****This document was last updated on April 18, 2008. A more recent version may be available as part of http://math.berkeley.edu/∼theojf/CstarAlgebras.pdf.****

1 April 18, 2008

Question from the audience: When we defined the lie group of a Lie Algebra, we said $X \in \mathfrak{g}$ iff $\exp X \in G$. Whenever I've seen this defined, the latter part was $exp(tX) \in G$. Are they equivalent?
Answer: No, you want tX. E.g. there are matrices so that $\exp X = 1$, but $\exp(X/2) \notin G$.

1.1 Smooth structures

Let G be a connected closed subgroup of $GL(n, \mathbb{R})$. In particular, we elide a course on Lie group theory, but G is a submanifold of GL. Let $\mathrm{Lie}(G) = \mathfrak{g} = \{X \in M_n(\mathbb{R}) : \exp(tX) \in G\, \forall t \in \mathbb{R}\}$. Then if $X, Y \in \mathfrak{g}$, then $[X, Y] = XY - YX \in \mathfrak{g}$.

The situation we were in: α is a strongly-continuous action of G on a Banach space B by bounded linear maps: $\alpha : G \to \mathcal{B}(B)$. (The same theory with semigroups comes up as well; when G is a group, clearly we map into the invertible maps.) We defined the derivative $D_X b$, if it exists. From there we could define multiple derivatives, and hence the class of C^∞ elements $B^\infty \subseteq B$. This is obviously a linear subspace of B.

Theorem: (Gärding)

B^∞ is a dense linear subspace in B.

Proof:

The buzzwords are "smoothly" and "molifies".

(**E.g.** Let α be an action of $\mathbb{R}$ on M a manifold, which could be chaotic evolution. (We won't define this, but see the paper: Lorenz just died, and he brought to life chaotic theory.) We get an action on $C_\infty(M) = B$, and there will be functions that are differentiable in this sense.)

Let $b \in B$ be given, and let $f \in C_c^\infty(G)$. Then we claim $\alpha_f(b) \in B^\infty$. But let f run over an approximate identity; then these will converge to b. Recall:

$$\alpha_f(b) \stackrel{\text{def}}{=} \int_G f(x)\, \alpha_x(b)\, dx$$

where dx is Haar measure; f has compact support, so this is a continuous B-values function.

So, why is $\alpha_f(b) \in B^\infty$? Go back to the definition: given $X \in \mathfrak{g}$, we look at

$$\frac{1}{t}\left(\alpha_{\exp(tX)}(\alpha_f(b)) - \alpha_f(b)\right)$$

Does this have a limit? For fixed t, we can pull into the integral sign, and commute past the number $f(x)$:

$$\begin{aligned}
\frac{1}{t}\left(\alpha_{\exp(tX)}(\alpha_f(b)) - \alpha_f(b)\right) &= \frac{1}{t}\left(\int_G f(x)\,\alpha_{\exp(tX)}\alpha_x(b)\,dx - \text{something}\right) \\
&= \frac{1}{t}\left(\int_G f(x)\,\alpha_{\exp(tX)x}(b)\,dx - \text{something}\right) \\
&= \frac{1}{t}\left(\int_G f(\exp(-tX)x)\,\alpha_x(b)\,dx - \text{something}\right) \\
&= \int_G \frac{f(\exp(-tX)x) - f(x)}{t}\,\alpha_x(b)\,dx
\end{aligned}$$

And the inside fraction is just a derivative of a scalar-function in the Lie group: it's just $D_X(f)(x) \stackrel{\text{def}}{=} \lim \frac{f(\exp(-tX)x)-f(x)}{t}$. We have Taylor series: $f(\exp(-tX)x) = f(x)+(D_Xf)(x)+\frac{1}{2}(D_X^2f)(x) + O(t^3)$. Thus the difference is

$$\frac{f(\exp(-tX)x) - f(x)}{t} - (D_Xf)(x) = (\text{continuous})(t) \underset{t\to 0}{\longrightarrow} 0$$

uniformly in x (we have compact support). Thus we can integrate, so

$$\int_G \frac{f(\exp(-tX)x) - f(x)}{t}\,\alpha_x(b)\,dx \underset{t\to 0}{\longrightarrow} \int_G (D_Xf)(x)\,\alpha_x(b)\,dx = \alpha_{D_Xf}(b)$$

But $D_Xf \in C_c^\infty(B)$, so $D_Y(\alpha_{D_Xf}(b)) = \alpha_{D_YD_Xf}(b)$. Iterate, and $\alpha_f(b) \in B^\infty$. $\square$

Definition: The *Gärding domain* is the linear span of $\{\alpha_f(b) : f \in C_c^\infty(B), b \in B\}$.

This is certainly dense in B, and contained in B^∞. Did everybody catch why? **Question from the audience:** Don't you need that your representation is nondegenerate? **Answer:** The representation is coming from an action. And such things are always nondegenerate, because we have approximate identities. For instance, let f approximate a delta function at the identity in G. Then

$$\begin{aligned}
\alpha_f(b) - b &= \int (f(x)\alpha_x(b) - b)dx \\
\|\alpha_f(b) - b\| &= \|\int (f(x)\alpha_x(b) - b)dx\| \\
&\leq \sup\{\|\alpha_x(b) - b\| : x \in \text{support}(f)\}
\end{aligned}$$

Question from the audience: These are unbounded operators. If we're in a C^* algebra, and give it the standard Hilbert structure, we can ask if these are adjointable? **Answer:** I don't know if that's been looked at. We can ask if the Gärding domain is equal to B^∞. Dixmier-Melhann **??** looked at a related question. They asked something like whether each element of $C_c^\infty(G)$ is a convolution of things in there. They found groups for which that's false, although any element is a finite sum of convolutions. There are places where knowing things like that it useful. But we will be working where the Lie algebra is abelian.

Theorem: On B^∞, $[D_X, D_Y] = D_X D_Y - D_Y D_X = D_{[X,Y]}$.

This is a basic and important fact, and takes some analysis.

Suppose that A is a Banach algebra, and α is a (strongly continuous) action of G on A by Banach-alg automorphisms. It's easy to prove (a la freshman calculus) that

$$D_X(ab) = (D_X(a))b + a(D_X(b))$$

for $a, b \in A^\infty$. **Cor:** A^∞ is a subalgebra of A.

Also, if A is a $*$-algebra and α is by $*$-automorphisms, then A^∞ is a $*$-subalgebra.

Question from the audience: It seems like our notion of smoothness depends on the group. Should we look for a maximal action of some sort to make sure we have the right functions? **Answer:** That seems like a good idea, but nobody knows how to do that. If you look at examples (interesting ones, nothing pathological), you find that they may have very few actions by Lie groups. Then there doesn't appear to be much differentiable structure. But the leap that Alain Connes has taken is to say that A is a C^*-algebra, and view $A \subseteq \mathcal{B}(\mathcal{H})$. Let D be an unbounded self-adjoint operator on $\mathcal{H}$. Let $U_t = e^{2\pi i t D}$, and let $\alpha_t(T) = U_t T U_t^*$ for $T \in \mathcal{B}(\mathcal{H})$. We can talk about smooth vectors $\mathcal{B}(\mathcal{H})^\infty$. There's no reason the action should carry the algebra into itself, but it may happen that $A \cap \mathcal{B}^\infty(\mathcal{H})$ is dense in A. $a \in A \cap \mathcal{B}^\infty(\mathcal{H})$ iff, more or less (only densely define), $[D, a]$ is a bounded operator (on a dense domain, so extends — well, this is once differentiability, so need to repeat. This picks out a smooth subalgebra of A. Then an operator is being used in this way, Connes calls this a *Dirac operator*. Because it matches the notion on a Riemannian manifold, and indeed we can recover the metric from the Dirac operator. So Connes says that this is the way to do non-commutative Riemannian geometry. **Question from the audience:** What is $\mathcal{H}$ in this manifold case? $L^2(M)$? **Answer:** No, it's L^2 with values in the spinor bundle.

1.2 Returning to our main example

Ok, let's return to the case at hand. We have A_θ, α, T^d. α is an action of T^d on a Banach space B. For $n \in \widehat{T^d} = \mathbb{Z}^d$, we have B_n defined by

$$B_n = \{b \in B : \alpha_x(b) = \langle x, n \rangle b\}$$

where, of course, $\langle x, n \rangle = e^{2\pi i x \cdot n}$. For fixed $b \in B_n$, its span is an invariant one-dimensional subspace in B_n.

On $\mathbb{R}^d$, we vie the Lie algebra and Lie group as

$$\begin{pmatrix} 0 & 0 & \vec{v} \\ 0 & \ddots & \\ 0 & & 0 \end{pmatrix} \overset{\exp}{\mapsto} \begin{pmatrix} 1 & 0 & \vec{v} \\ 0 & \ddots & \\ 0 & & 1 \end{pmatrix}$$

So really $\exp(X) = X$. Then for $b \in B_n$,

$$D_X(b) = \lim \frac{\alpha_{tX}(b) - B}{t} = \lim \frac{e^{2\pi i t X \cdot n} - 1}{t} b = (2\pi i X \cdot n) b$$

Question from the audience: Fourier transform takes differentiation to multiplication? **Answer:** Precisely.