Asymptotics of oscillating integrals via homological perturbation theory

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Abstract

The Batalin-Vilkovisky approach to integration converts the question of computing expectation values into a question in homological algebra, and reinterprets the asymptotics of oscillating integrals in terms of (quantum) deformations of (derived) intersections. The move to homological algebra makes these computations tractable by combinatorial means — a special case includes the Feynman-diagrammatic description of Gaussian integration. In this talk, I will try to explain both the derived geometry and the homological perturbation theory. Most of this story is known to experts, and a little of it is joint work with Owen Gwilliam.

Contents

For now, let X denote a compact (finite-dimensional) real manifold without boundary. More for convenience than necessity, I will also assume that X is oriented. Then let $\mu \in \Omega^{\mathrm{top}}(X)$ be a nowhere-vanishing smooth measure on X. Our primary goal is to understand the map that sends a function $f \in \mathscr{C}^\infty(X)$ to its *expectation value* $\langle f \rangle = \frac{\int f\mu}{\int \mu}$. By "understand" I mean a number of things. For example, often it happens that f, μ depend on some parameters, and I would like to compute the asymptotics as those parameters take limiting values. I would also like definitions and formulae that lend themselves well to generalizations (smooth Artin stacks, infinite-dimensional

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manifolds, singular varieties, etc.). We will approach this question of "understanding" from a homological-algebraic point of view.

1 Expectation values as homological algebra

The beginning of Batalin and Vilkovisky's story is well-elucidated by Witten in [Notes on the antibracket, 199?]. I will write top for $\dim X$. Recall that $\Omega^{\mathrm{top}}(X)$ is really the last term in a chain complex $(\Omega^\bullet(X), \mathrm{d})$, and that $\int_X : \Omega^{\mathrm{top}}(X) \to \mathbb{R}$ is a chain map, meaning that it vanishes on exact forms. (It vanishes on all non-top forms; that it vanishes on exact top forms is Stoke's formula, since $\partial X = \emptyset$.) It is convenient to regrade $\Omega^\bullet$ so that $\int_X$ be a degree-0 map, so we will work with a complex whose degree-k part is $\Omega^{\mathrm{top}-k}(X)$. In writing $-k$ rather than k, I have also switched the sign convention, so that I am working with *homological* rather than *cohomological* gradings: the differential d decreases degree. Nevertheless, I'll still write $\mathrm{H}^\bullet$ rather than, say, $\mathrm{H}_\bullet$, for the homology of any complex.

Anyway, the fact that $\int_X$ is a chain map almost pins it down. Indeed, suppose that X is connected. Then $\mathrm{H}^0(\Omega^{\mathrm{top}-\bullet}(X))$ is one-dimensional, and so $\int_X$ is determined up to multiplication by a real scalar by the dimensions alone.

Actually, we'd really like to study the map $f \mapsto \langle f \rangle$, which is built out of $\int_X$ and a choice of measure μ. So what we'd really like is a complex whose degree-0 part is $\mathscr{C}^\infty(X)$, rather than $\Omega^{\mathrm{top}}(X)$; it should be isomorphic to $(\Omega^{\mathrm{top}-\bullet}(X), \mathrm{d})$, with an isomorphism that extends the map $f \mapsto f\mu$ from $\mathscr{C}^\infty$ to Ω^{top}. I will describe this complex in a moment, but let's see first how it "solves" the problem of computing $\langle f \rangle$. If X is connected, then H^0(this complex) will be one-dimensional, and by construction $\langle \cdot \rangle$ will be a chain map to $\mathbb{R}$. So $\langle \cdot \rangle$ will be determined up to real constant, but by definition $\langle 1 \rangle = 1$, so $\langle \cdot \rangle$ is uniquely determined. If X is not connected, then still $\langle \cdot \rangle$ consists of "finitely much" information, namely the "probability" of being in each piece of X. (The un-normalized map $f \mapsto \int f\mu$ is determined by the same complex and the "finitely much information" consisting of the "volume" of each piece of X.)

So what is the complex we will write down? Recall the algebra $\mathrm{MV}^\bullet(X) = \Gamma(\mathrm{T}^{\wedge\bullet}X)$ of *anti-symmetric multivector fields* on X. It is a $\mathbb{Z}$-graded (super) commutative algebra with $\wedge$ as the multiplication. The degree-0 elements are functions on X; the degree-1 part is vector fields; etc. Then the measure μ determines an isomorphism as graded vector spaces of $\mathrm{MV}^\bullet(X) \overset{\sim}{\to} \Omega^{\mathrm{top}-\bullet}(X)$. For $\bullet = 0$, this map is "multiply by $\mu : \mathscr{C}^\infty(X) \to \Omega^{\mathrm{top}}(X)$". When $\bullet = \mathrm{top}$, the map takes a top-degree multivector field and pairs it with μ in the antisymmetric way extending the pairing $\Gamma(\mathrm{T}X) \otimes \Omega^1(X) \to \mathscr{C}^\infty(X)$. In general, we have a map

$$\mathrm{T}^{\otimes\bullet} \otimes (\mathrm{T}^*)^{\otimes \mathrm{top}} \to (\mathrm{T}^*)^{\otimes(\mathrm{top}-\bullet)}$$

by taking the antisymmetric average over all possible contractions, but this clearly factors through the antisymmetric parts

$$\mathrm{T}^{\wedge\bullet} \otimes (\mathrm{T}^*)^{\wedge \mathrm{top}} \to (\mathrm{T}^*)^{\wedge(\mathrm{top}-\bullet)}$$

and then we plug μ into the $(\mathrm{T}^*)^{\wedge \mathrm{top}}$ slot.

In local coodinates, $\mu = \mu(x)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{\mathrm{top}}$, and it takes the section $\partial_1 \wedge \cdots \wedge \partial_k \in \Gamma(\mathrm{T}^{\wedge k})$ to $\mu(x)\,\mathrm{d}x^{k+1} \wedge \cdots \wedge \mathrm{d}x^{\mathrm{top}}$, maybe with some signs and combinatorial factors. In a bit, we will want to work with partial derivatives, so rather than using the symbol $\partial_i \in \Gamma(\mathrm{T}X)$ for the section $\frac{\partial}{\partial x^i}$, I will call this section ξ_i.

So this is our desired isomorphism $\mu : \mathrm{MV}^\bullet(X) \overset{\sim}{\to} \Omega^{\mathrm{top}-\bullet}(X)$ of graded vector spaces, and we can use it to move the differential d to $\Delta_\mu = \mu^{-1} \circ \mathrm{d} \circ \mu$ on $\mathrm{MV}^\bullet(X)$. As we said above, if X is connected, then $\langle \cdot \rangle$ is (the restriction to MV^0 of) the unique chain map $(\mathrm{MV}^\bullet(X), \Delta_\mu) \to \mathbb{R}$ that sends 1 to 1. I want to point out that the map Δ_μ is invariant under changes $\mu \mapsto a\mu$ for $a \in \mathbb{R}^\times$, because we multiply and divide by it. I'll come back to this point in a bit.

2 Some Gerstenhaber geometry

The isomorphism is not a map of algebras, and Δ_μ is not a derivation of the wedge product on $\mathrm{MV}^\bullet$. So $(\mathrm{MV}^\bullet(X), \Delta_\mu)$ is not a dga. The remarkable fact is that Δ_μ is a *second-order* differential operator, which vanishes on 1 (since μ is d-closed) — this is why I call it "Δ", and in general it is the *BV Laplacian*. Indeed, its *principal symbol* (i.e. the symmetric biderivation defined by $\mathcal{P}(f, g) = \Delta_\mu(fg) - (\Delta_\mu(f)\,g + f\,\Delta_\mu(g))$, with the appropriate signs since we're in $\mathbb{Z}$-graded/super land) is precisely the Gerstenhaber / Nijenhuis-Schouten bracket on $\mathrm{MV}^\bullet$, which is the extension qua derivation of the bracket on vector fields.

Indeed, this can be checked directly in local coordinates. It is a theorem of Moser's that for any smooth measure μ there exist local coordinates $\{x^i\}$ with $\mu = \mathrm{d}x^1 \wedge \cdots \mathrm{d}x^{\mathrm{top}}$. Locally in these coordinates, $\mathrm{MV}^\bullet(X) = \mathscr{C}^\infty(x^1, \dots, x^{\mathrm{top}})[\xi_1, \dots, \xi_{\mathrm{top}}]$, where the ξ_i variables are in homological degree 1, and hence are fermionic. Then

$$\Delta_\mu\left(f(x)\prod_{i\in I}\xi_i\right) = \mu^{-1}\,\mathrm{d}\left(f(x)\left(\prod_{i\notin I}\mathrm{d}x^i\right)\right) = \mu^{-1}\left(\sum_j \frac{\partial f}{\partial x^j}\mathrm{d}x^j \prod_{i\notin I}\mathrm{d}x^i\right) =$$

$$= \sum_j \frac{\partial f}{\partial x^i} \prod_{i\in I,\, i\neq j} \xi_i = \sum_j \frac{\partial^2}{\partial x^j \partial \xi_j}\left(f(x)\prod_{i\in I}\xi_i\right)$$

The principal symbol of $\sum_j \frac{\partial^2}{\partial x^j \partial \xi_j}$ is the symmetric biderivation $\sum_j \left(\frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial \xi_j} + \frac{\partial}{\partial \xi_j} \otimes \frac{\partial}{\partial x^j}\right)$. A little thought shows that this defines the Nijenhuis-Schouten bracket in any coordinates. (If you shift $\bullet \rightsquigarrow \bullet - 1$, so that the bracket has homological degree 0, then to do the shifting you really tensored with a one-dimensional vector space in homological degree 1. But "symmetric" for odd vector spaces is "antisymemtric" for even ones, so the bracket really does have the statistics that you think it has on the underlying vector space, but "internal to the category of graded vector spaces" it is symmetric.)

I've been writing $\mathrm{MV}^\bullet(X)$, but of course anyone in our "Witten in the 80s" seminar will recognize it as the algebra of functions on the supermanifold $\pi\mathrm{T}^*X$, the *odd cotangent bundle*. You know that

T^*X is symplectic in a canonical way. Well, πT^*X is also "symplectic", except that the "Poisson bracket" is actually this Gerstenhaber bracket $\mathcal{P}$. Then Δ_μ is a second-order differential operator on $MV^\bullet(X)$.

I will leave as an exercise checking that Δ_μ, although not a derivation of $\wedge$, is a "Lie derivation", meaning that it is a derivation of the bracket $\mathcal{P}$.

Here is a question: what happens when you switch $\mu \mapsto e^s\mu$? Then $\Delta_{e^s\mu}$ also has to satisfy the conditions above: it is a second-order operator with principal symbol $\mathcal{P}$; it is a derivation of $\mathcal{P}$; it vanishes on 1. It follows that $\Delta_{e^s\mu} - \Delta_\mu$ is a vector field on πT^*X (i.e. a derivation of $MV^\bullet(X)$ as an algebra), which should be a "symplectic vector field". An easy calculation shows that this vector field is precisely the Hamiltonian vector field $\mathcal{P}(s, -)$.

Running this argument in reverse, we see that the space of second-order operators Δ satisfying the above axioms is an affine space, and is a torsor for the space of symplectic vector fields on πT^*X. We should now remember that not all symplectic vector fields are Hamiltonian, although locally they are. Similarly, not all *BV Laplacians* (second-order differential operators Δ satisfying $\Delta(1) = 0$, Δ is a derivation of $\mathcal{P}$, and the principal symbol of Δ is $\mathcal{P}$) on πT^*X come from measures on X — the ones that do are a torsor for the Hamiltonian fields (because any two positive measures are related by $e^{\text{something}}$). This isn't too surprising: Δ_μ only sees μ up to overall scalar, and so exists even for "measures" that change by a scalar as you move around X. We have almost proved a theorem of Koszul's: the space of BV Laplacians on πT^*X is canonically isomorphic to the space of flat connections on the line bundle $(T^*)^{\wedge \text{top}}X$. (A generalization for Gerstenhaber manifolds whose algebra of functions is generated in homological degrees 0 and 1 is due to Ping Xu.)

3 Some derived geometry

One problem that often comes up in physics is to understand integrals of the form $\int_X \exp(s/\hbar)\,\mu$ for some manifold X, measure μ, and *action* $s \in \mathscr{C}^\infty(X)$, where $\hbar$ is a physical parameter. (Depending on context, maybe I want $\hbar$ to be negative or pure imaginary or ....) More often, one wants to understand expectation values $f \mapsto \langle f \rangle = \int f e^{s/\hbar}\mu / \int e^{s/\hbar}\mu$. When X is connected, we have seen that to compute such expectation values, it suffices to compute the differential on $MV^\bullet(X) = \mathscr{C}^\infty(\pi T^*X)$ that corresponds to $e^{s/\hbar}\mu$. (Well, "compute" might be an overstatement, but we'll come back to that.) Suppose that Δ_μ is "known". Then we have seen that $\Delta_{\exp(s/\hbar)\mu} = \mathcal{P}(\hbar^{-1}s, -) + \Delta_\mu$.

All we really care about is $(MV^\bullet(X), \mathcal{P}(\hbar^{-1}s, -) + \Delta_\mu)$ as a chain complex. So if $\hbar$ is invertible, we can just as easily compute the complex with differential $\mathcal{P}(s, -) + \hbar\Delta_\mu$. If $\hbar \gg 1$, then $\hbar^{-1}\mathcal{P}(s, -) + \Delta$ seems like the correct differential to use. If $\hbar \ll 1$, then $\mathcal{P}(s, -) + \hbar\Delta_\mu$ seems more reasonable. Let's focus on the situation when $\hbar \ll 1$. For example, perhaps we'd like to compute explicitly the $\hbar \to 0$ asymptotics of $\langle \cdot \rangle$.

As a warm-up, what happens when $\hbar = 0$? I.e. what is the complex $(MV^\bullet(X), \mathcal{P}(s, -))$ for $s \in \mathscr{C}^\infty(X)$? Recall that $MV^\bullet(X) = \mathscr{C}^\infty(\pi T^*X)$ is a commutative (super) algebra. Since $\mathcal{P}(s, -)$

is a derivation (it is the Hamiltonian vector field for s), it makes $\pi\mathrm{T}^*X$ into a *Q-manifold*, and an object of (commutative) derived geometry, which is the "algebraic geometry" of commutative dgas.

The word *derived* in "derived geometry" means to do algebraic geometry, but every time you see a tensor product, take the left derived tensor product instead. What I mean is the following. Suppose that X, Y, Z are affine schemes, with closed embeddings $X \hookrightarrow Y$ and $Z \hookrightarrow Y$. Then the *intersection* $X \cap_Y Z$ is precisely the tensor product $\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z)$. Indeed, intersections are a form of fibered product, and we will start to conflate the two. In the derived world, you do not take intersections, but *derived intersections*.

I claim that $\pi\mathrm{T}^*X$ with the Q-structure $\mathcal{P}(s, -)$ is an example of a derived intersection of two Lagrangians in a symplectic manifold: namely, it is the intersection of the zero section $X \hookrightarrow \mathrm{T}^*X$ with the section $\mathrm{d}s : X \hookrightarrow \mathrm{T}^*X$. The derived intersection is precisely (spec of) the derived tensor product

$$\mathscr{C}^\infty(\{p = 0\}) \otimes^{\mathbb{L}}_{\mathscr{C}^\infty(\mathrm{T}^*X)} \mathscr{C}^\infty(\{p = \mathrm{d}s\})$$

where p is short for the fiber coordinate. To compute this, we should resolve one of the two tensorands by projectives.

As a warm-up, let's resolve $\{\mathrm{pt}\} \to V$ by projectives, where $\{\mathrm{pt}\}$ is mapping to the origin in the vector space V. This corresponds to the $\mathscr{C}^\infty(V)$-module structure on $\mathbb{R}$ in which the generators $v^1, \dots, v^{\dim}$ act by 0. (I have picked a basis $v_1, \dots, v_{\dim}$ of V, and the dual basis is $v^1, \dots, v^{\dim}$ in V^*.) Because I'm not very good at algebra, I'll just use free modules for my projective modules. Then in homological degree 0 I'll put $\mathscr{C}^\infty(V)$, the free module on the generator $1 \in \mathbb{R}$. I want the homology to be $\mathbb{R} = (v^i = 0)$, so in homological degree 1 I'll put the free $\mathscr{C}^\infty(v^1, \dots, v^{\dim})$-module on the basis $\phi^1, \dots, \phi^{\dim}$, where the ϕ^i are dummy variables with homological degree 1, and the differential I'll use is $\phi^i \mapsto v^i$. Ok, this still have kernel, of the form $x^i v^j - x^j v^i$. So in homological degree 2 I'll put the module generated by symbols $\{\phi^i\phi^j\}$, with the convention that $\phi^i\phi^j = -\phi^j\phi^i$, and the differential sends $\phi^i\phi^j = -\phi^j\phi^i$ to $x^i v^j - x^j v^i$. You can see where I'm going. At the end, my complex is precisely $\mathscr{C}^\infty(v^1, \dots, v^{\dim})[\phi^1, \dots, \phi^{\dim}]$ as a graded commutative algebra, with differential $\sum_j x^j \frac{\partial}{\partial v^j}$. This is precisely $\mathscr{C}^\infty(V \oplus \pi V)$.

You can do exactly the same argument for vector bundles. The inclusion $\{p = 0\} \hookrightarrow \mathrm{T}^*X$ has a resolution by projectives as the Q-manifold $(\mathrm{T}^* \oplus \pi\mathrm{T}^*)X$, with the obvious differential, and its obvious bundle structure over T^*X. Now it's clear how to intersect this with $\{p = \mathrm{d}s\}$. Write the coordinates on $(\mathrm{T}^* \oplus \pi\mathrm{T}^*)X$ as x on the base, p on the T^* fiber, and ξ on the $\pi\mathrm{T}^*$ fiber. Then intersecting with $\{p = \mathrm{d}s(x)\}$ kills the p coordinates, but leaves the ξ coordinates untouched. The differential was $\sum p_j \frac{\partial}{\partial \xi_j}$. Setting $p_i = \frac{\partial s}{\partial x^i}$ leaves us with the manifold $\pi\mathrm{T}^*X$ with differential $\sum \frac{\partial s}{\partial x^i} \frac{\partial}{\partial \xi_i} = \mathcal{P}(s, -)$.

As an aside, what's the intersection $\{\mathrm{pt}\} \cap_V \{\mathrm{pt}\}$? It is not $\{\mathrm{pt}\}$, but rather πV with the zero differential! You can tell that $\{\mathrm{pt}\}$ is the wrong answer, because when computing derived intersections, codimensions add exactly.

Anyway, the general picture of perturbative BV integration is this. We start with a symplectic manifold T^*X and two Lagrangians $\{p = 0\}$ and $\{p = \mathrm{d}s\}$. We take their derived intersection, getting a zero-dimensional derived manifold $(\pi\mathrm{T}^*X, \mathcal{P}(s, -))$ which comes naturally equipped with a Gerstenhaber bracket $\mathcal{P}$. Now we try to "deform" or "quantize" or "perturb" this manifold into something "noncommutative" (not even an associative algebra!) by modifying $\mathcal{P}(s, -)$ by adding a *BV Laplacian* $\hbar\Delta$, where the most important things are that Δ commute with the bracket and that it be a second-order operator with principal symbol $\mathcal{P}$. Choices of such Δ correspond, more or less, to choices of measures on X. Note that this story has a chance of making sense in vast generality: X could be a derived space itself (e.g. a derived quotient), or it could be infinite-dimensional (if you understood infinite-dimensional symplectic/Poisson geometry), or T^*X could be some old symplectic manifold with some Lagrangians in it, or $\pi\mathrm{T}^*X$ could be the basic data, or ....

4 Example of homological perturbation: Feynman diagrams

In this section I will describe joint work with Owen Gwilliam. Probably the experts know it already, but we could not find it written anywhere, so we wrote it up and put it online.

We consider the manifold $X = \mathbb{R}^n$ with action $s(x) = \sum_{i,j} a_{ij}x^ix^j/2 + \lambda b(x)$, where λ is some "small" parameter, a is nondegenerate, and b is cubic+higher at the origin. We would like to know everything there is to know about the (degree-zero part of the) corresponding complex $\mathscr{C}^\infty(x^i)[\xi_i]$, with differential $a_{ij}x^i\frac{\partial}{\partial \xi_j} + \lambda\frac{\partial b}{\partial x^i}\frac{\partial}{\partial \xi_i} + \hbar\frac{\partial^2}{\partial x^i\partial \xi_i}$. Note that the homological degree is the polynomial degree in the odd variables ξ_i.

Every degree-zero function is closed. Which are the exact ones? Given $f^j(x)\xi_j$, we compute:

$$\left(a_{ij}x^i\frac{\partial}{\partial \xi_j} + \lambda\frac{\partial b}{\partial x^i}\frac{\partial}{\partial \xi_i} + \hbar\frac{\partial^2}{\partial x^i\partial \xi_i}\right)\left[f^j(x)\xi_j\right] = a_{ij}x^i\, f^j(x) + \lambda\frac{\partial b}{\partial x^i}\, f^i(x) + \hbar\frac{\partial f^i(x)}{\partial x^i}$$

Since a is invertible, if $f(0) = 0$ then we can write $f(x) = a_{ij}x^i\, f^j(x)$ for some functions $f^j(x)$. It follows that for any $\mathscr{C}^\infty(x^i)[\xi_i]$ we have

$$f(x) \equiv f(0) + (\text{higher degree in } \lambda) + (\text{higher degree in } \hbar) \qquad \text{mod exact.}$$

One can write explicitly the higher-degree terms, and iterate the procedure. Then $f(x) \equiv$ some formal power series in $\lambda, \hbar$. In fact, this power series has the property that one can specialize either λ or $\hbar$ so long as you keep the other as a formal variable. When writing the power series, it is convenient to organize the combinatorics, and they are precisely (and somewhat automatically) organized by the usual Feynman diagrams. By a spectral sequence argument (or simply by inspection), the homology of the complex is one-dimensional. So this power series is the unique number $\langle f\rangle \in \mathbb{R}[\![\lambda, \hbar]\!]$ solving $\langle f\rangle \equiv f \mod \text{exact}$.

5 Using homological perturbation theory

There is a much more general machine, which is less combinatorial but also computes to all orders the asymptotics of solutions to $\langle f \rangle \equiv f \mod \text{exact}$.

Let X be a manifold and $s \in \mathscr{C}^\infty(X)$ an "action" function. As before, we are interested in the complex $\mathscr{C}^\infty(\pi \mathrm{T}^* X) = \mathrm{MV}^\bullet(X)$ with the differential $\mathcal{P}(s,-) + \hbar\Delta$ for some BV Laplacian δ. We expect this to be "controlled" by the derived critical locus $\big(\mathscr{C}^\infty(\pi\mathrm{T}^*X), \mathcal{P}(s,-)\big)$.

If the intersection $\{p = 0\} \cap \{p = \mathrm{d}s\}$ is transverse (or even if not, if you'll take the following as definitional), then

$$\text{restrict} : \big(\mathscr{C}^\infty(\pi\mathrm{T}^*X), \mathcal{P}(s,-)\big) \to \big(\mathscr{C}^\infty(\{\mathrm{d}s = 0\}), 0\big)$$

is a homomorphism of dgas, and a quasi-isomorphism. Since the codomain has no differential, this is a quasi-isomorphism of the complex onto its homology.

We can choose a splitting of this map as chain complexes (although not, in general, as dgas). For example, it suffices to trivialize the tubular neighborhood of $\{\mathrm{d}s = 0\}$ inside T^*X. Moreover, choose a homotopy. Then we find ourselves in the following position:

$$L \underset{\phi}{\overset{\iota}{\leftrightarrows}} (M, \partial) \circlearrowright \eta$$

where $L = \mathscr{C}^\infty(\pi\mathrm{T}^*X)$, $(M,\partial) = (\mathrm{MV}^\bullet, \mathcal{P}(s,-))$, ι is the restriction map, and ϕ, η are choices. We ask these to satisfy:

$$\iota\phi = \mathrm{id}_L, \quad \phi\iota = \mathrm{id}_M - [\partial, \eta], \quad \iota\eta = \eta^2 = \eta\phi = 0$$

The last few equations are called *side conditions*.

Suppose now that δ is any *small perturbation* of ∂, in the sense that $\partial + \delta$ is a differential for M and $(\mathrm{id} - \eta\delta)$ is invertible. Then the *homotopy perturbation lemma* says that

$$(L, \iota \circ \eta(\mathrm{id} - \delta\eta)^{-1} \circ \phi) \underset{(\mathrm{id}-\delta\eta)^{-1}\circ\phi}{\overset{\iota\circ(\mathrm{id}-\eta\delta)^{-1}}{\leftrightarrows}} (M, \partial + \delta) \circlearrowright \eta(\mathrm{id}-\delta\eta)^{-1}$$

is also a homotopy equivalence.

The homological perturbation implements the following geometry. Recall that $\iota : M \to L$ is the restriction to the critical locus, and we have seen already that the perturbation $\delta = \hbar\Delta$ encodes the measure. The deformation $\iota \circ (\mathrm{id} - \eta\delta)^{-1} : M \to L$ has the interpretation of "integrating over the fibers" if the homotopy corresponded to trivializing the tubular neighborhood. The new differential essentially amounts to a "measure" on the critical locus.

Put another way, we are interested in describing explicitly the (hopefully unique) chain map $(M, \partial + \delta) \to \mathbb{R}$. We have constructed a quasi-isomorphism $(M, \partial+\delta) \to (L, \iota\circ\eta(\mathrm{id} - \delta\eta)^{-1} \circ \phi)$, and generically the complex $L = \mathscr{C}^\infty(\pi\mathrm{T}^*\{\mathrm{d}s = 0\})$ is much smaller than $M = \mathscr{C}^\infty(\pi\mathrm{T}^* X)$. If through potentially some other means we can construct the map $L \to \mathbb{R}$, we win.

6 Why this is useful

The primary application, of course, is not to finite-dimensional integrals. In problems of physical interest, X is an infinite-dimensional manifold, and there is no hope of making analytic sense of a "measure" on X. What we can do, usually, is describe $\mathrm{MV}^\bullet(X)$, and sometimes it makes sense as a Poisson algebra. Then we might try to *define* a "measure on X" to be a BV Laplacian for $\mathrm{MV}^\bullet(X)$.

Moreover, often the infinite-dimensional manifold X comes equipped with an action whose Taylor coefficients are elliptic operators. Then the complex $L = \mathrm{H}_\bullet(\mathrm{MV}^\bullet(X), \text{contract with } \mathrm{d}s)$ is finite-dimensional. So if we have found a suitable BV Laplacian, we can determine the expectation values up to the finite-dimensional space of chain maps $L \to \mathbb{R}$. For many problems that you thought you needed an honest integral for, this is good enough.

We conclude by remarking that in addition to potential generalizability to infinite-dimensional spaces, the BV description of the problem of finding (asymptotic) expectation values makes perfect sense in the world of derived stacks. Recall that $\pi\mathrm{T}^* X$ was a "derived intersection", in the sense that in a certain construction, a tensor product was replaced by a left-derived tensor product. A *derived stack* is similar: to quotient a space by a group action can be written (on algebras of functions) in terms of hom, and we instead use the right-derived hom. In this way, BV integration is a "generalization" of BRST integrals (which we have not discussed today, but have discussed recently in the "Witten In The 80s" seminar at UC Berkeley). But it is my opinion that those who say that "BV is a souped up version of BRST" are mostly wrong: what's actually true is that BV and BRST are largely orthogonal, but both consist of generalizations of integration that have a chance of working for derived stacks (and potentially for infinite-dimensional manifolds), and that both rely on (or at least are most naturally worded in terms of) super/graded/derived geometry.