****This document was last updated on April 21, 2008. A more recent version may be available as part of http://math.berkeley.edu/~theojf/CstarAlgebras.pdf.****

1 April 21, 2008

We were in the situation of having T^d and α an action on B a Banach space. We had seen from general considerations of Lie groups that you can form the space B^∞ of smooth vectors, dense in B.

On the other hand, for $n \in \mathbb{Z}^d = \widehat{T^d}$, we had $B_n = \{b : \alpha_t(b) = \langle n, t\rangle b\}$. (Recall: $\langle n, t\rangle = e^{2\pi i t\cdot n}$.) Then if $X \in \mathbb{R}^d = \mathrm{Lie}(T^d)$, we defined

$$D_X b \overset{\text{def}}{=} \lim_{r\to 0} \frac{\alpha_{rX}(b) - b}{r}$$

and on $b \in B_n$ we have $D_X b = (2\pi i X \cdot n)b$.

Remember, on A_θ the noncom torus, we had a dual action

$$\alpha_t(U_n) \overset{\text{def}}{=} \langle n, t\rangle U_n$$

and so $(A_\theta)_n = \mathbb{C}U_n$ is just a one-dimensional span.

Question from the audience: What is α_{rX}? **Answer:** $\alpha_{\exp rX}$. But we're writing T^d additively.

So $D_{X_1} \dots D_{X_k} b = (2\pi i)^k \left(\prod_{j=1}^k n \cdot X_j\right) b$.

For $b \in B^\infty$, we expect to write incomponents: $b \sim \{b_n\}$. Then we expect $(D_{X_1} \dots D_{X_k} b)_n \sim D_{X_1} \dots D_{X_k} b_n = (2\pi i)^k \left(\prod_{j=1}^k n \cdot X_j\right) b_n$. But the right and left should be in b; taking norms:

$$(2\pi)^k \prod |n \cdot X_j| \, \|b_n\| \le \|D_{X_1} \dots D_{X_k} b\|$$

and the RHS is indep of n. The norm on the RHS is some constant, adjusting it we can say that

$$\|b_n\| \le \frac{c}{(1 + \prod |n \cdot X|^2)^n}$$

so the coefficients b_n must die faster than any polynomial.

We define the Schwartz space: $\mathcal{S}(\mathbb{Z}^d) = \{f : \mathbb{Z}^d \to \mathbb{C}$ such that $n \mapsto |f(n)p(n)$ is a bounded function for all polys $p\}$.

Theorem: B^∞ is the B-valued Schwartz space: it consists of all functions $c : \mathbb{Z}^d \to B$ such that $c(n) \in B_n$ and $\{n \mapsto \|c_n\|\} \in \mathcal{S}(\mathbb{Z}^d)$.

Then in particular $(A_\theta)^\infty$ "is" $\mathcal{S}(\mathbb{Z}^d)$ by $f \mapsto \sum f(n)U_n$. In any case, for p big enough, it's clear that $\mathcal{S}(\mathbb{Z}^d) \subseteq \ell^1(\mathbb{Z}^d)$, and this sum converges.

This relates to a well-known fact: for $g \in C(T^d)$, $g \in C^\infty(T^d)$ iff the Fourier transform $\hat{g} \in \mathcal{S}(\mathbb{Z}^d)$.

Proof of Theorem:

If we have $c \in$ RHS, we certainly have $\{n \mapsto \|c(n)\|\} \in \ell^1(\mathbb{Z}^d)$, so set $b = \sum c(n)$ converges just fine, and $b_n = c(n)$. Using the differential quotient, you find that $(D_X b)_n = (2\pi i)n \cdot X\, c(n)$, and the sum of these things since we're in the Schwartz space converges. So we have $\supseteq$. (We did not complete the proof, but it goes along pretty straightforwardly.)

In the opposite direction, let $b \in B^\infty$. ****We spend some time on a calculation we had done before, going the wrong direction.**** We're trying to show that $\{n \mapsto \|b_n\|\} \in \mathcal{S}(\mathbb{Z}^d)$. Then $D_X b$ exists, and we need to show that $(D_X b)_n = 2\pi i\, n \cdot X\, b_n$. Write $f(t) = e^{2\pi i\, n\cdot t}$. The limit $D_X b = \lim(\alpha_{rX}(b) - b)/r$ is a uniform limit, so:

$$\begin{aligned}
(D_X b)_n &= \int f(t)\,\alpha_t(D_X b)\,dt \\
&= \lim \int f(t)\,\alpha_t\left(\frac{\alpha_{rX}(b) - b}{r}\right)dt \\
&= \lim \frac{1}{r}\left(\int f(t)\,\alpha_t\alpha_{rX}(b)\,dt - \int f(t)\,\alpha_t(b)\,dt\right) \\
&= \lim \frac{1}{r}\left(\int f(t - rX)\,\alpha_t(b)\,dt - \int f(t)\,\alpha_t(b)\,dt\right) \\
&= \lim \int \frac{f(t - rX) - f(t)}{r}\alpha_t(b)\,dt \\
&= 2\pi i\, n \cdot X\, b_n, \text{ since } f = e^{2\pi i n\cdot t}
\end{aligned}$$

Thus $(D_{X_1} \ldots D_{X_n} b)_n = p(n)\, b_n$, and so take norms and observe that we're in $\mathcal{S}(\mathbb{Z}^d)$. $\square$

Let A be a C^* algebra, and let α be an action of T^d on A. (**E.g.** let α be an action of T^d on a locally compact space X, e.g. a manifold, so can take $A = C_\infty(X)$. Even this commutative case is interesting.) So we have all these spaces A_n, and let $a_m \in A_m$, $a_n \in A_n$, then $\alpha_t(a_m a_n) = \alpha_t(a_m)\alpha_t(a_n) = \langle m, t\rangle a_m\, \langle n, t\rangle a_n = \langle m+n, t\rangle a_{m+n}$, and the multiplication is graded: $A_m A_n \subseteq A_{m+n}$. Similarly, $A_m^* = A_{-m}$.

Let θ be given, and build the cocycle $c_\theta(m, n)$. Let $(\pi, U, \mathcal{H})$ be a faithful covariant representation of (A, T^d, α). T^d acts on $\mathcal{H}$, so we can factor

$$\mathcal{H} = \bigoplus_{n \in \mathbb{Z}^d} \mathcal{H}_n$$

You can check: if $a_m \in A_m$ and $\xi_n \in \mathcal{H}_n$, then $\pi(a_m)\xi_n \in \mathcal{H}_{m+n}$.

Now we will do something weird. Continuing to use these labels to tell you where things come from, define:

$$\pi^\theta(a_m)\xi_n \stackrel{\text{def}}{=} \pi(a_m)\xi_n c_\theta(m,n) \in \mathcal{H}_{m+n}$$

. Next time, we will explore this. We will find that all of this is well-defined on A^∞, and we're twisting this algebra by a cocycle.