How to quantize infinitesimally-braided symmetric monoidal categories

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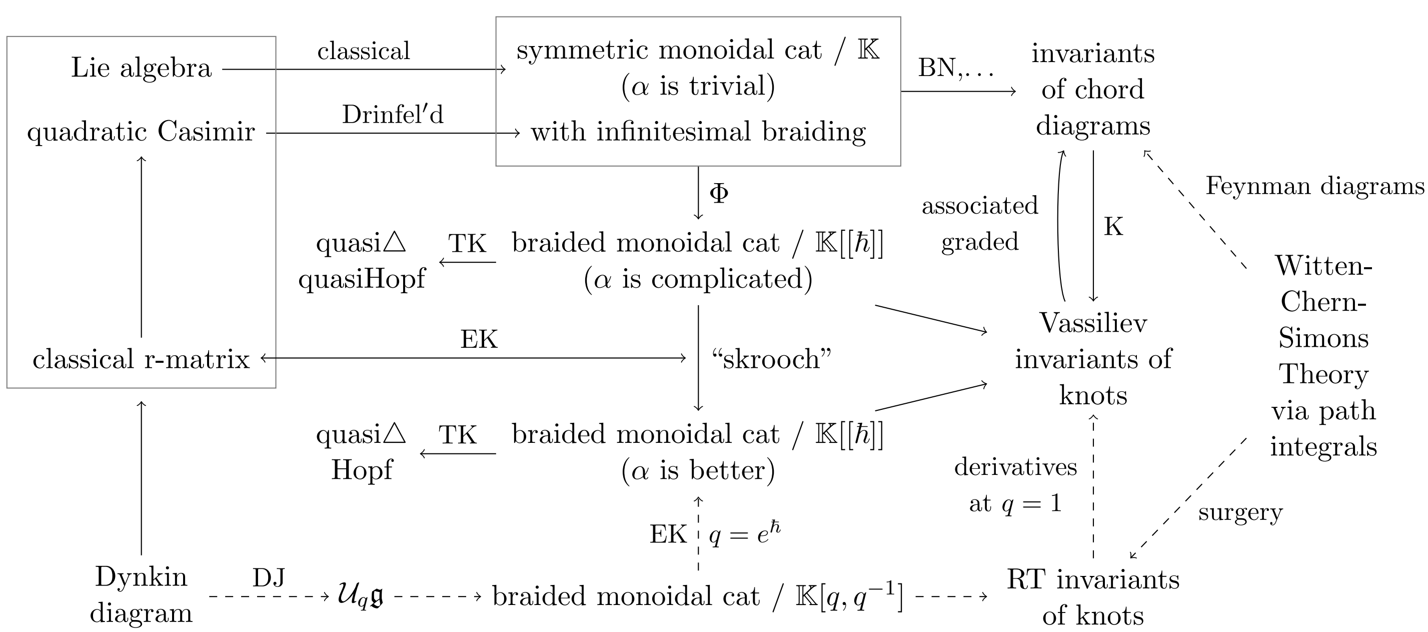
Abstract

An infinitesimal braiding on a symmetric monoidal category is analogous to a Poisson structure on a commutative algebra: both tell you a "direction" in which to "quantize". In this expository talk, I will tell a story that was completed by the end of the 1990s, concerning the quantization problem for infinitesimally-braided symmetric monoidal categories. Along the way, other main characters will include: a Lie algebra, a quadratic Casimir, and a classical R-matrix; braided monoidal categories, associators, and pentagons and hexagons; Tannakian reconstructions theorems and Hopf and quasiHopf algebras; and everyone's favorite knot invariants. I'll explain all these words, and try to explain how they're all part of a single story.

0 Outline



1 Introduction

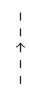
First things first. This is an expository talk — I certainly don't claim any of the material as my own. Indeed, almost everything I'm going to say has been known for almost two decades. Most of what I'll say is based on the last chapter of [21]. I'll try to give correct attribution for results, but

I will undoubtedly miss someone. In any case, experts can leave. The outline for the talk is given above: I will try to talk about all the solid arrows, but I've included the dashed arrows just to help situate the story I'm going to tell as part of a larger story.

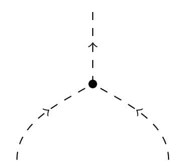
In any case, once and for all let's pick a field $\mathbb{K}$ of characteristic zero — actually, it's enough for this to be a ring containing $\mathbb{Q}$. In general our algebras, etc., will be infinite-dimensional, but a **representation** of some algebraic structure will always mean a finite-dimensional representation (when $\mathbb{K}$ is not a field, it should be a finitely generated (topologically) free $\mathbb{K}$-module). So VECT will always mean the category of finite-dimensional vector spaces over $\mathbb{K}$.

2 Casimir Lie algebras

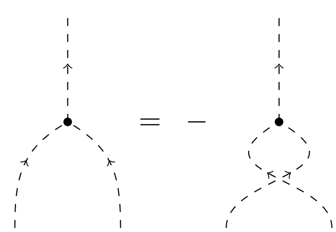
It will be convenient to adopt the usual graphical notation in VECT, and I will read time going up. Recall that a **Lie algebra** is a vector space, which I will draw as:



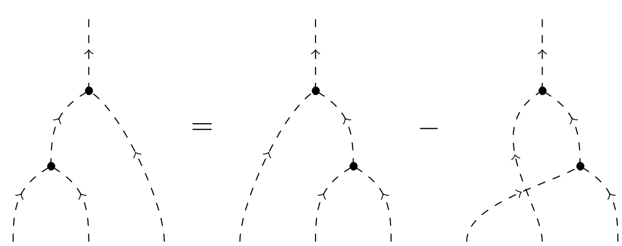
along with a bracket:



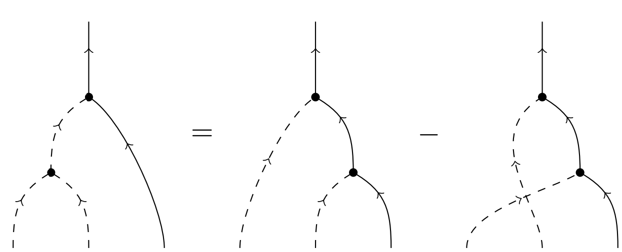
which should be **antisymmetric**:



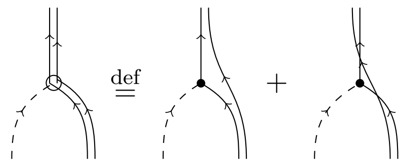
and also must satisfy the **Jacobi identity**:



A **representation** of a Lie algebra is a (finite-dimensional) vector space and a vertex as follows:

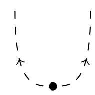


It's clear, anyway, that if $\mathfrak{g}$ is finite-dimensional, then it is a representation of itself. The tensor product of representations is defined by:

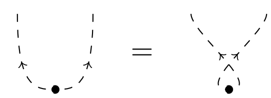


Then $\mathfrak{g}$-REP is a symmetric monoidal category. It also has duals, which are very important, but which I will probably not address in this talk. It is an abelian category, and the forgetful functor Forget : $\mathfrak{g}$-REP $\to$ VECT is exact and monoidal. We will come back to these comments later.

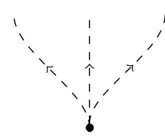
Let $\mathfrak{g}$ be a Lie algebra. A **quadratic Casimir** is an element $t \in \mathfrak{g}^{\otimes 2}$ so that (1) it is symmetric, and (2) its image under the multiplication map $\mathfrak{g}^{\otimes 2} \to \mathcal{U}\mathfrak{g}$ is central. We will draw it as:



Then condition (1) says that you can twist it without changing it:

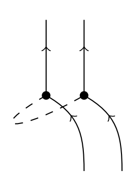


Condition (2) says that you can move trivalent vertices past the t vertex. So the idea is to use the Casimir to turn up any external edges, and drop the arrows on the Lie algebra. For example, the bracket becomes a totally antisymmetric trivalent vertex:



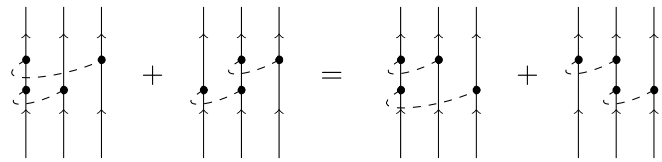
Dropping the arrows, the Jacobi identity becomes the equation that's usually called "$I = H - X$," and the rule for a representation is the "$S = T - U$" relation — I believe that the IHX relation owes its name to [5], and the STU name comes from physics, although of course they are really the same relation. Another way of expressing the Jacobi relation is that this trivalent vertex is central under $\mathfrak{g}^{\otimes 3} \to \mathcal{U}\mathfrak{g}$. Another way of expressing both the conditions on the Casimir and the Jacobi are that they are cocycles in the correct Lie cohomology theory.

What extra structure on $\mathfrak{g}$-REP does the chosen Casimir give? The Casimir is uniquely designed to act on pairs of representations, via:



It's standard to be sloppy in the notation, and lose track of whether the dashed line is hitting the solid line from the left or from the right. You can do this by always assuming that any vertex is a left action, although you may need to be a little careful with signs, especially if you want to think about dual representations. Moreover, since the Casimir t is central in $\mathcal{U}\mathfrak{g}$, the above is a morphism $X \otimes Y \to X \otimes Y$ in $\mathfrak{g}$-REP, and it's natural. Since t is symmetric, this natural transformation commutes with the flip map $\sigma : X \otimes Y \to Y \otimes X$.

Even better, the STU relation implies (in fact, is equivalent to) the **four-term relation**:



These conditions — that t is a natural transformation of $\otimes$ to itself, that it commutes with the flip map σ, and that it satisfies the four-term relation — make the category $(\mathfrak{g}\text{-REP}, \sigma, t)$ into a **infinitesimally-braided symmetric monoidal category**.

Incidentally, the way you'll see this in the literature is as follows. The idea is to pick a single representation $V \in \mathfrak{g}\text{-REP}$, restrict to the subcategory that it tensor-generates, and then see that this category is a representation of the **infinitesimal braid category**. This category is defined analogously to the **braid category**, which you probably know. The objects are natural number $n \in \mathbb{Z}_{\geq 0}$, and $\mathrm{Hom}(n,m) = \emptyset$ if $n \neq m$, and $\mathrm{Hom}(n,n) = \mathcal{B}_n$ is the Artin braid group on n strands. Similarly, the **infinitesimal braid group** on n strands is actually a Lie algbera (not a group at all), generated by $\binom{n}{2}$ elements $t^{ij} = t^{ji}$ for $i \neq j = 1, \dots, n$, satisfying $[t^{ij}, t^{kl}] = 0$ if i,j,k,l are all distinct, and $[t^{ij}, t^{ik} + t^{jk}] = 0$ if $|\{i,j,k\}| = 3$.

But, anyway, why the word "infinitesimally braided"? Here's the idea.

Proposition *Let $(\mathcal{C}, \otimes, \sigma, t)$ be an infinitesimally-braided symmetric monoidal category (abelian over $\mathbb{K}$), and let $\mathcal{C}[\hbar]/\hbar^2$ be the category with the same objects, but with $\mathrm{Hom}_{\mathcal{C}[\hbar]/\hbar^2}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes_{\mathbb{K}} \mathbb{K}[\hbar]/\hbar^2$. Let $s = \sigma \circ (1 + \hbar t/2)$. Then $(\mathcal{C}, \otimes, s)$ is a braided monoidal category.*

Conversely, let $(\mathcal{C}, \otimes, s)$ be a braided monoidal category over $\mathbb{K}[[\hbar]]$, and suppose that $\mathcal{C}/\hbar$ is symmetric, i.e. $s - s^{-1}$ is divisible by $\hbar$, so let $t = (s^2 - 1)/\hbar$. Then $(\mathcal{C}/\hbar, \otimes, [s], [t])$ is infinitesimmally-braided symmetric, where $[s], [t]$ are the images of s, t in $\mathcal{C} \twoheadrightarrow \mathcal{C}/\hbar$.

So this is all to say that infinitesimally-braided symmetric monoidal categories are *begging* to be deformation-quantized to braided categories.

Incidentally, we constructed an infinitesimally braided symmetric monoidal category from any Casimir Lie algebra. In fact, these are essentially all of them, but not quite [31]. See [29] for details: the idea is that actually you could have used a different symmetric structure, and define the Jacobi identity, etc., in terms of this different structure — e.g. take super vector spaces and super Lie algebras. Then that's basically it. Actually, there is some redundancy: different Casimir Lie algebras can give the same infinitesimally-braided categories [16].

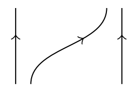
3 Quantization of infinitesimally-braided symmetric monoidal categories

We have explained the first two arrows in the outline. The next few arrows are essentially due to Drinfel′d [9, 10, 11]. The first step is to decide that we like exponentials better than polynomials, and so we use $\exp(\hbar t/2)$ in place of $1 + \hbar t/2$ — these agree up to $O(\hbar^2)$. So, let's start with an infinitesimally braided category $(\mathcal{C}, \otimes, \sigma, t)$, and write down the category $(\mathcal{C}[[\hbar]], \otimes)$, where we just extend everything by $\hbar$-linearity to formal power series. Now let's try to construct a non-trivial structure on it. The proposal is: let's just *declare* that the braiding is going to be $s = \sigma \circ \exp(\hbar t/2)$.

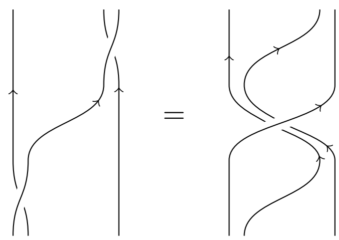
But this doesn't work. Recall that a braiding on a category must satisfy $(s_{X,Y} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_Y \otimes s_{X,Z}) = s_{X,Y\otimes Z}$ in order for the diagram where two strands cross one strand to be well-defined. But for $s = \sigma \circ \exp(\hbar t/2)$, this is the requirement that $\exp(\hbar t^{12}/2)\exp(\hbar t^{13}/2) =$

$\exp(\hbar(t^{12}+t^{13})/2)$, which is false since $[t^{12},t^{13}]\neq 0$ in general. I don't think that there is power series $f(A)=1+A+\ldots$ for which $f(A)f(B)=f(A+B)$ and $f(A)^{-1}f(B)^{-1}=f(A+B)^{-1}$ when $[A,B]\neq 0$ — the second equation is from passing a string the other way.

So are we stuck? Not quite. Drinfel′d's idea is to keep $s=\sigma\circ\exp(\hbar t/2)$, but to relax the *associator*. This is because really the law refers to the associator explicitly:



Then the braiding should satisfy two **hexagons**, one of which is:



So let's try to find a suitable associator.

Since we don't have much to work with, let's look for a power series $\Phi(A,B)$ in two noncommuting variables such that $\alpha_{XYZ}=\Phi(\hbar t_{XY},\hbar t_{YZ})$ is an associator on $(\mathcal{C}[[\hbar]],\otimes)$ and satisfies the above equations with the braiding given by $s_{XY}=\exp(\hbar t_{XY}/2)$. Such a power series is called a **Drinfel′d associator**. Precisely, the equations are:

$$\Phi(0,0)=1 \qquad \text{(so that when } \hbar\to 0,\ \alpha \text{ specializes to the usual associator)}$$
$$\Phi(t^{12},t^{23}+t^{24})\,\Phi(t^{13}+t^{23},t^{34})=\Phi(t^{23},t^{34})\,\Phi(t^{12}+t^{13},t^{24}+t^{34})\,\Phi(t^{12},t^{23}) \qquad (\pentagon)$$
$$\Phi(t^{23},t^{13})\,\exp\big((t^{13}+t^{12})/2\big)\,\Phi(t^{12},t^{23})=\exp(t^{13}/2)\,\Phi(t^{12},t^{13})\,\exp(t^{12}/2) \qquad (\hexagon_1)$$
$$\Phi(t^{12},t^{23})\,\exp\big(-(t^{13}+t^{23})/2\big)\,\Phi(t^{13},t^{12})=\exp(-t^{23}/2)\,\Phi(t^{13},t^{23})\,\exp(-t^{13}/2) \qquad (\hexagon_2)$$

Here t^{ij} are the generators of the infinitesimal braid group on 4 strands. Then $\pentagon$ is equivalent to the usual pentagon equation for α, and the two $\hexagon$s are the hexagons for a braided category.

The point is that Drinfel′d proves that such things exist, and better, it's of the form $\Phi(A,B)=\exp(\text{Lie series in } A,B)$. Not reading Russian, I'm not entirely sure how he came up with these guys — I think he was thinking about actual physics, and Khono's theorem. Anyway, I'll outline the proof. By studying a certain differential equation, Drinfel′d constructed explicitly a particular solution Φ_{KZ} (for "Knizhnik-Zamolodchikov") over $\mathbb{K}=\mathbb{C}$. In [23], it is given explicitly in terms of multiple zeta functions

$$\zeta(i_1,\ldots,i_k)=\sum_{0<m_1<\cdots<m_k\in\mathbb{N}}\frac{1}{m_1^{i_1}\ldots m_k^{i_k}}$$

and starts out:

$$\Phi_{\mathrm{KZ}}(A,B)=1+\frac{1}{24}[A,B]+\frac{\zeta(3)}{(2\pi i)^3}\big([[A,B],B]-[A,[A,B]]\big)+\ldots$$

But this is not the only solution. The complex conjugate is one, and there is a known one that's "even", i.e. real [1, 27]. But actually Drinfel′d proves that there exist rational solutions as well.

So take a Drinfel′d associator over $\mathbb{K}$, and use it as above: declare that $\alpha_{XYZ}=\Phi(\hbar t_{XY},\hbar t_{YZ})$. And declare the braiding to be $s_{XY}=\exp(\hbar t_{XY}/2)$. This gives $(\mathcal{C}[[\hbar]],\otimes)$ the structure of a braided monoidal category. It is a deformation of $(\mathcal{C},\otimes,s)$, and by a theorem of Deligne, it must have well-behaved duals ("tortile") [33].

4 Reconstruction theorems and skrooches

So, what have we done? We started with a Casimir Lie algebra $(\mathfrak{g}, t)$ and a Drinfel′d associator Φ and constructed a braided category $(\mathcal{C}[[\hbar]], \otimes, s)$ over $\mathbb{K}[[\hbar]]$. Actually, we have a bit more: we had a faithful exact functor $\text{Forget} : \mathfrak{g}\text{-}\textsc{rep} \to \textsc{Vect}_{\mathbb{K}}$, and this gives a faithful exact functor $\text{Forget} : \mathcal{C}[[\hbar]] \to \textsc{Vect}_{\mathbb{K}[[\hbar]]}$. And it still has a natural isomorphism (actually, the identity isomorphism) $\phi : \text{Forget}(X \otimes Y) \to \text{Forget}\, X \otimes \text{Forget}\, Y$.

Now recall the Tannaka-Krein philosophy, e.g. [20].

Theorem *Let $\mathcal{C}$ be an abelian K-linear category, for K a commutative ring, and let $F : \mathcal{C} \to \textsc{Vect}_K$ be a faithful exact functor, where "$\textsc{Vect}_K$" means "finitely-generated projective K-modules". Then there is a K-linear coalgebra A and $(\mathcal{C}, F)$ is equivalent as a category to $(A\text{-}\textsc{corep}, \text{Forget})$. Moreover, structure on $(\mathcal{C}, F)$ determines structure on A.*

The usual reconstruction is to get an *algebra* $A^* = \text{End}(F)$, the natural transformations $F \to F$. Actually, A^* has a distinguished predual that is a coalgebra. It is constructed by finding a representing object for $V \mapsto \text{Nat}(F, V \otimes F)$.

A typical version is:

Theorem *Suppose that $\mathcal{C}$ is monoidal, and there is a natural isomorphism $\phi : F(X \otimes Y) \to F(X) \otimes F(Y)$. Suppose furthermore that the two ways of getting from $F\big((X \otimes Y) \otimes Z\big) \to F(X) \otimes \big(F(Y) \otimes F(Z)\big)$ using ϕ and the associators are the same, so that (F, ϕ) is a* **monoidal functor***. Then A is actually a bialgebra. Furthermore, if $\mathcal{C}$ has duals, then A is a Hopf algebra. Furthermore, if $\mathcal{C}$ has a braiding, then A is coquasitriangular.*

I'll define coquasitriangularity in a moment.

But unfortunately, this isn't the situation we're in. The problem is that the associator α in our $\mathbb{C}[[\hbar]]$ is very complicated. The best we can do right now is the following theorem of Majid [24]:

Theorem *Given $(\mathcal{C}, \otimes, \alpha)$ and $F : \mathcal{C} \to \textsc{Vect}$ and a natural isomorphism $\phi : F(X \otimes Y) \to F(X) \otimes F(Y)$ that might not preserve the associator (a* **multiplicative functor***), you can reconstruct a quasiassociative coassociative bialgebra ("coquasiHopf algebra").*

I won't tell you what a "quasiassociative coassociative bialgebra" is. But I will tell you what an "associative quasicoassociative bialgebra" is, and then you can reverse all the arrows to move the "co" around.

Definition *An* **associative quasicoassociative bialgebra** *or* **quasiHopf algebra** *is an associative unital algebra A, along with an algebra homomorphism $\Delta : A \to A \otimes A$, and an invertible element $\psi \in A^{\otimes 3}$ such that for each $a \in A$:*

$$(\Delta \otimes \text{id})(\Delta(a)) = \psi^{-1} \cdot (\text{id} \otimes \Delta)(\Delta(a)) \cdot \psi$$

where $\cdot$ is the multiplication in $A^{\otimes 3}$, and such that ψ satisfies a pentagon:

$$(\text{id} \otimes \text{id} \otimes \Delta)(\psi) \cdot (\Delta \otimes \text{id} \otimes \text{id})(\psi) = (1 \otimes \psi) \cdot (\text{id} \otimes \Delta \otimes \text{id})(\psi) \cdot (\psi \otimes 1) \qquad (\pentagon)$$

where now $\cdot$ is the multiplication in $A^{\otimes 4}$.

Oh, and there should be rules about counits, and for "quasiHopf" also some sort of antipode. At the category level, ψ acts as an A-module homomorphism $(X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$.

Drinfel′d argues that quasiHopf algebras are more basic than Hopf algebras. Here's why. Let A be a (quasi)Hopf algebra, and pick $\varphi \in A^{\otimes 2}$ invertible. Then define:

$$\Delta^{\varphi}(a) = \varphi^{-1} \cdot \Delta(a) \cdot \varphi$$

Drinfel′d considered this because of some "gauge transformation" ideas from physics. He calls this a word that translates to "twisting", but "twist" is a massively overused word in this area — this is a different twist from the "twists" of Reshetikhin-Turaev. So instead, I'll follow Stasheff's suggestion [28] and simply transliterate the Russian to "skrooch": the above is **skrooching by** φ.

Then Drinfel′d's observation is that the skrooch of an honestly coassociative Hopf algebra is no longer coassociative, just quasicoassociative, but skrooching preserves quasiHopf algebras. Now I should mention that Drinfel′d's original construction from a Casimir Lie algebra to a quasiHopf algebra was directly algebraic, rather than going through this categorical language. Note that under skrooches, the representation theory stays the same: A-REP and A^{φ}-REP are equivalent as monoidal abelian categories.

Now that we have the notion of "skrooches", I can tell you what a quasitriangular Hopf algebra is. Remember that on the category $\mathfrak{g}[[\hbar]]$ we constructed not just a monoidal structure $(\otimes, \alpha)$, but also a braiding s. By [24], this braiding descends to a coquasitriangular structure on the coquasiHopf algebra. Again, I won't tell you what a coquasitriangular structure is: you can reverse the arrows yourself (or invent "coskrooches"). A **quasitriangular (quasi)Hopf algebra** is a (quasi)Hopf algebra (A, Δ) along with a skrooch $\rho \in A^{\otimes 2}$ such that:

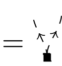
$$\Delta^{\rho} = \Delta^{\mathrm{op}}$$

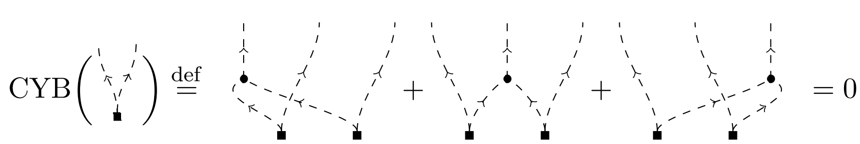
where $\Delta^{\mathrm{op}} = \sigma \circ \Delta$, and $\sigma : A \otimes A \to A \otimes A$ is the usual vector-space flip map. Also, ρ is required to satisfy some equations that correspond to the hexagons ($\hexagon$). Then it's clear that skrooching a quasitriangular quasiHopf algebra leads to a quasitriangular quasiHopf algebra.

Sometimes, though, you really do want an honestly coassociative Hopf algebra. Remember that from $(\mathfrak{g}, t)$ we constructed a quasiHopf algebra by quantizing $\mathfrak{g}$-REP, but leaving the forgetful functor untouched. The question is: can we find some other faithful exact functor $(\mathfrak{g}\text{-REP}[[\hbar]], \otimes, s) \to \mathrm{VECT}_{\mathbb{K}[[\hbar]]}$ that does preserve the associators? The answer is: yes, sometimes, by skrooching.

This is the topic of Etingof-Kazhdan [12]. Here's what they prove:

Theorem *Skrooches that take the quasitriangular quasiHopf algebra constructed above to an honestly coassociative quasitriangular Hopf algebra correspond bijectively with classical r-matrices for* $(\mathfrak{g}, t)$.

Then let me tell you what a classical r-matrix is. It is an element $r =$  $\in \mathfrak{g}^{\otimes 2}$ such that



and such that its symmetrization is the Casimir t:

I.e. $r + \sigma(r) = t$, where $\sigma : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 2}$ is the usual flip map. Then it follows that the antisymmetrization $d = r - \sigma(r)$ is such that $\mathrm{CYB}(d)$ is invariant under the $\mathfrak{g}$ action, i.e. a cocycle in some Lie algebra cohomology theory. Let $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ be given by $\delta(x) = [x, d]$. Then it is a coboundary, and in particular a cocycle, and $\mathrm{CYB}(d)$ being a cocycle is equivalent to δ satisfying the **coJacobi identity**, which I will let you write out or you can see e.g. [19, 14], and hence $(\mathfrak{g}, \delta)$ is a **Lie bialgebra**.

Let me give you one example of a classical r-matrix. Let $\mathfrak{g}$ be semisimple, pick a Cartan, a system of simple roots, etc. I.e. let $\mathfrak{g}$ come from a Dynkin diagram. Then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The usual killing form $\theta : \mathfrak{g}^{\otimes 2} \to \mathbb{K}$ is invertible as a map $\mathfrak{g} \to \mathfrak{g}^*$, and its inverse $\mathfrak{g}^* \to \mathfrak{g}$ is the usual Casimir $t \in \mathfrak{g}^{\otimes 2}$. It splits as

$$t = t_{\mathfrak{h}} + \sum_{\alpha \text{ a positive root}} (x_\alpha^+ \otimes x_\alpha^- + x_\alpha^- \otimes x_\alpha^+)$$

where $t_{\mathfrak{h}}$ is a quadratic Casimir for the abelian Lie algebra $\mathfrak{h}$,and $\{x_\alpha^\pm\}$ is the standard bases for $\mathfrak{n}^\pm$, at least up to some normalizing factors of 2. Anyway, the **standard Lie bialgebra structure on $\mathfrak{g}$ semisimple** is given by the r-matrix:

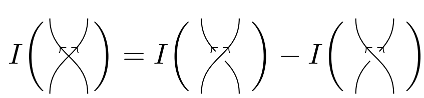
$$r = \frac{1}{2} t_{\mathfrak{h}} + \sum_{\alpha \text{ a positive root}} x_\alpha^+ \otimes x_\alpha^-$$

5 Vassiliev knot invariants

The last thing I'd like to say some words about are the Vassiliev knot invariants. The connection is essentially due to Cartier's [8]. And if you've read [5], it's almost obvious.

The first thing to point out is that a braided category is almost enough to define a knot invariant. By definition, braided categories give braid invariants, and if you have duals and some other coherency, you get knot invariants. The constructions above have duals and coherency, because you can't break this structure by deformation-quantizing [33], and $\mathfrak{g}$-REP has them. So given an element $V \in \mathfrak{g}$-REP, the structure $(\mathfrak{g}\text{-REP}[[\hbar]], \otimes, \alpha, s)$ turns the knot into a number in $\mathbb{K}[[\hbar]]$. To get $\mathbb{K}$-valued knot invariant, pick a number n and write down the coefficient on $\hbar^n$.

But it turns out that this coefficient is a Vassiliev invariant of degree n. Let me remind you what one of these is. Vassiliev says: given any (oriented) knot invariant I, extend it to an invariant of singular (oriented) knots by



A **Vassiliev invariant of degree** n is one that vanishes on singular knots with more than n singularities. The reason the coefficients on $\hbar$ are Vassiliev is because $s - s^{-1} = O(\hbar)$.

Suppose you have a Vassiliev invariant of degree n. Then on a singular knot with exactly n singular points, the invariant can't see any more of the topology besides the combinatorics of the singular points. These should be notated simply by chords on an oriented circle. In particular, a type-n invariant in particular induces invariants of chord diagrams with n chords. In fact, this invariant cannot distinguish diagrams that differ by the four-term relation, which can be seen by blowing up a triple intersection in two possible ways. Actually, denote a triple intersection by a trivalent vertex of chords: then the chord-and-trivalent vertices diagrams satisfy the STU relation. Actually, think about blowing up four-valent vertices in different ways: you get the IHX relations.

Thus, take the space $\mathcal{V}$ of Vassiliev invariants. It is a filtered algebra by degree. If you have two type-n invariants that differ by a type-$(n-1)$ invariant, then they give the same invariant of

n-chord diagrams. So there is a map [30, 5]:

$$\operatorname{gr}\mathcal{V} \to \{\text{invariants of chord diagrams}/4\text{T}\} = \{\text{invariants of trivalent diagrams}\}/\text{IHX, STU}$$

The LHS is the associated graded of the space of Vassiliev invariants. The RHS is a graded space, graded by the number of chords. Actually, it's much better, by a theorem of Kontsevich [22]:

Theorem *The above map is an isomorphism.*

One proof [8] is to go around the diagrams above: 4T is the same as the infinitesimal braid relations. I should mention that Kontsevich actually writes down an explicit integral expression for a knot invariant valued in chord diagrams, such that every Vassiliev invariant factors through it, and such that various things agree at the level of associated-graded.

Actually $\mathcal{V}$ is a commutative cocommutative Hopf algebra: you multiply knot invariants in $\mathbb{K}$ and comultiply by taking connect sum of knots. You can see the comultiplication at the Lie algebraic level, too: every dashed diagram realizes to a central element in $\mathcal{U}\mathfrak{g}$. This associated graded, when $\mathfrak{g}$ is semisimple, is nothing but the Duflo isomorphism. And in general taking associated graded of Hopf algebras should remind you of universal enveloping algebras. If it does, see [18].

6 The rest of the diagram

I'm totally out of time, so I will only briefly mention the dashed arrows on the diagram. Going along the bottom edge, Drinfel′d and Jimbo wrote down quasitriangular Hopf algebras over $\mathbb{K}(q)$ for each Dynkin diagram, and these specialize as $q \to 1$ to $\mathcal{U}\mathfrak{g}$ for $\mathfrak{g}$ coming from the Dynkin diagram, at least if you interpret "specialize" correctly. If you're very careful, you can set up a Hopf algebra over $\mathbb{K}[q, q^{-1}]$, and then a braided monoidal category with some extra structure. This is essentially what Reshetikhin and Turaev [26] do: the extra structure gives the interesting parts of the invariants. When q is specialized to a root of unity, they even construct three-manifold invariants [25]. The important point is that when $q = e^{\hbar}$, the Drinfel′d-Jimbo quantum groups, and hence the Reshetikhin-Turaev invariants, become the $\hbar$-deformations constructed by Etingof and Kazhdan [13]. It was known in [17, 7, 5] that the Jones, etc., polynomials were Vassiliev.

The very last few arrows, then, are about Chern-Simons-Witten theory, which is a long story unto itself, and was first introduced in [32]. Very briefly and extremely roughly, the story goes as follows. Pick a Lie group G with Lie algebra $\mathfrak{g}$, and consider the space $\mathcal{A}$ of all connections on the trivial G-bundle over the three-sphere S^3. This is an infinite-dimensional space isomorphic to $\Omega^1(S^3) \otimes \mathfrak{g}$. Let $A \in \mathcal{A}$; then it makes sense to wedge it with itself, etc. Suppose now that $\mathfrak{g}$ comes equipped with an invariant inner product $\langle,\rangle : \mathfrak{g}^{\vee 2} \to \mathbb{R}$. Then we can define the Chern-Simons action $\operatorname{CS}(A) = \int_{S^3} \big(\langle A \wedge dA\rangle + \frac{2}{3}\langle A \wedge [A \wedge A]\rangle\big)$. Here the notation is: wedge in the $\Omega^1(S^3)$ part, and $\langle,\rangle$ or $[,]$ in the $\mathfrak{g}$ part. This action is important because up to normalizing $\wedge$, the critical points are precisely the flat connections. Now let's pick a finite-dimensional representation X of $\mathfrak{g}$, and a knot $K : S^1 \hookrightarrow S^3$. Then we can define $\operatorname{hol}_{(X,A)}(K)$ to be the holonomy along the knot of the connection A in the representation X. I.e. it is: as you travel along the knot, the connection determines an element $g \in G$; take $\operatorname{tr}_X g$. Then Witten's idea is to consider, for each $(X, \mathfrak{g})$ and a knot K:

$$\int_{A\in\mathcal{A}} \operatorname{hol}_{(X,A)}(K) \exp\big(i\hbar^{-1} \operatorname{CS}(A)\big)$$

Actually, he didn't quite do this — he worked with non-trivialized principle bundles, and compact groups, and various things, because he felt it would make it more likely that the integral over the infinite-dimensional space could be defined. The argument is: since we've integrated out the dependence on the connection, the above is a knot invariant. In particular, when $\mathfrak{g} = \mathfrak{su}(2)$ and $X = \mathbb{C}^2$, Witten argues that the above knot invariant should be the Jones polynomial.

Integrals over infinite-dimensional spaces are something the physicists are very good at, ever since Feynman said that he could formulate quantum mechanics in terms of such an integral [15]. The traditional way to define such integrals is in terms of Feynman diagrams, which only make sense when $\hbar$ is a formal parameter. This definition has been worked out [6, 22]. In fact, it was by doing this that Kontsevich first wrote down his universal Vassiliev invariant.

Witten outlined another way to compute the above integral. To explain it requires explaining Witten's generalization. Actually, Witten considered arbitrary G-principle bundles on arbitrary three-manifolds M with boundary, and embedded tangles that end on the boundary. He argued that if an integral like the one above exists, then by the Fubini theorem it must be computable by cutting and pasting the manifolds. In particular, any three-manifold is achievable by surgery in well-understood ways, and Witten outlined how one would compute the above integral by surgery. This outline was realized and made rigorous in [26, 25]. For details, Witten's original paper [32] is highly readable.

In any case, I should mention one final thing. To define the Vassiliev invariants, etc., we started with a Casimir Lie algebra. But Witten starts with a **metric** Lie algebra: an algebra with an invariant metric. These are almost the same. If the metric is invertible, then its inverse is a quadratic Casimir, and conversely if the Casimir is invertible, then its inverse is a metric. This is the situation for simple Lie algebras: a simple Lie algebra has a unique quadratic Casimir, up to scalar, and a unique metric, up to scalar (the Killing form). But in general the (rigorous) algebraic story works with Casimirs, not metrics.

So what's going on? The deal is that near critical points, $\mathrm{CS}(A)$ is controlled first by its quadratic part, which is proportional to the metric. But you know from Gaussian integrals:

$$\int_{-\infty}^{\infty} x^{2k} \exp(-ax^2/2)\, dx = \sqrt{2\pi a^{-1}}\, a^{-k}(2k-1)(2k-3)\dots(3)(1)$$

What actually turns up in the integral are inverse powers of the quadratic part. So the final integral, at least perturbatively, should be a power series in the inverse to the metric, i.e. it's a power series in a Casimir, and makes sense even when said Casimir is not invertible. Some other perturbative Chern-Simons theory references: [2, 3, 4].

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