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1 April 23, 2008

I keep forgetting, you need to turn in a third problem set, and I don't want it at the end of the semester. Is it unreasonable to ask for it on Monday? If you won't be turning it in on Monday, please talk to me.

1.1 We were doing somewhat strange things

We have T^d and α an action on C^*-algebra A. This gives us a smooth algebra A^∞, which sort of looks like the Schwartz space, except for functions values in A_n. Then we get subspaces $A_n \subseteq A^\infty$ for $n \in \mathbb{Z}^d$. We had $(\pi, U, \mathcal{H})$ a covariant representation of (A, T^d, α), and we assume π is faithful. Then we have Hilbert spaces $\mathcal{H}_n$, and if $m \neq n$, then $\mathcal{H}_n \perp \mathcal{H}_m$. Exactly the same proof that eigenspaces of self-adjoint operators are orthogonal.

Question from the audience: The direct sum of A_ns is dense in A^∞? Is it all of it? **Answer:** A^∞ is the set of sequences $\{a_n\}$ where each $a_n \in A_n$, and where the map $\{n \mapsto \|a_n\|\} \in \mathcal{S}(\mathbb{Z}^d) \subseteq \ell^1(\mathbb{Z}^d)$. For any a we can get a sequence of a_n, but it's almost impossible to say what sequences come from general functions. So the direct sum is dense, but not complete in the Frechet topology; the direct sum is all the finite ones. **Question from the audience:** A^∞ is the sums of these sequences? **Answer:** Yes. Or define the space of these sequences as a graded algebra, and there's a bijection between A^∞ and these sequences, by summing in one direction, and in the other direction by taking Fourier modes.

In the non-commutative case, it's more complicated, but you still get a decomposition of the C^* algebra of functions via the group action. **Question from the audience:** With a non-compact Lie group? **Answer:** For each irreducible representation, you can define a space A_n. Where you get into trouble: if we multiply two characters, you get a character, but in higher dimension the tensor product of two irreducible representations may not be irreducible, and the bookkeeping gets harder.

So, given $\theta \in M_d(\mathbb{R})$ and cocycle c_θ. For any $\xi_n \in \mathcal{H}_n$, and $a_m \in A_m$, define

$$\pi^\theta(a_m)\xi_n \stackrel{\text{def}}{=} \pi(a_m)\xi_n c_\theta(m,n)$$

For any $\xi \in \mathcal{H}$, we have $\xi = \sum \xi_n$.

$$\begin{aligned}
\left\| \pi^\theta(a_m) \sum_{\|n\| \le M} \xi_m \right\|^2 &= \left\| \sum \pi(a_m)\xi_n c_\theta(m,n) \right\|^2 \\
&= \sum \|\pi(a_m)\xi_n c_\theta(m,n)\|^2 \text{ by orthogonality} \\
&\le \sum \|\pi(a_m)\| \, \|\xi_n\| \\
&= \|\pi(a_m)\| \, \|\xi\| \text{ where } \xi = \sum \xi_n
\end{aligned}$$

So $\sum_{n \in \mathbb{Z}^d} \pi^\theta(a_m)\xi_n$ converges, and we call the limit $\pi^\theta(a_m)\xi$.

$$\left\| \pi^\theta(a_m)\xi \right\| \le \|\pi(a_m)\| \|\xi\| \le \|a_m\| \|\xi\|$$

Thus for $a \in A^\infty$, set

$$\pi^\theta(a)\xi \stackrel{\text{def}}{=} \sum \pi^\theta(a_m)\xi$$

and $\pi^\theta(a_m)$ is ℓ^1. Then

$$\begin{aligned}
\pi^\theta(a_m)\pi^\theta(a_n)\xi_p &= \pi^\theta(a_m)\left(\pi(b_n)\xi_p c_\theta(n,p)\right) \\
&= \pi(a_m)\pi(b_n)\xi_p c_\theta(n,p) c_\theta(m, n+p) \\
&= \left(\pi(a_m)\pi(b_n)c_\theta(m,n)\right)\xi_p c_\theta(m+n,p)
\end{aligned}$$

So, on A^∞, we define a product

$$a \star_\theta b = \sum a_m \, b_n \, c_\theta(m,n)$$

and since these sequences in norm are ℓ^1, we see that this series converges without any difficulty. Then

$$\pi^\theta(a)\pi^\theta(b) = \pi^\theta(a \star_\theta b)$$

Moreover, the $*$:

$$\left(\pi^\theta(a_n)\right)^* = a_n^* c_\theta(n,n) \stackrel{\text{def}}{=} a_n^{*_\theta}$$

In any case, this gives a $*$-algebra structure on A^θ and a $*$-rep on $\mathcal{H}$. We want π faithful. Then we get a C^*-norm on A^∞. Complete this to get a C^*-algebra A^θ. This is not the same as A_θ from earlier.

Question from the audience: Why do you do this just for A^∞ and not all of A? **Answer:** The twisted C^* norm is not continuous for all of A. **E.g.** $A = C(T^d)$, then $A^\theta = A_\theta$, but the norm on A_θ is not equivalent to the sup norm on A, just on A^∞. Well, we could work on ℓ^1.

A bit of context: where does this come from? θ defines a "Poisson bracket" on A^∞ in the obvious sense: choose an orthonormal basis for $\mathbb{R}^d = \text{Lie}(T^d)$, which might as well be the "standard" basis $\{E_j\}$. Then the Poisson bracket of a and b is

$$\{a,b\}_{\theta,\alpha} \stackrel{\text{def}}{=} \sum \theta_{jk} D_{E_j}(a) D_{E_k}(b)$$

This is really best after we change so that $\theta^t = -\theta$ is skew-symmetric. If M is a manifold, and T^d acts smoothly on M, then T^d acts on $C_\infty(M)$, and it's very natural to define a Poisson bracket $\{f, g\} = \sum \theta_{jk} D_{E_j}(f) D_{E_k}(g)$. The first person to do our more general case carefully was a student of Alan Weinstein's, by the name of ****missed****.

When $A = C_\infty(M)$, we have A^θ, which we should view as a "quantization" of $C_\infty(M)$ in the direction of the Poisson bracket. To make this precise, we need to get Plank's constant $\hbar \in \mathbb{R}$ in here. Then $\hbar\theta$ is again skew-symmetric, so we can define $A^{\hbar\theta}$. I.e. $\hbar \mapsto A^{\hbar\theta}$ is a one-parameter family of algebras, and when $\hbar = 0$ we get the original algebra. Then for $a, b \in A^\infty$,

$$\left\| \frac{a \star_{\hbar\theta} b - b \star_{\hbar\theta} a}{\hbar} - 2i\{a, b\}_\theta \right\| \xrightarrow{\hbar \to 0} 0 \tag{1}$$

Says that the "semi-classical limit" of the $A^{\hbar\theta}$ is A equipped with the Poisson bracket $\{\}_\theta$.

So if you take the opinion that the world is quantum, then in the classical limit, the remnant of the quantum world is the Poisson bracket in the ordinary world.

Another way of putting equation (1):

$$a \star_{\hbar\theta} b = ab + i\hbar\{a, b\} + O(\hbar^2)$$

The limit (1) is often called the "correspondence principle".

For $\mathbb{R}^n$ acting on A, we again have A^∞, and for θ we can define $a \star_\theta b$. This is technically more difficult, because we don't have subspace A_n.

By the same formula as before, T^d acts on A^θ by multiplying by the corresponding character independent of θ. We see that $(A^\theta)^\infty = A^\infty$. And so given θ_1, θ_2, via the $\mathbb{R}^d$ action,

$$\left(A^{\theta_1}\right)^{\theta_2} = A^{\theta_1 + \theta_2}$$

and in particular we can twist by θ and then twist back by $-\theta$.

These are "uniform deformation formulas", also called "deformation quantization". There are other kinds of quantization, e.g. by approximating an algebra by an algebra of matrices. Want: for any Lie group G with "compatible" Poisson bracket, i.e. for any "Poisson Lie group", and any action α of G on a C^*-algebra A, we would want a construction to deform A^∞ in the direction of the Poisson bracket.

Most quantum groups people construct are made by doing this at the purely algebraic level, where a Lia algebra acts on an algebra. At this algebraic level, that's tough. It's even tougher in our analytical context. Some interesting papers exist, but it's presently under research.