This document was last updated on April 16, 2008. A more recent version may be available as part of http://math.berkeley.edu/∼theojf/CstarAlgebras.pdf.

1 April 16, 2008

1.1 Irreducible representations of algebras A_θ

Let $\mathcal{H} = L^2(\mathbb{R})$, and pick $\theta \in \mathbb{R} \smallsetminus \{0\}$. Let U be the operator that translates by θ:

$$(U\xi)(t) \stackrel{\text{def}}{=} \xi(t-\theta)$$

Let V be the operator that multiplies by a phase:

$$(V\xi)(t) \stackrel{\text{def}}{=} e^{2\pi i t}\xi(t)$$

Then the C^* algebra generated by V is $C(T)$, where T is the circle $T = \mathbb{R}/\mathbb{Z}$.

Question from the audience: Why? The closure is dense in the sup norm, not the operator norm. **Answer:** The sup norm is the operator norm for any of these pointwise multiplication, as long as your measure has full support.

Then, if $f \in C(T)$, we have $V_f = f \times (-)$. And $UV_f = V_{\alpha(f)}U$, where $(\alpha(f))(t) \stackrel{\text{def}}{=} f(t-\theta)$. In particular, taking $V_f = V$ itself, i.e. $f = e^{2\pi i t}$, then we conclude the commutation relation:

$$UV = e^{-2\pi i\theta}VU$$

So we let $W(p,q) \stackrel{\text{def}}{=} U^pV^q$, and $W(p,q)W(p',q') = U^pV^qU^{p'}V^{q'} = e^{2\pi i q p'\theta}W(p+p', q+q')$. This generates the C^* algebra:

$$C^*(\mathbb{Z}^2, \begin{pmatrix} 0 & 0 \\ \theta & 0 \end{pmatrix})$$

or perhaps the transpose of that matrix. But iterating the action α, it's clear that this algebra is a crossed product:

$$C(T) \times_\alpha \mathbb{Z}$$

Given a discrete group G, and α and action of G on M compact, we get an action α on $C(M)$. Hence, we can form $C(M) \times_\alpha G$. How can we construct *some* irreps of this algebra?

Well, pick some point $m_0 \in M$, and consider its orbit $\mathcal{O}_m$. We have a bijection $G/G_{m_0} \to \mathcal{O}_m$, where G_{m_0} is the stabilizer subgroup. Of course, the orbit might be infinite, so will have limit points. We form $\ell^2(G/G_{m_0})$, which we view as $\ell^2(\mathcal{O}_m)$, with the counting measure — this gives the measure of a compact space to be ∞. In any case, we can pull back continuous functions to bounded functions, and hence to bounded operators (multiplication):

$$\begin{tikzcd} C(M) \arrow[r] \arrow[rr, bend right, "\pi:\, f \mapsto f\times(-)"'] & C_b(\mathcal{O}_m) \arrow[r] & \mathcal{B}(\ell^2(G/G_{m_0})) \end{tikzcd}$$

Then the $\pi(f)$s separate points of G/G_{m_0}, and so we get a covariant rep of $C(M), G$. Exercise: this representation is irreducible.

E.g. θ is irrational. Then there are uncountably many different orbits, and each will give a different irrep of $C(T) \times_\alpha \mathbb{Z}$ above. Similarly, for $L^2(T, \text{Lebesgue})$; this is a different Hilbert space, but we can play the same game, so we get more irreps inequivalent to any of these. Classification theorem: you will never explicitly construct all irreducible representations.

****comment on von Neuman algebras, that I missed****

These algebras — $C(T) \times_\alpha \mathbb{Z}$ — are called *rotation algebras*. Even the rational rotation algebras are interesting, although not as much as the irrational ones. More generally, we can look at $C(T^m) \times_\theta \mathbb{Z}^n$ where these rotate at different speeds; this is a special case, because on each of $C(T^m)$ and $\mathbb{Z}^n$ have commuting generators.

Question from the audience: Does any measure on the circle give an irrep? **Answer:** No, I need it to be invariant under the rotations.

We saw that $U_m U_n U_m^* = \rho_\theta(m,n) U_n$, with perhaps a different convention last time, where ρ_θ is a bicharacter, and $\rho_\theta(m,n) = \alpha_{\rho_\theta(m,n)}$. Let $Z_{\rho_\theta} \stackrel{\text{def}}{=} \{m : \rho_\theta(m,n) = 1 \,\forall n\}$; then $U_m \in Z(A_\theta)$ the center iff $m \in Z_{\rho_\theta}$. ****Lecture uses the same symbol for the integers $\mathbb{Z}$ and the variable Z; either is reasonable in this context.**** U_n is central iff $\alpha_t(U_n) = U_n$ for any $t \in H_{\rho_\theta}$. Recall $\alpha_t(U_n) = \langle n, t \rangle U_n$. We defined

$$Q(a) = \int_{H_\rho} \alpha_t(a)\, dt$$

and so

$$Q(U_n) = \begin{cases} U_n, & n \in Z_{\rho_\theta} \\ 0, & n \notin Z_{\rho_\theta} \end{cases}$$

This requires a little bit of Fourier analysis. ****Recall that H is the closure of the image of Z^d in T^d under the pairing ρ_θ.****

In any case, Z_{ρ_θ} is a subgroup of $\mathbb{Z}^d$, and $\text{Range}(Q) \subseteq C^*(Z_{\rho_\theta}) \subseteq Z(A_\theta)$ the center, and if $a \in Z(A_\theta)$, then $Q(a) = a$. Hence $C^*(Z_{\rho_\theta})$ is exactly the center of A_θ.

Ok, so we now can decompose the algebra A_θ as a field of algebras over the center, and it turns out that each of the fibers is one of these simple algebras.

1.2 Differentiation

Smooth structures, in our experience, come from differentiation. We have A_θ and an action α of T^d. We have a surjection $\mathbb{R}^d \to T^d$. Let's generalize a little.

Let B be a Banach space, and α a strongly continuous action of $\mathbb{R}$ on B. We don't really need this, but for simplicity, let's think of this action as by isometries. Let $b \in B$, and look at $t \mapsto \alpha_t(b)$, which is norm-continuous on $\mathbb{R}$ with values in B. Is this function once-differentiable (at 0 is enough)? I.e., we want to know if

$$\lim_{t \to 0} \frac{\alpha_t(b) - b}{t}$$

exists for the norm $\| \cdot \|_B$ on B. In other words, does this limit equal some $c \in B$? Certainly, we'll want B to be complete. If the limit exists, we'll say that b is *differentiable*, and we'll write the limit at $D(b)$.

If $D(b)$ exists, we can ask whether $D(b)$ is differentiable. I.e. $D(D(b))$. And so on: does $D^n(b)$ exist?

****Picture this as $B = C(\mathbb{R})$ and α is by translation.****

Let V be a finite-dimensional vector space over $\mathbb{R}$ (which we think of as $\mathbb{R}^d$, but we don't want to be prejudicial about the basis). Let α be an action of V on B. For $v \in V$, we can ask for the *directional derivative* in the direction of v:

$$D_v(b) \stackrel{\text{def}}{=} \lim \frac{\alpha_{tv}(b) - b}{t}$$

if the RHS exists. Given $v_1, \ldots, v_n$, we can talk about $D_{v_n} \ldots D_{v_1} b$.

We won't need this generality, but it really does work: Let G be a connected Lie group, and take $G \subseteq GL(n, \mathbb{R})$ closed connected (we can do this up to a discrete subgroup). Then the Lie algebra $\mathfrak{g}$ of G is

$$\mathfrak{g} \stackrel{\text{def}}{=} \{X \in gl(n, \mathbb{R}) : \exp(X) \in G\}$$

There are substantial theorems about this. Then $t \mapsto \exp(tX)$ gives a 1-parameter subgroup of G for each X.

Let α be an action of G on B. Restrict to $t \mapsto \exp(tX)$. We can define, if it exists:

$$D_X(b) \stackrel{\text{def}}{=} \lim \frac{\alpha_{\exp(tX)}(b) - b}{t}$$

If $D_X(b)$ exists, we can ask about its differentiability, and so on, and let

$$B^\infty = \{b \in B : D_{X_n} \ldots D_{X_1} b \text{ exists for all } n \text{ and all } X_1, \ldots, X_n\}$$

Theorem: (Gärding ****sp?****)

B^∞ is dense in B.