1 March 7, 2008

We left off on this business of amenable groups. Another version:

Definition/Theorem: G is amenable if it satisfies Følner's condition:

$$\forall \epsilon > 0, \forall \text{ finite } K \subset G, \exists \text{ finite } U \in G \text{ s.t. } \forall x \in K, \frac{|xU \triangle U|}{|U|} < \epsilon$$

Where $\triangle$ is the symmetric difference "xor."

Finite groups and solvable groups are amenable, but F_n the free group and $SL(n, \mathbb{Z})$ for $n \geq 2$ are not.

1.1 Tensor products

Let A and B be C^*-algebras with 1. We want $A \otimes B$, which should include elements like $a \otimes b$, and $A \hookrightarrow A \otimes B$ via $a \mapsto a \otimes 1_B$. The multiplication has A and B commuting: $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$.

We now consider the $*$-algebra with generators $A \cup B$ and relations those of A and those of B and that $ab = ba$ if $a \in A$ and $b \in B$ (and that $1_A = 1_B$). Then we exactly get $A \overset{\text{alg}}{\otimes} B$. This is a $*$-algebra: $(a \otimes b)^\star = a^* \otimes b^*$.

We haven't yet introduced a norm. Does $A \overset{\text{alg}}{\otimes} B$ have any $*$-reps? There is a natural class: Let $(\pi, \mathcal{H})$ be a $*$-rep of A and $(\rho, \mathcal{K})$ a $*$-rep of B. We form $\mathcal{H} \overset{\text{alg}}{\otimes} \mathcal{K}$ the algebraic tensor product of vector spaces, with inner product $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle \overset{\text{def}}{=} \langle \xi_1, \xi_2 \rangle_{\mathcal{H}} \langle \eta_1, \eta_2 \rangle_{\mathcal{K}}$, extended to $\mathcal{H} \overset{\text{alg}}{\otimes} \mathcal{K}$ by (conjugate) linearity. Check that this result is positive definite (not hard by expressing everything in terms of an orthonormal basis). Then complete to get the Hilbert-space tensor product $\mathcal{H} \otimes \mathcal{K}$.

For $S \in \mathcal{B}(\mathcal{H})$, we define $(S \otimes \mathbb{1}_{\mathcal{K}})(\xi \otimes \eta) = (S\xi) \otimes \eta$, and extend by linearity to $\mathcal{H} \overset{\text{alg}}{\otimes} \mathcal{K}$, where it is a bounded operator, so extends by continuity to $\mathcal{H} \otimes \mathcal{K}$. Then we can check that $\|S \otimes \mathbb{1}_{\mathcal{K}}\| = \|S\|$ and $(S \otimes \mathbb{1})^* = S^* \otimes \mathbb{1}$. All this also works for $\mathbb{1}_{\mathcal{H}} \otimes T$. On the algebraic tensor product, these elements commute, so $S \otimes T$ is well-defined,

In any case, we can define $(\pi \otimes \rho)(a \otimes b) = \pi(a) \otimes \rho(b)$ on $\mathcal{H} \otimes \mathcal{K}$, which extends to $A \overset{\text{alg}}{\otimes} B$. If π is faithful then $\pi \otimes \rho$ is faithful.

Question from the audience: On $\mathcal{H} \otimes \mathcal{K}$, are all the bounded operators of this form? **Answer:** Oh, certainly not. Take finite linear combinations of elementary tensors, and close up in the weak-$*$ topology. By double-commutant theorem, these are dense.

For $t \in A \overset{\text{alg}}{\otimes} B$, we set

$$\|t\|_{\min} \overset{\text{def}}{=} \sup\{\|(\pi \otimes \rho)t\| : (\pi, \mathcal{H}), (\rho, \mathcal{K}) \text{ are representations of } A, B\}$$

This is a C^*-norm. It's a nice norm, but it's not the universal norm. We define the closure of $A \overset{\text{alg}}{\otimes} B$ with this norm to be $A \overset{\min}{\otimes} B$.

$\|t\|_{\max} \overset{\text{def}}{=} \sup\{\|(\pi \otimes \rho)t\| : \pi, \rho$ are reps of A, B on same $\mathcal{H}$ s.t. $\pi(a), \rho(b)$ commute $\forall a \in A, b \in B\}$

Then $\|t\|_{\min} \leq \|t\|_{\max}$, since we can always take $\mathcal{H}$ in the second definition to be $\mathcal{H} \otimes \mathcal{K}$ in the first, and we can complete with the latter to define $A \overset{\max}{\otimes} B$.

Are these the same? In 1959, Takesaki ****sp?**** said no: Let G be a discrete group, and let π be the left regular representation of $C_r^*(G)$ on $\ell^2(G)$. Let ρ be the right regular representation. Left and right representations commute, so $\pi \otimes \rho$ gives a representation of $C_r^*(G) \overset{\text{alg}}{\otimes} C_r^*(G)$ on $\ell^2(G)$. But this does not split as a tensor product of representations, and e.g. for $G = F_n$, $n \geq 2$, the free group on n generators, we have $\|\cdot\|_{\max} \gneqq \|\cdot\|_{\min}$.

A diagram, where $\dashrightarrow$ is $\pi \otimes \rho$ in the definition of $\|\cdot\|_{\max}$:

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr, "\pi"] & & \\
& A \overset{\text{alg}}{\otimes} B \arrow[r, dashed] & \mathcal{B}(\mathcal{H}) \\
B \arrow[ur] \arrow[urr, "\rho"'] & &
\end{tikzcd}$$

Everyone sort of assumed that tensor products were easy, until this example came along. Then min and max products are minimal and maximal in the appropriate sense, but there are many intermediate ones.

Definition: A C^*-algebra A is *nuclear* if for any C^*-algebra B we have $A \overset{\min}{\otimes} B = A \overset{\max}{\otimes} B$. (I.e. the two norms are the same.)

E.g. commutative, $\mathcal{B}_0(\mathcal{H})$, any GCR. There are others too.

Given a short-exact sequence

$$0 \to I \to B \to B/I \to 0,$$

we can show that

$$0 \to A \overset{\max}{\otimes} I \to A \overset{\max}{\otimes} B \to A \overset{\max}{\otimes} (B/I) \to 0$$

is exact. So we say that A is *exact* if for any exact $0 \to I \to B \to B/I \to 0$, we have

$$0 \to A \overset{\min}{\otimes} I \to A \overset{\min}{\otimes} B \to A \overset{\min}{\otimes} (B/I) \to 0$$

exact.

Then nuclear implies exact.

If G is discrete, then G is amenable iff $C^*(G)$ is nuclear (this fails for some non-discrete groups). Open question: does there exist G discrete with $C^*(G)$ not exact?

These matter for various differential-geometry questions.

Gromov: "Any statement you can make about all discrete groups is either trivial or false." This question is certainly not trivial; Gromov has some ideas of where to look for a counterexample.