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1: May 5, 2008

1.1 Vector bundles and projective modules

Theorem: (Swan, 1962)

> Let E be a ($\mathbb{R}$ or $\mathbb{C}$) vector bundle over X compact. Then $\Gamma(E)$ is a projective A module for $A = C(X)$. Conversely, suppose Ξ is a projective A module. Then by definition there exists Ξ_1 such that $\Xi \oplus \Xi_1 \cong (A^n)_A$. Let Q be the projection of A^n onto Ξ along Ξ_1. I.e., $Q \in \mathrm{End}_A(A^n) = M_n(A)$ and $Q^2 = Q$ (in this generality we don't have self-adjointness; in this case Q is called "idempotent"); view Q as a matrix of functions: $M_n(C(X \to k)) = C(X \to M_n(k))$ for $k =$ ground field $\mathbb{R}$ or $\mathbb{C}$. Define E a vector bundle by: the fiber E_x above x is the range of $Q(x)$ in k^n. This function is clearly continuous in x, so essentially it is a bundle. We check local triviality: if $b_1, \dots, b_k$ is a basis for range$(Q(x_0))$, then view $b_i \in k^n$, and then write down $Q(x)b_j$ and $(1 - Q(x))b_j$. We take the determinant of these n vectors; they're a basis at x_0, so this determinant is non-zero, and determinant is continuous in the coefficients, so it's a basis in a small neighborhood. □

Question from the audience: If X is not connected, there might be dimension change from component to component? **Answer:** Absolutely. Our definition of "vector bundle" allows for this

When X is not compact, the story is more complicated. But usually when we have non-compact spaces, we control the behavior at infinity by having in mind a particular compactification, and that throws us back into this story. For instance, we might use the one-point compactification; this makes our bundle trivial at infinity, i.e. there's a large enough compact set in X so that on the complement, the bundle is trivial.

For any ring R with 1, we can consider the finitely generated projective modules (if you're very careful, that's not a set, but you know how to deal with this), and we consider them up to isomorphism class: $\mathcal{S}(R)$ is the set of isomorphism classes. Given projective modules Ξ_1, Ξ_2, it's obvious that $\Xi_1 \oplus \Xi_2$. (Everywhere finitely generated, but I don't want to go into that. **Question from the audience:** Meaning n is finite? **Answer:** Well, a little more complicated. **Question from the audience:** Every kind of projective module we've defined is f.g. **Answer:** Yes) This sum interpreted as bundles is fiber-wise, called the "Whitney sum". This defines an addition on $\mathcal{S}(R)$, which is certainly commutative, and the 0 module is an identity element. So $\mathcal{S}(R)$ is a commutative semigroup with 0. This is an invariant of R. I.e. this is all functorial, but I'm glossing over that.

$\mathcal{S}(R)$ is interesting to calculate. As a teaser, let $\theta \in M_d(\mathbb{R})$, and build A_θ. If θ has at least one irrational entry, then we can describe $\mathcal{S}(A_\theta)$ in pretty explicit terms; indeed, up to isomorphism we can construct all the projective modules. The description depends on θ and is a little bit complicated. On the other hand, for $\theta = 0$, we have $C(T^d)$, and for $d \gtrsim 10$, $\mathcal{S}(C(T^d))$ is basically unknown: it corresponds to homotopy classes of something, but it's way too complicated. Similarly for the d-sphere above a certain dimension. In a surprising number of cases, the quantum world ends up being nicer like this than the classical world: the classical world ends up being "degenerate".

Let's indicate some of the obstructions. Last time, we gave some projective modules over the circle. Let's look at $S^2 \subseteq \mathbb{R}^3$ the unit two-sphere. Then we have the tangent bundle and cross-sections $\Gamma(TS^2) = \{\xi : S^2 \to \mathbb{R}^3 \text{ s.t. } \xi(x) \cdot x = 0 \, \forall x \in S\}$. We know the hairy ball theorem: this is not the trivial bundle, i.e. it's not A_A^2, where $A = C(S^2 \to \mathbb{R})$. We can also define the normal bundle $\Gamma(NS^2) = \{\xi : S^2 \to \mathbb{R}^3 \text{ s.t. } \xi(x) \in \mathbb{R}x\}$. This is the trivial bundle A_A. Well, $\Gamma(TS^2) \oplus \Gamma(NS^2) = A_A^3$ is a trivial (i.e. free) bundle. So $\Gamma(TS^2) \oplus A \cong A^2 \oplus A$, but $\Gamma(TS^2) \not\cong A^2$, so $\mathcal{S}(C(S^2 \to \mathbb{R}))$ is not cancelative. Even presenting semigroups in which cancelation fails is complicated. We can play the same game over $\mathbb{C}$, but have to get to $d \geq 5$ for cancelation in $\mathcal{S}(C(S^d \to \mathbb{C}))$ to fail. So the moral of the story: calculating $\mathcal{S}(R)$ can be hard.

On the other hand, in a paper some years ago by R., we show that in the noncommutative torus and a non-zero ****or non-rational, I didn't hear**** entry in θ, cancelation holds.

Given a semigroup S commutative with 0, force cancellation. I.e. consider $s \sim t$ if $\exists r$ with $s + r = t + r$. Check: then $S/\sim$ is a commutative unital cancelative semigroup. Call it cS, standing for cancelation. ****Board says "$C(S)$", but also "there are too many Cs around", so I'll use this notation.**** So we set $\mathcal{C}(R) = c\mathcal{S}(R)$. This is also an invariant of R, and can be a bit easier to calculate, but still possibly daunting.

Ok, remember how to construct the integers from the positive integers? That procedure works for any semigroup with cancelation. Recall: we look at pairs (m, n) which we think of as $m - n$, and consider $(m, n) \sim (m', n')$ if $m + n' = m' + n$. For a cancelative commutative semigroup C, we can embed it in an abelian group gC. ****"groupify"**** This procedure again loses information. We define $K_0(R) \stackrel{\text{def}}{=} g\mathcal{C}(R) = gc\mathcal{S}(R)$. This is the 0-group of K-theory, and finially gets us to a homology theory. For complicated examples, this can still be difficult to calculate. $\mathcal{C}(R)$ is a "positive cone" inside $K_0(R)$; it may be degenerate (e.g. it can be all of K_0). So denote $\mathcal{C}(R) = K^+(R)$, and we often see written the pair $(K_0(R), K^+(R))$, which of course has exactly the data of $\mathcal{C}(R)$.

Everything is functorial: Given rings R_1 and R_2 and a unital map $\phi : R_1 \to R_2$, we have $\mathcal{S}(\phi) : \Xi_{R_1} \mapsto \Xi_{R_1} \underset{R_1 R_1}{\otimes} (R_2)_{R_2}$, where we view R_2 as a left-R_1-module using ϕ. This extends to K_0. Given a short exact sequence

$$0 \to J \to R \to R/J \to 0$$

we want a long exact sequnce in K_*. We need to define $K_0(J)$. First we form $\tilde{J}$ by adjoining a unit. Then we have a homomorphism $\tilde{J} \to \mathbb{Z}$ (it's really better if everything is with algebras over a field k; certainly this works in that case, but probably works if $k = \mathbb{Z}$). Then we have $K_0(\tilde{J}) \to K_0(\mathbb{Z})$, and we define $K_0(J) = \ker(K_0(\tilde{J}) \to K_0(\mathbb{Z}))$. E.g. if $J = C_\infty(X)$, then $\tilde{J} = C(\tilde{X})$, where $\tilde{X}$ is the one-point compactification. Anyway, then we get

$$K_0(J) \to K_0(R) \to K_0(R/J)$$

but to extend that takes more work. This is an interesting direction, but not one we will pursue.