1 February 13, 2008

The first problem set is given out today. It's elementary, but explores positivity to prevent against wrong intuitions. Due next week.

For the homework, some notation: e.g. a^+ for a in a C^*-algebra is just the positive part of the function $a \in C^*(a, 1) = C(M)$.

1.1 We continue the discussion from last time

Prop: Let A be a $*$-normed algebra with two-sided approximate identity e_λ of norm 1. Let μ be a continuous positive linear funcitonal on A. Then

(a) $\mu(a^*) = \overline{\mu(a)}$

(b) $|\mu(a)|^2 \leq \|\mu\| |\mu(a^* a)|$

Proof:

(a) $\mu(a^*) = \lim \mu(a^* e_\lambda) = \lim \langle e_\lambda, a\rangle_\mu = \lim \overline{\langle a, e_\lambda \rangle} = \lim \overline{\mu(e_\lambda^* a)} = \overline{\mu(a)}$.

(b) $\underbrace{|\mu(e^* a)|^2}_{\to |\mu(a)|^2} = |\langle a, e_\lambda\rangle_\mu|^2 \overset{C.S.}{\leq} \langle a, a\rangle_\mu \langle e_\lambda, e_\lambda\rangle_\mu = \mu(a^* a)\mu(e_\lambda^* e_\lambda) \leq \mu(a^* a)\|\mu\|$.

$\square$

Question from the audience: Do we assume positivity of e_λ? Self-adjointness? **Answer:** Positivity only makes sense in a C^*-algebra. We can assume self-adjointness: if e_λ is two-sided, then e_λ^* is also a two-sided approximate identity, so $(e_\lambda + e_\lambda^*)/2$ is a self-adjoint approximate identity.

Prop: Let A be a $*$-normed algebra (without identity). Let μ be a positive linear functional on A. Suppose that μ satisfies the two results of the previous proposition. Let $\tilde{A}$ be A with 1 adjoined, and define $\tilde{\mu}$ on $\tilde{A}$ by $\tilde{\mu}(a + z1) = \mu(a) + z\|\mu\|$. Then $\tilde{\mu}$ is positive.

Proof:

$$
\begin{aligned}
\tilde{\mu}\left((a+z1)^*(a+z1)\right) &= \mu(a^*a)+\mu(a^*z)+\mu(\bar{z}a)+\mu(\bar{z}z) \\
&= \mu(a^*a)+\underbrace{z\,\underbrace{\mu(a^*)}_{=\overline{\mu(a)}}+\bar{z}\mu(a)}_{2\Re(\bar{z}\mu(a))}+|z|^2\|\mu\| \\
&\geq \mu(a^*a)-2|z||\mu(a)|+|z|^2\|\mu\| \\
&\geq \mu(a^*a)-2|z|\|\mu\|^{1/2}\mu(a^*a)^{1/2}+|z|^2\|\mu\| \\
&= \left(\mu(a^*a)^{1/2}-|z|\|\mu\|^{1/2}\right)^2 \\
&\geq 0
\end{aligned}
$$

$\square$

Cor: Let A be a $*$-normed algebra with approximate identity of norm 1. Let μ be a continuous positive linear functional on A. Define $\tilde{\mu}$ on $\tilde{A}$ by $\tilde{\mu}(a+z1)=\mu(a)+z\|\mu\|$. Then $\tilde{\mu}\geq 0$.

Theorem: Let $(\pi,\mathcal{H},\xi_0)$ be the GNS representation for $\tilde{A},\tilde{\mu}$. Then when π is restricted to A, we might worry that it is degenerate. But in fact it is non-degenerate. In particular, $\xi_0\in\overline{\text{span}\{\pi(A)\mathcal{H}\}}$. Also, $\mu(a)=\langle\pi(a)\xi_*,\xi_*\rangle$.

Proof:

Let $\{a_j\}$ be a sequence of states of A such that $\|a_j\|\leq 1$ and $\mu(a_j)\to\|\mu\|$. Then view these in the GNS Hilbert space.

$$
\begin{aligned}
\|\underbrace{\xi_{a_j}}_{=a_j\text{ in }\mathcal{H}}-\xi_*\|^2=\left\langle\xi_{a_j}-\xi_*,\xi_{a_j}-\xi_*\right\rangle_{\tilde{\mu}} &= \mu(a_j^*a_j)-\mu(a_j^*)-\mu(a_j)+\tilde{\mu}(1) \\
&\leq \|\mu\|-\overline{\mu(a_j)}-\mu(a_j)+\|\mu\| \\
&\to 0
\end{aligned}
$$

$\square$

This completes the GNS business.

1.2 Irreducible representations

Prop (do earlier): Let A be a $*$-algebra with 1 (or really a $*$-set, e.g. a group where $x^*\overset{\text{def}}{=}x^{-1}$). Let $(\pi,\mathcal{H})$ be a non-degenerate representation of A (i.e. it assignes $*\mapsto$ adjoint). Then $(\pi,\mathcal{H})$ is the direct sum (possibly vast) of cyclic representations.

Definition: A subspace $\mathcal{K}$ of $\mathcal{H}$ is π-*invariant* if $\pi(A)\mathcal{K}\subseteq\mathcal{K}$.

Prop: If $\mathcal{K}$ is π-invariant, then so is $\mathcal{K}^\perp$. So $\pi=\pi|_{\mathcal{K}}\oplus\pi|_{\mathcal{K}^\perp}$.

Proof:

Let $\xi \in \mathcal{K}^\perp$, $a \in A$, with $\pi(a)\xi \in \mathcal{K}^\perp$. Let $\eta \in \mathcal{K}$. Then $\langle \pi(a)\xi, \eta\rangle = \langle \xi, \underbrace{\pi(a^*)\eta}_{\in\mathcal{K}}\rangle = 0$. $\square$

So we have a sort of "semi-simplicity".

Proof of "do earlier" Prop:

Choose $\xi \in \mathcal{H}$, $\xi \neq 0$. Let $\mathcal{H}_1$ be $\overline{\pi(A)\xi}$. This is obviously π-invariant (by continuity). So $\mathcal{H}_1^\perp$ is π-invariant. Choose $\xi_2 \in \mathcal{H}_1^\perp$, $\xi_2 \neq 0$, so $\mathcal{H}_2 \stackrel{\text{def}}{=} \overline{\pi(A)\xi_2}$. Then $\mathcal{H}_2 \perp \mathcal{H}_1$. Then $(\mathcal{H}_1 \oplus \mathcal{H}_2)^\perp$ is π-invariant. Choose $\xi_3, \ldots$. Really, use Zorn. $\square$

In the separable case, we choose a countable dense "basis", and work with that sequence, eschewing Choice.