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1 April 7, 2008

1.1 Some group cohomology

We had looked at the Heisenberg commutation relations in the form that Herman Weyl gave:

U a rep of G (Abelian) on $\mathcal{H}$, V a rep of the dual group $\hat{G}$ on $\mathcal{H}$; we declare

$$V_x U_s = \langle s, x\rangle U_s V_x$$

We saw, and basically gave the proof, that when G is $\mathbb{R}^n$, and more generally when $\hat{\hat{G}} = G$, there's one irreducible representation ("Schrodinger representation") on $L^2(G)$, and every representation comes from one of these.

Looking at this in a slightly different way, define unitary W on $G\times\hat{G}$ by $W_{(x,s)} \overset{\text{def}}{=} V_x U_s$. Then

$$W_{(x,s)} W_{(y,t)} = \langle s, y\rangle W_{(x,s)+(y,t)}$$

and $\langle s, y\rangle \in T \overset{\text{def}}{=} \{e^{i\theta} \in \mathbb{C} : \theta \in \mathbb{R}\}$.

For any group G (e.g. $G\times\hat{G}$), we can consider $W : G \to \mathcal{U}(\mathcal{H})$ the unitary operators on $\mathcal{H}$ such that $W_x W_y = c(x, y)W_{xy}$ for $c(x, y) \in T$. The associativity in G implies that c is a T-valued 2-cocycle, meaning

$$c(x, yz)\, c(y, z) = c(xy, z)\, c(x, y)$$

It's natural to assume $W_e = \mathbb{1}_{\mathcal{H}}$: $c(x, e) = 1 = c(e, x)$.

There's a homology theory of groups ("group cohomology"). We're looking at $[c] \in H^2(G, T)$, which we won't really define. For a function of one variable $h : G \to T$, we define the boundary of h by

$$\partial h(x, y) \overset{\text{def}}{=} h(x)\, h(y)\, \overline{h(xy)}$$

Definition: W is a *projective* representation of G on $\mathcal{H}$ with cocycle c

Since T is abelian, $H^2(G, T)$ is a group. If G is topological, we do not demand that c be continuous. This machinery works best when G is second-countable locally-compact, and then we want c to be measurable. Such c correspond to extensions:

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] & E_c \arrow[r] & G \arrow[r] \arrow[l, bend left, "c"] & 0 \end{tikzcd}$$

E.g. $\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & T \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd}$

This is important for physics. In QM, $\mathcal{B}_0(\mathcal{H})$, with pure state the vector states. Because the only irrep is the one on the Hilbert space. But for any algebra, the pure states via GNS give an irreducible rep for which the state is a vector state. An operator gives a projection:

$$T \mapsto \frac{\langle T\xi, \xi\rangle}{\langle \xi, \xi\rangle}$$

These give one-dim subspaces of $\mathcal{H}$, and $\mathcal{PH}$, the projective space, is the space of states. Automorphisms of QM are automorphisms of $\mathcal{PH}$.

Theorem: (Wigner, 1930s)

> This are given by unitary or anti-unitary (conjugate-linear but length-preserving) operators (unique up to multiplication by an element of T.

If we have a one-parameter family of automorphisms of $\mathcal{PH}$, then for each auto, U^2 is linear. So anti-unitary operators only come up in discrete situations, usually as autos of order 2. For example:

- Charge conjugation C
- Parity P — weak force does not respect parity
- Time reversal T

Then $C^2 = P^2 = T^2 = 1$, and CPT often comes up.

In any case, we've found irreducible projective representations with non-trivial cocycle: $H^2(\mathbb{R}^2, T) \neq \{0\}$.

Theorem: For G a semi-simple connected simply-connected Lie group, then $H^2(G, T) = \{0\}$.

For example, $SO(3) \to \mathrm{Aut}(\mathcal{PH})$. The double cover $SU(2) \xrightarrow{2} SO(3)$ is simply connected and semi-simple. So any projective representation of $SO(3)$ gives an ordinary representation of $SU(2)$. Similarly, the (connected component of the) Lorentz group $\mathcal{L}$ is covered by simply-connected $SL(2, \mathbb{C})$, so has the same story. And it's much easier to work with ordinary representations than with projective representations. (The story does not work with $\mathbb{R}^{2n}$, which has an infinite irreducible projective representation, even though any ordinary irrep is one-dimensional.)

Question from the audience: How do we get a cocycle? **Answer:** We have $\alpha : G \to \mathrm{Aut}(\mathcal{PH})$. For each x, chose U_x implementing $\alpha(x)$. This is only defined up to scalar multiple. $U_x U_y = c(x, y)U_{xy}$. Associativity in Aut implies the cocycle condition ****and the unknown scalars are the boundaries****. When G is topological, you cannot make this choice continuous, but you'd like to make it at least measurable. If $\mathcal{H}$ is separable, Aut can be given topology of a complete metric space (not locally compact), and from that there are theorems that can go and chose c to be measureable.

Incidentally, the complete metric space for Aut makes it into a *Polish space*; these do not have Haar measure, but the homology was worked out nicely by Prof Moore in our department.

We should mention another aspect of this story. Given G and a cocycle $c : G \to T$, we can define the convolution

$$(f \star_c g)(x) \stackrel{\text{def}}{=} \int f(y)\, g(y^{-1}x)\, c(y, y^{-1}x)\, dy$$

This is associative iff c is a 2-cycle almost everywhere. So we get a $C^*(G, c)$, and if c' and c are homologous, then the corresponding algebras are isomorphic (indeed, the boundary tells how to build the isomorphism). Look at $c(s,t) = e^{2\pi i s t}$ on $\mathbb{R}^2$; then $C^*(\mathbb{R}^2, c) \cong \mathcal{B}_0(L^2(\mathbb{R}))$. We can, of course, stick in a constant, and promote the product to a dot-product: $c(s,t) = e^{2\pi\hbar i\langle s,t\rangle}$. This is one view on what we've been doing. Even more generally, we can build $C^*(G, A, \alpha, c)$ where c is an A-valued cocycle and α a representation. There is a very nice treatment in this language of the Quantum Hall Effect.

For the last five minutes, some special example. Let $G = \mathbb{Z}^d$ (we use m, n for elements of G, not the dimension). Let $\theta \in M_d(\mathbb{R})$ be a $d \times d$ matrix. Define

$$c_\theta(m,n) \stackrel{\text{def}}{=} e^{2\pi i (m\cdot\theta n)}$$

This is a bicharacter, i.e. for n fixed, $m \mapsto c_\theta(m,n)$ is a character (element of $\widehat{\mathbb{Z}^d}$). An easy check: a bicharacter is a 2-cocycle. We will not prove:

Theorem: Every 2-cocycle on $\mathbb{Z}^d$ with values in T is homologous to a bicharacter.

Now we will study $C^*(\mathbb{Z}^d, c_\theta)$. For $\theta = 0$ (or all integers), this is just $C^*(\mathbb{Z}^d) = C(T^d)$ continuous functions on the d-dim torus. In general, $C^*(\mathbb{Z}^d, c_\theta)$ are called *non-commutative tori* (or "quantum tori"). These are the easiest examples of non-commutative differentiable manifolds.