Problem Set 1
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Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$, *for* $s, t \geq 0$.

1. *Determine for which* s, t *we have* $B \geq A$.

   By "$B \geq A$" we mean "$B - A \geq 0$". Since $B - A$ is symmetric with real entries, it is self-adjoint, so we can simply measure its eigenvalues (positivity is independent of the ambient C^* algebra).

$$\begin{aligned} B - A &= \begin{pmatrix} s & -1 \\ -1 & t \end{pmatrix} \\ \text{with eigenvalues } \lambda_\pm &= \frac{s+t}{2} \pm \sqrt{\left(\frac{s-t}{2}\right)^2 + 1} \end{aligned}$$

   Then $\lambda_- \geq 0$ if and only if $st \geq 1$.

2. *Determine for which* s, t *we have* $B \geq A^+$.

   We compute the spectrum of A in order to find A^+:

$$\begin{aligned} A &: \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \mapsto & +1 \\ \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \mapsto & -1 \end{cases} \\ A^+ &: \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \mapsto & +1 \\ \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \mapsto & 0 \end{cases} \\ A^+ &= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \end{aligned}$$

   Then $B - A^+ = \frac{1}{2}\left(\tilde{B} - A\right)$ where $\tilde{B}$ is B with $s \mapsto 2s - 1$ and $t \mapsto 2t - 1$. So $B \geq A^+$ if and only if $(2s-1)(2t-1) \geq 1$, i.e. iff $2st \geq s + t$.

3. *Find values of* s, t *for which* $B \geq A$, $B \geq 0$, *and yet* $B \not\geq A^+$. *(So be careful about false proofs.)*

   $B \geq 0$ provided that both $s, t \geq 0$. We are looking for $s, t \geq 0$ so that $st \geq 1$ and $2st \not\geq s + t$. For example, $s = 3$ and $t = 1/3$ works, as does any $t = 1/s$ for $s \neq 1$ (if $st = 1$, then $s + t \geq 2$ by the AM-GM inequality, and $s + t = 2$ only if $s = t = 1$).

4. *Find values of* s, t *such that* $B \geq A^+ \geq 0$ *and yet* $B^2 \not\geq (A^+)^2$. *(So again be careful.)*

We have $B^2 = \tilde{B}$, where $\tilde{B}$ is B with $s \mapsto s^2$ and $t \mapsto t^2$. A^+ is a projection operator: $(A^+)^2 = A^+$. So we are looking for s, t so that $2st \geq s + t$, but $2s^2t^2 \not\geq s^2 + t^2$. I.e. we are looking to satisfy

$$\text{RMS} = \sqrt{\frac{s^2 + t^2}{2}} > st \geq \frac{s+t}{2} = \text{AM}$$

Then the root-mean-squared is strictly greater than the arithmetic mean iff $s \neq t$, so we pick $s \neq t$ arbitrarily and then rescale s and t to squeeze $st = (s+t)/2$. For example, $s = 2/3$ and $t = 2$ works, since then the inequalities become $2\sqrt{5}/3 > 4/3 \geq 4/3$.

5. *Can you find values of s, t such that $B \geq A^+$ and yet $B^{1/2} \not\geq (A^+)^{1/2}$?*

   Conversely, by the RMS-AM inequality (i.e. by C-S), we cannot reverse the inequalities above.

6. *For 2×2 matrices T and P such that $T \geq 0$ and P is an orthogonal projection, is it always true that $PTP \leq T$?*

   Guessing $P = A^+$ and $T = B$, then the eigenvalues of $B - A^+BA^+$ are

   $$\lambda_{\pm} = (s+t) \pm \sqrt{(s+t)^2 + 4(s-t)^2}$$

   and then $\lambda_- < 0$ if $s \neq t$. So if $s \neq t$, then $A^+BA^+ \not\leq B$.