1 Problem Set 2: Due March 14, 2008

****The problem set was given out typed. I've retyped it, partly so I could submit my answers set between the questions. I have corrected some typos, and no doubt introduced even more. In doing so, I have changed the formatting slightly.****

A. Fields of C^*-algebras. Anytime the *center* of a C^*-algebra (i.e. the set of its elements which commute with all elements of the algebra) is more than one-dimensional and acts non-degenerately on the algebra, the C^*-algebra can be decomposed as a field of C^*-algebras over the maximal ideal space of the center (or of any non-degenerate C^*-subalgebra of the center). For simplicity we deal here with unital algebras, but all of this works without difficulty in general. So let A be a C^*-algebra with 1, and let C be a C^*-subalgebra of the center of A with $1 \in C$. Let $C = C(X)$, and for $x \in X$ let J_x be the ideal of functions vanishing at x. Let $I_x = AJ_x$ (closure of linear span), an ideal in A. Let $A_x = A/I_x$ ("localization"), so $\{A_x\}_{x\in X}$ is a field of C^*-algebras over X. For $a \in A$ let a_x be its image in A_x.

1. Prove that for any $a \in A$ the function $x \mapsto \|a_x\|_{A_x}$ is upper-semi-continuous. (So $\{A_x\}$ is said to be an upper-semi-continuous field.)

2. If $x \mapsto \|a_x\|_{A_x}$ is continuous for all $a \in A$, then the field is said to be continuous. For this part assume that A is commutative. Note that then one gets a continuous surjection from $\hat{A}$ onto $\hat{C}$. Find examples of As and Cs for which $x \mapsto \|a_x\|$ is not continuous. In fact, find an attractive characterization of exactly when the field is continuous, in terms of the surjection of $\hat{A}$ onto $\hat{C}$ and concepts you have probably met in the past. (It can be shown that an analogous characterization works in the non-commutative case, using the primitive ideal space of A.)

3. Let

$$A_1 \stackrel{\text{def}}{=} \left\{ f : [0,1] \to M_2 \text{ continuous, with } f(1) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right\}$$
$$A_2 \stackrel{\text{def}}{=} \left\{ f : [0,1] \to M_2 \text{ continuous, with } f(1) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right\}$$

   and let $C_i \stackrel{\text{def}}{=} Z(A_i)$ be the center of A_i. Are the corresponding fields continuous? Are all the fiber algebras A_x isomorphic? Show that A_1 and A_2 are very simple prototypes of behavior that occurs often "in nature", but with higher-dimensional algebras, and more complicated boundary behavior.

4. Determine the primitive ideal space of each of these two algebras, with its topology.

B. An important extension theorem. (This will be used in the lectures.) Prove that if I is a $*$-ideal of a $*$-normed algebra A, and if I has an approximate identity of norm one for itself, then every non-degenerate $*$-representation of I extends uniquely to a non-degenerate representation of A.

C. The non-commutative Stone-Čech compactification. (At a few points in the course I may use the results of this problem.) Motivation: If the locally compact space X is an open subset of the compact space Y, then $C_\infty(X)$ "is" an ideal in the C^*-algebra $C_b(Y)$ of bounded continuous functions on Y. Then X is dense in Y exactly if $C_\infty(X)$ is an essential ideal in $C(Y)$, where by definition, an ideal I in an algebra B is *essential* if there is no non-zero ideal J in B with $IJ = 0$ or $JI = 0$. Thus the Stone-Čech compactification of an algebra A without 1 should be a "maximal" algebra with 1 in which A sits as an essential ideal. If B is any algebra in which some (probably non-unital) algebra A sits as an ideal, then each $b \in B$ defines a pair (L_b, R_b) of operators on A defined by $L_b a = ba$, $R_b a = ab$. These operators satisfy, for $a, c \in A$, $L_b(ac) = (L_b(a))c$, $R_b(ca) = c(R_b(a))$, and $a(L_b(c)) = (R_b(a))c$.

Definition: By a *double centralizer* (or *multiplier*) on an algebra A we mean a pair (S, T) of operators on A satisfying the above three conditions. Let $M(A)$ denote the set of double centralizers of A.

1. Using the example of A as ideal in B as motivation, define operations on $M(A)$ making it into an algebra, with a homomorphism of A onto an ideal of $M(A)$.

2. Sow that if A is a Banach algebra with approximate identity of norm one, and if we require S and T to be continuous (which actually is automatic), then $M(A)$ can be made into a Banach algebra in which A sits isometrically as an essential idea. (This is quite useful for various Banach algebras which are not C^*-algebras. For example, if $A = L^1(G)$ for a locally compact group G, then it can be shown that $M(A)$ is the convolution measure algebra $M(G)$ of G.) Show that if A is a $*$-Banach algebra, then its involution extends uniquely to make $M(A)$ a $*$-algebra. Note then that the theorem of problem B above says that every nondegenerate $*$-representation of A extends to $M(A)$.

3. Show that if A is a C^*-algebra, then so is $M(A)$.

4. Let A be a C^*-algebra, and let $X = A_A$ as a right A-module, with A-valued inner product as defined in class. Let $B_A(X)$ be the algebra of all continuous (which actually is automatic) A-module endomorphisms of X that have a continuous enomorphism as adjoint for the A-valued inner product (which is not automatic). Show that in a very natural way $M(A) = B_A(X)$.

5. For A a C^*-algebra, show that if B is any C^*-algebra in which A sits as an essential ideal, then B can be identified as a subalgebra of $M(A)$, so $M(A)$ is maximal in this sense, and thus ban be considered to be the Stone-Čech compactification of A.

6. Determine $M(A)$ when $A = C_\infty(X)$, and when $A = \mathcal{B}_0(\mathcal{H})$, the algebra of compact operators on a Hilbert space $\mathcal{H}$.

D. Morphisms. If X and Y are locally compact spaces and ϕ is a continuous map from X to Y, then ϕ determines a homomorphism from $C_\infty(Y)$ to $C_b(X)$, the algebra of bounded continuous functions.

1. Give a characterization of those homomorphisms from $C_\infty(Y)$ to $C_b(X)$ which arise in

this way from maps from X to Y. Your characterization should be phrased so that it makes sense for non-commutative C^*-algebras. (Hint: recall the definition of a representation being non-degenerate.) Such homomorphisms are then called "morphisms". That is, define what is meant by a morphism from a (non-commutative) C^*-algebra A "to" a C^*-algebra B.

2. For the non-commutative case explain how to compose morphisms.