****This document was last updated on May 12, 2008. A more recent version may be available as part of http://math.berkeley.edu/∼theojf/CstarAlgebras.pdf.****

# 1: May 12, 2008

****I was a little late.****

## 1.1 Guest lecture by W. Arveson: Operator Spaces, "Quantized Functional Analysis"

Lecture notes are available at http://math.berkeley.edu/∼arveson/Dvi/opSpace.pdf.

Let $\mathcal{S} \subseteq \mathcal{B}(H)$ be a linear subspace that is $\|\cdot\|$-closed. We have a notion of *completely contractive* maps, which form a category.

We will see some examples, which illustrate the non-commutativity in finite-dimensional setting.

Consider $\mathcal{B}(\mathbb{C}^p)$ for $p = 1, 2, \dots$. We have two particular operator spaces, the "row"-space $\mathcal{R}$ and the "column" space $\mathcal{C}$:

$$\mathcal{R} = \left\{ \begin{pmatrix} z_1 & z_2 & \dots & z_p \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{pmatrix} z_1 & 0 & \dots & 0 \\ z_2 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ z_p & 0 & \dots & 0 \end{pmatrix} \right\}$$

Given $z = (z_1, \dots, z_p) \in \mathbb{C}^p$, we have $R_z$ and $R_c$ as above. Then $\|R_z\| = \|R_z R_z^*\|^{1/2} = \|z\| = \|C_z\|$. Let $\phi : \mathcal{R} \to \mathcal{C}$ be this isometry.

Now, what is $M_n(\mathcal{R})$? Well, they are $n \times n$ matrices with entries in $\mathcal{R}$, but equivalently they are

$$M_n(\mathcal{R}) = \left\{ \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} : A_i \in M_n(\mathbb{C}) \right\}$$

and similarly for $M_n(\mathcal{C})$. We can extend $\phi$ to $\phi_n$:

$$\phi_n : \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \mapsto \begin{pmatrix} A_1 & 0 & \dots & 0 \\ A_2 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ A_p & 0 & \dots & 0 \end{pmatrix}$$

Is $\phi_n$ an isometry?

$$\left\| \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \right\| = \sqrt{\|A_1 A_1^* + \dots + A_p A_p^*\|}$$

and it's important that the $*$s are on the right. On the other hand, in $M_n(\mathcal{C})$, the $*$s are on the left. And if $n$ is large enough, in particular if $n \geq p$, we can make these different. E.g. if $n = p$, and if $A_i$ are rank-one partial isometries with mutually orthogonal matrices $A_i : e_1 \mapsto e_i$, then $\{A_i A_i^*\}$ are mutually orthogonal projections, so their sum has norm 1. On the other hand, $A_i^* A_i$ is the projection onto $e_1$, so the sum has norm $p$.

Incidentally, we can do the same thing with $\phi^{-1}$. In particular, $\|\phi_n\| \geq \sqrt{p}$ and $\|\phi_n^{-1}\| \geq \sqrt{p}$. And so $\phi$ is not a *complete isometry*. Remark: this does not show that $\mathcal{R}$ and $\mathcal{C}$ are not completely isometric. But an easy generalization of the above calculation does show that $\mathcal{R}$ and $\mathcal{C}$ are not completely isometric. So even a finite-dimensional Hilbert space can be realized in many different ways as an operator space: "The same Banach space has many quantizations."

Recall the basic tool of functional analysis: Hahn-Banach. We need such a theorem in this context; this is what makes the theory fly:

**Theorem:** The non-commutative Hahn-Banach theorem

> Let $\mathcal{S} \subset \mathcal{B}(H)$ be an operator system and $\phi : \mathcal{S} \to \mathcal{B}(K)$ be an operator map. Then this map has a completely bounded norm — it might be a complete contraction — as we defined and erased. Then there exists an extension
>
> $$\tilde{\phi} : \mathcal{B}(H) \to \mathcal{B}(K)$$
>
> with $\|\tilde{\phi}\|_{\mathrm{CB}} = \|\phi\|_{\mathrm{CB}}$.

In particular, any completely contractive map can be extended to a completely contractive map of the ambient space:

$$\begin{tikzcd} \mathcal{S}_1 \arrow[r, hook] \arrow[d, "\text{c.c. } \phi"'] & \mathcal{S}_2 \arrow[r, hook] \arrow[dl, dashed, "\tilde{\phi} \text{ c.c.}"] & \mathcal{B}(H) \\ \mathcal{B}(K) \end{tikzcd}$$

Let's take a moment to talk about Stinespring's Theorem. We have a GNS construction. On the other hand,

**Theorem:** (Sz.-Nagy)

> Suppose $\phi : C(X) \to \mathcal{B}(H)$ is linear with $f \geq 0 \Rightarrow \phi(f) \geq 0$ (hence $\phi$ is bounded). Then there exists a representation $\pi : C(X) \to \mathcal{B}(K)$ and $V : H \to K$ such that $\phi(f) = V^* \pi(f) V$. I.e.
>
> $$\langle \phi(f)\xi, \eta \rangle_H = \langle \pi(f) V\xi \, V\eta \rangle_K$$

This looks a lot like the GNS construction: if $\phi : A \to \mathbb{C}$ is positive linear, then we get a representation $\pi$ such that $\phi(a) = \langle \pi(a)\xi, \xi \rangle$. Our formula is a lot like that, except involves $V$ because of the non-commutativity.

**Theorem:** (Stinespring, 1955)

> Let $A$ be a unital $C^*$-algebra, and $\phi : A \to \mathcal{B}(H)$ a completely positive map. (A *positive* map is a map that takes self-adjoint positive elements of $A$ to the same; *completely positive* maps are positive on all $M_n(A)$.) Then there exists a rep $\pi : A \to \mathcal{B}(K)$ and $V : H \to K$ such that $\phi(a) = V^* \pi(a) V$ for all $a \in A$.

By the way, the converse is true: if there is such a $\pi$ and a $V$, then $\phi$ is completely positive. Stinespring generalizes both GNS and Sz.-Nagy. Even though he assumes more (complete positivity)? Yes, because he proved in the same paper that a positive linear function is completely positive (i.e. if $H = \mathbb{C}$). And he proves that a positive linear map from a commutative $C^*$-algebra to an operator space is completely positive.

This was the first penetration into the area of non-commutative functional analysis. There was no further work for many years. A. was assigned the paper be thesis advisor, and it was beautiful, but no one really understood it. In late 1970s, this theory started to take hold, and has become popular in recent years.

All of this is discussed in more detail in the notes.