Way too long. Think about how to cut down

Categorified Algebraic Closure

ATCAT 15 Feb 2022, Theo Johnson-Freyd

Given a com ring $\mathbb{K}$, recall that a ^com^ $\mathbb{K}$-algebra is simply a com ring R equiped with a map $\mathbb{K} \to R$.
Given a com $\mathbb{K}$-alg R and an element $r \in R$, we get a canonical map $\mathbb{K}[x] \xrightarrow{ev_r} R$, $x \mapsto r$.

<u>Defn:</u> $r \in R$ is <u>algebraic</u> over $\mathbb{K}$ if ev_r has kernel.
Algebra R is <u>algebraic</u> over $\mathbb{K}$ if every elt is.
In other words, r "solves a nontrivial poly equation".
In other words, algebraic algebras are built out of proper quotients $\mathbb{K}[x]/I$. Better: R is a colimit of rings of the form $\mathbb{K}[x]/(f)$ where you adjoin roots of polys.

A usual course on Galois thy is about such extensions where both $\mathbb{K}, R$ are fields. In a usual course, special emphasis is placed on the <u>separable</u> extensions, which is when $\forall r \in R$, "r solves an eqn w/ no repeated roots". Here is an equivalent condition: R is <u>separable</u> over $\mathbb{K}$ if the multiplication map

$$m : R \underset{\mathbb{K}}{\otimes} R \to R \qquad \text{i.e. } m\Delta = id_R$$

admits a one-sided inverse $\Delta : R \to R \underset{\mathbb{K}}{\otimes} R$ which is <u>R-bilinear</u>. — i.e. $\Delta(abc) = (a\otimes 1)\Delta(b)(1\otimes c)$.
In other words, R is <u>Frobenius</u>. (Exercise: Δ is coassoc.)
So we have categories

$$\text{sep'l exts} \subset \text{Algebraic extensions} \subset \mathrm{Com}_{\mathbb{K}}$$

Initial object $= \mathbb{K}$. Terminal object $= 0$.