Joint work (in progress) with Dan Berwick-Evans.

1. Review: Lie algebroids and Q-manifolds

manifolds : algebras :: Q-manifolds : dg algebras

Defn: A *Q-manifold* is a ($\mathbb{Z}/2$) supermanifold along with a bosonic vector field $\mathbf{n}$ and a fermionic vector field $\mathbf{q}$, such that eigenvalues of $\mathbf{n}$ are in $\mathbb{Z}$, $[\mathbf{q},\mathbf{q}] = 0$, and $[\mathbf{n},\mathbf{q}] = \mathbf{q}$, plus technical conditions. $\mathbf{n}, \mathbf{q}$ generate (left) action by Lie supergroup $\mathbb{R}^\times \ltimes \mathbb{R}^{0|1}$.

$\mathbf{n}$-eigenvalue = *cohomological degree*.

If X is a Q-manifold, $\mathscr{C}^\infty(X)$ is $\mathbb{Z}$-graded (by $\mathbf{n}$) and $\mathbf{q}$ makes it a dg algebra. Define $\mathrm{H}^\bullet(X) = \mathrm{H}^\bullet(X,\mathbf{q}) = \mathrm{H}^\bullet(\mathscr{C}^\infty(X),\mathbf{q})$.

Eg: If X is a manifold, $\mathrm{spec}(\Omega^\bullet_{\mathrm{dR}}(X), \mathrm{d})$ is Q-manifold.

Defn: A morphism $f : X \to Y$ of Q-manifolds is a *quasi-isomorphism* if $\mathrm{H}^\bullet(f)$ is an isomorphism. $f : X \leftrightarrow Y : g$ are *quasi-inverse* if $\mathrm{H}^\bullet(f \circ g) = \mathrm{id}_{\mathrm{H}^\bullet(Y)}$ and $\mathrm{H}^\bullet(g \circ f) = \mathrm{id}_{\mathrm{H}^\bullet(X)}$.

Eg: $\underline{\mathrm{Aut}}(\mathbb{R}^{0|1}) = \mathbb{R}^\times \ltimes \mathbb{R}^{0|1} \Rightarrow \mathbb{R}^{0|1}$ is Q-manifold. $\emptyset \to \mathbb{R}^{0|1}$ is quasi-isomorphism but not quasi-invertible.

Lie algebras : groups :: Lie algebroids : groupoids

Defn: A *Lie algebroid* is a vector bundle $A \to X$, a Lie algebra structure over $\mathbb{R}$ on $\Gamma(A)$, and a vector bundle morphism $\rho : A \to \mathrm{T}X$, such that the *Leibniz rule* holds:

$$[a, fb] = f[a,b] + \rho(a)[f]\, b, \quad a, b \in \Gamma(A),\ f \in \mathscr{C}^\infty$$

Eg: Bundles of Lie algebras. Lie algebra actions: $\mathfrak{g} \curvearrowright X \rightsquigarrow A = \mathfrak{g} \times X$. Integrable distibutions.

Defn (Beilinson–Bernstein): A *module* of A is (roughly) a vector bundle $V \to X$ and a Lie algebra action $\Gamma(A) \curvearrowright \Gamma(V)$, with a Leiniz rule. (Better: Use quasicoh sheaves.)

Eg: Trivial line $\mathscr{C}^\infty \to X$ is A-module. $A \to X$ is (usually) not an A-module (no "adjoint action").

Defn: The *coarse quotient* X/A = spec(A-invariant functions on X) = $\mathrm{spec}(\mathrm{Hom}_{A\text{-mod}}(\mathscr{C}^\infty, \mathscr{C}^\infty))$. The *derived quotient* $X//A$ is what you get when you replace Hom with its right-derived functor. It is some sort of dg space, defined only up to quasi-inverse.

Fact: $X//A$ can be presented by a Q-manifold, with underlying supermanifold ΠA = the "parity-reverse total space" (make the fibers fermionic). (Idea: Chevalley-Eilenberg complex.) $A \to \Pi A$ is full faithful functor {Lie algebroids} → {Q-manifolds}.

This is a version of Quillen's theory relating CDGAs to DGLAs, and is a version of Koszul duality.

Eg: $X_{\mathrm{dR}} = \Pi \mathrm{T}X = \mathrm{spec}(\Omega^\bullet(X))$ presents $X//\mathrm{T}X$. $\mathrm{d} \Leftrightarrow$ canonical vector field on $\mathrm{T}X$.

2. What should be an integral on $X//A$?

Eg: G = compact Lie group, X = compact manifold, $G \curvearrowright X$. Then expect $\int_{X/G} = \frac{1}{\mathrm{Vol}\, G}\int_X$. So a "measure" on X/G is a ratio: G-invariant measure on X / Haar measure on G (=ad-invariant measure on $\mathfrak{g}$).

Fact (Weinstein): If $A \to X$ is a Lie algebroid, then the line $\bigwedge^{\mathrm{rank}\, A} A \otimes \bigwedge^{\dim X} \mathrm{T}^* = \frac{\bigwedge^{\dim X} \mathrm{T}^*}{\bigwedge^{\mathrm{rank}\, A} A^*}$ is an A-module.

Defn (Weinstein): A *measure* on a Lie algebroid $A \to X$ is an A-invariant global section of $\bigwedge^{\mathrm{rank}\, A} A \otimes \bigwedge^{\dim X} \mathrm{T}^*$.

Fact: A measure on $A \to X$ determines a $\mathbf{q}$-invariant Berezinian measure on ΠA.

But it is defective:

1. Not $\mathbf{n}$-invariant: cohomological degree $= -\,\mathrm{rank}\, A$. So integrates all functions on X/A to 0.
2. Want to study $\int \exp(\frac{1}{\hbar}s)$ in limit as $\hbar \to 0$ by localizing near critical points of $s \in \mathscr{C}^\infty(X/A)$. Even if s has unique nondegenerate critical point in smooth part of X/A, critical points of s extended ($\mathbf{q}$-invariantly) to ΠA comprise a ($\mathrm{rank}\, A | \mathrm{rank}\, A$)-dimensional sub-supermanifold of ΠA. Unacceptable in infinite dimensions.

2'. Integrals of $\mathbf{q}$-invariant functions on ΠA, if they were not zero, never converge absolutely.

3. The BRST argument: solving defects 2 and 2'

Becchi–Rouet–Stora 1974-6, Tyutin 1975.

Suppose we have Q-manifold $(B, \mathbf{n}, \mathbf{q})$ presenting $X//A$, with a $\mathbf{q}$-invariant $\mathbf{n}$-invariant Berezinian measure μ "encoding" the chosen Lie algebroid measure. Suppose $\mathscr{C}^\infty(B)$ includes functions in cohomological degree $= -1$.

We choose an *action* $s \in \mathscr{C}^\infty(B)$ with $\mathbf{q}s = \mathbf{n}s = 0$ (i.e. $s \in \mathrm{H}^0(B) = \mathscr{C}^\infty(X/A)$). We want $\int_B \exp(s)\,\mu$.

Arbitrarily choose $f \in \mathscr{C}^\infty(B)$ with $\mathbf{n}f = (-1)f$.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_B \exp(s+\lambda\mathbf{q}f)\,\mu = \int_B \frac{\mathrm{d}}{\mathrm{d}\lambda}\exp(s+\lambda\mathbf{q}f)\,\mu$$

$$= \int_B \exp(s+\lambda\mathbf{q}f)\,(\mathbf{q}f)\,\mu$$

$$= \int_B \mathcal{L}_{\mathrm{q}}\big(\exp(s+\lambda\mathbf{q}f)\,\mu\big) \quad \text{by } \mathbf{q}\text{-invariance of } s, \mu$$

$$= 0 \quad \text{by Stoke's theorem.}$$

(Last line requires conditions "at ∞".)

It often happens that even if s has large critical locus and $\int \exp(s)\mu$ fails to converge absolutely, nevertheless can find f so that $s + \mathbf{q}f$ has isolated critical points and $\int \exp(s+\mathbf{q}f)\mu$ converges absolutely.

Defn: $s_{\mathrm{GF}} = s + \mathbf{q}f$ is the *gauge-fixed action.*

4. The FP construction: solving defect 1

Faddeev–Popov 1967.

Fact: $X \mapsto X_{\mathrm{dR}} = (\Pi \mathrm{T}X, \mathbf{q}{=}d)$ is right-adjoint to "forget $\mathbf{q}$" : {Q-manifolds} $\to$ {$\mathbb{Z}$-graded manifolds}.

Defn (Mackenzie): $f : Y \to X$ is submersion of $\mathbb{Z}$-graded manifolds, ΠA is Q-manifold, $\Pi A \to X$ is map of $\mathbb{Z}$-graded manifolds. The *Q-pullback* (*double pullback*, *Lie algebroid pullback*) of ΠA along f is the pullback $B = Y_{\mathrm{dR}} \times_{X_{\mathrm{dR}}} \Pi A$ in {Q-manifolds}:

$$\begin{array}{ccc} & & \Pi A \\ & & \downarrow \\ Y & \xrightarrow{f} & X \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} B & \xrightarrow{\tilde f} & \Pi A \\ \downarrow & & \downarrow \\ Y_{\mathrm{dR}} & \xrightarrow{\Pi \mathrm{T} f} & X_{\mathrm{dR}} \end{array}$$

Locally, $Y = F \times X \Rightarrow Y_{\mathrm{dR}} = F_{\mathrm{dR}} \times X_{\mathrm{dR}} \Rightarrow B = F_{\mathrm{dR}} \times \Pi A$.

Fact: If $\Pi A \to X$ does come from Lie algebroid and Y is classical, then B comes from Lie algebroid over Y.

Fact (Mackenzie, García-Saz–Mehta): If $Y \to X$ is vector bundle, then so is $B \to \Pi A$ ("Type 1 VBLA").

Fact (—, DBE): If $Y \to X$ is ($\mathbb{Z}$-graded) vector bundle, then $B \to \Pi A$ has quasi-inverse the zero section.

Cor: ΠA presents $X//A \Rightarrow$ so does B.

$\mathrm{T}F \to F$ has canonical Lie algebroid measure $\Rightarrow \Pi \mathrm{T}F$ has canonical Berezinian $\Rightarrow B$ has product measure $=$ canonical on $F_{\mathrm{dR}} \times \mu$ on A.

Given Lie algebroid $A \to X$ with measure, denote by $\Pi^{-1}A$ the supermanifold ΠA with opposite $\mathbf{n}$. Let $Y \to X$ be a graded vector bundle locally isomorphic to $\Pi^{-1}A \to X$. Then the $\mathbf{q}$-invariant Berezinian measure on B is also $\mathbf{n}$-invariant.

A choice ($Y, f \in \mathscr{C}^\infty(B)$ in coh degree $= -1$) is a *gauge-fixing* of $\int_{X//A} \exp(s)$.

Defn: The *Faddeev–Popov gauge-fixing* presents $X//A$ as the Q-pullback of $\Pi A \to X$ along $\Pi^{-1}A^* \to X$.

A section $f \in \Gamma(A)$ determines $f \in \mathscr{C}^\infty(\Pi^{-1}A^*)$ in cohomological degree $= -1$. The critical locus of $s_{\mathrm{GF}} = s + \mathbf{q}f$ is the intersection (in X) of the critical locus of s with the zero locus of f. Formally (does not converge absolutely, and probably should include $\sqrt{-1}$):

$$\int_B \exp(s_{\mathrm{GF}}) = \int_X \exp(s) \times \delta(f) \times \text{Jacobi}.$$

5. $\mathrm{Vol}(X//\mathrm{T}X)$ and Chern–Gauss–Bonnet

$X//\mathrm{T}X$ is some sort of "zero-dimensional stack", and so should have "counting measure". $\mathrm{Vol}(X//\mathrm{T}X) = \int e^0$ for this measure. $\mathrm{T}X \to X$ does have its canonical Lie algebroid measure. We gauge fix this integral.

Let $X =$ compact manifold. Choose $Y = \Pi^{-1}\mathrm{T}X$. Then $B = Y_{\mathrm{dR}} = \Pi \mathrm{T}(\Pi^{-1}\mathrm{T}X)$. There is canonical map $B \to (\Pi \mathrm{T} \oplus \Pi^{-1}\mathrm{T})X$; fiber is affine modeled on $\mathrm{T}X$.

How to find $f \in \mathscr{C}^\infty(B)$ in coh degree $= -1$? B has another vector field $\bar{\mathbf{q}} =$ de Rham d on $\Pi^{-1}\mathrm{T}X$, satisfying $[\mathbf{n}, \bar{\mathbf{q}}] = -\bar{\mathbf{q}}$. So $\bar{\mathbf{q}}(f)$ is in coh degree $= -1$ if f in coh degree $= 0$.

Choose (positive-definite) metric g on X. It defines a pairing $\Pi \mathrm{T} \otimes \Pi^{-1}\mathrm{T} \to \mathscr{C}^\infty$, and hence a function "$\frac{1}{2} g(x)\psi\bar\psi$" on $(\Pi \mathrm{T} \oplus \Pi^{-1}\mathrm{T})X$, which pulls back to B. One calculates:

$$\frac{1}{2}\mathbf{q}\bar{\mathbf{q}}(g(x)\psi\bar\psi) = -\frac{1}{2}g(x)b^2 + (\star)\psi\bar\psi b + (\star)\psi\bar\psi\psi\bar\psi$$

b is affine coordinate on the fiber of $B \to (\Pi \mathrm{T} \oplus \Pi^{-1}\mathrm{T})X$, and $(\star)$s depend on derivatives of g.

Thus fiber integral $\int \exp(s_{\mathrm{GF}}) \mathrm{d}b$ converges absolutely. Gaussian integrals are easy, and

$$\int_B \exp(s_{\mathrm{GF}}) = \int_{(\Pi \mathrm{T} \oplus \Pi^{-1}\mathrm{T})X} \exp(R\psi\bar\psi\psi\bar\psi)\sqrt{g}$$

$R =$ Riemann curvature 4-tensor. Integrating out the odd fibers gives:

$$= \int_X \mathrm{Pf}(R).$$

Alternately, try to use Faddeev–Popov style gauge fixing. Any $\alpha \in \Omega^1(X)$ defines coh-degree $= -1$ function "$\alpha(x)\bar\psi$" on B. One calculates:

$$\mathbf{q}(\alpha(x)\bar\psi) = \alpha(x)b + \partial\alpha(x)\psi\bar\psi$$

Formally $\int_B \exp(\mathbf{q}(\alpha(x)\bar\psi))$ counts zeros of α (via delta function argument).

This integral does not converge absolutely, but

$$\int_B \exp(\hbar\mathbf{q}\bar{\mathbf{q}}(g(x)\psi\bar\psi) + \mathbf{q}(\alpha(x)\bar\psi))$$

converges absolutely for $\hbar > 0$. In $\hbar \to 0$ limit, $\int$ localizes at zeros of α. If zeros of α are nondegenerate,

$$\int_B \exp(s_{\mathrm{GF}}) = \sum_{\alpha(x)=0} \mathrm{sign}(\partial\alpha)$$

Thus we've proved:

Fact (Chern–Gauss–Bonnet):

$$\mathrm{Vol}(X//\mathrm{T}X) = \int_X \mathrm{Pf}(R) = \sum_{\alpha(x)=0} \mathrm{sign}(\partial\alpha).$$

Credit where it's due: above calculations (when $\alpha = \mathrm{d}h$ for h a Morse function, whence $\partial\alpha = \mathrm{hess}(h)$) entirely due to DBE in his thesis. `http://math.berkeley.edu/~devans/CGB_Draft.pdf`

Heuristic argument: $X//\mathrm{T}X$, hence $\mathrm{Vol}(X//\mathrm{T}X)$, is a homotopy invariant. $\mathrm{Vol}(\mathrm{pt}//\mathrm{Tpt}) = 1$. Properties of integrals: inclusion/exclusion formula for unions and multiplicative for fiber bundles. $\Rightarrow \mathrm{Vol}(X//\mathrm{T}X) =$ Euler characteristic of X. $\square$