1 March 10, 2008

Last time we defined the tensor product of C^* algebras. We also have a free product:

Definition: Given C^*-algebras A and B, we define $A * B$ to be the free algebra with all relations in A and all those in B, and that $\mathbb{1}_A = \mathbb{1}_B$, but we do not require that the algebras commute.

Then a representation is just a pair of non-commuting representations on the same Hilbert space. There is also a reduced product $A *_r B$, which we will not go into.

E.g. $C(S^1) * C(S^1) = C^*(F_2)$, because $C(S^1)$ is the C^* algebra generated freely by one unitary operator.

1.1 C^*-dynamical systems

Let A be a C^* algebra, G a discrete group, and $\alpha : G \to \mathrm{Aut}(A)$. **E.g.** Let M be a locally compact space, $\alpha : G \to \mathrm{Homeo}(M)$. Set $A = C_\infty(M)$; then $(\alpha_x(f))(m) \stackrel{\text{def}}{=} f(\alpha_x^{-1}(m))$ where $\alpha : x \in G \mapsto \alpha_x$.

The first discussion of what we are about to say came from quantum physics, where the *observables* of a system are self-adjoint operators (possibly unbounded, but we will duck that question, as well as the philosophy of physics), i.e. they are in some C^*-algebra A. We have already defined "states" for an algebra, and we will continue that notion here. Symmetries of the system form a group G ****usually a Lie group****.

The physicists want everything acting on a Hilbert space, which in fact is a useful way to understand groups acting on algebras of operators. So we will represent A on a Hilbert space $\mathcal{H}$, via a $*$-rep π, and let's ask for U to be a unitary representation of G on $\mathcal{H}$. What about the action α? From the physicists' point of view, α should be unitarily represented.

Setting $\beta_x(T) = U_x(T)U_{x^{-1}}$ gives an action $G \to \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$, as inner representations. So we demand what the physicists call the *covariance condition*:

$$\pi(\alpha_x(a)) = U_x \pi(a) U_x^{-1}$$

Definition: We say that (π, U) is a *covariant representation* of (A, G, α) if this condition holds.

We can use the generators of G and A and their relations, along with the covariance relation, which can be rewritten as $xa = \alpha_x(a)x$, and the requirement that $x^* = x^{-1}$. But this says that any word in the generators can be rearranged into normal form with all the xs on the right and all the as on the left (just about everyone seems to use this convention); but then we can multiply adjacent xs and adjacent as. So the $*$-algebra is just finite linear combinations of ax, i.e. sums of the form $\sum f(x)x$ where $f(x) \in A$.

So f contains the data of the element, and so we define operations on $C_c(G, A)$ (= functions of finite support with values in A):

$$\begin{aligned}
\left(\sum f(x)\, x\right)\left(\sum g(y)\, y\right) &= \sum_{x,y} f(x)\, x\, g(y)\, y \\
&= \sum f(x)\, \alpha_x(g(y))\, xy \\
&= \sum_{x,y} f(x)\, \alpha_x(g(x^{-1}y))\, y \\
&= \sum_y \left(\sum_x f(x)\, \alpha_x(x^{-1}y)\right)
\end{aligned}$$

So we define the *twisted convolution* ****the standard notation, using $*$ for both the convolution and the adjoint, is unfortunate; I will use $\star$ for convolution****:

$$(f \star g)(y) = \sum f(x)\, \alpha_x(g(x^{-1}y))$$

We also have a $*$ operation:

$$\begin{aligned}
\left(\sum f(x)\, x\right)^* &= \sum x^*\, f(x)^* \\
&= \sum x^{-1}\, f(x)^* \\
&= \sum \alpha_x^{-1}(f(x)^*)\, x^{-1} \\
&= \sum \alpha_x(f(x^{-1})^*)\, x
\end{aligned}$$

So, every covariant representation (π, U) of (A, G, α) will give a representation of $(C_c(G, A), \star, ^*)$. For $f \in C_c(G, A)$, we set $\sigma_f \overset{\text{def}}{=} \sum \pi(f(x)) U_x$; then σ is a $*$-rep of $(C_c(G, A), \star, ^*)$.

Then we can estimate norms:

$$\|\sigma_f\| \leq \sum \|f(x)\|_A \overset{\text{def}}{=} \|f\|_1$$

where $\|\cdot\|_1$ is the "ℓ^1" norm in A.

In general, we define $\|f\|_{C^*(G,A,\alpha)}$ to be the supremum over all such representations, but it's not clear that there are any.

We can make the following comments. In a suitable sense, $A \hookrightarrow C_c(G, A, \alpha)$ by $a \mapsto a\delta_{1_G}$. If A has an identity element, then $G \hookrightarrow C_c(G, A, \alpha)$ by $x \mapsto 1_A \delta_x$. If A does not have a unit, then $G \to M(C_c(G, A, \alpha))$ where this is the algebraic multiplier algebra, in the sense as on the problem set. All of this works for $*$-normed algebras

Why are there plenty of covariant representations? We need representations on A, which for generic $*$-normed algebras might be few and far between. But for each rep ρ of A on $\mathcal{K}$, form the *induced* *covariant representation* of (G, A, α). (This is induced from $\{e\} \subseteq G$; we can induce from any subgroup.) In particular, we take $\mathcal{H} = \ell^2(G, \mathcal{K}) = \ell^2(G) \otimes \mathcal{K}$. Then the actions are by

$$\begin{aligned}(U_x\xi)(y) &\stackrel{\text{def}}{=} \xi(x^{-1}y) \\ (\pi(a)\xi)(y) &\stackrel{\text{def}}{=} \rho(\alpha_y^{-1}(a))\xi(y)\end{aligned}$$

We check the covariance conditions, and sure enough it passes.

Then we define the *reduced norm*:

$$\|f\|_{C_r^*(G,A,\alpha)} = \sup\{\|\pi(f)\| \text{ for all induced covariant reps}\}$$

If we start with a faithful representation of A, then our induced representation is faithful on the functions of compact support, so this is a norm. The full norm:

$$\|f\|_{C_r^*(G,A,\alpha)} = \sup\{\|\pi(f)\| \text{ for all covariant reps}\}$$