1 February 20, 2008

This piece we're doing right now is the last piece of the purely theoretical bit. We will soon start edging into examples, with theory along the way.

We are trying to show that for unital $*$-normed algebras, that the state space (a compact convex subset of the dual) has extreme points which, per the GNS construction, result in irreducible representations. It has been convenient to think not in terms of states but in terms of positive linear functional.

Let A be a unital $*$-normed algebra, with positive linear functionals μ, ν with $\mu \geq \nu \geq 0$. Let $T \in \mathrm{End}_A(\mathcal{H}_\mu) = \mathcal{B}(L^2(A,\mu))$ and $0 \leq T \leq \mathbb{1}$. Then for example with can set

$$\nu : a \mapsto \langle Ta\xi_0, \xi_0 \rangle_\mu. \tag{1}$$

Conversely, given ν, we consider:

$$|\nu(b^*a)| = |\langle a, b\rangle_\nu| \leq \nu(a^*a)^{1/2}\nu(b^*b)^{1/2} \leq \mu(a^*a)^{1/2}\mu(b^*b)^{1/2} = \|a\xi_\mu\|_\mu \|b\xi_\mu\|_\mu \tag{2}$$

Thus, if we set $\langle a\xi_\mu, b\xi_\mu\rangle_\nu \stackrel{\text{def}}{=} \nu(b^*a)$, is this well-defined? Yes, by equation (2), because if the difference on is the 0-vector, then the RHS of above is 0. So then $a\xi_\mu \mapsto \langle a\xi_\mu, b\xi_\mu\rangle_\nu$ is a continuous linear functional for $\|\cdot\|_\mu$, so it extends to $\mathcal{H}_\mu$ (the vectors of the form $a\xi_\mu$ are dense in $\mathcal{H}$ = the completion). But on a complete Hilbert space, every positive linear functional comes from a vector. So there is a vector $T^*b\xi_m u$ so that $\langle a\xi_\mu, b\xi_\mu\rangle_\nu = \langle a\xi_m u, T^*b\xi_\mu\rangle_\mu$. This defines T^* for vectors of the form $b\xi_m u$, but also by equation (2), we see that $b\xi_\mu \mapsto T^*b\xi_m u$ is continuous for $\|\cdot\|_\mu$ so T^* extends to $\mathcal{H}_\mu$, and chasing constants gives $\|T\| \leq 1$. So we see that $\nu(b^*a) = \langle Ta\xi_\mu, b\xi_\mu\rangle_\mu$. Letting $b = 1$ gives $\nu(a) = \langle Ta\xi_\mu, \xi_\mu\rangle_\mu$.

Checking that T is in fact an endomorphism over A:

$$\langle TL_a b\xi_\mu, c\xi_\mu\rangle = \langle T(ab)\xi_\mu, c\xi_\mu\rangle = \nu(c^*(ab)) = \nu((a^*c)^*b) = \langle Tb\xi_\mu, a^*c\xi_\mu\rangle = \langle L_a Tb\xi_\mu, c\xi_\mu\rangle$$

This checks that T commutes with L_a on a dense subspace, so T is in fact an endomorphism. Chasing inequalities gives $0 \leq T \leq \mathbb{1}$. Thus every $0 \leq \nu \leq \mu$ is of the form (1).

Moreover, if we have two different Ts giving the same positive linear functional, then taking their difference gives the zero positive linear functional ($T \mapsto \nu$ in (1) is linear), so $\nu \mapsto T$ is injective. Thus, there is a bijection between $\{\nu : \mu \geq \nu \geq 0\}$ and $\{T \in \mathrm{End}_A(\mathcal{H}_\mu) : 0 \leq T \leq \mathbb{1}\}$.

Definition: A positive linear functional μ is *pure* if whenever $\mu \geq \nu \geq 0$, then $\nu = r\mu$ for some $r \in [0,1]$.

Theorem: For positive linear functional μ, its GNS representation $(\pi_\mu, \mathcal{H}_\mu, \xi_\mu)$ is irreducible if and only if μ is pure.

Proof:

If the GNS representation is not irreducible, then there exists $P \in \mathrm{End}_A(\mathcal{H}_\mu)$ a proper projection. (So $0 \leq P \leq \mathbb{1}$.) Use $T = P$ in equation (1), and then $\nu \leq \mu$ but $\nu \notin [0,1]\mu$, because $r\mu$ gives the same GNS representation as μ (except that $\xi_{r\mu} = r\xi_m u$), whereas ν gives a different one (shrunk by P). Thus μ is not pure.

Conversely, if μ is not pure, then there is a positive $\nu \leq \mu$ with $\nu \notin [0,1]\mu$, so $T_\nu \notin \mathbb{C}\mathbb{1}$, so $\mathrm{End}_A(\mathcal{H}_\mu) \neq \mathbb{C}\mathbb{1}$. So by Schur's lemma, the GNS representation is not irreducible. I.e. T_ν will map onto a proper invariant subspace. $\square$

Now we want to convert this statement about positive linear functionals into one about states.

Reminder: For a convex set C, a point μ is an *extreme* point if whenever $\mu = t\nu_1 + (1-t)\nu_0$ and $0 < t < 1$, then $\nu_0 = \nu_1 = \mu$.

Theorem: (Krein-Milman)

A compact convex set is the closed convex hull of its extreme points.

Question from the audience: Any topological vector space? **Answer:** Locally convex. **Question from the audience:** Not just a Banach space? **Answer:** No. We are applying it to the dual of a hairy space, so not necessarily Banach.

Even in the finite-dimensional case, the set of extreme points need not be closed.

Theorem: For a $*$-normed algebra A with 1, and a state μ, the GNS representation for μ is irreducible if and only if μ is an extreme point of $S(A)$ = state space.

Since "pure" has fewer syllables than "extreme", we refer to extreme points as pure points.

Proof:

Suppose μ is extreme. It is sufficient to show that it is pure.

Suppose $\mu > \nu > 0$. Then

$$\mu = \nu + (\mu - \nu) = \|\nu\| \overbrace{\frac{\nu}{\|\nu\|}}^{\in S(A)} + \|\mu - \nu\| \overbrace{\frac{\mu - \nu}{\|\mu - \nu\|}}^{\in S(A)}$$

But $\|\nu\| + \|\mu - \nu\| = \nu(1) + (\mu - \nu)(1) = \mu(1) = 1$ by positivity, and since μ is extreme, we must have $\frac{\nu}{\|\nu\|} = \mu$, so $r = \|\nu\|$ and μ is pure.

Conversely, if μ is pure, we should show that it is extreme. Consider $\mu = t\nu_1 + (1-t)\nu_0$ with $0 < t < 1$ and $\nu_0, \nu_1 \in S(A)$. Then $\mu - t\nu_1 = (1-t)\nu_0 \geq 0$, so $\mu \geq t\nu_1$, so $t\nu_1 = r\mu$, but $\nu_1(1) = 1 = \mu(1)$, so $r = t$, so $\nu_1 = \mu$ (and same argument works for $1-t$ and ν_0). $\square$

Question from the audience: Are these the point measures? **Answer:** Yes, exactly. These are the δ_x on $C(X)$, and $L^2(X, \delta_x)$ are the irreducible representations. In the noncommutative case, things are more complicated.

We have two minutes left, and will talk about non-unital algebras.

For A non-unital $*$-normed with approximate two-sided identity of norm 1, we define a quasi-state space $\mathcal{Q}S$. We drew a picture ****perhaps I'll add later: a cone with 0 at the vertex and $S(A)$ at the base****, and the extreme points of $\mathcal{Q}S(A)$ are exactly the extreme points of $S(A)$ together with 0. Certainly 0 does not give an interesting representation (the 0 Hilbert space). Even in this non-unital case, the extreme points of the now non-closed $S(A)$ are almost enough; $\mathcal{Q}S$ is weak closed.