1 March 17, 2008

****I was a little late.****

Theorem: Let $0 \to I \xrightarrow{i} A \xrightarrow{p} A/I \to 0$ be an exact sequence of C^*-algebras. Let α be an action of G on A, which caries I into itself; i.e. i is *equivariant*, and also α drops to action on A/I. Then

$$0 \to I \times_\alpha G \xrightarrow{i_*} A \times_\alpha G \xrightarrow{p_*} (A/I) \times_\alpha G \to 0$$

is exact.

(This can fail for $\times^r_\alpha$.) **Question from the audience:** Can you get half-exactness? **Answer:** Yes, somewhat.

Proof:

$p_* : C_c(A, G, \alpha) \to C_c(A/I, G, \alpha)$ has dense range. When G is not discrete, this is not an immediate fact, but it is true. We approximate functions $f : G \to A/I$ by $f \sim \sum h_j a_j$, which we can do for any continuous function of compact support into a Banach space, for the L^1 norm. Thus $p_* : A \times_\alpha G \to (A/I) \times_\alpha G$, which has dense range, and a homomorphism of C^* algebras, but those have closed image, so this must be onto by denseness.

If $f \in C_c(I, G)$, then $p_*(i_*(f))$ is clearly 0, just by following image values. Extending by continuity we still have $p_* \circ i_* = 0 : I \times_\alpha G \to (A/I) \times_\alpha G$.

So, why is i_* injective? And, we've shown that the image of i_* is contained in the kernel of p_*; why are these equal?

For exactness at $I \times_\alpha G$, let $(\sigma, \mathcal{H})$ be a (non-degenerate) representation of $I \times_\alpha G$. From what we sketched last time, this must come from a covariant representation: let $(\pi, U, \mathcal{H})$ be the corresponding covariant representation of (I, G, α). Then π is non-degenerate. Using the extension theorem from the problem set, let $\tilde{\pi}$ be the unique extension of π to a representation of A. Claim: $(\tilde{\pi}, U, \mathcal{H})$ is a covariant representation of (A, G, α). Because:

$$\begin{aligned}
U_x(\tilde{\pi}(a))(\pi(d)\xi) &= U_x(\pi(ad)\xi) \\
&= \pi(\alpha_x(ad))U_x\xi \\
&= \pi(\alpha_x(a)\alpha_x(d))U_x\xi \\
&= \tilde{\pi}(\alpha_x(a))\pi(\alpha_x(d))U_x\xi \\
&= \tilde{\pi}(\alpha_x(a))U_x(\pi(d)\xi)
\end{aligned}$$

where $d \in I$, and the linear span of these things is dense by the nondegeneracy.

Question from the audience: Why is π nondegenerate? **Answer:** That was something quick from last time. In the correspondence, $\sigma(f)\xi \stackrel{\text{def}}{=} \int \pi(f(x))U_x\xi$, and if this is nondegenerate, then the places where it was zero would be invariant, so we'll have nondegenerateness of σ exactly when we have it for π.

So let $\tilde{\sigma}$ be the integrated form of $(\tilde{\pi}, U, \mathcal{H})$ a rep of $A \times_\alpha G$. Then $\tilde{\sigma}|_{I \times_\alpha G} = \tilde{\sigma} \circ i_* = \sigma$. If σ is faithful on $I \times_\alpha G$, then i_* has kernel 0.

Now we want exactness at $A \times_\alpha G$, i.e. that the kernel of p_* is the range of i_*. Since we know that i_* is injective, we should think of the range as $I \times_\alpha G \subseteq A \times_\alpha G$ as an ideal (we didn't do this part, but it's not hard that at the level of functions, this is an ideal). We saw at the outset that $\supseteq$ is easy. Here we will need the full force of C^* algebras. We look at $(A \times_\alpha G)/(I \times_\alpha G)$, which is a C^*-algebra, so it has a faithful representation σ on $\mathcal{H}$. (I.e. the kernel of σ is exactly $I \times_\alpha G$). Then σ is the integrated form of some $(\pi, U, \mathcal{H})$ where π is a rep of A. If $d \in I$, then for any $h \in C_c(G, \mathbb{C})$, we have $dh \in C_c(G, I)$. So $0 = \sigma(hd)\xi = \int h(x)\pi(d)U_x\xi dx$ for all h and ξ. So $\pi(d) = 0$. So $\pi(I) = 0$. Thus we can look at $(\pi, U, \mathcal{H})$ as a covariant rep of $(A/I, U, \mathcal{H})$, with integrated form $\tilde{\sigma}$, a representation of $(A/I) \times_\alpha G$.

Then

$$\begin{tikzcd}
A \times_\alpha G \arrow[rr, "p_*"] \arrow[dr, "\sigma"'] & & (A/I) \times_\alpha G \arrow[dl, "\tilde{\sigma}"] \\
& \mathcal{B}(\mathcal{H}) &
\end{tikzcd}$$

□