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1.1 Further comments on deformation quantization

$SO(n+1)$ acts on S^n in a natural way, and so long as $n \geq 3$, we can find inside $SO(n+1)$ two or more copies of the rotation group:

$$\begin{pmatrix} \cos 2\pi r & \sin 2\pi r & & & \\ -\sin 2\pi r & \cos 2\pi r & & & \\ & & \cos 2\pi s & \sin 2\pi s & \\ & & -\sin 2\pi s & \cos 2\pi s & \\ & & & & \ddots \end{pmatrix}$$

So we have roughly a $(n+1)/2$-dimensional torus acting on S^n, so pick a θ and build $(S^n)^\theta$. M.R. had described our deformation quantization in general; Alain Connes became interested in examples, and in particular the quantum spheres $(S^n)^\theta$.

Moreover, M.R. gave the prescription for building quantum groups like $(SO(n+1))^\theta$, a quantum group, but others did the examples, and $(SO(n+1))^\theta$ acts on $(S^n)^\theta$. This is a relatively tame situation.

Question from the audience: So when you deform a group to get a quantum group, you have different multiplication by same comultiplication? **Answer:** Well, there are different versions. If we have a compact group, we use

$$\begin{aligned} C(G) &\overset{\Delta}{\to} C(G) \underset{C^*}{\otimes} C(G) = C(G \times G) \\ f &\mapsto (\Delta f)(x,y) \overset{\text{def}}{=} f(xy) \end{aligned}$$

Then the comultiplication encodes the group structure, and a quantum group is some algebra with a coassociative comultiplication.

1.2 Differential forms

Let G be a Lie group, α and action on A. Then we have A^∞, $\alpha_X = D_X$ for $X \in \mathfrak{g} = \mathrm{Lie}(G)$. Given $a \in A^\infty$, define $da : \mathfrak{g} \to A^\infty$, i.e. $da \in \mathfrak{g}' \otimes A^\infty$ (where $\mathfrak{g}'$ is the dual algebra of $\mathfrak{g}$ **why not use $\hat{\mathfrak{g}}$?** by

$$(da)X \overset{\text{def}}{=} \alpha_X(a)$$

Then

$$d(ab)\,X = \alpha_X(ab) = \alpha_X(a)b + a\alpha_X(b) = \left((da)b + a(db)\right)(X)$$

Definition: For any algebra A, a *first-order differential calculus* over A is a pair (Ω^1, d) where Ω^1 is an A-bimodule and d is a map $d : A \to \Omega^1$ satisfying the Leibniz rule $d(ab) = a\,db + da\,b$.

Often, we require additionally that Ω^1 be generated as a bimodule by $d(A)$. In this case, if $1 \in A$, the Leibniz rule provides that Ω^1 is generated by $d(A)$ as a left (or as a right) module.

Every A (with 1) has a universal first-order calculus: we let Ω^1 is (a subspace of) the algebraic tensor product $A \otimes A$, where $da \stackrel{\text{def}}{=} 1 \otimes a - a \otimes 1$. Then in particular $a\,db = a \otimes b - ab \otimes 1$. We have the algebra multiplication a map $m : A \otimes A \to A$, and then $m(a\,db) = 0$. You can check: $\text{span}\{a\,db\} = \ker(m)$. Sometimes people define Ω^1 as this kernel, but this, at a philosophical level, is from convenience rather than general principles.

Question from the audience: In what notion is this universal? **Answer:** Any other first-order differential calculus is a quotient of this one.

In any case, above (Lie group) is an example. In non-commutative geometry, the notion of "tangent space" becomes less useful. Any algebra A might have lots of derivations, but the space of derivations is not really a module over A. By non-commutativity, if D is a derivation, then aD probably is not.

But here we have cotangent spaces: differential forms. Indeed, we have a proliferation of them. Without getting too deep, we certainly have higher-order differential calculi:

$$A \longrightarrow \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \cdots$$

where we demand that $d^2 = 0$. Once we have this type of structure, we can define a cohomology for our differential calculus: $Z^n \stackrel{\text{def}}{=} \ker(\Omega^n \xrightarrow{d} \Omega^{n+1})$ and $B^n \stackrel{\text{def}}{=} \ker(\Omega^{n-1} \xrightarrow{d} \Omega^n)$, and $H^n \stackrel{\text{def}}{=} Z^n/B^n$.

For the universal calculus, we can take Ω^2 as the span of symbols of the form $a_0\,da_1\,da_2$, manipulated in the obvious way (where we be careful about keeping the order in tact, as we are noncommutative). For G and an action α on A, we can use $\Omega^n = (\bigwedge^n \mathfrak{g}') \otimes A^\infty$. **Question from the audience:** Normally we let the wedge product be anti-commutative. In the non-commutative case, shouldn't this be worse? **Answer:** Well, we want n-linear alternating A-valued forms on $\mathfrak{g}$. The Leibniz rule is complicated:

$$d(\omega_p\omega) = (d\omega_p)\omega + (-1)^p\omega_p(d\omega)$$

where ω can be any form, and ω_p is homogeneous of degree p.

Well, this is all somewhat weird. I'm sure you've heard that even in ordinary differential geometry, as soon as you get to dimension 7, the 7-sphere and higher have exotic differential structures. This happens in non-com-land, e.g. for non-commutative tori, when $n \geq 4$: On T^4, we can have θ_1 and θ_2 where $A_{\theta_1} \cong A_{\theta_2}$ but $A^\infty_{\theta_1} \not\cong A^\infty_{\theta_2}$. Finding the right invariants to show all this is hard, and gets into K-Theory. It is in the direction we want to go in.

1.3 Vector Bundles

We won't assume that you know too much about vector bundles in detail, but the general picture is that you have some space M and a bundle of vector spaces over M ****a standard picture****

$$\begin{array}{c} E \\ \downarrow \\ M \end{array}$$

so that locally $E \cong \mathcal{O} \times \mathbb{R}^n$ or $\mathbb{C}^n$, i.e. local triviality. We can think about smooth cross-sections. We write $\Gamma(E)$ for the space of continuous cross-sections, and by local triviality we can take bump functions, giving lots of continuous cross-sections (certainly we also have the 0-section).

Take M compact for simplicity. Then we have $C(M)$, and given $f \in C(M)$ and $\xi \in \Gamma(E)$, we can define $f\xi$ in the obvious way. By looking at an open neighborhood of each point, it's clear that this again is a continuous cross-section. Written briefly: $\Gamma(E)$ is a module over $C(M)$.

In fact, these are somewhat special modules. We know from working with vector spaces that it's useful to have inner products. Since we're assuming compactness, we can cover M with a finite number of open sets $\mathcal{O}_1, \dots, \mathcal{O}_k$ over which E is trivial. Then we can find a continuous partition of unity $\{\phi_j\}$ subordinate to $\{\mathcal{O}_1, \dots, \mathcal{O}_k\}$: i.e. the support of ϕ_j is contained in $\mathcal{O}_j$ for each j, and $\sum \phi_j = 1$ and $0 \leq \phi_k \leq 1$.

Then for $\xi, \eta \in \Gamma(E)$, we look at E over $\mathcal{O}_j$, over which it looks like $\mathcal{O}_j \times \mathbb{R}^n$ (or perhaps $\mathbb{C}^n$), then we can view $\xi, \eta|_{\mathcal{O}_j}$ as living in $\mathcal{O}_j \times \mathbb{R}^n$, and then we can form the standard inner product in terms of our choice of trivialization and get a function $\langle \xi, \eta \rangle_{\mathbb{R}^n}$. Multiplying by ϕ_j gives us a function that's 0 near the boundary, and so extends to the whole space. Then we can get a global inner product:

$$\langle \xi, \eta \rangle_{C(M)} \stackrel{\text{def}}{=} \sum_j \phi_j \langle \xi|_j, \eta|_j \rangle_{\mathbb{R}^n} \tag{1}$$

This is a continuous function, i.e. it is an element of $C(M)$. This is a good example of an "A-valued inner product" (for $A = C(M)$) on $\Gamma(E)$. In the real case, these are called "Riemannian metrics" on the bundle, and in the complex case called "Hermetian metrics". A good neutral term is *bundle metric*.

We should set this machinery up to avoid the following stupid possibility: $A = C([0,1])$ and $E = [0,1] \times \mathbb{R}^n$, and we could set $\langle \xi, \eta \rangle_A(t) = t\langle \xi(t), \eta(t) \rangle$. This is an inner product, and has the property that if $\langle \xi, \xi \rangle = 0$ then $\xi = 0$. So this A-valued inner product satisfies all the right conditions for an inner product, but it seems wrong to have the zero inner product even at a point. What's wrong is that it's not self-dual. Our earlier inner product (1) is *self-dual* in the sense that if $F \in \mathrm{Hom}_A(\Gamma(E), A)$ then there is (unique) $\eta \in \Gamma(E)$ such that $F(\xi) = \langle \xi, \eta \rangle_A$ for every ξ. (We are in the commutative case, so the order we write in doesn't really matter.) These modules are called *projective*.