1 February 29, 2008

****I was 10 minutes late.****

We prove the theorem stated at the end of lecture last time. ****Proof omitted, since I was late. The proof follows from the results stated in last time's lecture.****

Definition: (Kaplansky 1950s)

For A a C^*-algebra,

(a) A is CCR ("completely continuous representation", not "canonical commutation relations") if for every irreducible representation $(\pi, \mathcal{H})$ of A, we have $\pi(A) = \mathcal{B}_0(\mathcal{H})$. In Dixmier's ****sp?**** book, these are called *liminal*.

(b) A is GCR ("generalized") if for every irreducible representation, $\pi(A) \subseteq \mathcal{B}_0(\mathcal{H})$. Also called *postliminal*.

(c) A is NCR ("not") if $\pi(A) \cap \mathcal{B}_0(\mathcal{H}) = 0$ for all irreducible representations. Also "antiliminal". (These might be "NGR" rather than "NCR".)

E.g. Let $K = \bigcap\{\ker(\pi) : \pi(A) \cap \mathcal{B}_0(\mathcal{H}) \neq 0\}$, then K is GCR and A/K is NCR.

Theorem: (Harish-Chandra, 1954)

For G a semi-simple Lie group (e.g. $SL(n, \mathbb{R})$), then $\pi(L^1(G)) \subseteq \mathcal{B}_0(\mathcal{H})$, i.e. $C^*(G)$ is CCR.

Theorem: (Dixmier)

If G is a nilpotent Lie group (i.e. closed connected subgroups of $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ — every Lie group satisfying ****some conditions**** is a discrete quotient of one of these), then $C^*(G)$ is CCR.

Many solvable groups — $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ — are GCR. However, Mantner showed by 1950s, there exists solvable groups that are not GCR. **E.g.** Take $\mathbb{C}^2$, and let α be the action of $\mathbb{R}$ on $\mathbb{C}^2$ by $\alpha_t : (z, w) \mapsto (e^{it}z, e^{it\theta}w)$ with θ irrational. This doesn't change the lengths of vectors, so e.g. if $z, w \in S^1 \subseteq \mathbb{C}$, they are preserved. I.e. we have an orbit that, since $\theta \notin \mathbb{Q}$, is dense in the torus $T^2 = S \times S$. So let $G = \mathbb{C}^2 \rtimes_\alpha \mathbb{R}$. Topologically this is $\mathbb{R}^5$, but the group is not GCR.

Prop: Let A be a unital ∞-dimensional simple (no proper ideals) C^*-algebra. Then A is NCR.

E.g. Canonical anti-commutation relations:

$$M_2(\mathbb{C}) \longrightarrow M_4(\mathbb{C}) \longrightarrow M_8(\mathbb{C}) \longrightarrow \cdots$$

$$T \longmapsto \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \longmapsto \cdots$$

Taking $M_3 \to M_9 \to \dots$, or $2 \mapsto p$ for other primes, give "ultra hyperfinite" algebras. Jim Glimm studied these in his doctoral thesis, and then went off into QFT, working with Arthur Jaffey, and then wandered into PDEs and shock waves, then numerical analysis.

Also, $C_r^*(F_n)$, where F_n is a free group, is an interesting example for $n \geq 2$.

Theorem: (Thoma, 1962)

> For a discrete group G, if $C^*(G)$ is GCR, then G has an Abelian normal subgroup N such that G/N is finite.

Definition: For a $*$-algebra A, an ideal is *primitive* if it is the kernel of an irreducible $*$-representation on a Hilbert space. (Or omit the $*$ and look at Banach spaces.)

Notation: For a $*$-normed algebra A, we write $\hat{A}$ for the set of unitary equivalence classes of irreducible representations on Hilbert space.

Cor: For a GCR C^*-algebra, there is a bijection between $\hat{A}$ and the set of primitive ideals of A.

Theorem: (Machey, Dixmier, Fell, and final hard part by Glimm)

> Let I be a primitive ideal of A with irreducible representation $(\pi, \mathcal{H})$ (i.e. $\ker \pi = I$) with $\pi(A) \cap \mathcal{B}_0(\mathcal{H}) = 0$. (So-called "bad case.") Then there is an uncountable number of inequivalent irreducible representations of A with kernel I, and they are unclassifiable (in a specific technical sense stated by Machey).

Proof idea:

> Let $B = A/I$, and look at irreducible faithful reps. Can find a Hilbert space $\mathcal{H}$ on which each rep can be realized. So . . . ****out of time**** $\square$