1 February 1, 2008

Today we present a few simple consequences of this theorem of positivity. Notation: we say $a \leq b$ is $b - a \in A^+$ and a and b are self-adjoint.

Prop: If $0 \leq a \leq b$, then $\|a\| \leq \|b\|$.

Proof:

We can assume $1 \in A$. Then $b \leq \|b\|1$, by looking at $C^*(b, 1)$. I.e. $\|b\|1 - b, b - a \in A^+$, and add, so $\|b\|1 - a \in A^+$. Then $\|a\| \leq \|b\|$ by looking at $C^*(a, 1)$. $\square$

Prop: If $a \leq b$, then for any $c \in A$, we have

$$c^*ac \leq c^*bc$$

Proof:

$b - a \in A^+$ so $b - a = d^2$ for $d \geq 0$. Then $c^*bc - c^*ac = c^*(b - a)c = c^*d^2c = (dc)^*(dc) \geq 0$. $\square$

1.1 Ideals

We now turn our attention to two-sided ideals of C^* algebras. These generally do not have identity elements; we will talk about approximate identities. Recall: for a normed algebra A without 1, a *left approximate identity* for A is a net $\{a_\lambda\}$ in A such that $e_\lambda a \to a$ for all $a \in A$. These don't always exist — stupid counterexample is a Banach space with multiplication always 0. The notions of "right approximate identity" and "two-sided approximate identity" are obvious. We can also ask if our approximate identity is "bounded": does there exist a k so that $\|e_\lambda\| \leq k$ for every λ. Similarly for "approximate identities of norm 1": $\|e_\lambda\| \leq 1 \forall \lambda$.

In the commutative case, $C_c(X)$ of fns with compact support is a dense ideal in $C_\infty(X)$. In the non-commutative case, e.g. $B_0(\mathcal{H})$ of compact operators, there are trace-class operators $\mathcal{L}^1$, Hilbert-Schmidt $\mathcal{L}^2$, etc. Alain Connes has advocated viewing compact operators, and specifically elements of these ideals, as infinitesimals. In our department, we have a leading expert: Voidjitski ****spell?****. These are non-closed ideals, which are important, but in this course, we will mostly focus on closed ideals.

Theorem: Let L be a left ideal (not necessarily closed) in a C^*-algebra A. Then L has a right approximate identity $\{e_\lambda\}$ with $e_\lambda \geq 0$ and $\|e_\lambda\| \leq 1$.

We can even arrange that if $\lambda \geq \mu$, then $e_\lambda \geq e_\mu$, but we will not take the time to show that. Davidson does this for closed ideals. Indeed, what he shows is that $\{a \in A^+, \|a\| < 1\}$ with the usual ordering really is a net, and that for A a complete C^* algebra without identity, then this is an approximate identity — closed ideals are complete C^* algebras without identity.

If L is separable, then we can have an approximate identity given by a sequence (no need for nets), although for us, we allow any cardinality.

Proof of Theorem:

We can assume that A has an identity element.

Choose a dense subset S of L (e.g. $S = L$, or if L is separable, we can take S countable). Set $\Lambda =$ finite subsets of S ordered by inclusion. For $\lambda = \{a_1, \ldots, a_n\} \in \Lambda$, set $b_\lambda = \sum a_j^* a_j$. Well, L is a left ideal, so this is in L, and is positive.

Well, look at $\left(\frac{1}{n}1 + b_\lambda\right)$, which is certainly a strictly-positive function, and hence invertible, so we define

$$e_\lambda = \left(\frac{1}{n}1 + b_\lambda\right)^{-1} b_\lambda \in L \cap A^+$$

Then checking norms of both multiplicands, we see that $\|e_\lambda\| \leq 1$.

Claim: this is a right approximate identity. First we show for $a \in S$, and then by boundedness and density, we will have the desired result. We examine

$$\begin{aligned}
\|a - ae_\lambda\|^2 &= \|(a - ae_\lambda)^*(a - ae_\lambda)\| \\
(a - ae_\lambda)^*(a - ae_\lambda) &= (1 - e_\lambda)^* a^* a (1 - e_\lambda) \\
&\leq (1 - e_\lambda)^* b_\lambda (1 - e_\lambda) \\
&= \left(1 - \left(\frac{1}{n} + b_\lambda\right)^{-1} b_\lambda\right)^2 b_\lambda \\
&= \frac{1}{n} \frac{\frac{1}{n}}{\frac{1}{n} + b_\lambda} \frac{b_\lambda}{\frac{1}{n} + b_\lambda} \\
\|\text{RHS}\| &\leq \frac{1}{n}
\end{aligned}$$

Where we have assumed that $a \in \lambda$, and hence $b_\lambda \geq a^* a$. $\square$

Cor: And C^*-algebra A has a two-sided approximate identity, self-adjoint with norm 1.

Proof:

Let $\{e_\lambda\}$ be a right approximate identity. But it is self-adjoint, so it's a left approximate identity for a^*. $\square$

Cor: Let I be a closed two-sided ideal in A. Then I is closed under $*$, so it is a C^* algebra.

Proof:

Let $\{e_\lambda\}$ be a right approximate identity, and look at $e_\lambda a^* - a^*$. Because it's a two-sided ideal, $e_\lambda a^* \in I$. But $\|e_\lambda a^* - a^*\| = \|ae_\lambda - a\| \to 0$, so since I is closed, $a^* \in I$. $\square$

Cor: If I is a closed two-sided ideal in A, and J is a closed two-sided ideal in I, then J is a closed two-sided ideal in A.

Proof:

Let $d \in J$ and $a \in A$; we want to show $ad \in J$. Well, take $\{e_\lambda\}$ an approximate identity for I. Then $\underbrace{\underbrace{ae_\lambda}_{\in I} d}_{\in J} \to ad$. $\square$

Cor: If I, J are closed two-sided ideals in A, then $I \cap J = IJ \overset{\text{def}}{=}$ closed linear span of products.

Proof:

$\supseteq$ is clear. On the other hand, let $d \in I \cap J$. Consider $de_\lambda \in IJ$, where $\{e_\lambda\}$ is an approximate ideal for J. But IJ is closed, so $de_\lambda \to d$ must be in IJ. $\square$

Next time: quotient spaces.