Combinatorial Calculus:
From Taylor Series to Feynman Diagrams

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Here are all the definitions and exercises from class. Certainly, there are too many exercises here for you to do. But try your hand at some. These notes are available at http://math.berkeley.edu/∼theojf/CombinatorialCalculusExercises.pdf.

Day 1: Taylor Series

Definition 1.1. A function is *continuous at* 0 if it satisfies

$$f(x) = f(0) + o(1)$$

Definition 1.2. We say that $f(x) = o(g(x))$ if $\lim_{x\to 0} f(x)/g(x) = 0$

Exercise 1.1. If $f = o(g)$ and a is any non-zero real number, then $f = o(ag)$. If $f = o(g)$ and $g = o(h)$, then $f = o(h)$.

Exercise 1.2. If $f(x)$ is any function, not necessarily continuous, then there's at most one real number a_0 such that

$$f(x) = a_0 + o(1)$$

Definition 1.3. A function $f(x)$ is *differentiable* at x if the *derivative* $f'(x)$ exists:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x+0)}{h}$$

If f is differentiable in a neighborhood of x (an open interval containing x), then $f'(x)$ is a function defined in that neighborhood, and its derivative at x, or in a whole neighborhood of x, might exist. If it does, the function f is called *twice differentiable at* x or *twice differentiable near (in a neighborhood of)* x. Similarly, we have functions that are k*-times differentiable near* x, etc.

Exercise 1.3. If $f(x)$ is any function such that $f(x) = a_0 + a_1 x + o(x)$, then $a_1 = f'(0)$.

Exercise 1.4. Given a function $f(x)$, there is at most one polynomial $a(x) = a_0 + a_1 x + \cdots + a_n x^n$ of degree n such that

$$f(x) = a(x) + o(x^n)$$

Definition 1.4. The *nth Taylor polynomial of f* is the degree-n polynomial $a(x)$, if it exists, such that

$$f(x) = a(x) + o(x^n)$$

Exercise 1.5. Let $f(x)$ be differentiable and $f(0) = 0$. Show that if $f'(x) = o(x^n)$, then $f(x) = o(x^{n+1})$. The converse is harder: if f' is continuous in a neighborhood of 0 and $f(x) = o(x^n)$, then $f'(x) = o(x^{n-1})$.

Exercise 1.6. *Taylor's theorem:* Let $f(x)$ be n-times differentiable. Then $f(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n)$ where $a_k = f^{(k)}(0)/k!$. (Hint/Outline: If f is differentiable, by definition $f(x) = a_0 + a_1 x + o(x)$. Use the previous exercise to show that if $f'(x) = a(x) + o(x^k)$ for some polynomial $b(x)$ of degree k, then there is a polynomial $B(x)$ of degree $k+1$ such that $f(x) = b(x) + o(x^{k+1})$. Hence, by induction, show that if f is n-times differentiable, then $f(x) = a(x) + o(x^n)$ for some polynomial $a(x)$ of degree n. Either by finding a formula relating $b(x)$ and $B(x)$ above, or using the previous exercise, prove the formula for the Taylor coefficients.)

Exercise 1.7. Here's an alternate description of the coefficients of the Taylor polynomial:

1. Define $f^{[0]}(x) = f(x)$, and define $f^{[k]}(x)$ by induction:

   $$f^{[k]}(x) = \begin{cases} \dfrac{f^{[k-1]}(x) - f^{[k-1]}(0)}{x}, & x \neq 0 \\ \lim\limits_{x \to 0} \dfrac{f^{[k-1]}(x) - f^{[k-1]}(0)}{x}, & x = 0 \end{cases}$$

   Show that if $f(x) = o(x^n)$ then $f^{[k]}(x)$ exists for $k \leq n$.

2. Show that if $f(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n)$ is continuous, then $f^{[k]}(x)$ exists for $k \leq n$, and that $a_k = f^{[k]}(0)$.

Definition 1.5. A *formal function* or *formal power series* is a sequence of numbers, thought of as a series.

Exercise 1.8. Use Exercise 1.6 to find the Taylor series for e^x, using the fact that the derivative of e^x is e^x. Use Exercise 1.7 to find the Taylor series for $1/(1-x)$.

Definition 1.6. Let $f(x) = \sum f_{(n)}\, x^n/n! = \sum f_{[n]}\, x^n$ and $g(x) = \sum g_{(n)}\, x^n/n! = \sum g_{[n]}\, x^n$ be formal functions. The *sum* and *product* are given by

$$\begin{aligned}
(f+g)_{(n)} &= f_{(n)} + g_{(n)} &(1)\\
(f+g)_{[n]} &= f_{[n]} + g_{[n]} &(2)\\
(f\cdot g)_{(n)} &= \sum_{k=0}^{n} \binom{n}{k} f_{(k)} \cdot g_{(n-k)} &(3)\\
(f\cdot g)_{[n]} &= \sum_{k=0}^{n} f_{[k]} \cdot g_{[n-k]} &(4)
\end{aligned}$$

The *derivative* of f is given by

$$(f')_{(n)} = f_{(n+1)} \qquad\qquad (f')_{[n]} = (n+1) f_{[n+1]} \tag{5}$$

Exercise 1.9. Justify the above formulas. (Be sure to show that they are consistent: we gave two formulas for each concept.)

Exercise 1.10. Show that the formal power series 0 given by $0_{[n]} = 0_{(n)} = 0$ is an additive identity of formal power series. Explain how to subtract formal power series. Show that the formal power series 1 given by $1_{(0)} = 1_{[0]} = 1$ and $1_{(n)} = 1_{[n]} = 0$ for $n \geq 1$ is a multiplicative identity.

Exercise 1.11. Using Exercise 1.8, show that the Taylor series f for e^x satisfies $f' = f$, and that the Taylor series g for $1/(1-x)$ satisfies $(1-x)\cdot g = 1$.

Exercise 1.12. Let f be a formal function. Find a formula for the coefficients of the nth power f^n of f; find formulas using each notion of coefficients.

Exercise 1.13. If f is a formal power series with $f_{(0)} = f_{[0]} \neq 0$, find a formula for the coefficients of f^{-1}, the multiplicative inverse of f. (Hint: Long division works.)

Day 2: Chain Rule

Exercise 2.1. If f and g are honest functions with $f(x) = a(x) + o(x^n)$ and $g(x) = b(x) + o(x^n)$ where a and b are polynomials, show that $f(x)\,g(x) = a(x)\,b(x) + o(x^n)$.

Exercise 2.2. Prove that if f and g are differentiable in a neighborhood of 0, then so is the product $f \cdot g$. Argue by induction that if f and g are n-times differentiable in a neighborhood of 0, then so is the product $f \cdot g$.

Exercise 2.3. By approximating f and g by Taylor polynomials of degree n, prove the generalized product rule:

$$\frac{d^n}{dx^n}\left[f(x)\,g(x)\right]\bigg|_{x=0} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(0)\, g^{(n-k)}(0) \tag{6}$$

Definition 2.1. If $m_1, \ldots, m_k$ are non-negative integers with $m_1 + \cdots + m_k = n$, then the *multinomial coefficient* is the number of ways of sorting n labeled objects into k boxes, such that the jth box has m_j numbers in it. The formula is given by

$$\binom{n}{m_1, \ldots, m_k} = \binom{n}{\vec{m}} = \frac{n!}{m_1! \ldots m_k!}$$

If $k = n = 0$, so that the string $\vec{m} = (m_1, \ldots, m_k)$ is the empty string, then we declare that

$$\binom{0}{} = 1$$

Exercise 2.4. Generalize the product rule further: Let $f_1, f_2, \ldots, f_k$ be k different smooth (infinitely differentiable) functions. Show that the nth derivative of the product is given by

$$\frac{d^n}{dx^n}\left[\prod_{j=1}^{k} f_j\right]\Bigg|_{x=0} = \sum_{\substack{\vec{m}\geq 0 \text{ s.t.} \\ \sum m_j = n}} \binom{n}{\vec{m}} \prod_{j=1}^{k} f^{(m_j)}(0) \tag{7}$$

The sum ranges over strings of length k of nonnegative integers such that the sum of the string is n.

Definition 2.2. If f and g are formal functions with $g_{[0]} = 0$, then the *composition* $f \circ g$ is defined by

$$(f \circ g)_{[n]} = \sum_{\substack{\vec{m}>0 \text{ s.t.} \\ \sum m_j = n}} f_{[|\vec{m}|]} \prod_{j=1}^{|\vec{m}|} g_{[m_j]} \tag{8}$$

The sum ranges over strings $\vec{m} = (m_1, m_2, \ldots, m_k)$ of positive integers of any length. $\sum m_j$ is the sum of the terms, and $|\vec{m}|$ is the number of terms.

Exercise 2.5. When $g_0 = 0$, find a formula for $(f \circ g)_{(n)}$ in terms of $f_{(j)}$ and $g_{(k)}$. Interpret it combinatorially.

Exercise 2.6. Define $\ln(1+x)$ to be for formal power series with $\ln_{[0]} = 0$ and $\ln_{[n]} = (-1)^{n-1}/n$. Compute $\exp(\ln(1+x))$. Also, $\exp(x) - 1$ has $(\exp - 1)_0 = 0$. Compute $\ln(1 + (\exp(x) - 1))$.

Exercise 2.7. Check the following calculuation:

$$\begin{aligned} \frac{d^n}{dx^n}\left[f(g(x))\right]\Big|_{x=a} &= \left(\tilde{f} \circ \tilde{g}\right)_{(n)} \\ \text{where } \tilde{f}(x) &= f(x + g(a)) \\ \text{and } \tilde{g}(x) &= g(x+a) - g(a) \end{aligned}$$

Theorem 2.1 (Generalized Chain Rule)**.** If $f(x)$ and $g(x)$ are n-times differentiable functions everywhere, then

$$\frac{d^n}{dx^n}\left[f(g(x))\right]\Big|_{x=a} = \sum_{\substack{\vec{m}>0 \text{ s.t.} \\ \sum m_j = n}} \frac{1}{|\vec{m}|!}\binom{n}{\vec{m}} \frac{d^{|\vec{m}|}f}{dx^{|\vec{m}|}}\Big|_{x=g(a)} \prod_{j=1}^{|\vec{m}|} \frac{d^{m_j}g}{dx^{m_j}}\Big|_{x=a} \tag{9}$$

Exercise 2.8. Another way to describe a partition is by counting how many boxes of each size. If $\vec{b} = (b_1, b_2, \ldots, b_n)$ is a string of non-negative integers such that $b_1 + 2b_2 + \cdots + nb_n = n$ and $b_1 + b_2 + \cdots + b_n = k$, then we can construct a partition in the previous sense by setting $m_1 = m_2 = \cdots = m_{b_1} = 1$, and in general setting the next b_j boxes to size j. Prove the following alternate formula for the chain rule (this was the one originally found by Faà di Bruno):

$$\frac{d^n}{dx^n}\left[f(g(x))\right]\Big|_{x=a} = \sum_{\substack{\vec{b}\geq 0 \text{ s.t.} \\ \sum j b_j = n}} \frac{n!}{1!^{b_1}\, b_1!\, 2!^{b_2}\, b_2! \, \ldots \, n!^{b_n}\, b_n!} f^{(\sum b_j)}(g(a)) \prod_{j=1}^{n} \left(g^{(j)}(a)\right)^{b_j} \tag{10}$$

Here the sum ranges over strings $(b_1, \ldots, b_n)$ of non-negative integers with $b_1 + 2b_2 + \cdots + nb_n = n$. One really ought to come up with good notation for that fraction of factorials.

Day 3: Feynman Diagrams

Definition 3.1. A *Feynman Diagram* is a directed graph, with vertices labeled by formal functions.

Definition 3.2. To *evaluate* a Feynman Diagram consists of two steps:

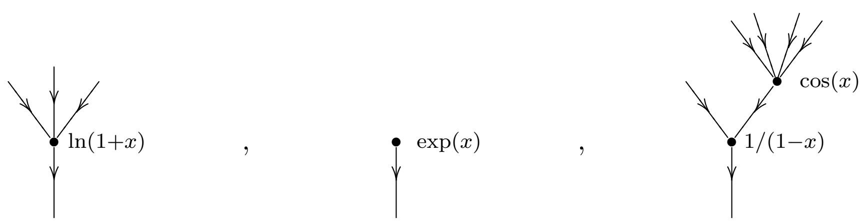
1. For each vertex in the diagram, write down the corresponding number: a vertex labeled by $f(x) = \sum f_{(n)} x^n/n!$ with n incoming edges and one outgoing edge is assigned the number $f_{(n)} = f^{(n)}(0)$. The function(s) f used in the diagram are often called “Feynman Rules”.

2. All the numbers in a diagram are multiplied together, and then divided by the number of symmetries of the diagram.

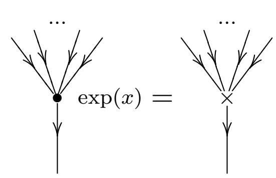
3. Given a set of diagrams, the evaluation of the set is the sum of the evaluations. So usually we will just write $\sum$ for “set of”.

Usually, we will confuse Feynman Diagrams and their evaluations.

Exercise 3.1. Evaluate the following diagrams:

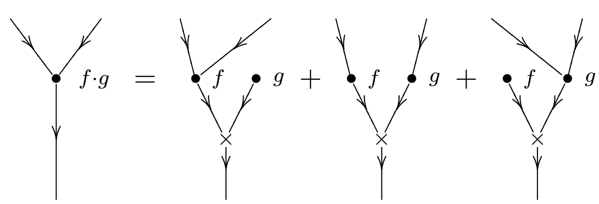


Definition 3.3. Because we will use it so much, write $\times$ for any vertex labeled by exp:

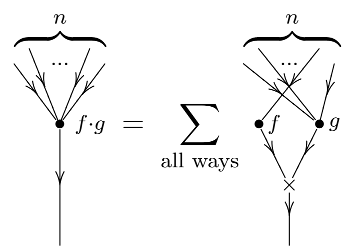


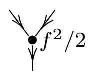
Exercise 3.2. Why is this a reasonable notation?

Exercise 3.3. By evaluating the diagrams, check the following equation:

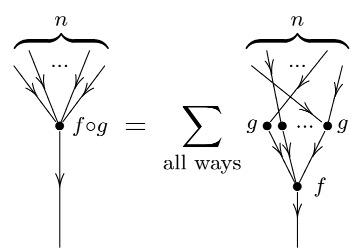


Exercise 3.4. Generalize to any vertex labeled by the product $f \cdot g$. You should end up convincing yourself of the following formula:



Exercise 3.5. Write  as a sum of diagrams. Why does the "/2" matter?

Exercise 3.6. Convince yourself of the following formula, when $g_0 = 0$:



Exercise 3.7. Let $a > 0$ be a real number. Show that:

$$\int_{-\infty}^{\infty} \frac{x^n}{n!}\, e^{-ax^2/2}\, dx = \begin{cases} 0, & n \text{ is odd} \\ \sqrt{\frac{2\pi}{a}}\, \frac{a^{-k}}{2^k\, k!} & n = 2k \end{cases}$$

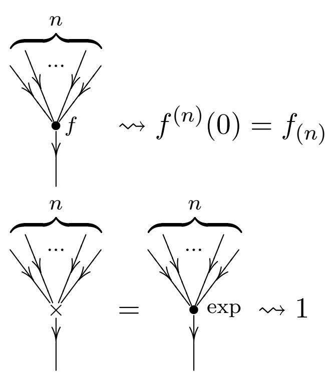
Days 4 and 5: Integration

Definition 5.1. Given a formal function f, we define a formal function $\frac{1}{\int 1}\int f$ by

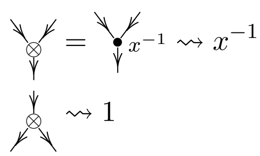
$$x \mapsto \frac{\displaystyle\int_{-\infty}^{\infty}\sum_{n=0}^{\infty} f_{(n)}\frac{t^n}{n!}\,e^{-x^{-1}t^2/2}\,dt}{\int_{-\infty}^{\infty} e^{-x^{-1}t^2/2}\,dt} = \sum_{k=0}^{\infty} f_{(2k)}\frac{x^k}{2^k\,k!}$$

Remark. When x is positive, the integral $\int_{-\infty}^{\infty} e^{-x^{-1}t^2/2}\,dt$ converges to $\sqrt{2\pi x}$. Thus the numerator can be understood as a formal power series in $\sqrt{x}$. But this is inelegant, and the division yields an answer that does not depend on the measure. Thus, this formula can be generalized to settings that do not have measures.

Definition 5.2. Recall that a *Feynman rule* is any entry in the dictionary explaining how to translate vertices in diagrams into numbers. We already saw two entries:



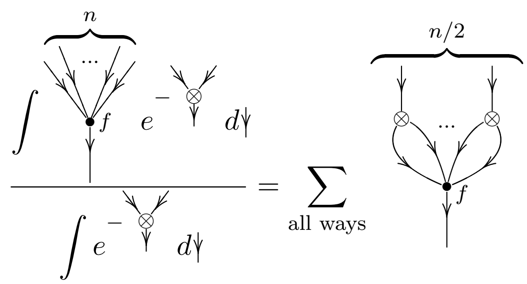
To these we add two more, which are related:



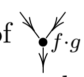
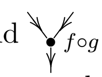
Remark. The notation $\otimes$ is meant to connote two different things. In the upside-down Y, the tensor symbol should suggest that we think of x, coming in the top, as splitting into a "tensor product" of two numbers going out the bottom, undoing a multiplication. In the standard-orientation Y, the vertex $\otimes$ suggests the letter x, although really a value of x^{-1} is attached. However, x is a dummy variable. In the standard-orientation Y, it's best to think of x^{-1} as coming up the bottom string: it's a wrong-way facing input, rather than a right-way facing output. Don't necessarily read to much into this pneumonic.

Nevertheless, I hope the notation suggests that these two vertices are somewhat like inverses. If we were to label every external edge by a vertex, then incoming edges would receive the name of the input variable. For composition to behave correctly, an outgoing edge should be assigned the inverse of its variable. Thus, when we hook an outgoing edge of one vertex to an incoming edge of another vertex, the variables on the now-internal edge cancel.

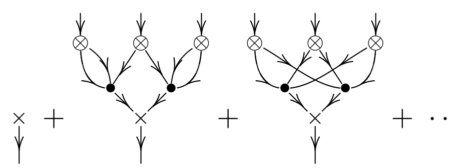
Exercise 5.1. In diagrammatic notation, integration is given by



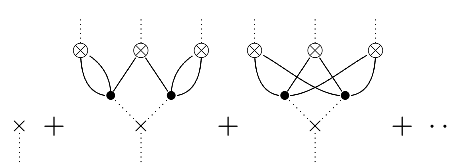
Make sure you understand the above formula/notation.

Exercise 5.2. Compute the integrals of  and  by first expanding each vertex as a sum of diagrams, and then integrating each summand.

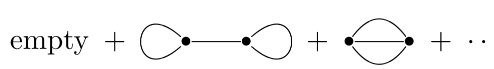
Exercise 5.3. Write out the first few diagrams in $\frac{1}{\sqrt{2\pi x}}\int_{-\infty}^{\infty}\exp(t^3/3!)\,e^{-x^{-1}t^2/2}\,dt$. For example, the first few non-zero diagrams (using $\bullet$ for the $t^3/3$ vertices) are:



Highlight the edges that connect $\otimes$ vertices to $\bullet$ vertices, and make faint the incoming, outgoing, and $\bullet{-}\times$ edges. Also let's suppress the arrows:



Now interpret just the $\bullet$ vertices and highlighted edges as making trivalent graphs in their own right:



Prove that for each trivalent graph, there is exactly one graph in the original sum of diagrams that yields it upon highlighting. Prove that the number of symmetries of the original graph equals the number of symmetries of the highlighted graph.

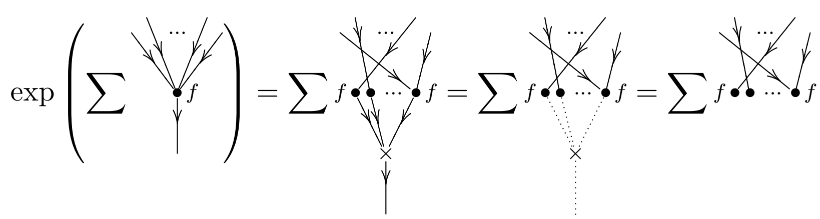
Exercise 5.4. By Exercise 5.3, we can write the integral $\frac{1}{\sqrt{2\pi x}} \int_{-\infty}^{\infty} \exp(t^3/3!)\, e^{-x^{-1}t^2/2}\, dt$ in terms of trivalent graphs. Explain how to evaluate each trivalent graph. (The instructions should always be to convert each component into a number, multiply the numbers, and divide by symmetry. Does this work? What are the values of the vertices? We have suppressed the incoming edges — how do you know how many xs to multiply each graph by to get a term in power series?)

Exercise 5.5. By counting trivalent graphs, show that the formal power series in Exercise 5.4 has zero radius of convergence.

Exercise 5.6. Let f be a formal function that starts in degree 3: i.e., $f_0 = f_1 = f_2 = 0$. Repeat Exercises 5.3 and 5.4 to write out the first few terms of the formal power series for

$$\frac{1}{\sqrt{2\pi x}} \int_{-\infty}^{\infty} \exp(f(t))\, e^{-x^{-1}t^2/2}\, dt$$

Definition 5.3. While we're erasing edges, let's look more closely at $\exp(f(t))$. By the chain rule, we have

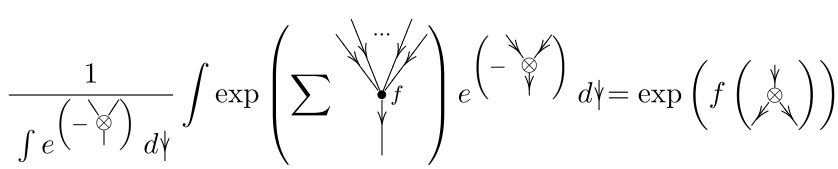


The sum on the left-hand side consists of a collection of connected diagrams; the sum on the right can have disconnected ones. So we sum up this calculation by:

$$\exp\left(\sum_{\substack{S \text{ a set of} \\ \text{connected} \\ \text{diagrams}}}\right) = \sum_{\substack{\text{diagrams with all} \\ \text{components in } S}}$$

Of course, no diagram in S should have zero incoming edges (although the incoming edges might be invisible if we are writing the diagrams in the above "highlighted" way).

Exercise 5.7. Make sense of and justify the following formula:



Exercise 5.8. What happens when you use this method to compute

$$\frac{1}{\sqrt{2\pi x}} \int_{-\infty}^{\infty} \exp(at^2/2)\, e^{-x^{-1}t^2/2}\, dt$$

for some number a? Use the power series expansions of $1/(1-x)$ and $\ln(1+x)$.

Day 6: The Final Lecture

Definition 6.1. Let f be a formal function starting in degree three. By highlighting some edges and writing others in invisible ink, we have:

$$\frac{1}{\sqrt{2\pi x}} \int_{-\infty}^{\infty} \exp(f(t))\, e^{-x^{-1}t^2/2}\, dt = \exp\left(\sum_{\substack{\text{connected} \\ \text{graphs}}} \right) = \sum_{\text{graphs}} \tag{11}$$

The first sum ranges over connected graphs of degree three and higher; the second over all graphs of degree three or higher. We have the following Feynman rules: (1) Each vertex of degree $n \geq 3$ is assigned the value $f^{(n)}(0)$. (2) Each edge is labeled by the formal variable x. As always, we multiply the values of all components of a graph, and divide by the number of symmetries of the graph.

Exercise 6.1. Let f and g be formal functions — f must start in degree three, but g can be anything. Explain how to evaluate the following integral as a sum of graphs:

$$\frac{1}{\sqrt{2\pi x}} \int_{-\infty}^{\infty} g(t)\, \exp(f(t))\, e^{-x^{-1}t^2/2}\, dt$$

Exercise 6.2. Let $\hbar$ be a non-zero number, and consider the integral in which we have divided every exponent by $\hbar$.

$$\frac{1}{\sqrt{2\pi\hbar x}} \int_{-\infty}^{\infty} \exp\left(\hbar^{-1} f(t)\right)\, e^{-x^{-1}\hbar^{-1}t^2/2}\, dt$$

Thus, we divide every vertex and multiply each x by $\hbar$. Assuming that the coefficients of f are non-zero only in degree 3 and higher, show that there are only finitely many graphs in each power in $\hbar$. What is the power? I.e., without knowing anything about f, how can you tell what power of $\hbar$ is in each graph?

Exercise 6.3. If f starts in degree three, then by the previous exercise the sum over graphs

$$\frac{1}{\sqrt{2\pi\hbar x}} \int_{-\infty}^{\infty} \exp\left(\hbar^{-1} f(t)\right)\, e^{-x^{-1}\hbar^{-1}t^2/2}\, dt$$

has only finitely many terms in each power in $\hbar$. Hence, let's change our minds: x is any non-zero number, and $\hbar$ is the formal variable. Thus, let $F(t) = -x^{-1}t^2/2 + f(t)$ be any formal power series that starts in degree two and whose degree-two coefficient F_2 is non-zero. What are the rules to evaluate

$$\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp\left(\hbar^{-1} F(t)\right)\, dt$$

as a sum of graphs?

Exercise 6.4. Let g be any formal function, and F a formal function such that $F_0 = F_1 = 0$ and F_2 is non-zero. How do you express the following integral as a sum of diagrams?

$$\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} g(t)\, \exp\left(\hbar^{-1} F(t)\right)\, dt$$

Exercise 6.5. How many graphs are there where every vertex has degree four? Let F_2 and F_4 be non-zero numbers, and write

$$\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp\left(\hbar^{-1}\left(F_2 t^2/2 + F_4 t^4/4!\right)\right)\, dt$$

as a sum of diagrams, and hence as a formal power series in $\hbar$. What is the radius of convergence of this series? For what $\hbar$, F_2, and F_4 does the above integral converge as a Riemann integral?

Exercise 6.6. This problem is primarily for those who've done calculus in multiple variables. Everything we've done works for formal functions of many variables. A formal function in two variables, for instance, is a power series $f(x, y) = \sum_{n,m=0}^{\infty} f_{n,m} x^n y^m / n!m!$. Then edges should come in two colors: x edges and y edges. (Alternately, one can develop the theory of tensors.) A formal function from $\mathbb{R}^2$ to $\mathbb{R}^2$, then, is a pair (f, g) of formal functions. The chain and product rules continue to apply just as stated above.

The notion of integration needs only one modification. The second-derivative of a formal function f of two variables is a triple of numbers, corresponding to the quadratic terms $f_{2,0}x^2/2 + f_{1,1}xy + f_{0,2}y^2$. These can be put together into a matrix $x = \begin{pmatrix} f_{2,0} & f_{1,1} \\ f_{1,1} & f_{0,2} \end{pmatrix}$; then x^{-1} still makes sense, and the only change in the formulas for integration is that $\sqrt{2\pi x}$ should be replaced by $\sqrt{\det(2\pi x)}$.

Go back through these exercises, reinterpreting everything in multiple variables. In particular, see if you can work out explicitly the generalized chain rule in multiple variables. To read more about the multiple-variables chain rule and its combinatorial interpretation, see the paper by Michael Hardy: http://arxiv.org/abs/math/0601149.