THE MORITA THEOREMS

by

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Introduction

Virtually all algebraic notions in category theory are a parody of their parents in the most "classical" of categories, ${}_A\mathfrak{M}$, the category of (left) A-modules. Granting the interest and importance of categories, therefore, - perhaps the most presumptuous of our assumptions, to many - it is natural to ask, when is ${}_A\mathfrak{M} \cong {}_B\mathfrak{M}$? Isomorphism problems of this generality generally admit either a trivial, or no humanly manageable, solution. The present instance, however, must be regarded either as exceptional, or, on closer scrutiny, as acquiring depth in virtue of the formidable mélange of trivialities which the solution meaningfully organizes. Indeed, the solution is so overwhelmingly complete that it permits a classification of all isomorphisms from ${}_A\mathfrak{M}$ to ${}_B\mathfrak{M}$. and, in particular, a computation of Aut $({}_A\mathfrak{M})$.

So natural a question as the above awaits category theory only to be asked properly, but not to be answered, at least in part. Historically the theory emerged piecemeal, first in the Wedderburn structure theory for simple algebras, then in the theory of the Brauer group, in the arithmetic of hypercomplex systems (see, e.g., Chevalley's monograph, "L'Arithmetique dans les algebras de matrices"), in the Brauer-Osima construction of "basic rings", in the study of quasi-Froberius rings, and in the new theory of the Brauer group over a commutative ring. Auslander and Goldman developed a good deal of the present theory independently for the needs of the Brauer group, and simultaneously S. Schanuel and S. Chase (unpublished) elaborated a large portion of it. However, the problem had already received a complete and systematic treatment in full generality by Morita. The exposition here is based on that of Chase-Schanuel, and it is adapted to applications in the theory of the Brauer group, in contrast with Morita's prevailing concern with rings with minimum condition. However, all of what follows is, for the patient reader, an easy consequence of results in Morita's work.

Some Remarks on Categories and Isomorphisms

Recall that a category, $\mathcal{C}$, consists of objects, obj $\mathcal{C}$, morphisms, $\mathrm{Hom}_{\mathcal{C}}(M,N)$, for $M, N \in \mathrm{obj}\,\mathcal{C}$, and a law of composition. If K is a commutative ring then we call $\mathcal{C}$ a K-category if $\mathrm{Hom}_{\mathcal{C}}(M,N)$ is always a K-module and the composition is K-bilinear. If $K = \mathbb{Z}$ $\mathcal{C}$ is called an additive category. If A is a K-algebra, then ${}_A\mathfrak{M}$ and $\mathfrak{M}_A$, the categories of left and right A-modules, are K-categories. If $T: \mathcal{C} \longrightarrow \mathcal{C}'$ is a functor* on K-categories we call T a K-functor when the induced maps

$$T: \mathrm{Hom}_{\mathcal{C}} \longrightarrow \mathrm{Hom}_{\mathcal{C}'}$$

are K-homomorphisms. For example, if P is an A-B-bimodule, where A and B are K-algebras, a situation we denote by ${}_AP_B$, then a necessary and sufficient condition that

$$P\otimes_B \;:\; {}_B\mathfrak{M} \to {}_A\mathfrak{M}$$

be a K-functor is that, whenever $a \in K$ and $x \in P$, $ax = xa$.

If $S: \mathcal{Q} \longrightarrow \mathcal{B}$ is a functor we call S an isomorphism when there is a functor $T: \mathcal{B} \to \mathcal{Q}$ such that $ST \cong I_{\mathcal{B}}$ and $TS \cong I_{\mathcal{Q}}$, where $I_{\mathcal{Q}}$ and $I_{\mathcal{B}}$ are the identity functors, and $\cong$ denotes a natural equivalence of functors. A property of an object or morphism in $\mathcal{Q}$ will be called categorical if it is shared by the image under any isomorphism. We give below definitions of several notions which make it manifest (or an easy exercise) that they are categorical:

* All functors in what follows are covariant.

I. For morphisms:

(a) f is an <u>epimorphism</u>: $fg_1 = fg_2 \Rightarrow g_1 = g_2$.

f is a <u>monomorphism</u>: $g_1 f = g_2 f \Rightarrow g_1 = g_2$

(b) f is a proper <u>monomorphism</u>: f is a monomorphism but not an isomorphism (i.e. has no inverse).

(c) $f \leq g$: f and g are monomorphisms and $f = gh$ for some h.

This defines a partial ordering on the monomorphisms into a fixed object M.

II. Fixed objects:

(a) M is <u>projective</u>: If $N \to N'$ is an epimorphism, $\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(M,N')$ is surjective.

(a') M is <u>injective</u>: dualize.

(b) M is a <u>generator</u>: If $g_1 \neq g_2$: $N \to N'$ there exists f: $M \to N$ such that $g_1 f \neq g_2 f$.

(b') M is a <u>cogenerator</u>: dualize.

(c) M is <u>finitely generated</u>: If f_v is a totally ordered family of proper monomorphisms into M there is a proper monomorphism f into M such that $f_v \leq f$ for all v. I.e. the proper monomorphisms into M are inductively ordered.

<u>Defn</u>. $P \in \mathrm{obj}\,\mathfrak{A}$ is called a <u>progenerator</u> for $\mathfrak{A}$ if P is a finitely generated projective generator in $\mathfrak{A}$.

<u>Corollary.</u> If T: $\mathfrak{A} \to \mathfrak{B}$ is an isomorphism and P is a progenerator for $\mathfrak{A}$, then T(P) is a progenerator for $\mathfrak{B}$.

Finally, we define

<u>Center</u> $\mathfrak{C}$ = End (I),

where I is the identity functor of $\mathfrak{C}$ and End (I) is the set of natural transformations $I \to I$

<u>Proposition</u> Center ${}_A\mathfrak{M} \cong$ center of A, for any ring A. (

(The "homothetic" defined by an element of the center of A establishes the above isomorphism)

<u>Proof</u>. Let $\phi \in \operatorname{End} I_A$. Then there exists in particular an A-homomorphism $\phi(A)$: $I_A(A) \to I_A(A)(=A)$, and we write $(1)\phi(A) = a_\phi$ (Note map convention) Then $(a)\phi(A) = aa_\phi$ for all $a \in A$. Now let M be any A-module, $x \in M$, x arbitrary and consider the map $f: A \to M$ sending $a \in A$ onto $ax \in M$. This is an A-homomorphism, and, in identifying $I_A(M)$ with M, $I_A(f)$ with f, we see that

$$\begin{array}{ccc} A & \xrightarrow{\phi(A)} & A \\ \downarrow f & & \downarrow f \\ M & \xrightarrow{\phi(M)} & M \end{array}$$

must commute, i.e. $(1)\phi(A)f = (1)f\phi(M)$; this means that $x\phi(M) = a_\phi x$.

Now $\phi(M)$ is an A-endomorphism of M, and the equation $x\phi(M) = a_\phi x$ asserts that $\phi(M)$ is induced by left multiplication by a_ϕ for all A-modules M. Clearly this can happen if and only if a_ϕ is in the center of A. Retracing the above steps we see that left multiplication by an element of the center of A does indeed induce an endomorphism of the identity functor. Finally we see that left multiplication by distinct elements of a and b of the center of A induces distinct endomorphisms ϕ_a, ϕ_b, since in particular $\phi_a(A) \neq \phi_b(A)$.

<u>The Morita Context</u>

We adopt the following standard notation:

${}_AM$: M is a left A-module

M_B : M is a right B-module

${}_AM_B$: M is an A-B-bimodule: i.e. ${}_AM$, M_B, and $(ax)b = a(xb)$ for $a \in A$, $x \in M$, $b \in B$.

If A^o denotes the opposite ring of A, ${}_AM \Leftrightarrow M_{A^o}$.

All rings in what follows will be understood to be K-algebras **for a fixed commutative ring K.** Moreover when we write ${}_AM_B$ we shall understand further that $tx = xt$ for all $x \in M$, $t \in K$

Finally, we adopt the following, non standard, convention, which will facilitate our notation considerably: homomorphisms will be written opposite the scalars. For right modules this is standard. However if $u: M \longrightarrow M'$ is a homomorphism of left A-modules, we write xu instead of $u(x)$, and A-linearity becomes $(ax)u = a(xu)$. If $u: M' \longrightarrow M''$ is another, $x(uu') = (xu)u'$. This convention has several consequences, of which the following are notable: (i) commutativity becomes associativity; (ii) a number of opposite rings tend to disappear. In regard to (ii) we shall take care to arrange the statements of the theorems so as to agree with standard usage, but we invoke the convention systematically in discussion, as illustrated now.

Given ${}_AM$, set $B = \mathrm{Hom}_A(M,M)^o$. Then ${}_AM_B$. For if $b_1, b_2: M \longrightarrow M$ then, by our convention b_1b_2 denotes their product in B. Let $M^* = \mathrm{Hom}_A(M,A)$. Then ${}_BM^*{}_A$: for $b \in B$, $u \in M^*$, $a \in A$ we have

$$bua: M \longrightarrow A$$

defined for $x \in M$ by $x(bua) = ((xb)u)a$. Now since the expression xu, $x \in M$, $u \in M^*$ is B-bilinear and A - A linear, it defines an A-A-homomorphism

$$(\ ,\ ) = (\ ,\ )_M : M \otimes_B M^* \longrightarrow A;\ x \otimes u \longrightarrow (x,u) = xu.$$

Next observe that if, for $x \in M$, $u \in M^*$ we let $[u,x] \in B$ be defined, for $y \in M$, by

I. $$y[u,x] = (y,u)x$$

then this defines a B - B-homomorphism

$$[\ ,\ ] = [\ ,\ ]_M : M^* \otimes_A M \longrightarrow B.$$

Finally, note that if $x, y \in M$ and $u,v \in M^*$ then

$$(y,[u,x]v) = (y[u,x],v) = ((y,u)x,v) = (y,u)(x,v) = (y,u(x,v))$$

and, hence,

II. $$[u,x]v = u(x,v)$$

The abave considerations provide an example of what we now define, abstractly, to be a Morita context: this consists of K-algebras A and B, bimodules ${}_AP_B$ and ${}_BQ_A$, an A-A-homomorphism

$$(\ ,\ ) : P\otimes_B Q \longrightarrow A$$

and a B - B - homomorphism

$$[\ ,\ ] : Q\otimes_A P \longrightarrow B$$

subject to the two axioms,

I. $y[u,x] = (y,u)x$ for $x,y \in P$, $u \in Q$.

II. $[u,x]v = u(x,v)$ for $x \in P$, u, $v \in Q$.

When discussing the Morita Context, and/or the example of the Morita Context already given, we shall consistently denote elements of A by a, of B by b, of P by x,y,z, and of Q by u,v,w.

Lemma 1: In the category ${}_A\mathfrak{M}$ of left A-modules the following are equivalent:

(1) P is a generator for ${}_A\mathfrak{M}$ (see page 3 for definition)

(2) $\sum_{u \in P^*} Pu = A$ (i.e. $(,)_P$ is an epimorphism).

(3) A is a direct summand of a direct sum of copies of P.

Proof. 1) $\Rightarrow$ 2: Let $\mathfrak{U} = \sum_{u \in P^*} Pu \subset A$. We note that $\mathfrak{U}$ is a left ideal and we consider the canonical projection π: $A \to A/\mathfrak{U}$. If $\pi \neq 0$, then there exists g: $P \to A$ such that $P \xrightarrow{g} A \xrightarrow{\pi} A/\mathfrak{U}$ is not zero. But $g \in P^*$ and hence $g\pi$ must be zero. Hence $\pi = 0$ and $\mathfrak{U} = A$.

2) $\Rightarrow$ 3: Define $S = \sum_{u \in P^*}^{\oplus} P_u$ (Note that P^* is being used as an index set; S is a direct sum of copies of P.). The map f: $S \to A$ given by $f|_{P_u} = u$ is well-defined and surjective by 2). But A is a projective A-module and hence A is a direct summand of S.

3) $\Rightarrow$ 1: Let $\sum_i^{\oplus} P_i = A \oplus L$, and let f: $M \to N$ be a non-zero map. Then there exists a map $\bar{g}$: $A \to M$ such that $\bar{g}f$ is non-zero. This map $\bar{g}$ can be extended to a map $g: \sum_i^{\oplus} P_i \to M$ and clearly gf is non-zero. Hence there exists at least one i such that $gf|_{P_i}$ is non-zero, and $g|_{P_i}$ is the desired map.

Lemma (Dual Basis Lemma)

A left A-module P is projective iff there exist $x_i \in P$ and $u_i \in P^*$, $i \in I$, such that

(a) For $x \in P$, $xu_i = 0$ for almost all i and

(b) For $x \in P$, $x = \sum_{i \in I} (xu_i)x_i$.

Moreover, in this case, the families $\{x_i\}$ which may occur in this way are precisely the generating sets of P.

Proof: If P is projective with generators $\{x_i\}_{i \in I}$, define $F = \sum_{i \in I}^{\oplus} AX_i$ and define π: $F \to P$ via $\pi(X_i) = x_i$, extending by linearity. P is a direct summand of F, since it is projective, and hence can be considered to be a submodule of F. If $x \in P$,

then $x = \sum a_i X_i$ and the a_i are certainly well-defined functions of x; indeed the maps u_i: $P \longrightarrow A$ given via $xu_i = a_i$, $i \in I$, are A-homomorphisms, i.e. $u_i \in P^*$. Now if $x \in P$, then
$\pi x = x = \sum_i a_i x_i = \sum_i (xu_i) x_i$; since $P \subset F$, a direct sum of copies of A, one sees that the condition that xu_i is almost always 0 A is satisfied. Note finally that $\{x_i\}_{i \in I}$ was an arbitrary generating set of P.

Conversely, given the $x_i \in P$ and $u_i \in P^*$, $i \in I$, we form the free module $\sum_{i \in I} \oplus AX_i$, and we define a map π: $F \to P$ via

$\pi(X_i) = x_i$, for all $i \in I$, and extend by linearity. π is clearly an epimorphism (note that $x_i \in P$ for all $i \in I$). We now define a map ϕ: $P \longrightarrow F$, such that $P \xrightarrow{\phi} F \xrightarrow{\pi} P$ is the identity on P; this is done by setting, for $x = \sum_i (xu_i) x_i \in P$, $x\phi = \sum_i (xu_i) X_i$. This is clearly a homomorphism and $\phi \circ \pi = 1_P$; hence P is a direct summand of F and hence projective. We note finally that the $\{x_i\}$ is a generating set for P.

<u>Corollary</u> If $B = \mathrm{Hom}_A(P,P)^\circ$ and $[\,,\,]_P$: $P^* \otimes_A P \longrightarrow B$ is defined as in the last section, then P is a finitely generated projective A-module iff $[\,,\,]_P$ is an epimorphism.

Recalling that we call P a <u>progenerator</u> for ${}_A\mathfrak{M}$ when P is a finitely generated projective generator in ${}_A\mathfrak{M}$, we can combine this corollary with Lemma 1 to conclude,

<u>Proposition</u> P is a progenerator for ${}_A\mathfrak{M}$ iff both $(\,,\,)_P$ and $[\,,\,]_P$ are epimorphisms.

<u>Proof of Corollary.</u> If P is projective and finitely generated, say by $x_i \ldots x_n$, then by the Lemma
$x = \sum_i (xu_i) x_i = \sum_i x[u_i, x_i]_P = x \sum_i [u_i, x_i]_P$ for all $x \in P$ and

suitable $u_1 \ldots u_n \in P^*$. Hence $\sum [u_i, x_i]_P = [\,,\,]_P(\sum u_i \otimes x_i) = 1 \in B$.

Conversely, if, for $u_i \in P^*$ and $x_i \in P$, $i = 1 \ldots n$. we have $[\,,\,]_P\ (\sum u_i \otimes x_i) = 1 \in B$, then $x = x \sum [u_i, x_i] = \sum (xu_i)x_i$ for all $x \in P$, and P is thus finitely generated and, by the Lemma, is projective.

<u>Morita I.</u>

<u>Theorem</u> Suppose given a Morita Context: A, B, ${}_AP_B$, ${}_BQ_A$, $(\,,\,)$, and $[\,,\,]$ with the latter both epimorphisms. Then:

(1) P is a progenerator for ${}_A\mathfrak{M}$ and $\mathfrak{M}_B$.
Q is a progenerator for $\mathfrak{M}_A$ and ${}_B\mathfrak{M}$.

(2) $(\,,\,)$ and $[\,,\,]$ are isomorphisms

(3) $Q \cong \mathrm{Hom}_A({}_AP,\ {}_AA) \cong \mathrm{Hom}_B\ (P_B,\ B_B)$

and

$P \cong \mathrm{Hom}_A(Q_A, A_A) \cong \mathrm{Hom}_B({}_BQ,\ {}_BB)$

as bimodules.

(4) $A \cong \mathrm{Hom}_B\ ({}_BQ,\ {}_BQ)^o \cong \mathrm{Hom}_B(P_B,\ P_B)$

and

$B \cong \mathrm{Hom}_A(Q_A, Q_A) \cong \mathrm{Hom}_A\ ({}_AP\ ,\ {}_AP)^o$

as K-algebras.

(5) $P\otimes_B : {}_B\mathfrak{M} \to {}_A\mathfrak{M}$ and $Q\otimes_A : {}_A\mathfrak{M} \to {}_B\mathfrak{M}$ are inverse K-isomorphisms.
Similarly for $\otimes_A P : \mathfrak{M}_A \to \mathfrak{M}_B$ and $\otimes_B Q : \mathfrak{M}_B \to \mathfrak{M}_A$.

(6) The lattice of A-submodules of P is isomorphic to the lattice of left ideals in B, with A-B-submodules corresponding to two sided ideals. Similar statements follow from symmetry. In particular, A and B have isomorphic lattices of two sided ideals.

(7) Center of $A \cong$ Center of B.

Proof: (as before, we shall write consistently, $x, y \in P$, $u, v \in Q$, $a \in A$, $b \in B$)

(1) Each element $u \in Q$ induces an A-homomorphism $(,u): P \to A$. Since $(,)$ is surjective, there exist $x_i \in P$, $u_i \in Q$ such that $\sum (x_i, u_i) = 1 \in A$. By Lemma 1 we see that P is a generator for ${}_A\mathfrak{M}$. In exactly the same way, exchanging $[,]$ for $(,)$, we see that P is a generator for $\mathfrak{M}_B$. Since $[,]$ is surjective, there exist $v_i \in Q$, $y_i \in P$, such that $\sum_k [v_k, y_k] = 1 \in B$. Hence, for $x \in P$, $x = x \cdot 1 = \sum_k (x, v_k) y_k$, and by Lemma 2 P is a finitely-generated projective left A-module. In exactly the same way, exchanging $(,)$ for $[,]$, one sees that P is a finitely generated projective right B-module. This proves the assertion for P, and that for Q is proved by symmetry.

[Let us, for the rest of the proof, hold fast to the notation $\sum_i (x_i, u_i) = 1 \in A$, $\sum_k [v_k, y_k] = 1 \in B$]

(2) We show that Ker $(,)$ is zero. Indeed, if $z_j \in P$, $w_j \in Q$ s.t. $\sum_j (z_j, w_j) = 0$, then $\sum_j z_j \otimes w_j = \sum_j z_j \otimes w_j \sum_i (x_i, u_i)$

$$= \sum_{i,j} z_j \otimes [w_j, x_i] u_i = \sum_{i,j} z_j [w_j, x_i] \otimes u_i = \sum_i \Big(\sum_j (z_j, w_j) x_i \otimes u_i\Big) = 0$$

It is clear that an exactly analogous proof shows that Ker $[,]$ is zero. Since $(,)$ and $[,]$ are onto by hypothesis, we see that $[,]$ and $(,)$ are isomorphisms as asserted.

(3) We show that $Q \cong \mathrm{Hom}_A({}_AP, {}_AA)$, via $u \to (,u)$: that $(,u): P \to A$ is an element of $\mathrm{Hom}_A({}_AP, {}_AA)$ is clear by hypothesis.

We show that the map $u \longrightarrow (\ ,u)$ is an isomorphism.

Suppose $(x,u) = 0 \ \forall x \in P$. Then $u = 1 \cdot u = \sum_k [v_k, y_k] u$
$= \sum_k v_k (y_k, u) = 0$. Hence the map is injective.

Let $f \in \mathrm{Hom}_A({}_AP, {}_AA)$; we show $f = (\ , \sum_k v_k (y_k f))$.

Indeed $(x, \sum_k v_k (y_k f)) = \sum_k (x, v_k)(y_k f) = (\sum_k (x, v_k) y_k) f =$
$(x \sum [v_k, y_k]) f = xf$. Hence the map is surjective. Finally note that this is a $B - A$-homomorphism, since $(x, bua) = (xb, u)a$.
In exactly the same way isomorphisms are established

between Q and $\mathrm{Hom}_B(P_B, B_B)$ via $u \to [\ u,]$,
" P and $\mathrm{Hom}_B({}_BQ, {}_BB)$ via $x \to [\ ,x]$,
and " P and $\mathrm{Hom}_A(Q_A, A_A)$ via $x \to (x, \ )$.

(4) There is a map from A to $\mathrm{Hom}_B(P_B, P_B)$ given by left multiplication. We show it is an isomorphism.

<u>Injective</u>: If $ax = 0$ for all $x \in P$ then $a = a \cdot 1 = a \sum (x_i . u_i) =$
$\sum (ax_i, u_i) = 0$

<u>Surjective</u>: Let $f \in \mathrm{Hom}_B(P_B, P_B)$; we show that f is given by left multiplication by $\sum (fx_i, u_i)$. Indeed $\sum (fx_i u_i) x =$
$\sum f x_i [u_i, x] = f(\sum x_i [a_i, x]) = f(\sum (x_i, u_i) x) = fx.$

Hence $A \cong \mathrm{Hom}_B(P_B, P_B)$, and similarly $B \cong \mathrm{Hom}_A(Q, Q)$, via left multiplication by elements of B.

Finally, there is a map from A to $\mathrm{Hom}_B({}_BQ, {}_BQ)^{o}$ given by right multiplication, just as explained following the definition of the Morita Context. We see, just as in the previous two cases, that $A \cong \mathrm{Hom}_B({}_BQ, {}_BQ)^{o}$ and $B \cong \mathrm{Hom}_A({}_AP, {}_AP)^{o}$. The commuting of elements of K with those of P and Q guarantees that the above are K-algebra isomorphisms.

(5) Let $S = Q\otimes_A \ : {}_A\mathfrak{M} \to {}_B\mathfrak{M}$ and

$T = P\otimes_B \ : {}_B\mathfrak{M} \to {}_A\mathfrak{M}$.

Then for M a left A-module.

$$
\begin{aligned}
TS(M) &= P\otimes_B (Q\otimes_A M) \\
&= (P\otimes_B Q)\otimes_A M \\
&\cong A \otimes_A M \quad \text{using the isomorphism, } (,) \\
&\cong M.
\end{aligned}
$$

Similarly $ST = I_B$. S and T are K-isomorphisms since the elements of K commute with those of P and Q.

(6) Since T is an isomorphism , the (left) B-submodules of of B are lattice isomorphic to the A-submodules of $T(B) = P$, with the B-B submodules of B corresponding to A-B-submodules of P

(7) Using an earlier proposition

Center $A \cong$ Center ${}_A\mathfrak{M} \cong$ Center ${}_B\mathfrak{M} \cong$ Center B.

This concludes the proof of Morita I.

Morita II

Theorem Let $S: {}_A\mathfrak{M} \longrightarrow {}_B\mathfrak{M}$ and $T: {}_B\mathfrak{M} \longrightarrow {}_A\mathfrak{M}$ be K-isomorphisms. Then if $P = T(B)$ and $Q = S(A)$ we have ${}_AP_B$, ${}_BQ_A$, and

$$S \cong Q\otimes_A \quad \text{and } T \cong P\otimes_B$$

as functors.

Proof: P is a progenerator for ${}_A\mathfrak{M}$ since ${}_BB$ is a progenerator for ${}_B\mathfrak{M}$ and T is an isomorphism.
Moreover

$$Q \cong \mathrm{Hom}_B(\;B,\; Q) \cong \mathrm{Hom}_A(T(B),\; T(Q))$$
$$\cong \mathrm{Hom}_A\;(P, TS(A) \cong \mathrm{Hom}_A\;(P, A),$$

and these are all bimodule isomorphisms. Further T defines an isomorphism

$$B \cong \mathrm{Hom}_B({}_BB, {}_BB)^o \xrightarrow{T} \mathrm{Hom}_A(P,P)^o \;,$$

the isomorphism thus being given by right multiplication. Now since P is a progenerator for ${}_A$ we can apply Morita I to P, $\mathrm{Hom}_A(P,P)^o \cong B$, and $\mathrm{Hom}_A(P,A) \cong Q$ and conclude, in particular, that

$$Q \cong \mathrm{Hom}_B\;(P,B)$$

as a bimodule. We thus obtain the following isomorphisms of functors ${}_B\mathfrak{M} \rightarrow {}_A\mathfrak{M}$: let M be a left B-module then

$$P\otimes_B M \cong P\otimes_B \mathrm{Hom}_B(B,M)$$
$$\cong \mathrm{Hom}_B(\mathrm{Hom}_B(P,B),\; M), \text{ since P is a finitely generated projective B-module*}$$

* We are using the following identity in the situation P_B, ${}_CM_B$, ${}_CN$. The natural transformation $P\otimes_B\mathrm{Hom}_C(M,N) \xrightarrow{\phi} \mathrm{Hom}_C(\mathrm{Hom}_B(P,M),N)$ defined by $\phi(p\otimes f)(g) = fgp$ is clearly an isomorphism for $P = B$, and hence for P finitely generated projective right B-module, by additivity.

$\cong \mathrm{Hom}_B(Q,M)$, by the remark above

$\cong \mathrm{Hom}_A(T(Q),T(M))$, via T

$\cong \mathrm{Hom}_A(A,T(M))$, since $T(Q)=T(S(A)) = A$.

$\cong T(M)$.

Morita III.

Defn: If A and B are K-algebras, an A - B-progenerator is a module ${}_AP_B$ such that P is a progenerator for ${}_A\mathfrak{M}$ and right multiplication defines an isomorphism $B \longrightarrow \mathrm{Hom}_A(P,P)^{\circ}$. (As always, elements of K commute with those of P.) It follows from Morita I that this definition is in fact symmetrical.

Theorem : ${}_AP_B \longrightarrow P\otimes_B$ defines a bijection between the (isomorphism types of) A-B-progenerators and the (isomorphism types of) K-isomorphisms ${}_B\mathfrak{M} \to {}_A\mathfrak{M}$. Composition of isomorphisms corresponds to tensor products of progenerators.

Proof: Clearly an A-B isomorphism $P \longrightarrow P'$ induces an isomorphism $P\otimes_B \longrightarrow P'\otimes_B$ of functors ${}_B\mathfrak{M} \to {}_A\mathfrak{M}$. Moreover, by Morita I. $P\otimes_B$ is an isomorphism for P an A-B-progenerator. Hence our mapping is well defined. By Morita II it is surjective. Finally, it is an easy exercise to conclude from an isomorphism of $P\otimes_B$ with $P'\otimes_B$ as functors ${}_B \quad {}_A$ an A-B-isomorphism of P with P'. and this shows the mapping is injective.

Applications:

We now proceed to indicate how to subsume certain classical theorems in the theory of algebras and their arithmetics in the preceding circle of ideas.

Wedderburn Structure Theory :

Let A be a simple Artinian ring. Then every non zero finitely generated A-module P is projective (A is semi-simple) and a generator (since Im $(,)_P$ is a non zero two sided ideal and A is simple). Hence, if $D = \mathrm{Hom}_A(P,P)^o$ we may conclude from Morita I that:

(a) P is a D-progenerator (condition 1)

(b) D is a simple Artinian ring (condition 6)

(c) Center D $\cong$ center A (condition 7)

(d) $A \cong \mathrm{Hom}_D(P,P)$ (condition 4)

Theorem 1: If A is a division ring the ring of n×n matrices over A is a simple Artinian ring with the same center as A.

Proof: P is a vector space of some dimension, n, and D is the ring of n×n matrices over A.

Theorem 2. If A is a simple Artinian ring and P is a simple left A-module, then $D = \mathrm{Hom}_A(P,P)^o$ is a division ring (Schur), P is a finite dimensional D-space, and $A \cong \mathrm{Hom}_D(P,P)$.

The Brauer Group:

Let K be a commutative ring, A a K-algebra. We write $A^e = A \otimes_K A^o$. Then a left A^e-module is just an A-A-bimodule. (K, as always, commutes with everything.) In particular A is a left A^e-module and we call A a separable K-algebra if it is

A^e-projective. A is <u>central</u> if center $A = K\cdot 1 \cong K$.

If M is an A^e-module, then $f \longrightarrow f(1)$ defines a natural isomorphism of $\mathrm{Hom}_{A^e}(A,M)$ with the "fixed points" of $M = \{m \in M \mid am = ma \text{ for all } a \in A\}$. In particular, if M and N are left A-modules then $\mathrm{Hom}_K(M,N)$ is an A-A-bimodule whose fixed points are $\mathrm{Hom}_A(M,N)$: i.e.

$$\mathrm{Hom}_{A^e}(A, \mathrm{Hom}_K(M,N)) \cong \mathrm{Hom}_A(M,N).$$

It follows that if K is a field (so Hom_K is exact), and if A is separable (so $\mathrm{Hom}_{A^e}(A,)$ is exact), then Hom_A is exact, so A is a semi-simple Artinian ring. If, in addition, A is central, therefore, A must be simple:

<u>Proposition</u>: If K is a field, a central separable K-algebra is central simple.

Now suppose A is central and separable, but K is arbitrary. Then Auslander and Goldman show, using this Proposition, that every two sided proper ideal in A^e is contained in $\mathfrak{m}A^e$ for some maximal ideal $\mathfrak{m} \subset K$, and from this it follows that $\mathrm{Im}\ (,)_A$ is the unit two sided ideal in A^e.

<u>Proposition:</u> (Auslander-Goldman) A central separable K-algebra A is an A^e-progenerator.

Finally, to apply Morita I, we state the following lemma, which is contained in the remarks above in fixed points.

<u>Lemma</u> For any K-algebra A,

$$\mathrm{Hom}_{A^e}(A,A) \cong \text{center } A.$$

Putting this together, if A is a central separable K-algebra then A is an A^e-progenerator and $\mathrm{Hom}_{A^e}(A,A) = K$. We apply Morita I and conclude,

<u>Theorem:</u> Let A be a central separable K-algebra. Then

(a) A is a K-progenerator.

(b) $A^e \cong \mathrm{Hom}_K(A,A)$

(c) $K = \text{center } K \cong \text{center } A^e$

(d) $\mathfrak{A} \longrightarrow \mathfrak{A}A^e$ defines a bijection between the ideals of K and the two sided ideals of A^e.

(e) $A\otimes_K : {}_K\mathfrak{M} \rightarrow {}_{A^e}\mathfrak{M}$ is an isomorphism of categories which converts $\otimes_K$ into $\otimes_A$.

<u>The Automorphisms of ${}_A\mathfrak{M}$</u>

Let A be a K-algebra. We shall write

$$\text{Sk-N}(A) = \text{K-Aut}(A)/\text{In Aut}(A)$$

where K-Aut(A) is the group of K-automorphisms of A, and In Aut (A) the normal subgroup of inner automorphisms. Further, let

$$\text{K-Aut}({}_A\mathfrak{M})$$

denote the group of isomorphism types of K-automorphisms of the K-category ${}_A\mathfrak{M}$. This is indeed a group (the only issue being that it is a set) since, by Morita III,

<u>Theorem</u> $\text{K-Aut}({}_A\mathfrak{M})$ is isomorphic to the group of isomorphism classes of A-A-progenerators, P, where the group operation is induced by $\otimes_A$, the identity is the class of A, and the inverse of the class of P is that of $\mathrm{Hom}_A({}_AP, {}_AA)$, which, by Morita I, is A-A-isomorphic to $\mathrm{Hom}_A(P_A, A_A)$.

Recall that, as always, K commutes with the elements of the modules P as above. Hence, an A-A-bimodule with this property is just a left A^e-module, where $A^e = A\otimes_K A^o$.

Now let S be an automorphism of ${}_A\mathfrak{M}$, and suppose $P = S(A)$ is isomorphic to A as a left A-module. Such a left A-isomorphism may fail to be an A-A-isomorphism, and this failure is measured by an automorphism, α , of A, since A, via right multiplication, is isomorphic to both. $\mathrm{Hom}_A({}_AA, {}_AA)$ and $\mathrm{Hom}_A({}_AP, {}_AP)$. It is straight-forward to verify that if another left A-isomorphism $A \longrightarrow P$ is chosen then α varies only in its coset modulo In Aut(A) and that A and P are A-A-isomorphic iff $\alpha \in$ In Aut(A). Now let T be another automorphism of ${}_A\mathfrak{M}$ such that $Q = T(A)$ is left isomorphic to A. Then $P = S(A)$ is left isomorphic to $S(Q)$ and if $\rho \in$ K-Aut(A) measures the failure of A and Q being A-A- isomorphic, it does likewise for P and $S(Q)$. Hence $\rho\alpha$ measures the failure of A and $ST(A)$ being isomorphic.

The above remarks are intended as a sketch of the proof that

Theorem: The group of K-automorphisms of ${}_A\mathfrak{M}$ which fix the left isomorphism type of A is isomorphic to Sk-N(A). Moreover the cosets of this group in K-Aut(${}_A\mathfrak{M}$) is in bijective correspondence with the left isomorphism classes of A-A-progenerators.

Remarks: (1) The argument above yielded an anti-isomorphism. but any group is anti-isomorphic to itself.

(2) We have viewed K-Aut(${}_A\mathfrak{M}$) as acting on the left isomorphism types of A-A-progenerators and identified Sk-N(A) with the stability subgroup of ${}_AA$. Hence the cosets correspond to the orbit of ${}_AA$ under this action, so the last assertion follows since the group operates transitively: given P, $P \otimes_A$ carries A into P. Moreover, this observation shows that Sk-N corresponds to a well defined conjugacy class of subgroups of K-Aut(${}_A\mathfrak{M}$).

Now suppose A is a central separable K-algebra, and let P be an A-A-progenerator. We know by the Brauer group theorem (condition (e1)) that $A\otimes_K: {}_K\mathfrak{M} \to {}_{A^e}\mathfrak{M}$ is an isomorphism of categories. Hence, as a left A^e-module, $P \cong A\otimes_K E$ for a projective K-module E. determined uniquely up to isomorphism, and for which

$$\mathrm{Hom}_K(E,E) = \mathrm{Hom}_{A^e}(P,P) = K.$$

$\mathrm{Hom}_{A^e}(P,P) = K$ follows from the fact that any A^e-endomorphism of the progenerator P is induced by a right multiplication by a unique $a \in A$ (Since an A^e-endomorphism is, a fortiore, a left A-endomorphism), which must commute with all right multiplications; hence $a \in K$.

Conversely, if E is a finitely generated projective K-module with $\mathrm{Hom}_K(E,E) \cong K$, then $P = A\otimes_K E$ is clearly an A-A-progenerator.

Finally, since the category isomorphism converts $\otimes_A$ into $\otimes_K$ it defines a group isomorphism

$$K\text{-Aut}\,({}_A\mathfrak{M}) \longrightarrow \mathfrak{P}(K)$$

by $P \longrightarrow E$, where $\mathfrak{P}(K)$ is the group (under $\otimes_K$) of isomorphism types of finitely generated projective K-modules with endomorphism ring K. In particular:

> $K\text{-Aut}({}_A\mathfrak{M})$ is abelian, and depends only on K, for A a central separable K-algebra.

Hence, in the theorem above, we see that the subgroup Sk-N(A) must be normal, and therefore that $\otimes_A$ induces a group structure on $\mathfrak{P}_K(A)$, the set of left isomorphism classes of A-A-progenerators. Moreover the theorem above then yields

<u>Theorem</u>: (Rosenberg-Zelinsky) If A is a central separable K-algebra then there is an exact sequence

$$1 \longrightarrow \text{Sk-N}(A) \longrightarrow \mathcal{P}(K) \longrightarrow \mathcal{P}_K(A) \longrightarrow 1.$$

<u>Corollary</u>: (Skolem-Noether) If $\mathcal{P}(K) = \{1\}$ all K-automorphisms of a central separable K-algebra are inner.

<u>The Commutative Case</u>

Let A be a commutative K-algebra. Then there are no inner automorphisms, so

$$\text{Sk-N}(A) = \mathcal{G}(A/K) \;,$$

the "galois group" of K-automorphisms of A. If P is an A-A-progenerator we call P <u>symmetric</u> if ax = xa for all a A. There is an obvious procedure for writing an arbitrary P uniquely in the form

$$P \cong Q \otimes_A A_\alpha$$

as a bimodule, where Q is symmetric, $\alpha \in \mathcal{G}(A/K)$, and A_α is an A-A-module left isomorphic to A on which A right multiplies via α. This decomposition amounts to exhibiting a split exact sequence

$$1 \longrightarrow \mathcal{P}_K(A) \longrightarrow \text{K-Aut}(A) \longrightarrow \mathcal{G}(A/K) \longrightarrow 1$$

(Note that this is a reversal of the Rosenberg-Zelinski sequence.) Moreover, it is easy to verify that the action of $\mathcal{G}(A/K)$ on $\mathcal{P}_K(A)$ is defined by letting an α "twist" the operation of A on a projective left A- module P.

As an example, let $K = \mathbb{Z}$ and let A be the algebraic integers in a number field L over $\mathbb{Q}$. Then $\mathcal{P}_{\mathbb{Z}}(A)$ is known to be the class group, C(L), and $\mathcal{G}(A/Z) = \mathcal{G}(L/\mathbb{Q})$. Moreover the action above is the usual action of the galois group on the class group.

We conclude:

<u>Theorem:</u> If $\underset{\sim}{A}$ is the ring of integers in an algebraic number field L, then

$$\mathrm{Aut}\,({}_{A}\mathfrak{M})$$

is the split extension of the class group, C(L), of L, by the absolute Galois group, $\mathfrak{G}(L/\mathbb{Q})$, where C(L) is viewed as a $\mathfrak{G}(L/\mathbb{Q})$ module in the usual way.

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