****This document was last updated on May 9, 2008. A more recent version may be available as part of http://math.berkeley.edu/∼theojf/CstarAlgebras.pdf.****

1: May 9, 2008

MR will not be here next time; Prof. A will give the final lecture, on not this material but stuff related to this course.

1.1 Continuation of last time

We make a correction from last time — a $*$ was mis-placed in the bookkeeping — and simplify to $M = \mathbb{R}$ (the general case is just like $\mathbb{R} \times \mathbb{Z}/p$, but even there the bookkeeping is hard). So $M \times \hat{M} \cong \mathbb{R}^2$, and $L^2(M) = L^2(\mathbb{R})$, and we have $\theta \in \mathbb{R}$ with $\theta \neq 0$ (we can have $\theta = 1, 2, \ldots$; then we get non-trivial bundles on the commutative torus).

We have $\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ generated by two generators $\pi_{(1,0)}$ and $\pi_{(0,1)}$. We pick $(1, 0) \mapsto (\theta, 0) \in \mathbb{R}^2$ and $(0, 1) \mapsto (0, 1) \in \mathbb{R}^2$. We write $e(s) \overset{\text{def}}{=} e^{2\pi i s}$, and take $\xi \in \mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$. Then

$$(\pi_{(m,n)}\xi)(t) = e(nt)\,\xi(t - m - \theta)$$

and hence

$$\beta((m, n), (p, q)) = \bar{e}(mq\theta)$$

For $f \in C_c(\mathbb{Z}^2)$, or more generally in $\mathcal{S}(\mathbb{Z}^2)$, we have

$$\begin{aligned}
(\xi \cdot f)(t) &= \sum_{m,n} (\pi^*_{m,n}\xi)(t)\, f(m, n) \\
&= \sum \beta((m, n), (m, n))\, (\pi_{-m,-n}\xi)(t)\, f(m, n) \\
&= \sum \bar{e}(mn\theta)\, \bar{e}(nt)\, \xi(t + m\theta)\, f(m, n) \\
&= \sum \bar{e}((t + m\theta)n)\, \xi(t + m\theta)\, f(m, n) \\
&= \sum_m \xi(t + m\theta) \sum_n \bar{e}((t + m\theta)n)\, f(m, n)
\end{aligned}$$

This looks like a Fourier mode. For $g \in \mathcal{S}(\mathbb{Z}^2)$, we set

$$\grave{g}(m, t) \overset{\text{def}}{=} \sum_n \bar{e}(nt)\, g(m, n)$$

which is periodic in t with period 1. (The grave accent is half a hat, because we're only transforming one variable.) Then

$$(\xi \cdot f)(t) = \sum_m \xi(t + m\theta) \sum_n \bar{e}((t + m\theta)n)\, f(m, n) = \sum_m \xi(t + m\theta)\, \grave{f}(m, t + m\theta)$$

Ok, then we have an action like $C^\infty(T) \times_\alpha \mathbb{Z}$. We have an inner product:

$$\begin{aligned}
\langle \xi, \eta \rangle\grave{}_A(m,t) &= \sum \bar{e}(nt)\overline{\langle \xi, \pi_{m,n}\eta\rangle_{L^2(\mathbb{R})}} \\
&= \sum_n \bar{e}(nt) \int_{\mathbb{R}} \bar{\xi}(s)\, e(ns)\, \eta(s - m\theta)\, ds \\
&= \sum_n \int \bar{\xi}(s)\, \eta(s - m\theta)\, \bar{e}((t-s)n)\, ds \\
&= \sum_n \int \bar{\xi}(s+t)\, \eta(s + t - m\theta)\, e(sn)\, ds
\end{aligned}$$

We think of $\bar{\xi}(s+t)\,\eta(s+t-m\theta)$ as some function $h(s)$. Then we have

$$\sum_n \int_{\mathbb{R}} h(s)\, e(ns)\, ds = \sum_n \hat{h}(n) = \sum_n h(n)$$

by the Poisson summation formula. So

$$\begin{aligned}
\langle \xi, \eta \rangle\grave{}_A(m,t) &= \sum_n \int \bar{\xi}(s+t)\, \eta(s + t - m\theta)\, e(sn)\, ds \\
&= \sum_n \bar{\xi}(n+t)\, \eta(n + t - m\theta)
\end{aligned}$$

is obviously periodic in t.

Question from the audience: The Poisson summation formula just expresses that Fourier transform is an isometry? **Answer:** No, it is more subtle. For instance, L^2 functions aren't defined at points, so plugging in n doesn't work; it uses that we are in Schwartz space, and generalizes slightly.

Continuing on:

$$\begin{aligned}
(f \star_\beta g)\grave{}\,(m,t) &= \sum_n \bar{e}(nt) \sum_{p,q} f(p,q)\, g(m-p, n-q)\, e(p(n-q)\theta) \\
&= \sum_{n,p,q} f(p,q)\, \bar{e}(qt)\, g(m-p, n-q)\, \bar{e}((n-q)t)\, e(p(n-q)\theta) \\
&= \sum_{n,p,q} f(p,q)\, \bar{e}(qt)\, g(m-p, n-q)\, \bar{e}((n-q)(t-p\theta)) \text{ sum in } n \\
&= \sum_{p,q} f(p,q)\, \bar{e}(qt)\, \grave{g}(m-p, t-p\theta) \text{ sum in } q \\
&= \sum_p f(p,t)\, \grave{g}(m-p, t-p\theta)
\end{aligned}$$

This is exactly the cross-product formula for $C(T) \times_{\alpha^\theta} \mathbb{Z}$, $(\alpha_p^\theta \phi)(t) \overset{\text{def}}{=} \phi(t - p\theta)$:

$$(\grave{f} \star \grave{g})(m,t) = \sum_p \grave{f}(p,t)\, \grave{g}(m-p, t-p\theta)$$

Now we take a leap of faith, and ask if we can find $\xi \in \mathcal{S}(\mathbb{R})$ so that $\langle \xi, \xi \rangle_{A_\theta}$ is a projection in A. If we have the ordinary torus, then there are no projections. Suppose that $0 < \theta < 1$. Then we take ξ to be a bump on $[0,1]$ that is 0 at 0, 1 at θ, and 0 again at 1 and 2θ. Then when translated by θ, ξ doesn't intersect itself. So

$$\langle \xi, \xi \rangle_A(p,t) = \sum \bar{\xi}(t+n)\, \xi(t+n-p\theta)$$

has support only at $p = -1, 0, 1$.

Now look for projections in $C(T) \times_{\alpha^\theta} \mathbb{Z}$ of the form $P = \delta_{-1}\phi + \delta_0\psi + (\delta_{-1}\phi)^*$ with $\psi = \bar{\psi}$, and $\psi, \phi \in C(T)$. Then

$$\begin{array}{rclcccl} P^2 & = & \delta_{-1}\phi\delta_{-1}\phi & + & \delta_{-1}\phi\, \delta_0\psi + \delta_0\psi\, \delta_{01}\psi & + & \text{adjoints} \\ & = & \delta_{-2}\, \alpha_1^\theta(\phi)\, \phi & + & \delta_{-1}\left(\phi\psi + \psi\alpha_1^\theta(\phi)\right) + \delta_0\,() & + & \dots \\ & & \text{need} = 0 & & \text{want} = \phi & & \end{array}$$

****I got a little lost in this next remark.**** Then we have A_θ with a tracial state τ — given $f(p,t)$, we have $\tau(f) = \int_T f(0,t)$, and we can graph ϕ and ψ, and what we discover is that $\tau(P) = \int \psi(t) = \Theta$. All of these projections correspond to projective modules.