1 Problem 1

a) Convince yourself that there is only one ground state in the quantum mechanics of a particle in a double well potential. The Hamiltonian is

$$\begin{aligned} H &= -\frac{d^2}{dx^2} + V(x) \ , \\ V(x) &= (x^2 - a^2)^2 \ . \end{aligned} \tag{1.1}$$

b) Estimate the energy splitting between the two lowest energy states. (Hint: instantons)
c) Try to convince yourself that quantum mechanics of a single particle moving within any bounded potential $V(x) \in L^2_{loc}(\mathbb{R})$, $\lim_{x\to\pm\infty} V(x) = +\infty$ has only one ground state.

2 Problem 2

Consider a generic Hamiltonian with parametrized by n parameters. Determine the codimension of the level-crossing loci.

3 Problem 3

Consider the quantum mechanics of a particle on a ring with magnetic field. The Euclidean Lagrangian is

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 + \frac{i\theta}{2\pi}\dot{q} \ , \tag{3.1}$$

where $q \sim q + 2\pi$.
a) Convince yourself $\theta \sim \theta + 2\pi$.
b) Solve the spectrum and show that the spectrum at $\theta = 0$ is the same one as $\theta = 2\pi$.
c) Compare how charge conjugation $q(t) \to -q(t)$ acts at $\theta = 0$ and $\theta = \pi$.
d) Show that the $O(2)$ symmetry acts a $Pin^+(2)$ group on the Hilbert space at $\theta = \pi$.
e) Now modify the Lagrangian by adding a potential

$$\lambda \cos(2q) \ . \tag{3.2}$$

Show that at $\theta = 0$, the ground state degeneracy lifted by order λ while at $\theta = \pi$, the ground state degeneracy is not lifted.

4 Problem 4

Consider $2N$ number of Majorana fermions γ_i, which obey the algebra

$$\{\gamma_i, \gamma_j\} = \delta_{ij} \ . \tag{4.1}$$

a) Construct a 2^N dimensional Hilbert space using this Majorana fermions.
b) Show that time-reversal symmetry $\gamma_i \to -\gamma_i$ forbids quadratic terms in the Hamiltonian.
c) Show that when $N = 2$, one can pick a Hamiltonian such that there are only 2 degenerate ground states out of 4 states.
d) Show that when $N = 4$, one can construct a Hamiltonian such that there is a unique ground state.

5 Problem 5

Let us revisit the particle on a ring example. Coupling the Euclidean Lagrangian to $U(1)$ background gauge field A for the $U(1)$ symmetry: $q \to q + \alpha$. We get

$$\mathcal{L} = \frac{1}{2}(\dot{q} - A)^2 + \frac{i\theta}{2\pi}(\dot{q} - A) + ikA\,, \tag{5.1}$$

where the last term is a counterterm.
a) How does the Lagrangian transforms under the charge conjugation symmetry at $\theta = \pi$?
b) Show that k has to be an integer in order to preserve the $U(1)$ gauge symmetry of $A \to A + d\alpha$.
c) Convince yourself that with nontrivial A, preserving the background gauge symmetry of A breaks the charge conjugation symmetry at $\theta = \pi$. And, preserving the charge conjugation symmetry at $\theta = \pi$ breaks the background gauge symmetry of A.
d) Show that charge conjugation symmetry and the background gauge symmetry of A can be preserved by coupling the quantum mechanics to a 1+1d bulk with a classical Lagrangian

$$\mathcal{L}_{1+1} = \frac{i}{2}\int F\,. \tag{5.2}$$

e) Convince yourself that because of charge conjugation symmetry, we can not continuously deform the classical Lagrangian (5.2) to a trivial one.
d) Now let us add a potential term $\cos(2q)$ to Lagrangian. Convince yourself that the $U(1)$ symmetry is broken to $\mathbb{Z}_2$. And the $\mathbb{Z}_2$ symmetry have a mixed anomaly with the charge conjugation symmetry. What is the classical bulk Lagrangian that cancels the anomaly?

6 Problem 6

Show that every 2-manifold have a $Pin^-(2)$ structure

$$w_2 = \text{Euler number mod 2} = w_1 \cup w_1\,. \tag{6.1}$$

Hint: it can be checked explicitly

Which 2-manifold has $Pin^+(2)$ structure?

7 Problem 7

Show that 2d Majorana fermion has a time-reversal symmetric mass term with $T^2 = 1$ where T is the time-reversal symmetry.

8 Problem 8

Construct an $SO(3)$ bundle on T^4 with fractional instanton.