1 February, 27 2008

One more comment on the theory of bounded operators:

Naimark Conjecture: (1940s)

> Let A be a C^*-algebra. Suppose A has the property that, up to equivalence, A has only one irreducible representation. Then $A \cong \mathcal{B}_0(\mathcal{H})$ for some $\mathcal{H}$.

Rarely do we come across algebras that we don't know are $\mathcal{B}_0$, but for neatness, this would be nice to know. The answer is "Yes" if A is separable. In 2003, however, there was a surprise: Chuck Akerman and Nick Weaver (both Ph.D. students of Prof. Bade, at UC-Berkeley) showed that the answer depends on the axioms of set theory. For usual axioms ****ZF definitely, and probably ZFC?****, the question is undecidable. More specifically, If we assume the Diamond Principle, we can construct a counterexample to Naimark's conjecture, and Diamond is consistent with usual axioms.

Question from the audience: What does separable mean? **Answer:** As a Banach space, it is separable: there is a countable norm-dense subset. **Question from the audience:** If A acts on a separable Hilbert space, does that imply A is separable? **Answer:** I don't know. I'd expect that that does imply the result, but I did not try to understand the paper, since I don't know set theory.

1.1 Continuing from last time

We stated this Burnside theorem:

Theorem: If A is a C^*-subalgebra of $\mathcal{B}_0(\mathcal{H})$ that acts irreducibly on $\mathcal{H}$, then $A = \mathcal{B}_0(\mathcal{H})$.

Proof:

> The action must clearly be non-degenerate. A is a C^*-subalgebra, and $\mathcal{H}$ is of non-zero dimension, so A has non-zero elements, and T^*T is non-zero if T is. So we pick out $T \in A$ with $T \neq 0$, $T \geq 0$. Then T is compact — specturm is discrete, except perhaps 0 could be an accumulation point —, by the spectral theorem of compact self-adjoint operators. So T has a non-zero eigenvalue, and the projection onto the eigensubspace is in A. Thus A contains proper projections onto finite-dimensional subspace.
>
> So we look at all projections, and find a minimal one: Let $P \in A$ be a projection of minimal positive dimension (of range of P). Then for any $T \in A$ with $T = T^*$, we look at PTP, which is clearly self-adjoint with finite dimension. So it has spectral projections, and it's obvious that the spectral projections must be smaller than P (in the strongest sense: they are onto subspaces of range of P). These spectral projections are certainly still in A, since they are polynomials in PTP. But P is minimal, so the only possible spectral projections for PTP are 0 and P. Thus, there exists $\alpha(T) \in \mathbb{R}$ (self-adjoint implies real eigenvalues) such that

$PTP = \alpha(T)P$. By splitting operators into their real and imaginary parts, we can extend this from self-adjoint T to all T: for any $T \in A$, we have $\alpha(T) \in \mathbb{C}$ so that $PTP = \alpha(T)P$.

Let $\xi, \eta \in$ range of P, with $\|\xi\| = 1$ and $\eta \perp \xi$. We'd like to show that $\eta = 0$, since we're trying to show that range of P is one-dimensional. Well, for $T \in A$,

$$\begin{aligned}
\langle T\xi, \eta\rangle &= \langle TP\xi, P\eta\rangle \text{ since } P\xi = \xi, \text{ etc.} \\
&= \langle PTP\xi, \eta\rangle \text{ since } P^* = P \\
&= \langle \alpha(T)\xi, \eta\rangle \\
&= 0
\end{aligned}$$

But $\overline{\{T\xi : T \in A\}}$ is A-invariant, so $= \mathcal{H}$, so $\eta = 0$.

So A contains a rank-1 projection P on $\mathbb{C}\xi$. Then $\{T\xi\}$ are dense in $\mathcal{H}$, so TP is rank-≤ 1 taking ξ to $T\xi$. Since A is norm-closed, if we take any vector t$\eta \notin \{T\xi\}$, we can approximate it by such, and then look at corresponding TP, which will converge to the rank-one operator on η. I.e., for any $\eta \in \mathcal{H}$, the rank-one operator $\langle \eta, \xi\rangle_0$ is in A. But A is closed under $*$, so $\langle \xi, \zeta\rangle_0$ is also in A for all $\zeta \in \mathcal{H}$. Multiplying gives $\langle \eta, \zeta\rangle_0 \in A$, so all rank-one operators in $\mathcal{B}_0(\mathcal{H})$ are in A, and so all of $\mathcal{B}_0(\mathcal{H})$ is in A. (All rank-one are in, so all finite-rank, and we defined $\mathcal{B}_0$ to be the closure of finite-rank. Remember that you have to look fairly far to find a Banach algebra where the compact operators are not the closure of finite-rank ones, but there are some examples, but in C^*-land they all are.) $\square$

Question from the audience: This is a converse of Schur's lemma. **Answer:** In some sense.

1.2 Relations between irreducible representations and two-sided ideal

We don't need the full strength of a C^*-algebra.

Prop: Let A be a $*$-normed algebra, I a two-sided ideal, and assume that I has a two-sided bounded approximate identity (for I). Let $(\pi, \mathcal{H})$ be a continuous irreducible representation of A. Then either

(a) $\pi(I) = 0$, or

(b) $\pi|_I$ is irreducible.

Proof:

If $\pi(I) \neq 0$, then look at $\overline{\{\pi(I)\mathcal{H}\}} \neq 0$ (meaning linear span), which is clearly A-invariant. So it is all of A, and hence $\overline{\{\pi(I)\mathcal{H}\}} = \mathcal{H}$, i.e. $\pi|_I$ is non-degenerate. Let $\{e_j\}$ be an approximate identity for I. We showed that $\pi(e_j)\xi \to \xi$ for all $\xi \in \mathcal{H}$. Let $\mathcal{K}$ be a closed $\pi|_I$-invariant subspace. Then $\mathcal{K}$ is A-invariant, because: given $\xi \in \mathcal{K}$ and $a \in A$, and switching to module notation, $a\xi = \lim a(e_j\xi) = \lim(ae_j)\xi$. But $ae_j \in I$, so $ae_j\xi \in \mathcal{K}$, and since $\mathcal{K}$ is closed, $a\xi \in \mathcal{K}$. $\square$

Prop: Let A be a $*$-normed algebra, I an ideal with approximate identity. Let $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ be two irreducible representations of A. If $\pi(I), \rho(I) \neq 0$ (so $\pi|_I$ and $\rho|_I$ are irreducible), and if $\pi|_I$ is unitarily equivalent to $\rho|_I$, then π and ρ are unitarily equivalent.

Proof:

Let $U : \mathcal{H} \to \mathcal{K}$ be a unitary equivalence over I. I.e. $U\pi(d) = \rho(d)U$ for $d \in I$, and U unitary. Then for $a \in A$, we have $U\pi(a)\xi = \lim U\pi(a)\pi(e_j)\xi = \lim U\pi(ae_j)\xi = \lim \rho(ae_j)U\xi = \lim \rho(a)\rho(e_j)U\xi = \rho(a)U\xi$. $\square$

Theorem: Let A be a C^*-algebra and $(\pi, \mathcal{H})$ an irreducible representation of A. If $\pi(A)$ contains at least one non-zero compact operator, then $\pi(A)$ contains all compact operators. In this case, moreover, any irreducible representation of A with the same kernel as π is unitarily equivalent to $(\pi, \mathcal{H})$.

We will give the proof next time.