1 February 11, 2008

We were speaking in generalities about representations, and were in the midst of contemplating

$$\mathcal{H} = \bigoplus \mathcal{H}_\lambda$$

indexed by a vast set $\lambda \in \Lambda$. Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a family of $*$-representations of a $*$-algebra A. We want to define the direct sum $\bigoplus \pi_\lambda$ on $\bigoplus \mathcal{H}_\lambda$: the obvious answer is

$$\pi_\lambda(a)\xi = (\pi_\lambda(a)\xi_\lambda)_\lambda$$

How do we know that the RHS is square-integrable? This construction works if there is a constant K such that

$$\|\pi_\lambda(a)\| \leq K\|a\| \, \forall \lambda$$

Prop: Let A be a Banach $*$-algebra and π a $*$-homomorphism into a C^*-algebra (e.g. $\mathcal{B}(\mathcal{H})$) (or A is a $*$-normed algebra with each π continuous, so that π extends to the Banach completion). Then $\|\pi(a)\| \leq \|a\|$ for each $a \in A$.

Proof:

Adjoin identity elements. This is a little bit funny: A may well have an identity element, but the homomorphism need not be identity-preserving. Even if A has an identity elements, you can still adjoin another, in such a way as to make the homomorphism unital.

For each $a = a^*$, we have $\|\pi(a)\| = \rho(\pi(a))$ = the spectral radius (since $\pi(a)$ is in a C^* algebra). But if $A \overset{\text{unital}}{\longrightarrow} B$, then we have only introduced more inverses, so $\rho(\pi(a)) \leq \rho(a) \leq \|a\|$.

For general a, do the usual thing: $\|\pi(a)\|^2 = \|\pi(a^*a)\| \leq \|a^*a\| \leq \|a\|^2$. $\square$

Question from the audience: When you add a unit to a unital algebra, I think of this as compactifying an already compact space? **Answer:** yes; the original unit is an idempotent, so you are just adding a point that has nothing to do with the rest of the space.

Question from the audience: Are we assuming, in the continuous non-Banach case, that π has unit norm? **Answer:** no, that's a corollary.

Theorem: Any abstract C^*-algebra is isomorphic to a concrete C^*-algebra.

Proof:

Namely, let A be a C^*-algebra, and adjoin 1 if necessary. Let $S(A)$ be the state space. For $\mu(S(A))$, let $(\pi_\mu, \mathcal{H}_\mu)$ be the GNS representation. Let

$$(\pi, \mathcal{H}) = \bigoplus_{\mu \in S(A)} (\pi_\mu, \mathcal{H}_\mu)$$

This is a faithful representation of A.

Given self-adjoint a, there exists μ with $|\mu(a)| = \|a\|$. On the other hand, $|\mu(a)| = \langle \pi_\mu(a)\xi_\mu, \xi_\mu\rangle$ where ξ_μ is a cyclic element from GNS. This implies that $\|\pi(a)\| \geq \|a\|$. For general a, do the usual squaring. By the previous Prop, we have $\|\pi(a)\| = \|a\|$. $\square$

If A is separable, then we can use a countable number of states, so we can get $\mathcal{H}$ separable.

Lurking in the background, we used Choice to get all these states.

Question from the audience: What is the advantage of finding pure states, and working just with those? Davidson does this. **Answer:** We haven't talked about pure states yet. They are exactly the ones that give irreducible representations. We can sometimes get a smaller Hilbert space by working just with pure states.

Prop: Let A be a $*$-algebra, and let $(\pi_j, \mathcal{H}_j, \xi_j)$ for $j = 1, 2$ be two cyclic representations of A. Let $\mu_j = \langle \pi_j(a)\xi_j, \xi_j\rangle$ be the corresponding positive linear functionals on A. If $\mu_1 = \mu_2$, then there is a unique unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ with $\xi_1 \mapsto \xi_2$ and intertwining the A-action (i.e. $\pi_2(a)U = U\pi_1(a)$, i.e. U is an A-module homomorphism).

Proof:

Try to define U by

$$U(\pi_1(a)\xi_1) = \pi_2(a)\xi(2)$$

since the $\pi_j(a)\xi_j$ are dense. It's not clear that this is well-defined. Ducking this for a moment,

$$\begin{aligned}
\langle U(\pi_1(a)\xi_1), U(\pi_1(b)\xi_1)\rangle_{\mathcal{H}_2} &= \langle \pi_2(a)\xi_2, \pi_2(b)\xi_2\rangle \\
&= \langle \pi_2(b^*a)\xi_2, \xi_2\rangle \\
&= \mu_2(b^*a) \\
&= \mu_1(b^*a) \\
&= \dots \\
&= \langle \pi_1(a)\xi_1, \pi_1(b)\xi_1\rangle_{\mathcal{H}_1}
\end{aligned}$$

So U is certainly length-preserving, so extends to all of $\mathcal{H}_1$.

But if the RHS is 0, so must be the LHS, so U is well-defined and isometric and unitary. $\square$

Thus, for $*$-normed algebras, there is a bijection between {continuous positive linear functionals} and {pointed cyclic representations} (i.e. has a specific choice of cyclic vector). ****The board says "isomorphism classes", but if we have a unique isomorphism as thingies between two thingies, then I say that as thingies they are the same thingy.****

If μ is a positive linear functional on a $*$-algebra A, do we have $\mu(a^*) = \overline{\mu(a)}$? No, e.g. $A =$ polynomials vanishing at 0 on $[0, 1]$. Then let $\mu(p) = ip'(0)$.

On the other hand, sesquilinearity and positivity guarantee that $\langle a, b\rangle_\mu = \overline{\langle b, a\rangle_\mu}$ for any positive linear functional. If A is unital, we can let $b = 1_A$, so $\mu(1) = \langle a, 1\rangle_\mu = \overline{\langle 1, a\rangle_\mu} = \overline{\mu(a^*)}$.

Remark: "An approximate identity is enough."

Let A be a $*$-normed algebra with $\{e_\lambda\}$ a two-sided approximate identity of norm 1. (We do not require these to be self-adjoint; if it is two-sided, then $\{e_\lambda^*\}$ is also a two-sided approximate identity.) And let μ be a continuous positive linear funcitonal on A. Then

(a) $\mu(a^*) = \overline{\mu(a)}$

(b) $|\mu(a)|^2 \leq \|\mu\| \mu(a^* a)$