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Any questions? No. Today we will look seriously at positive linear functionals.

## 1.1 Positive Linear Functionals

If you don't have an identity elements, things are more complicated, and we will have to deal with that; as always, we begin with the unital situation.

**Prop:** Let $A$ be a unital $*$-normed algebra. Let $\mu$ be a positive linear functional on $A$. If either

(a) $\mu$ is continuous (with respect to norm). This is situation in many examples.

(b) $A$ is complete (i.e. a Banach algebra). (If (a) holds, then $\mu$ extended to the completion of $A$, so reduces to (b), whereas if (b) holds, then $\mu$ is automatically continuous, as we will show.)

Then $\|\mu\| = \mu(1)$. (In particular, it is continuous.)

**Proof:**

We assume, by parenthetical remark above, that we are in case 2. above. We do not assume continuity. Certainly $1^* = 1 = 1^* \times 1$, so $\mu(1)$ is a nonnegative real number.

Consider first $a$ with $\|a\| < 1$ and $a = a^*$. Claim: then $1 - a = b^*b$ for some $b$ in $A$. Why? Consider $\sqrt{1-z} : \mathbb{C} \to \mathbb{C}$ which is holomorphic near 0, so has power series $\sum r_n z^n$, converging absolutely and uniformly on any disk about 0 with radius less than 1. Thus $b = \sum r_n a^n$ converges in $A$, by completeness. The $r_n \in \mathbb{R}$, so $b = b^*$, and $b^2 = 1 - a$.

Then $\mu(1) - \mu(a) = \mu(b^*b) \geq 0$, so $\mu(a) \in \mathbb{R}$, so $\mu(1) \geq \mu(a)$, and also $\mu(1) \geq \mu(-a)$, and hence $\mu(1) \geq |\mu(a)|$.

For general $a$ with $\|a\| < 1$ (but no longer considering $a = a^*$, we consider, using Cauchy-Schwartz ($\mu$ is positive, and C.S. does not require definiteness), that

$$|\mu(a)|^2 = |\mu(1a)|^2 = |\langle 1, a\rangle_\mu|^2 \overset{\text{C.S.}}{\leq} |\langle 1, 1\rangle_\mu||\langle a, a\rangle_\mu| = \mu(1)\mu(a^*a) \leq \mu(1)^2$$

where the last inequality follows from the previous paragraph, since $\|a^*a\| \leq \|a^*\|\|a\|$, and we assume that $*$ is isometric. (This is part of the word "$*$-normed".) This completes the proof ($\|\mu\|$ is the sup of $|\mu(a)|$ for $\|a\| \leq 1$). $\square$

**Prop:** Let $A$ be a $C^*$-algebra with 1, and let $\mu$ be a continuous linear functional on $A$. If $\mu(1) = \|\mu\|$, then $\mu$ is positive. I.e., this condition characterizes positivity.

**Proof:**

Let $a \in A$. We must show that $\mu(a^*a) \geq 0$. We can write $a^*a = b^2$ for some $b = b^*$, so suffice to show that $\mu(b^2) \geq 0$. Let $B = C^*(b, 1) = C(\sigma(b))$, and restrict $\mu$ to $B$. Thus, we can verify the result in the commutative case.

So, we need to show: if $A = C(M)$ with $M$ compact, and $\mu$ is a linear functional on $A$ with $\mu(1) = \|\mu\|$, then $\mu$ is positive. By dividing, we can assume that $\mu(1) = \|\mu\| = 1$.

If $f^* = f$, then $\mu(f) \in \mathbb{R}$. Why? Let $\mu(f) = \alpha + i\beta$. Then $|\mu(f + it1)|^2 = |\alpha + i(\beta + t)|^2 = \alpha^2 + \beta^2 + 2\beta t + t^2$. On the other hand, $|\mu(f + it)|^2 \leq \|f + it\|^2 \leq \|f\|^2 + t^2$ since $f$ is $\mathbb{R}$-valued. Then for every $t$, $\underbrace{\alpha^2 + \beta^2}_{|\mu(f)|} + 2\beta t \leq \|f\|^2$. Thus, $\beta = 0$, and we see that $f \geq 0$ implies that $\|f - \|f\|1\| \leq \|f\|$, so $|\mu(f) - \|f\|| = |\mu(f - \|f\|1)| \leq \|f\|$, so $\mu(f) \geq 0$. $\square$

**Theorem:** Let $A$ be a $C^*$-algebra with 1. For any $a \in A$ with $a = a^*$, and for any $\lambda \in \sigma(a) \subseteq \mathbb{R}$, there is a state $\mu$ n $A$ such that $\mu(a) = \lambda$.

**Proof:**

Let $B = C^*(a, 1) = C^*(\sigma(a))$, and let $\mu_0$ on $B$ be the $\delta$-function at $\lambda$. Then $\mu_0(a) = \lambda$. Then $\|\mu_0\| = 1 = \mu_0(1)$.

We invoke the Hahn-Banach theorem (big, mysterious, uses Choice). This extended $\mu_0$ to $\mu$ on $A$, with $\|\mu\| = \|\mu_0\|$. But $\|\mu_0\| = 1 = \mu_0(1) = \mu(1)$. So by the previous proposition, $\mu \geq 0$, and hence a state (positive linear functional of norm 1). $\square$

In the commutative case, we care about this kind of thing because we want, e.g., $\ell^\infty(\mathbb{Z}) = C(?)$, where ? = maximal ideals, or something. In separable case, we can get states in a more hands-on way. In normed Banach spaces, we don't know that there are states, but, e.g., for $\ell^1(G)$ or $\ell^2(G)$, we can see there are some.

Next time, we will use this result, and dig into the GNS construction.