1 Problem Set 3: Due April 11, 2008

****The problem set was given out typed. I've retyped it, partly so I could submit my answers set between the questions. I have corrected some typos, and no doubt introduced even more. In doing so, I have changed the formatting slightly.****

1. (a) Show that the following C^*-algebras are isomorphic:

    i. The universal unital C^*-algebra generated by two (self-adjoint) projections

    ii. The universal C^*-algebra generated by two self-adjoint unitary elements

    iii. The group algebra $C^*(G)$ for $G = \mathbb{Z}_2 * \mathbb{Z}_2$, the free product of two copies of the 2-element group.

    iv. The crossed-product algebra $A \times_\alpha G$ where $A = C(T)$ for T the unit circle in the complex plane, $G = \mathbb{Z}_2$, and α is the action of taking complex conjugation. (So T/α exhibits the unit interval as an "orbifold", i.e. the orbit-space for the action of a finite group on a manifold, and $A \times_\alpha G$ remembers where the orbifold comes from.) Hint: In $\mathbb{Z}_2 * \mathbb{Z}_2$ find a copy of $\mathbb{Z}$.

    (b) Determine the primitive ideal space of the above algebra, with its topology.

    (c) Use the center of the algebra above to express the algebra as a continuous field of C^*-algebras.

    (d) Use part (c) to prove that if p and q are two projections in a unital C^*-algebra such that $\|p - q\| < 1$, then they are unitarily equivalent, that is, there is a unitary element u in the algebra (in fact, in the subalgebra generated by p and q) such that $upu^* = q$.

    (e) Use part (d) to show that in a unital separable C^*-algebra the set of unitary equivalence classes of projections is countable.

2. For any $n \times n$ real matrix T define an action α of $\mathbb{R}$ on the group $\mathbb{R}^n$ by $\alpha_t = \exp(tT)$ acting in the evident way. Let $G = \mathbb{R}^n \times_\alpha \mathbb{R}$. Then G is a solvable Lie group. For the case of $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ determine the equivalence classes of irreducible unitary representations of G, i.e. the irreducible representations of $C^*(G)$. Determine the topology on $\mathrm{Prim}(C^*(G))$. Discuss whether $C^*(G)$ is CCR or GCR, and why.