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Any questions? We hand back some problem sets from before.

1.1 A landscape sketch

Last time we defined the Poincaré group $\mathbb{R}^4 \times_\alpha L$, where L is the Lorentz group **or rather the connected component**. If you want a quantum mechanics, you would like a Hilbert space. An elementary particle, for quite some time, was a representation of this group — so that physics would be symmetric — and irreducible — so that the particle was really just a unique particle. In 1939, Wigner listed the representations (that are non-trivial as representations of $\mathbb{R}^4$), which consists of understanding the C^*-algebra $C^\infty(\widehat{\mathbb{R}}) \times_\alpha L$. In fact, we want to use the simply connected cover of L, which is $SL(2, \mathbb{C})$.

We can understand the orbits of L **picture of concentric hyperbolas, with the degenerate one labeled the "light cone"**. But the action of L is not free on these orbits; rather, if G acts on M via α, we can construct the *stability subgroup* $G_m = \{x \in G : \alpha_x(m) = m\}$; then the orbit of $m \in M$ bijects to G/G_m by $\alpha_x(m) \leftrightarrow x$. The stability subgroup of L is $SO(3) \subseteq L$; this lifts to $SU(2) \subseteq SL(2, \mathbb{C})$. So the orbits are $SL(2, \mathbb{C})/SU(2)$.

Given G acting via α on M, and $m \in M$, we understand what happens when $G_m = \{1_G\}$, and if $G_m = G$, everything is trivial. We want to understand the intermediate case. But the orbit $G/G_m = \mathrm{Orbit}_\alpha(m) \subseteq M$ can be very bad. **E.g.** $M = T = S^1$ and $G = \mathbb{Z}$ acting by rotation by an irrational multiple of π. The action is free, and the orbit is dense in T — countable dense subsets are very bad from the point of view of the things we've been doing. We'll look at that later.

The good situation: we want the orbit, with the relative topology, to be locally compact. (Contemplate pulling back the topology of the circle to the integers.) For the experts:

Theorem: Given M locally compact and $S \subseteq M$, then S is locally compact for the relative topology if and only if S is open in its closure.

If G is second countable **and we are in this good case**, we can use Baire Category to show that $G/G_m \to$ Orbit is a homeomorphism. **E.g.** $\mathbb{R}_\delta$ is $\mathbb{R}$ with the discrete topology, and is not second-countable; it acts on $\mathbb{R}$ in the natural way, but we don't have a homeomorphism.

We can look at the ideal of functions on M that vanish on the closure of the orbit, and if the open orbit is dense in its closure, we can generalize the picture we had last time of $\mathbb{R} \cup \{+\infty\}$. We reduce to considering G action on G/G_m.

In general, for H a closer subgroup of G, G acts on G/H (call action α) and we consider the covariant representations. From our point of view, we're curious to understand $C(G/H) \times_\alpha G$.

Question from the audience: So G acts on M, which we decompose into orbits. So we're just looking at one orbit at a time? **Answer:** We showed that irreducible representations live on orbit-closures. As long as your orbit is open, and so locally compact, the irreducible representations all come from this mechanism. To be specific: an irreducible representation comes from an orbit closure, and if there is a dense open set, that gives us an essential ideal. And if you have the right orbit closure for the representation, this essential ideal cannot act as zero, so the representation comes from the cross-product with that ideal.

Mackey worked out the general theory measure-theoretically. It involves induced representations — Frobenius understood these for finite groups in the late 1800s, but Mackey had to work it out for locally-compact groups. Assume for simplicity that there is a G-invariant measure on G/H. (Otherwise, you have to introduce modular functions, and the bookkeeping is too complicated for this exposition.) Let $(\rho, \mathcal{K})$ be a representation of H. (We think of $H = G_m$; the phycisists call these "little groups".) Set

$$\mathcal{H} \overset{\text{def}}{=} \{\xi : G \to \mathcal{K} \text{ measureable s.t. } \xi(xs) = \rho_s^{-1}(\xi(x)) \text{ for } s \in H, x \in G\}$$

We have $\langle \xi(xs), \eta(xs) \rangle_{\mathcal{K}} = \langle \xi(x), \eta(x) \rangle_{\mathcal{K}}$ as functions on G/H, so we define the inner product on $\mathcal{H}$:

$$\|\xi\|^2 \overset{\text{def}}{=} \int_{G/H} \langle \xi(x), \xi(x) \rangle_{\mathcal{K}} \, dx$$

where dx is the invariant measure on G/H.

Let π be the action of $C_\infty(G/H)$ on $\mathcal{H}$ by pointwise multiplication. Let U be the action of G by translation; then $(\pi, U, \mathcal{H})$ is a covariant representation. Thus, as before, we get a representation of $C_\infty(G/H) \times_\alpha G$. If $(\rho, \mathcal{K})$ is irreducible, then so is $(\pi, U, \mathcal{H})$. **Question from the audience:** Sorry, what is α? **Answer:** The action by translation. **$\alpha = U$?**

Theorem: (Mackey, phrased in terms of L^∞, not C^*-algebras)

Every irreducibly representation arises in this way:

$$\{\text{Irreducible representations of } C_\infty(G/H) \times G\} \leftrightarrow \{\text{irreps of } H\}$$

E.g. Every irrep of the Poincaré group has an inherent *spin*, which is the representation of $SU(2)$.

We can reformulate this, which is a nice way to do it because M.R. did it.

Theorem: (M.R.)

$C_\infty(G/H) \times_\alpha G$ and $C^*(H)$ are strongly morita equivalent.

To say this, we need bimodules. We think of $C_c(H)$ with measure dx_H, which acts on the right by convolution $\star$ on $C_c(G)$. But $C_\infty(G/H) \times_\alpha G$ acts on the left by point-wise multiplication, so at least at the level of functions, we have $C_c(G)$ as a bimodule. But on the level of inner products? If we have $f, g \in C_c(G)$, we can define $\langle f, g \rangle_{C^*(H)}$? For better or for worse, it pays off to work with continuous rather than measurable functions, because if H is a null-set, restricting a measurable function doesn't make sense. In the unimodular case, we can define

$$\langle f, g \rangle_{C^*(H)} \stackrel{\text{def}}{=} f^* \star g|_H$$

In fact, everything fits together nicely, although there are many things to check, and in the non-unimodular case you have to sprinkle in modular functions.

So, basically, and you have to define things right: if you have a representation of $B = C^*(H)$, you can tensor it with this bimodule to get a representation of $A = C_\infty(G/H) \times_\alpha G$. What you need to check is that $\langle f, g \rangle_A h = f \langle g, h \rangle_B$.

And the long and the short of it is that the representation theories are the same.

Question from the audience: So if you embed H as a subgroup of different groups G, you can look at their relationships? **Answer:** Yes, there are many interesting games you can play. For instance, you can sort out that $C_\infty(G/H) \times_\alpha K$ is Morita equivalent to $C_\infty(G/K) \times_\alpha H$ if $H, K \subseteq G$.

We've been looking at an action G, α on a space M. This leads to a simple class of groupoids, although we won't tell you what a groupoid is. Much of the above story generalizes to groupoids. (Groupoids come from gluing together group-type things and space-type things, and arise in many interesting places. Locally compact ones have a C^*-algebra, and by the now there is a very substatially developed theory, which imitates the theory of groups acting on spaces.)

1.2 Heisenberg commutation relations

Planck wrote his paper in 1909; in 1926, Heisenberg basically suggested that, for $\mathbb{R}^n$ (really $\mathbb{R}^{3n}$) one do the following.

We have unbounded self-adjoint (in a sense not made precise by Heisenberg) "position" operators $Q_1, \ldots, Q_n$ and "momentum" operators $P_1, \ldots, P_n$ all on a Hilbert space $\mathcal{H}$. Since these operators are unbounded and hence only defined on dense domains, we will need to make this precise or avoid the problems entirely. Even saying these operators commute is hairy. But, naively, we want that P_j, Q_k commute if $j \neq k$. For the same index:

$$[P_j, Q_j] = i\hbar \mathbb{1}_{\mathcal{H}}$$

where $i = \sqrt{-1}$, and $\hbar$ is an experimental fudge factor and $\mathbb{1}_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. Shortly thereafter, Weyl suggested how to bypass some of the difficulties here.

****I prefer to use slightly more tensorial notation. We have what a physicists would call a "vector" of operators Q^i and a "covector" P_j. Then the canonical commutation relations are**

$$[P_j, Q^k] = i\hbar\delta_j^k \mathbb{1}_{\mathcal{H}}$$

where δ_j^k is Kronecker.**