Problem Set 2
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1. (a) *Show that the following C^*-algebras are isomorphic:*

   i. *The universal unital C^*-algebra generated by two (self-adjoint) projections*

   ii. *The universal C^*-algebra generated by two self-adjoint unitary elements*

   iii. *The group algebra $C^*(G)$ for $G = \mathbb{Z}/2 * \mathbb{Z}/2$, the free product of two copies of the 2-element group.*

   iv. *The crossed-product algebra $C(T) \times_\alpha \mathbb{Z}/2$ where T is the unit circle in the complex plane, and α is the action of taking complex conjugation. (So T/α exhibits the unit interval as an "orbifold", i.e. the orbit-space for the action of a finite group on a manifold, and $A \times_\alpha G$ remembers where the orbifold comes from.) Hint: In $\mathbb{Z}/2 * \mathbb{Z}/2$ find a copy of $\mathbb{Z}$.*

Let A_j be the algebra in j. above, as j ranges in $\{i, ii, iii, iv\}$. Then, for example, A_{ii} and A_{iii} match identically: a self-adjoint unitary element squares to the identity in the subgroup of unitary elements, and the usual construction of $C^*(G)$ embeds G as unitary elements; thus, the presentations when fully written out of A_{ii} and A_{iii} match.

On the other hand, in $A_{ii} = A_{iii}$, let x and y be the generators of $G = \mathbb{Z}/2 * \mathbb{Z}/2 = (x, y : x^2 = y^2 = 1)$. Then $(1+x)/2$ and $(1+y)/2$ are projections, whereas if p and q are the projections generating A_i, then $2p-1$ and $2q-1$ are self-adjoint unitary elements. So we have a unital bijection (both algebras are free subject to their relations, which are preserved) between A_i and $A_{ii} = A_{iii}$.

Analyzing $G = \mathbb{Z}/2 * \mathbb{Z}/2$ more, we see that G is the dihedral group $D_\infty = (z, x : x^2 = (zx)^2 = 1)$ by $z = yx$. Then z generates a copy of $\mathbb{Z}$, and $\mathbb{Z}_2 = (x)$ acts by inverting z: $G = \mathbb{Z} \rtimes \mathbb{Z}_2$. But $C(T) = C^*(\mathbb{Z})$, and since the group algebra uses the group inverse as the adjoint, we see that the action $z \mapsto z^{-1}$ lifts to $C^*(\mathbb{Z})$ as the adjoint $Z \mapsto Z^*$. Thus $C^*(G) \cong C^*(\mathbb{Z}) \times_\alpha \mathbb{Z}_2$, where α is this action. From here on, I will write A for the algebra $A_i = \cdots = A_{iv}$ above.

Thus we can describe A from a very hands-on perspective. Let T be the unit circle in the complex plane; then $C^*(\mathbb{Z}) = C(T)$ where we consider z as the embedding $T \hookrightarrow \mathbb{C}$. Then any polynomial $p(z, z^{-1})$ can be interpreted as a function $t \mapsto p(t, t^{-1})$. By thinking of $G = D_\infty$ as a semi-direct product $\mathbb{Z} \cup \mathbb{Z}x$, we can write A as a direct sum: $A = C(T) \oplus C(T)x$, with multiplication given by $x\, f(t) = f(t^{-1})\, x$, i.e.:

$$(a_1(t) + a_2(t)\, x)\,(b_1(t) + b_2(t)\, x) = \big(a_1(t)\, b_1(t) + a_2(t)\, b_2(t^{-1})\big) + \big(a_1(t)\, b_2(t^{-1}) + a_2(t)\, b_1(t^{-1})\big)\, x$$

This corresponds to matrix mutliplication, where think of

$$a_1(t) + a_2(t)\, x \rightsquigarrow \begin{pmatrix} a_1(t) & a_2(t) \\ a_2(t^{-1}) & a_1(t^{-1}) \end{pmatrix}$$

The data here is four functions on the upper-half-circle with boundary conditions: $a_i(\pm 1) = a_i(\pm 1^{-1})$. Hence,

$$A = \left\{ f : [0,1] \to M_2 \text{ s.t. } f(\pm 1) = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right\}$$

We think of the interval as, e.g., the upper half circle, or better as the orbifold T/α.

(b) *Determine the primitive ideal space of the above algebra, with its topology.*

The irreps of A, a group algebra, are exactly the unitary irreps of the group $G = \mathbb{Z} \rtimes \mathbb{Z}/2 = (x, z : x^2 = 1, xzx = z^{-1})$. There are exactly four one-dimensional (i.e. commutative) representations: $x \mapsto \pm 1$, $z \mapsto \pm 1$. In general, a representation $(\phi, \mathcal{H})$ of G induces a representation of $\mathbb{Z} \subseteq G$; then $\mathcal{H}$ splits as a direct sum $\mathcal{H} = \bigoplus \mathbb{C}_k$, where $\mathbb{C}_k$ is the representation of $\mathbb{Z}$ with eigenvalue z_k. Let e_k be the eigenvector for the corresponding $\mathbb{C}$, and write $X = \phi(x)$ and $Z = \phi(z)$. Then $ZX = XZ^{-1}$, and so $ZXe_k = XZ^{-1}e_k = z_k^{-1}Xe_k$. So Xe_k is an eigenvector of Z. But $X^2 = 1$, so e_k and Xe_k span an invariant subspace of $\mathcal{H}$. Hence all other irreps are 2-dimensional:

$$Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad\qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

If $\zeta = \pm 1$, then this is not irreducible. The representation is determined by the *unordered* pair $\{\zeta, \zeta^{-1}\}$ of eigenvalues; the unitarity or the representation assures that $\zeta = e^{i\theta}$ for real θ. Hence the two-dimensional irreps of G correspond to points in the orbifold T/α, where α is the action of $\mathbb{Z}/2$ on $T =$ unit circle in $\mathbb{C}$ by complex conjugation.

So the full primitive ideal space is T/α along with four more points, two at $\zeta = 1 \in T/\alpha$ and two at $\zeta = -1$.

(c) *Use the center of the algebra above to express the algebra as a continuous field of C^*-algebras.*

The center of A is (the closure of) the span of the elements $1, z + z^{-1}, z^2 + z^{-2}, \ldots$. An element is central if and only if it is fixed by conjugation by x and by z. Conjugation by x switches z with z^{-1}; conjugation by z sends $x \mapsto z^2x$. Thus the coefficients of $z^{2n}x$ must be all the same for any central element, and $\sum_{n=-\infty}^{\infty} z^{2n}x$ does not converge. Thus, the center C of A is generated by $w = z + z^{-1}$, which as a function on T returns twice the real part; hence $C = C(I)$, where we think of $I =$ spectrum of w.

A possibly better description is in terms of A as a matrix algebra $A = \{f : T/\alpha \to M_2 \text{ s.t. boundary conditions}\}$. Since the center of M_2 is $\mathbb{C}$ thought of as a diagonal matrix, it's easy to check that the center of A is $C = C(T/\alpha)$. Then for each $s \in T/\alpha$, we have the ideal $I_s \subseteq C$ of functions that vanish at s, and hence $J_s \subseteq A$ of (matrix-valued) functions vanishing at s; and $A_s = A/J_s = M_2$ (if $s \neq \pm 1$) or $\mathbb{C}[\mathbb{Z}/2]$ (if $s = \pm 1$). Given $a(t) \in A$, the image $a_s \in A_s$ is given by $a_s = a(s)$; since the norm on M_2 is continuous in the coefficients, this field is continuous except maybe at ± 1. But indeed the inverse in M_2 of an element in $\mathbb{C}[\mathbb{Z}/2]$ is again in $\mathbb{C}[\mathbb{Z}/2]$, so the norm is continuous at ± 1 as well.

Lastly, we remark that, for $a \in A \subseteq \{f : T/\alpha \to M_2\}$, we have

$$\begin{aligned} \|m\| &= \sup\{|\alpha| : m - \alpha \text{ is not invertible}\} \\ &= \sup\{|\alpha| : m(t) - \alpha \text{ is not invertible for some } t \in T/\alpha\} \\ &= \sup_{t \in T/\alpha} \sup\{|\alpha| : m_t - \alpha \text{ is not invertible}\} \\ &= \sup_{t \in T/\alpha} \|m_t\| \end{aligned}$$

(d) *Use part (c) to prove that if P and Q are two projections in a unital C^*-algebra such that $\|P - Q\| < 1$, then they are unitarily equivalent, that is, there is a unitary element U in the algebra (in fact, in the subalgebra generated by P and Q) such that $UPU^* = Q$.*

We let $p, q \in A$ be the projections

$$p = \frac{1+x}{2} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \qquad q = \frac{1+zx}{2} = \begin{pmatrix} 1/2 & t/2 \\ t^{-1}/2 & 1/2 \end{pmatrix}$$

If $t = e^{i\theta}$ with $\theta \in [0, \pi]$, then an easy computation gives $\|p_t - q_t\| = \|p(t) - q(t)\|_{M_2} = \sin(\theta/2)$.

We let self-adjoint P, Q with $\|P - Q\| = \alpha < 1$ generate a unital C^*-subalgebra $B = C^*(P, Q)$ of their ambient algebra; we can surject $A \overset{\phi}{\twoheadrightarrow} B$ since A is free on self-adjoint projections. We write $Z = \phi(z)$ and $X = \phi(x)$; then X and Z are still unitary (the surjection is a unital $*$-map) and $X^2 = XZXZ = 1$. Thus inside B we can find a commutative unital algebra $C^*(Z)$ generated by Z: it is the algebra of continuous functions on the spectrum of Z. Since Z is unitary, its spectrum $\sigma(Z)$ is a symmetric (under complex-conjugation) subset of $T \subseteq \mathbb{C}$ (and since $C^*(Z)$ is unitary, $\sigma(Z)$ is compact), and X acts by complex conjugation as above. Then B is the semidirect product $C(\sigma(Z)) \rtimes_\alpha \mathbb{Z}/2$, where the $\mathbb{Z}/2$ is generated by X, except in the trivial case when $Z = X = \pm 1$. We go through the same steps as above, write $S = \sigma(Z)/\alpha \subseteq T/\alpha$, discover that

$$B = \{f : S \to M_2 \text{ s.t. } f(\pm 1) \in \mathbb{C}[\mathbb{Z}/2]\}$$

where the condition is vacuous if $\pm 1 \notin \sigma(Z)$. The surjection $A \twoheadrightarrow B$ is the pull-back of the injection $S \hookrightarrow T/\alpha$.

In particular, $\|P - Q\| = \sup_{t \in S} \|P(t) - Q(t)\| = \sup_{t \in S} \sin(\theta/2)$, where $t = e^{i\theta}$. If this number is less than 1, then in particular $-1 \notin \sigma(Z)$, and we can take the unitary element $U \in B$ given by

$$u(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

Then, since P and Q have the same formulas as p and q above, it is easy to check that $UPU^* = Q$.

(On the other hand, if $\|P - Q\| = 1$ — it cannot be longer, since C^*-maps like $A \twoheadrightarrow B$ are norm-non-increasing — then $-1 \in \sigma(Z)$, and there is no unitary transformation in $B_{-1} = \mathbb{C}[\mathbb{Z}/2] = \mathbb{C} \oplus \mathbb{C}X$ between X and $-X$, since this algebra is abelian. Hence P and Q are not unitarily equivalent in B.)

(e) *Use part (d) to show that in a unital separable C^*-algebra the set of unitary equivalence classes of projections is countable.*

In a unital separable algebra, within the space of projections I can find a countable sense subset. But then each projection is within distance $\epsilon < 1$ of some projection in my choice of dense subset, and hence by (d) unitarily equivalent to that projection. Unitary equivalence is an equivalence relation.

2. *For any $n \times n$ real matrix T define an action α of $\mathbb{R}$ on the group $\mathbb{R}^n$ by $\alpha_t = \exp(tT)$ acting in the evident way. Let $G = \mathbb{R}^n \times_\alpha \mathbb{R}$. Then G is a solvable Lie group. For the case of $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ determine the equivalence classes of irreducible unitary representations of G, i.e. the irreducible representations of $C^*(G)$. Determine the topology on $\mathrm{Prim}(C^*(G))$. Discuss whether $C^*(G)$ is CCR or GCR, and why.*

We have

$$\begin{tikzcd} 1 \arrow[r] & (\mathbb{R}^2, +) \arrow[r] & G \arrow[r] & (\mathbb{R}, +) \arrow[r] \arrow[l, bend right, "\alpha"'] & 1 \end{tikzcd}$$

The (irreducible) representations of $\mathbb{R}^2$, which we think of as a space of column vectors, are one-dimensional — $\mathbb{R}^2$ is commutative — and indexed by the $\mathbb{R}^2 = \widehat{\mathbb{R}^2}$ of row vectors:

$$\vec{p} : \vec{v} \mapsto e^{i\vec{p}\cdot\vec{v}} \times$$

A representation of G induces a representation of $\mathbb{R}^2$, which splits. The trivial representations of $\mathbb{R}^2$ sit below one-dimensional representations of $\mathbb{R}$ pulled back to representations of G. Otherwise, $\mathbb{R}$ acts freely on $\mathbb{R}^2$: by tracing how generators of G act on a basis of the representation, we see that $\mathbb{R}$ acts on $\widehat{\mathbb{R}^2}$ by multiplication by $\alpha_{-t} = \exp(-tT)$ from the left. Hence each irrep looks like functions on the line, and these ∞-dimensional irreps form an orbifold $\mathbb{R}^2/\mathbb{R}$, where the $\mathbb{R}$-action is along hyperbolas.

Thus the primitive ideal space consists of four rays emanating from the origin, four points at the origin (one for each axis), and $\mathbb{R}$-many one-dimensional representations at the origin.

In any case $C^*(G)$ is not GCR, so certainly not CCR. We take a generic ∞-dimensional representation ϕ of G on functions on a line $p \in \mathbb{R}_{>0}$; for each real $c \neq 0$ (there are four like this when $c = 0$, whence we act by either v_1 or v_2), we have two such representation:

$$\phi(\vec{v})f(p) = e^{i(v_1 p + v_2 c/p)} f(p) \text{ or } e^{-i(v_1 p + v_2 c/p)} f(p), \text{ and } \phi(t)f(p) = f(e^{-t}p)$$

But the action of $\vec{v}$ is not by a compact operator.

In question 1.(c), I use results from the exercise in the previous problem set that I did not work. I do it now:

Fields of C^*-algebras. *Let A be a C^*-algebra with 1, and let C be a C^*-subalgebra of the center of A with $1 \in C$. Let $C = C(X)$, and for $x \in X$ let J_x be the ideal of functions vanishing at x. Let $I_x = AJ_x$ (closure of linear span), an ideal in A. Let $A_x = A/I_x$ ("localization"), so $\{A_x\}_{x\in X}$ is a field of C^*-algebras over X. For $a \in A$ let a_x be its image in A_x.*

1. *Prove that for any $a \in A$ the function $x \mapsto \|a_x\|_{A_x}$ is upper-semi-continuous. (So $\{A_x\}$ is said to be an upper-semi-continuous field.)*

   We recall that the quotient norm is given by

   $$\|a_x\| \overset{\text{def}}{=} \inf_{j\in J_x} \|a - j\|$$

   We pick a metric on X and let $f_x : y \to \operatorname{dist}(x, y)$; more generally, we just need some continuous choice of functions f_x that vanish only at x. Then I_x is generated as an ideal of C by f_x, and similarly J_x is all A-multiples of f_x. Hence we have:

   $$x \mapsto \|a_x\| = \inf_{b\in A} \|a - bf_x\|$$

   But $\|a - bf_x\|$ is obviously continuous in each of the variables (X is compact, and so this continuity is uniform in x), and hence the infimum is upper-semi-continuous.

2. *If $x \mapsto \|a_x\|_{A_x}$ is continuous for all $a \in A$, then the field is said to be continuous. For this part assume that A is commutative. Note that then one gets a continuous surjection from $\hat{A}$ onto $\hat{C}$. Find examples of As and Cs for which $x \mapsto \|a_x\|$ is not continuous. In fact, find an attractive characterization of exactly when the field is continuous, in terms of the surjection of $\hat{A}$ onto $\hat{C}$ and concepts you have probably met in the past.*

   When A is commutative, it is the space of functions on its maximal (i.e. primitive) ideal space, which consists exactly of the non-zero elements of the dual space $\operatorname{Hom}(A, \mathbb{C})$. We let C be a sub-C^*-algebra of commutative A. Thus, write $C = C(X)$ and $A = C(Y)$ with $Y = \hat{A}$ and $X = \hat{C}$. In any case, we assume that C, A are unital, i.e. X and Y are compact.

   At the purely algebraic level, we have a surjection $\pi : Y \twoheadrightarrow X$: if $y \in Y$ is a maximal ideal of A, then $y \cap C$ is a maximal ideal of C, and conversely if $x \in X$ is a maximal ideal of C, then either x contains elements that in A (but not in C) are invertible, or x is contained in a maximal ideal in A. The former cannot happen through some C^* magic involving spectrums being the same when we make algebras bigger: the unital algebra generated by the ostensibly invertible element in x knows everything there is to know about the element, and in particular knows whether it is invertible. In any case, the surjection $Y \twoheadrightarrow X$ pulls back to the injection $C \subseteq A$, and the surjection is continuous for the Zariski topology.

Moving on, in this continuous case, we have a surjection $\pi : Y \twoheadrightarrow X$ of compact (Hausdorff, etc.), and the norm is easy to express:

$$\|a_x\| = \sup_{y \in \pi^{-1}(x)} |a(y)|$$

I claim that $x \mapsto \|a_x\|$ is continuous for every $a \in A$ if and only if π is open, i.e. if for $U \subseteq Y$ we have $\pi(U) \subseteq X$ open. Indeed, let π be open, and pick any $\epsilon > 0$. There is some $y \in \pi^{-1}(x)$ with $|a(y)| > \|a_x\| - \epsilon/2$, and $|a(y)|$ is continuous, so there is some $U \ni y$ so that $|a(y')| > |a(y)| - \epsilon/2$ for every $y' \in U$. Now π is open, and $\pi(U) \ni x$; then for any $x' \in \pi(U)$, there is some $y' \in \pi^{-1}(U)$ so that $|a(y')| > \|a_x\| - \epsilon$: namely, take the (any) $y' \in U$. This shows lower-semi-continuity, and upper-semi-continuity is true for any surjection.

Conversely, let $U \subseteq Y$ be open, and pick a function $a \in A$ that vanishes outside of U and is strictly positive on U. The inverse image under the map $x \mapsto \|a_x\|$ of the open interval $(0, \infty)$ will be exactly those $x \in X$ so that there's a $y \in Y$ with $\pi(y) = x$ and $a(y) > 0$, i.e. those x in $\pi(U)$. Hence if $x \mapsto \|a_x\|$ is commutative, then $\pi(U)$ is open (the inverse image of an open set under a continuous map). Thus π must be an open map.

3. *Let*

$$A_1 \stackrel{\text{def}}{=} \left\{ f : [0,1] \to M_2 \textit{ continuous, with } f(1) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right\}$$
$$A_2 \stackrel{\text{def}}{=} \left\{ f : [0,1] \to M_2 \textit{ continuous, with } f(1) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right\}$$

and let $C_i \stackrel{\text{def}}{=} Z(A_i)$ be the center of A_i. Are the corresponding fields continuous? Are all the fiber algebras A_x isomorphic? Show that A_1 and A_2 are very simple prototypes of behavior that occurs often "in nature", but with higher-dimensional algebras, and more complicated boundary behavior.

In each case, the center if $C_i = C = C([0,1])$, thought of as

$$t \mapsto \begin{pmatrix} f(t) & 0 \\ 0 & f(t) \end{pmatrix}$$

Then for each $x \in [0,1]$ we have $I_x =$ functions that vanish at x, and $J_x =$ matrix-valued functions that vanish at x. Then

$$\|a_x\| = \|a(x)\|$$

for any $a \in A_i$, where on the LHS it is the norm in the quotient algebra, and on the RHS it is the operator norm for two-by-two matrices

$$\|M\| = \sqrt{\text{largest eigenvalue of } M^* M}$$

But for finite-dimensional matrices, the eigenvalues vary continuously in the coefficients of the matrix. So these fields are continuous.

For $x \neq 0$, the fibers $(A_i)_x$ are all isomorphic: they are each the algebra of square matrices. But $(A_1)_0 = \mathbb{C}$, whereas $(A_2)_0 = \mathbb{C}^2$. I don't know what occurs in nature, although I can artificially construct similar examples and pass them off as natural?

4. *Determine the primitive ideal space of each of these two algebras, with its topology.*

Given an irrep $\phi : A_i \to \mathrm{End}(\mathcal{H})$, we compose with the injection $C \subseteq A_i$ to get a representation of C, which must split as a direct sum of irreps $\mathcal{H} = \bigoplus \mathbb{C}_k$, where $\mathbb{C}_k$ is a one-dimensional irrep, i.e. $f \in C = C([0,1])$ acts by multiplication by $f(t)$ for some $t(k)$ depending on the representation. Let e_k be the basis element in $\mathbb{C}_k$; then multiplying e_k by any $f \in C$ with $f(t) = 1$ will not change e_k. So multiplying e_k by any $f \in A_i$ with $f(t) = 1 \in M_2$ will not change e_k; hence all that matters is $f(t)$. Thus the irreps of A_i are indexed by t, and correspond exactly to those irreps of $(A_i)_t = (A_i)/J_t$. When $t \neq 0$, $(A_i)_t = M_2$, which is simple; it's only irrep is its action on $\mathbb{C}^2$. When $t = 0$, $(A_1)_0 = \mathbb{C}$, which has just the one irrep. On the other hand, $(A_2)_0 = \mathbb{C}^2$, which has two irreps: each component. Hence $\mathrm{Prim}(A_1)$ is the interval, whereas $\mathrm{Prim}(A_2)$ is the interval with a double point at the origin.

We can check the topology: given a set of points in the interval, each is the ideal of (M_2-valued) functions that vanish at that point; the intersection of this ideal is the ideal of all functions that vanish at all the points. But functions in this ideal vanish exactly at the closure of the set, and so the ideal contains those primitive ideals corresponding to the points in the closure. Hence the primitive ideal space inherits exactly the topology it should: the description at the end of the previous paragraph is as topological spaces, not just as sets.