1 February 22, 2008

(We begin by handing back the homework turned in last time.)

We begin with a few comments to wind things up.

Let A be a $*$-normed algebra with approximate identity of norm 1. Then for any continuous $*$-representation $(\pi, \mathcal{H})$, of A, then we know that $\|\pi(a)\| \leq \|a\|$. So set

$$\|a\|_{C^*} = \sum \{\|\pi(a)\| : (\pi, \mathcal{H}) \text{ is a cont's } *\text{-rep of } A\}. \tag{1}$$

So $\|a\|_{C^*} \leq \|a\|$, and you can check that $\|a^*a\|_{C^*} = (\|a\|_{C^*})^2$, so the completion of A with respect to $\|\cdot\|_{C^*}$ is a C^*-algebra, called the "universal C^*-enveloping algebra of A". It has the property that there is a natural bijection of

$$\{\text{continuous } *\text{-reps of } A\} \leftrightarrow \{*\text{-reps of } C^*(A)\}$$

Comment: If $(\pi, \mathcal{H})$ is a representation, look at

$$\underbrace{\overline{\pi A \mathcal{H}}}_{\substack{\text{rep is} \\ \text{non-degen}}} \oplus \underbrace{\left(\overline{\pi A \mathcal{H}}\right)^{\perp}}_{0-\text{rep}}.$$

Question from the audience: Is there some diagram that goes with this? **Answer:** Yes:

$$\begin{tikzcd}
C^*(A) \arrow[dr, dashed] & \\
A \arrow[u] \arrow[r, "\pi"] & \mathcal{B}(\mathcal{H})
\end{tikzcd}$$

Now, we form

$$\bigoplus_{\mu \in S(A)} (\pi_\mu, \mathcal{H}_\mu)$$

which has the same norm, since we could add the word "cyclic" to eqn (1).

Instead, let's look at the "atomic" representation:

$$(\pi, \mathcal{H}) \overset{\text{def}}{=} \bigoplus_{\mu \text{ a pure state of } A} (\pi_\mu, \mathcal{H}_m u)$$

$$\|a\|_{\text{atomic}} \overset{\text{def}}{=} \|\pi(a)\| = \sup\{\|\pi(a)\| : (\pi, \mathcal{H}) \text{ is an irreducible rep of } A\}$$

This is a good construction, because there is more than a *set* of irreducible representations, but only a set of pure states.

Prop: $\|a\|_{\text{atomic}} = \|a\|_{C^*}$. (These are both obviously C^*-seminorms.)

Proof:

> $\leq$ is clear. For $\geq$, by C^*-identity, it suffices to show for a^*a, i.e. we can assume that $a \geq 0$.
>
> If $\mu(a) \leq c$ for all pure states μ (so certainly for any convex combination of pure states, so also for the closure of the convex combinations), then by the Krein-Milman ****sp?**** theorem, we have $\mu(a) \leq c$ for any μ. Take a^*a and look at the commutative C^*-algebra it generates, and we can find a state that returns the norm, and use the Hahn-Banach theorem to extend this state to the whole algebra. So there is a state μ with $\|a^*a\|_{C^*} = \mu(a^*a)$, so $\|a^*a\|_{C^*} \leq c$. This gives us the reverse inequality. $\square$

Getting a little ahead of ourselves: if G is locally compact, with a left Haar measure on G, we can form $L^1(G)$ under convolution. This has a faithful (i.e. injective) representation as operators on $L^2(G)$ (again by convolution). Then every vector in L^2 is a state in L^1 — it has enough states to separate the points. So L^1 has lots of pure states, and hence G has lots of irreducible unitary representations. In 1943, Gelfand and Raikov showed this. We will go into this in more detail later; this is just foreshadow.

If you respond "Ok, it has lots, show me some", then that's hard. The Krein-Milman theorem is non-constructive — it needs Axiom of Choice — so this is all very encouraging, but it doesn't give you any real technique for finding these representations. How to find things depends on how the example is presented. There's an enormous literaturs on, say, $GL(3, \mathbb{R})$, which is understood, but $GL(3, \mathbb{Z})$ is not.

1.1 Compact operators

For a Hilbert space $\mathcal{H}$, let $\mathcal{B}_0(\mathcal{H})$ be the C^*-algebra of compact operators on $\mathcal{H}$. We should ask what definition we mean for "compact operators". In Banach land, there is a definition, and you have to look fairly far before you come across an example where the compact operators are not just the closure of finite-rank operators. Hilbert spaces are more constrained; even finite-dimensional normed spaces can be bewildering. For Hilbert spaces, compact is nice: $\mathcal{B}_0(\mathcal{H}) \stackrel{\text{def}}{=}$ the closure of finite-rank operators.

Under the operator norm, this is a C^*-algebra. Moreover, $\mathcal{B}_0(\mathcal{H})$ is topologically simple: there are no proper closed 2-sides ideals. It has lots of important non-closed 2-sided ideals (trace-class, Hilbert-Schmidt, etc.; these are in fact ideals of $\mathcal{B}(\mathcal{H})$). But any ideal that isn't 0, then we can compress it between two rank-1 projections, so there are rank-1 operators in the ideal, so all are, so all finite-rank operators are in the ideal.

Theorem: (Shows up lots of places, e.g. uniqueness of Heisenberg commutation relations.)

> Up to unitary equivalence, $\mathcal{B}_0(\mathcal{H})$ has exactly one irreducible representation, namely the one on $\mathcal{H}$. Furthermore, every non-degenerate representation is a direct sum of copies of $\mathcal{H}$.

Proof:

Let $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ be linear in the second variable ****this is how I like it****, and put the scalars on the right. For $\xi, \eta \in \mathcal{H}$, define $\langle \xi, \eta \rangle_0 \in \mathcal{B}_0(\mathcal{H})$. This will be a familiar object: $\langle \xi, \eta \rangle_0 \stackrel{\text{def}}{=} \{\zeta \mapsto \xi \langle \eta, \zeta \rangle_{\mathcal{H}}\}$. I.e. we have a bimodule ${}_{\mathcal{B}_0(\mathcal{H})}\mathcal{H}_{\mathbb{C}}$.

For $T \in \mathcal{B}_0(\mathcal{H})$ (or in fact in $\mathcal{B}(\mathcal{H})$), we have

$$T\langle \xi, \eta \rangle_0 = \langle T\xi, \eta \rangle_0$$

and you can check that $(\langle \xi, \eta \rangle_0)^* = \langle \eta, \xi \rangle_0$. The consequence is that $\langle \xi, \eta \rangle_0 T = \langle \xi, T^* \eta \rangle_0$. (Of course, $\langle \cdot, \cdot \rangle_0$ is $\mathbb{C}$-linear in the first variable.) $\square$

(Time is up; we will continue this discussion next time.)