****This document was last updated on May 1, 2008. A more recent version may be available as part of http://math.berkeley.edu/~theojf/CstarAlgebras.pdf.****

1: March 3, 2008

****Again I was late to class.****

Theorem: (Glimm)

> If A has an irreducible representation π such that $\pi(A) \cap \mathcal{B}_0(\mathcal{H}) = 0$, then the irreducible representations are not classifiable.

To "classify" the irreducible representations means to find a countable number of real-valued Borel functions on $\mathbb{R}$ that are constant on the equivalence classes.

Because of the analogy with von Neuman algebras, GCR is also called "type I".

1:.1 Some topology and primitive ideals

For non-GCR algebras, don't look for all irreducible representations. Look instead at the kernels of irreducible representations.

Definition: For a C^* algebra A, a two-sided ideal is *primitive* if it is the kernel of an irreducible representation.

We write $\mathrm{Prim}(A)$ for the set of primitive ideals, although this is an unfortunate notation, as it looks like "prime". ****I will use "**$\mathrm{Spec}(R)$**" for the primes of a ring** R**.**** (Recall, J a two-sided ideal of R is *prime* if whenever $K_1K_2 \subseteq J$ and K_1 and K_2 are two-sided ideals, then at least one of the K_i is in J.) In fact, if A is C^*, then any primitive ideal is prime. **Proof:** Let K_1, K_2 be subideals of primitive ideal J (with associated representation $(\pi, \mathcal{H})$) with $K_1K_2 \subseteq J$. Since J is closed, we can assume that K_1 and K_2 are. Assume $K_1 \not\subseteq J$, and then $\pi(K_1) \neq 0$, and so by irreducibility $\overline{\pi(K_1)\mathcal{H}} = \mathcal{H}$. Ditto for K_2, and everything is continuous, so $\mathcal{H} = \overline{\pi(K_2)\mathcal{H}} = \overline{\pi(K_2)\overline{\pi(K_1)\mathcal{H}}} = \overline{\pi(K_2K_1)\mathcal{H}} = 0$. $\square$

Rhetorical Question: Is every prime ideal primitive? **Answer:** No. Consider $A = C([0,1])$ and $J = \{f \in A : f = 0 \text{ in a nbhd of } 1/2\}$. This is not a closed ideal, but it is prime. There are more complicated examples as well. But primitive ideals are closed. **Rhetorical Question$'$:** Is every closed prime ideal primitive? **Answer:** If A is separable, yes, using Baire category theorem. In 2001, Nik Weaver constructed a (large) counterexample using transfinite induction.

For any ring R (with 1), on the set $\mathrm{Spec}(R)$ of prime ideals we have the "hull-kernel" or "Jacobson" topology (for commutative rings called "Zariski" topology). We can take its restriction to any subset M of prime ideals. Misusing the word ker, given $S \subseteq M$, we set $\ker_M(S) = \bigcap\{J \in S\}$. Conversely, given $I \in \mathrm{Spec}(R)$, we set $\mathrm{hull}_M(I) = \{J \in M : J \supseteq I\}$. Then we declare our topology: For $S \subseteq M$, we define the closure $\overline{S} \stackrel{\text{def}}{=} \mathrm{hull}_M \ker_M(S)$.

The Kuratowski closure axioms tell us when a definition of "closure" defines a topology.

1. $\overline{\emptyset} = \emptyset$
2. $\overline{S} \supseteq S$ ****original notes said $\subseteq$, but that is surely wrong, and I probably mistranscribed from the board****
3. $\overline{\overline{S}} = \overline{S}$
4. $\overline{S \cup T} = \overline{S} \cup \overline{T}$

Only this last property requires any thinking. If $S \subseteq T$, then $\overline{S} \subseteq \overline{T}$, clearly, and thus we have $\overline{S \cup T} \supseteq \overline{S} \cup \overline{T}$. Why $\subseteq$? This requires primality. Let $L \in \overline{S \cup T}$. Then $L \supseteq \bigcap\{J \in S \cup T\} = \bigcap\{J \in S\} \cap \bigcap\{J \in T\} = \ker(S) \cap \ker(T) \supseteq \ker(S)\ker(T)$ (multiplication as ideals). But L is prime, so $L \subseteq \ker(S)$ or $\ker(T)$, and so $L \in \overline{S} \cup \overline{T}$.

So, on $\mathrm{Prim}(A)$ (for A a C^*-algebra), put on the hull-kernel topology. $\mathrm{Prim}(A)$ is not in general Hausdorff. But for C^*-algebras, it is locally compact, in the sense that every point has a closed neighborhood such that any open cover of the nbhd has a finite subcover. Now, Baire category theorem works in separable land, and also in this case. $\mathrm{Prim}(A)$ is T_0, i.e. if $J \in \overline{\{K\}}$ and $K \in \overline{\{J\}}$, then $K = J$.

E.g. Let $A = \mathcal{B}_0(\mathcal{H})^\sim$, i.e. $\mathcal{B}_0(\mathcal{H})$ adjoin an identity element. Then we have (only) two closed ideal 0 and $\mathcal{B}_0(\mathcal{H})$, and $\overline{\{0\}} = \{0, \mathcal{B}_0(\mathcal{H})\}$, and $\{\mathcal{B}_0(\mathcal{H})\}$ is closed.