Goal: Describe integration algebraically. Use as *definition* for generalized manifolds (stacks, ∞-dim, etc.).

1. Expectation values as homological algebra

[Witten, A Note on the Antibracket Formalism, 1990]

X = compact manifold. Choose measure μ.

Want *expectation value* $\langle f\rangle_\mu = \frac{\int_X f\mu}{\int_X \mu}$, $f \in \mathscr{C}^\infty(X)$.

Observation: $\int$: Chains$(X) \to \mathbb{R}$ is almost completely determined by requirement that it be a chain map.

Defn: $\mathrm{MV}^\bullet(X) = \Gamma(\bigwedge^\bullet \mathrm{T}X)$, *multivector fields*.

Fact: If μ nowhere-vanishing, then "contract with μ": $\mathrm{MV}^\bullet \to \Omega^{\dim X - \bullet}$ is iso of graded vector spaces.

Defn: $\Delta_\mu = \mu^{-1} \circ \mathrm{d} \circ \mu : \mathrm{MV}^\bullet(X) \to \mathrm{MV}^{\bullet-1}(X)$, *divergence with respect to μ*.

Fact: $(\mathrm{MV}^\bullet(X), \Delta_\mu)$ is a model of Chains(X).

Cor: $f \mapsto \int_X f$ is almost completely determined by requirement that it extends to chain map $\mathrm{MV}^\bullet \to \mathbb{R}$.

Cor: If X is connected then $\langle\cdot\rangle_\mu$ is determined by requirement that it be a chain map and that $\langle 1\rangle = 1$.

Remark: $\mu \mapsto \Delta_\mu$ looses data: $\Delta_{a\mu} = \Delta_\mu$ for $a \in \mathbb{R}^\times$.

2. Some Gerstenhaber geometry

Fact: $\mathrm{MV}^\bullet(X)$ is a $\mathscr{C}^\infty(X)$-module. Δ_μ is a derivation of $\mathscr{C}^\infty(X)$ modules.

Question: $\mathrm{MV}^\bullet(X)$ is a graded commutative algebra. Is $(\mathrm{MV}^\bullet, \Delta_\mu)$ a dga? I.e. is Δ_μ a derivation of $m = \wedge$?

Answer: No. Let $f, g \in \mathrm{MV}^\bullet$ be homogeneous. The *failure of Δ_μ to be a derivation* is $[\Delta_\mu, m]$:

$$f \otimes g \mapsto \Delta_\mu(fg) - \big((\Delta_\mu f)g + (-1)^{|f|\cdot|\Delta_\mu|} f(\Delta_\mu g)\big)$$

(Extended linearly. Note: $|\Delta_\mu| = -1$.) Since m is (graded) commutative, $[\Delta_\mu, m]$ is (graded) symmetric.

Fact: $[\Delta_\mu, m]$ is a *biderivation*:

$$[\Delta_\mu, m](fg, h) = (-1)^{|f|\cdot|[\Delta_\mu, m]|} f\,[\Delta_\mu, m](g,h) + \\ + (-1)^{|g|\cdot|h|}[\Delta_\mu, m](f,h)\,g$$

I.e. Δ_μ is a *second-order diff. op.* on $(\mathrm{MV}^\bullet, m = \wedge)$. $[\Delta_\mu, m]$ is its *principal symbol*.

Fact: You've met $[\Delta_\mu, m]$ before: it is the *Gerstenhaber* or *Schouten–Nijenhuis bracket* $\mathcal{P}$ on $\mathrm{MV}^\bullet$, i.e. the extension (as a biderivation) of $[,] : \mathrm{MV}^1 \otimes \mathrm{MV}^1 \to \mathrm{MV}^1$ to all of $\mathrm{MV}^\bullet$. $\mathcal{P}$ satisfies Jacobi identity. $|\mathcal{P}| = -1$.

manifolds : commutative algebra :: supermanifolds : graded commutative algebra

Eg: $\Pi\mathrm{T}^*X$ = "manifold" with $\mathscr{C}^\infty(\Pi\mathrm{T}^*X) = \mathrm{MV}^\bullet(X)$. S–N bracket = "symplectic" structure on $\Pi\mathrm{T}^*X$.

For hands-on understanding, choose coordinates x^i on X. Get local sections $\partial_i = \pi_i \in \Gamma(\mathrm{T}X) = \mathrm{MV}^1(X)$. These are "linear functions" on fibers of $\Pi\mathrm{T}^*X$. Algebraically, S–N bracket is

$$\mathcal{P} = \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial \pi_i} + \frac{\partial}{\partial \pi_i} \otimes \frac{\partial}{\partial x^i}.$$

If we choose x^i so that $\mu = \mathrm{d}x^1 \cdots \mathrm{d}x^{\dim X}$, then

$$\Delta_\mu = \frac{\partial^2}{\partial x^i \partial p_i}.$$

Defn: A *BV Laplacian* is a second-order diff. op. Δ on $\Pi\mathrm{T}^*X$ such that:

0. $[\Delta, 1] = \Delta(1) = 0$
1. $[\Delta, m] = \mathcal{P}$
2. $[\Delta, \mathcal{P}] = 0$
3. $[\Delta, \Delta] = 0$

(Should also require $|\Delta| = -1$; then 4. is automatic for $\mathrm{MV}^\bullet$. We include it in case X is already "super".)

Cor: By 0. and 1., two BV Laplacians differ by a vector field (= derivation of $\mathrm{MV}^\bullet$). By 2. this v-field is "symplectic v-field." For classical X, 3. is then automatic.

Thm (Koszul): There is canonical bijection

$$\{\text{BV Laplacians}\} = \{\text{flat connections on } \textstyle\bigwedge^{\dim}\mathrm{T}^*\}.$$

(Flat $\Leftrightarrow \Delta^2 = 0$. $\{$connections$\}$ = satisfies 0,1,2.)

3. Some derived geometry

How does Δ_μ change under $\mu \rightsquigarrow \exp(s)\mu$? Must change by symplectic vector field. Not too surprisingly:

Fact: $\Delta_{\exp(s)\mu} = \mathcal{P}(s,-) + \Delta_\mu$.

Often want to understand $\langle\cdot\rangle$ against measure $\exp(\frac{1}{\hbar}s)\mu$ (maybe with some $\sqrt{-1}s$). If $\hbar$ is invertible, homology for $\mathcal{P}(\frac{1}{\hbar}s,-) + \Delta_\mu$ is the same as for $\mathcal{P}(s,-) + \hbar\Delta_\mu$. The latter feels better if $\hbar \ll 1$.

In limit $\hbar \to 0$, consider $\big(\mathrm{MV}^\bullet, \mathcal{P}(s,-)\big)$. Note: $\mathcal{P}(s,-)$ is derivation, so $\big(\mathrm{MV}^\bullet, \mathcal{P}(s,-)\big)$ is dg commutative algebra, i.e. makes $\Pi\mathrm{T}^*X$ into *Q-manifold*.

Fact: $\mathrm{H}^0\big(\mathrm{MV}^\bullet, \mathcal{P}(s,-)\big) = \mathscr{C}^\infty(\{\mathrm{d}s = 0\})$.

Fact: $\big(\Pi\mathrm{T}^*X, \mathcal{P}(s,-)\big)$ is the *derived critical locus* of s, i.e. the *derived* intersection $\{p = \mathrm{d}s\} \cap_{\mathrm{T}^*X} \{p = 0\}$.

Why? Intersection $\leftrightarrow \otimes$, and derived intersection uses left-derived functor of $\otimes$. We should "resolve" the zero section $X \hookrightarrow \mathrm{T}^*X$, and then tensor with $\mathscr{C}^\infty(\{p = \mathrm{d}s\})$. One resolution is $X \simeq (\mathrm{T} \oplus \Pi\mathrm{T})X$, with dg structure $p_i \frac{\partial}{\partial \pi_i}$ (i.e. identity: $\mathrm{T} \to \Pi\mathrm{T}$). Intersection is

$$\mathscr{C}^\infty\big((\mathrm{T}^* \oplus \Pi\mathrm{T}^*)X, p_i\tfrac{\partial}{\partial\pi_i}\big) \underset{\mathscr{C}^\infty(\mathrm{T}^*X)}{\otimes} \mathscr{C}^\infty\big(\{p_i = \tfrac{\partial s}{\partial x^i}\}\big)$$
$$= \mathscr{C}^\infty(\Pi\mathrm{T}^*X) \text{ with dg structure } \tfrac{\partial s}{\partial x^i}\tfrac{\partial}{\partial \pi_i} = \mathcal{P}(s,-).$$

Expect that $\mathcal{P}(s,-) + \hbar\Delta$ is "controlled" by $\mathcal{P}(s,-)$, when $\hbar \approx 0$. (E.g. related by spectral sequence.) This is a version of statement that oscillating integrals localize near critical points.

4. Feynman diagrams

[Gwilliam and —, `http://math.berkeley.edu/~theojf/BVexample.pdf`]

X = formal manifold, i.e. $\mathscr{C}^\infty(X) = \mathbb{R}[\![x^1,\dots,x^N]\!]$. Suppose s has nondegenerate critical point at 0, i.e. $s = -\frac{1}{2}a_{ij}x^ix^j + b(x)$ with: $b \in I^3$, I = ideal gen. by $\{x^1,\dots,x^n\}$; a = invertible matrix; $\mu = \mathrm{d}x^1\dots\mathrm{d}x^N$.

$$D = \mathcal{P}(s,-) + \hbar\Delta_\mu = -a_{ij}x^i\frac{\partial}{\partial\pi^j} + \frac{\partial b}{\partial x^i}\frac{\partial}{\partial \pi_i} + \hbar\frac{\partial^2}{\partial x^i\partial\pi_j}$$

acts on $\mathbb{R}[\![x^i,\pi_i,\hbar]\!]$ with $|x^i| = |\hbar| = 0$, $|\pi_i| = 1$.

Write $[f]$ for class of f in homology. Ansatz (spectral sequence): $\mathrm{H}^0\big(\mathbb{R}[\![x^i,\pi_i,\hbar]\!], D\big) \cong \mathbb{R}[\![\hbar]\!]$. $\langle f\rangle = [f]/[1]$.

Eg ($N=1$, $b=0$): Then $\mathrm{MV}^\bullet = \mathbb{R}[\![x,\hbar]\!]\pi \oplus \mathbb{R}[\![x,\hbar]\!]$. Differential is $D = -ax\frac{\partial}{\partial\pi} + \hbar\frac{\partial^2}{\partial x\partial\pi}$.

ker D? $f \in \mathbb{R}[\![x,\hbar]\!]$, then $f\pi \in \ker D$ iff $-axf' + \hbar f = 0$ iff $f = \exp(ax^2/2\hbar) \notin \mathbb{R}[\![x,\hbar]\!]$.

im D? Profinite closure of D(polys). $D(x^n\pi) = -ax^{n+1} + \hbar n x^{n-1} \Rightarrow [x^{n+1}] = \frac{\hbar}{a}n[x^{n-1}] \Rightarrow [x^{2n+1}] = 0$,

$$\langle x^{2n}\rangle = \left(\frac{\hbar}{a}\right)^n(2n-1)!! = \left(\frac{\hbar}{a}\right)^n\frac{(2n)!}{2^n\,n!}.$$

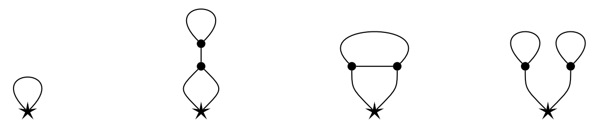
This is *Wick's formula*. □

Eg ($N=1$, $a=1$, $b=x^3/6$): $[D(x^n)] = 0 \Rightarrow$

$$[x^{n+1}] = \frac{1}{2}[x^{n+2}] + \hbar\, n\,[x^{n-1}].$$

$[x] = \frac{1}{2}[x^2] = \frac{1}{2}\big(\frac{1}{2}[x^3] + \hbar[1]\big) = \frac{1}{2}\big(\frac{1}{2}\big(\frac{1}{2}[x^4] + 2\hbar[x]\big) + \hbar[1]\big) = \dots$ so $\langle x\rangle = \frac{\hbar}{2} + \dots$ and converges in profinite topology. To organize the combinatorics:

A *Feynman diagram for* $\langle x^n\rangle$ is a connected finite graph Γ with a distinguished n-valent vertex (totally ordered incident half-edges) and $v(\Gamma)$ 3-valent vertices (unordered incident half-edges). *1st Betti number* is $\beta(\Gamma) = (v(\Gamma)+n)/2$. The graphs for $\langle x^2\rangle$ with $\beta = 1,2$ are:



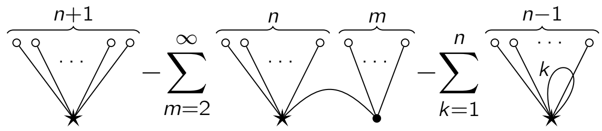
$|\mathrm{Aut}| = 1$ $|\mathrm{Aut}| = 2$ $|\mathrm{Aut}| = 2$ $|\mathrm{Aut}| = 4$

Claim: $\displaystyle\langle x^n\rangle = \sum_{\substack{\Gamma\text{ a Feynman}\\ \text{diagram for }\langle x^n\rangle}} \frac{\hbar^{\beta(\Gamma)}}{|\mathrm{Aut}\,\Gamma|}.$

Proof: Write c_n for RHS. $c_0 = 1$. So must verify recursion $c_{n+1} = \frac{1}{2}c_{n+2} + \hbar n c_{n-1}$. Let Γ be diagram for $\langle x^{n+1}\rangle$. Walk along last half edge from $\star$: either (a) you return to $\star$ or (b) you hit a vertex. If (a), delete this half-edge, producing diagram for $\langle x^{n-1}\rangle$ — there were n ways to produce said diagram, and it costs $\hbar$. If (b), unzip this edge, producing diagram for $\langle x^{n+2}\rangle$ — costs factor of 2 in count-with-symmetry. □

Eg (general case): Can use similar diagrams. To compute $\langle f\rangle$ for $f \in \mathbb{R}[\![x^i]\!]$, allow vertices {$\star$ for Taylor coef of f, $\bullet$ for Taylor coefs of b, $\circ$ for x} and "cap" edges for a^{-1}. $\Gamma \mapsto \frac{\mathrm{ev}(\Gamma)\hbar^{\beta\Gamma}}{|\mathrm{Aut}(\Gamma)|}$, ev = contract tensors.

$D\big((a^{-1})^{ij}f_{j,\vec{k}}x^{\vec{k}}\pi_i\big) =$



is boundary. In final diagram, self-loop connects kth and $(n+1)$th half-edges on marked vertex.

Now play Hercules' game of the *many-headed hydra*: chop off the last head by either attaching it to another head (increase power of $\hbar$) or by producing a new vertex with more heads (increase power of x). Game converges to something in $\mathbb{R}[\![\hbar]\!]$ (the only hydrae with no heads to chop). Game produces all such hydrae. □

5. Homological perturbation theory

If $\{p = \mathrm{d}s\} \cap_{\mathrm{T}^*X} \{p = 0\}$ is clean, then

$$\mathrm{H}^\bullet\big(\mathscr{C}^\infty(\Pi\mathrm{T}^*X), \mathcal{P}(s,-)\big) = \big(\mathscr{C}^\infty(\Pi\mathrm{T}^*\{\mathrm{d}s = 0\}), 0\big).$$

Write $M = \mathrm{MV}^\bullet$, $L = \mathrm{H}^\bullet$, $\partial = \mathcal{P}(s,-)$. Choose ι, ϕ, η to form a *retraction*:

$$(\star)\qquad (L,0) \underset{\phi}{\overset{\iota}{\rightleftarrows}} (M,\partial) \circlearrowleft \eta \qquad \begin{aligned}\iota\phi &= \mathrm{id}_L\\ \phi\iota &= \mathrm{id}_M - [\partial,\eta]\end{aligned}$$

Can always achieve *side conditions:* $\iota\eta = \eta^2 = \eta\phi = 0$.

$\partial^2 = \mathcal{P}(s,-)^2 = 0$ is the *classical master equation*. Set $\delta = \hbar\Delta$. $(\partial+\delta)^2 = 0$ is the *quantum master equation*. The *homotopy perturbation lemma* says:

Thm: If $(\star)$ is a retraction, so is:

$$(\tilde\star)\qquad (L,\tilde\delta) \underset{(\mathrm{id}-\delta\eta)^{-1}\circ\phi}{\overset{\iota\circ(\mathrm{id}-\eta\delta)^{-1}}{\rightleftarrows}} (M,\partial+\delta) \circlearrowleft \eta(\mathrm{id}-\delta\eta)^{-1}$$

$\tilde\delta = \iota\circ\eta(\mathrm{id}-\delta\eta)^{-1}\circ\phi$. Remark: don't need $\hbar$ formal, just "small" enough for $(\mathrm{id}-\eta\delta)^{-1}$ to exist.

One choice of splitting comes from trivializing tubular nbhd of $\{\mathrm{d}s = 0\}$. $\iota \leftrightarrow$ "restriction to critical locus". $\iota\circ(\mathrm{id}-\eta\delta)^{-1}$ = integrate out the fibers. New differential $\tilde\delta$ on $L = \mathscr{C}^\infty(\Pi\mathrm{T}^*\{\mathrm{d}s = 0\})$ encodes remaining measure on critical locus (the *effective action*).

Put another way: we are interested in the (hopefully unique) chain map $\langle\cdot\rangle : (M,\partial+\delta) \to \mathbb{R}$ sending $1 \mapsto 1$. To construct it, factor through L, which is usually much smaller than M.

Difficulties: If X is stacky, $(\partial+\delta)^2 = 0$ is not automatic for $\delta = \hbar\Delta_\mu$, $\partial = \mathcal{P}(s,-)$. If X is ∞-dim'l, defining $\mathcal{P}$ requires *renormalization theory*.

C.f. [Crainic, 2004], [Costello–Gwilliam, 2011].