(\mathrm{B}G) \neq \mathcal{I}_\tau^\bullet(\mathrm{B}G, \mathrm{pt}) \oplus \mathcal{I}^\bullet(\mathrm{pt})$. Rather, there is a long exact sequence

$$\cdots \to \mathcal{I}^{\bullet-1}(\mathrm{pt}) \to \mathcal{I}_\tau^\bullet(\mathrm{B}G, \mathrm{pt}) \to \mathcal{I}_\tau^\bullet(\mathrm{B}G) \to \mathcal{I}^\bullet(\mathrm{pt}) \to \dots$$

Warning: There is more data in a G-action of $\mathcal{I}^\bullet$ then just the action on homotopy groups $I^\bullet = \pi_0\mathcal{I}^\bullet$. For example, there are two different time-reversing actions on $\{$fermionic invertible phases$\}$, corresponding to $T^2 = +1$ or $T^2 = (-1)^f$. The difference appears in the d_2 differential for the AHSS.

Thank you!

Further details:

[arxiv:1712.07950] JF–Gaiotto. Symmetry protected topological phases and generalized cohomology. *JHEP*, May 2019.

[these slides] http://categorified.net/ASQC4.pdf

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