WHAT IS AN OPERATOR SPACE?

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Abstract. These are notes for a lecture delivered on 12 May, 2008, in a graduate course on operator algebras in Berkeley. The intent was to give a brief introduction to the basic ideas of operator space theory. The notes were hastily written and have not been carefully checked for accuracy or political correctness.

1. An overview of operator spaces

Operator spaces are a subtle refinement of the notion of Banach spaces. They embody the notion of noncommutativity in an essential way.

An *operator space* is a complex-linear space $\mathcal{S} \subseteq \mathcal{B}(H)$ of operators on some Hilbert space that is closed in the norm of $\mathcal{B}(H)$. Operator spaces are the objects of a category, but we have not yet defined the maps of this category. In particular, we have not said precisely when two operator spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are considered *equivalent*.

For example, one might consider $\mathcal{S}_1 \subseteq \mathcal{B}(H_1)$ and $\mathcal{S}_2 \subseteq \mathcal{B}(H_2)$ to be equivalent if they are isometrically isomorphic as Banach spaces. This is the notion of equivalence that results from declaring the maps of $\hom(\mathcal{S}_1, \mathcal{S}_2)$ to be linear mappings $L : \mathcal{S}_1 \to \mathcal{S}_2$ such that $\|L(A)\| \leq \|A\|$ for all $A \in \mathcal{S}_1$. As we will see, however, that category is only a disguised form of the more familiar category of Banach spaces (with contractions as maps).

In fact, once we have made a proper definition of the maps of the category of operator spaces, we will find that operator spaces are a subtle refinement of Banach spaces, whose objects $\mathcal{S}$ carry along with them nonclassical features that are connected with noncommutativity of operator multiplication. Some analysts like to think of operator space theory as "quantized functional analysis" in the sense that the resulting category is a noncommutative refinement of the classical category of Banach spaces.

2. Completely contractive maps

We now turn to the issue of properly defining the maps of the category of operator spaces so as to cause these remarks to have some concrete meaning.

Let $\mathcal{S} \subseteq \mathcal{B}(H)$ be an operator space. For every $n = 1, 2, \dots,$ we can form the direct sum $n \cdot H = H \oplus \cdots \oplus H$ of n copies of H. If one considers vectors of $n \cdot H$ as column vectors of height n with entries in H, then operators on $n \cdot H$ can be realized as $n \times n$ matrices with entries in $\mathcal{B}(H)$ in the usual

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way. In particular, we can define an operator space $M_n(\mathcal{S}) \subseteq \mathcal{B}(n \cdot H)$ as the space of all $n \times n$ operator matrices with entries in $\mathcal{S}$.

This is a very significant step. Starting with a single operator space $\mathcal{S} \subseteq \mathcal{B}(H)$, we have associated an entire sequence of operator spaces

$$M_n(\mathcal{S}) \subseteq \mathcal{B}(n \cdot H), \qquad n = 1, 2, \dots,$$

and each term of the hierarchy $M_n(\mathcal{S})$ is endowed with the norm it inherits from $\mathcal{B}(n \cdot H)$. Notice that there is no way of constructing such a hierarchy if we had started with simply a Banach space $\mathcal{S}$ that was not presented as a subspace of some $\mathcal{B}(H)$.

Given a linear map $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ between operator spaces $\mathcal{S}_k \subseteq \mathcal{B}(H_k)$ and given an $n = 1, 2, \dots$, we can define a linear map $\phi_n : M_n(\mathcal{S}_1) \to M_n(\mathcal{S}_2)$ by applying ϕ element-by-element to a matrix $(A_{ij}) \in M_n(\mathcal{S})$:

$$\phi_n : (A_{ij}) \mapsto (\phi(A_{ij})).$$

Definition 2.1. A linear map $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ is said to be *completely contractive* if $\|\phi_n\| \leq 1$ for every $n = 1, 2, \dots$.

One checks that a composition of completely contractive maps is completely contractive, and hence we obtain a category of operator spaces by declaring the morphisms to be completely contractive maps.

There is a broader category of operator spaces that is associated with *completely bounded* maps, which means $\|\phi\|_{cb} < \infty$ where

$$\|\phi\|_{cb} = \sup_{n \geq 1} \|\phi_n\| < \infty.$$

Notice that the completely contractive maps are those with $\|\phi\|_{cb} \leq 1$. To keep this discussion as simple as possible, we will confine attention to the category associated with completely contractive maps. The isomorphisms of this category are linear isomorphisms $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ such that $\|\phi_n\|$ is an isometry for every n, and such maps are called *complete isometries* of operator spaces. Perhaps it is unnecessary to point out that there is no notion of complete isometry in the classical category of Banach spaces; and this is the fundamental difference between operator spaces and Banach spaces.

3. Examples

Fix $p = 1, 2, \dots$, consider the p-dimensional Hilbert space $\mathbb{C}^p$, and consider the "row" and "column" operator spaces $\mathcal{R}, \mathcal{C} \subseteq \mathcal{B}(\mathbb{C}^p)$:

$$\mathcal{R} = \{\begin{pmatrix} z_1 & z_2 & \dots & z_p \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} : z_k \in \mathbb{C}\}, \quad \mathcal{C} = \{\begin{pmatrix} z_1 & 0 & \dots & 0 \\ z_2 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ z_p & 0 & \dots & 0 \end{pmatrix} : z_k \in \mathbb{C}\}.$$

Simple calculation shows that the operator norm of a row operator in $\mathcal{R}$ is the same as the operator norm of the corresponding column operator in $\mathcal{C}$,

$$\| \begin{pmatrix} z_1 & z_2 & \dots & z_p \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \| = \| \begin{pmatrix} z_1 & 0 & \dots & 0 \\ z_2 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ z_p & 0 & \dots & 0 \end{pmatrix} \| = \sqrt{|z_1|^2 + \cdots + |z_p|^2},$$

and in particular both $\mathcal{R}$ and $\mathcal{C}$ are isometrically isomorphic to the p-dimensional Hilbert space $\mathbb{C}^p$. In particular, $\mathcal{R}$ and $\mathcal{C}$ *are indistinguishable at the level of Banach spaces.*

On the other hand, we now show that they are far from being completely isometric at the level of operator spaces. For $z = (z_1, \dots, z_p) \in \mathbb{C}^p$, let us write R_z and C_z for the row and column operators associated with z

$$R_z = \begin{pmatrix} z_1 & z_2 & \dots & z_p \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \qquad C_z = \begin{pmatrix} z_1 & 0 & \dots & 0 \\ z_2 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ z_p & 0 & \dots & 0 \end{pmatrix},$$

and consider the map $\phi : \mathcal{R} \to \mathcal{C}$ defined by $\phi(R_z) = C_z$, $z \in \mathbb{C}^p$. Obviously, ϕ is isometric. Is it a complete isometry?

In order to answer that question we have to pass through the matrix hierarchies over $\mathcal{R}$ and $\mathcal{C}$. Fix $n = 1, 2, \dots$. We can realize the operator spaces $M_n(\mathcal{R})$ and $M_n(\mathcal{C})$ as subspaces of operators acting on a direct sum of p copies of $\mathbb{C}^n$ as follows

$$M_n(\mathcal{R}) = \{ \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \}, \quad \mathcal{C} = \{ \begin{pmatrix} A_1 & 0 & \dots & 0 \\ A_2 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ A_p & 0 & \dots & 0 \end{pmatrix} \},$$

where $A_1, \dots, A_p$ are complex $n \times n$ matrices - i.e., operators on $\mathbb{C}^n$. This is simply a matter of choosing an appropriate orthonormal basis and looking at operators as operator matrices. Notice that the map ϕ_n carries a row matrix with entries $A_1, \dots, A_p$ to the corresponding column matrix with entries $A_1, \dots, A_p$.

In order to show that ϕ_n is not isometric, we calculate the norm of a typical operator in the first space as follows. Since in any C^*-algebra we have $\|x\|^2 = \|xx^*\|$, we take x for the row operator with entries $A_1, \dots, A_p$ to obtain

$$\| \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \|^2 = \|A_1 A_1^* + \cdots + A_p A_p^*\|.$$

Similarly, using $\|y\|^2 = \|y^*y\|$ and taking y to be the column operator with entries $A_1, \dots, A_p$ we obtain

$$\| \begin{pmatrix} A_1 & 0 & \dots & 0 \\ A_2 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ A_p & 0 & \dots & 0 \end{pmatrix} \|^2 = \|A_1^*A_1 + \dots + A_p^*A_p\|.$$

So if n is sufficiently large ($n \geq p$ will do), then we can find partial isometries $A_1, \dots, A_p \in \mathcal{B}(\mathbb{C}^p)$ such that $A_1^*A_1, \dots, A_p^*A_p$ are mutually orthogonal projections of the same rank and $A_1A_1^* = \dots = A_pA_p^*$ is a single projection (of the same rank). With this choice of $A_1, \dots A_p$ it follows that

$$\|A_1A_1^* + \dots + A_pA_p^*\| = p, \qquad \|A_1^*A_1 + \dots + A_p^*A_p\| = 1.$$

We conclude that ϕ_n carries an operator of norm $\sqrt{p}$ to an operator of norm 1. By replacing each partial isometry A_k with its adjoint A_k^*, we obtain another operator of norm 1 in $M_n(\mathcal{R})$ that is mapped to an operator of norm $\sqrt{p}$ in $M_n(\mathcal{C})$.

Conclusion: If $n \geq p$ then we have $\|\phi_n\| \geq \sqrt{p}$ and $\|\phi_n^{-1}\| \geq \sqrt{p}$.

This particular isometry $\phi : \mathcal{R} \to \mathcal{C}$ is not a complete isometry. But it is conceivable that some other isometry of $\mathcal{R}$ to $\mathcal{C}$ is a complete isometry. It is a good exercise to show how the above construction can be adapted (hint: there is an easy way) to show that this is not true, and in fact *The operator spaces $\mathcal{R}$ and $\mathcal{C}$ are not completely isometric.*

We conclude from this discussion that, while viewed through a "classical" lens the spaces $\mathcal{R}$ and $\mathcal{C}$ are indistinguishable because they are both isometric to a p dimensional Hilbert space, they are actually very different at the operator space level. In particular, a given Banach space may have many (inequivalent) realizations as an operator space. Moreover, as the above examples show in a rather concrete way, *the reason why operator spaces are different from Banach spaces arises ultimately from the noncommutativity of operator multiplication*. This represents a dramatic shift in perspective, and the consequences of the operator space point of view are still being worked out today.

4. Stinespring's theorem

The first penetration into noncommutative Banach space theory was made (perhaps inadvertently) by W. Forrest Stinespring in the mid-fifties [Sti55]. Stinespring wanted to explain two rather different representation theorems in terms of a more general construction. His theorem was seen as a nice bit of work, but a piece of work that was peculiar enough that while many functional analysts learned it in their graduate courses in Chicago, Berkeley, UCLA and Penn, they did not really take it in as part of their toolkit. Indeed, this theorem was not fully appreciated for fifteen years, and even after it was fully understood, it was slow to make an impact in the larger community. I have heard that Stinespring was motivated to look for the theorem he found because of the basic mathematical values that motivate all of us - namely a desire to find the "right" proof of several intriguing ancillary results that seem connected. In the end, however, it provided the stepping-off point of noncommutative functional analysis - an event that was unforseen in the 1950s and would not emerge for many years.

In order to discuss Stinespring's theorem, we return to a general setting in which A is a C^*-algebra with unit $\mathbf{1}$. The GNS construction implies that every positive linear functional $\rho : A \to \mathbb{C}$ can be represented in the form

$$\rho(a) = \langle \pi(a)\xi, \xi\rangle, \qquad a \in A,$$

where $\pi : A \to \mathcal{B}(K)$ is a $*$-representation of A on a Hilbert space K and ξ is a vector in K. A related theorem of Bela Sz.-Nagy asserts that if X is a compact Hausdorff space and

$$\phi : C(X) \to \mathcal{B}(H)$$

is a linear map satisfying $f \geq 0 \implies \phi(f) \geq 0$, then ϕ can be represented in the form

$$\phi(f) = V^*\pi(f)V, \qquad f \in C(X),$$

where $\pi : C(X) \to \mathcal{B}(K)$ is a $*$-representation and $V : H \to K$ is a bounded operator. These two theorems are both special cases of a more general, as we now describe.

Returning to the general setting in which A is a unital C^*-algebra, recall that for every $n = 1, 2, \ldots$ there is a unique C^*-norm on the $*$-algebra $M_n(A)$ of $n \times n$ matrices over A. Note too that for every operator valued linear map $\phi : A \to \mathcal{B}(H)$ we can form a sequence of operator valued linear maps

$$\phi_n : M_n(A) \to \mathcal{M}_n(\mathcal{B}(H)) \cong \mathcal{B}(n \cdot H), \qquad n = 1, 2, \ldots$$

exactly as we did in the previous section.

Definition 4.1. ϕ is said to be *completely positive* if for every $n = 1, 2, \ldots,$ ϕ_n maps positive elements of $M_n(A)$ to positive operators in $\mathcal{B}(n \cdot H)$.

Here is the essential statement of Stinespring's theorem [Sti55]:

Theorem 4.2. *Every completely positive map $\phi : A \to \mathcal{B}(H)$ can be represented in the form*

$$\phi(a) = V^*\pi(a)V, \qquad a \in A, \tag{4.1}$$

where π is a representation of A on some other Hilbert space K and V is a bounded operator from H to K.

Sketch of proof. Consider the algebraic tensor product of complex vector spaces $A \otimes H$, and consider the sesquilinear form $\langle \cdot, \cdot \rangle$ that is uniquely defined on it by requiring

$$\langle a \otimes \xi, b \otimes \eta\rangle = \langle \phi(b^*a)\xi, \eta\rangle, \qquad a, b \in A, \quad \xi, \eta \in H.$$

Because ϕ is completely positive, one can check that this is a (semidefinite) inner product, and after dividing out by the subspace

$$\{u \in A \otimes H : \langle u, u \rangle = 0\}$$

and completing, one obtains a complex Hilbert space K. For every $a \in A$ there is a unique operator $\pi(a) \in \mathcal{B}(K)$ that extends the left multiplication operator $b \otimes \xi \mapsto ab \otimes \xi$ (one must check the details carefully here, but there are no surprises), and similarly one can define an operator $V : H \to K$ by defining $V\xi$ to be the coset of $\mathbf{1} \otimes \xi$ in K. Finally, it is a consequence of these definitions that

$$\langle \pi(a)V\xi, V\eta \rangle = \langle \phi(a)\xi, \eta \rangle, \qquad a \in A, \quad \xi, \eta \in H,$$

and (4.1) follows. □

It is an instructive exercise to show that, conversely, every linear map ϕ that can be represented in the form (4.1) is completely positive. Notice that the GNS representation of states follows from the special case in which $H = \mathbb{C}$; indeed, in that case the operator V maps the vector $1 \in \mathbb{C}$ to some vector $\zeta \in K$, and (4.1) becomes the familiar formula

$$\phi(a) = \langle \pi(a)\zeta, \zeta \rangle, \qquad a \in A.$$

It is a more challenging exercise to show (as Stinespring did in his original paper) that a positive linear map of $C(X)$ into $\mathcal{B}(H)$ is already completely positive; and therefore Sz.-Nagy's theorem also follows from Theorem 4.2.

5. Operator systems and complete positivity

The fundamental tool that makes functional analysis useful is the Hahn-Banach theorem: A linear functional defined on a subspace of a Banach space can be extended to a linear functional on the ambient space *without increasing its norm*. We now show that the theory of operator spaces has a corresponding tool. That fact is based on an extension theorem for positive linear maps that we now describe.

An *operator system* is an operator space $\mathcal{S} \subseteq \mathcal{B}(H)$ with two additional properties that allow one to speak of positivity:

(i) (Self-adjointness) $\mathcal{S}^* = \mathcal{S}$.
(ii) (Identity operator) $\mathbf{1} \in \mathcal{S}$,

where of course, $\mathbf{1}$ denotes the identity operator of $\mathcal{B}(H)$. Notice that in any operator system $\mathcal{S}$, it makes sense to speak of positive operators in $\mathcal{S}$. For example, every operator of the form $\mathbf{1} - X$ with X a self adjoint operator in $\mathcal{S}$ satisfying $\|X\| \leq 1$ is a positive operator in $\mathcal{S}$. If one is given a unital C^*-algebra A, then every self-adjoint linear subspace $\mathcal{S} \subseteq A$ that contains the identity of A can be viewed as an operator system.

There is some subtlety in that last comment. A C^*-algebra A can be represented in many ways as a concrete C^*-algebra of operators acting on a Hilbert space H. But the fact is that all such representations of A have the property that when they are faithful (i.e., have trivial kernel), they are completely isometric, and they are also complete isomorphisms with respect to the operator order. Hence the operator space class of $\mathcal{S} \subseteq A$ is perfectly well-defined, independently of how A is realized concretely as a C^*-subagebra of $\mathcal{B}(H)$.

Theorem 5.1. *Let $\mathcal{S}$ be an operator system contained in a C^*-algebra A and let $\phi : \mathcal{S} \to \mathcal{B}(K)$ be completely positive linear map. Then ϕ can be extended to a completely positive linear map of A into $\mathcal{B}(K)$.*

This is a theorem of [Arv69]; see [Pau02] for the details. The following extension theorem for maps of operator spaces provides an exact counterpart of the Hahn-Banach theorem for the category of operator spaces. It is possible to deduce Theorem 5.2 below from Theorem 5.1 by a simple but powerful device using 2×2 operator matrices that was discovered by Vern Paulsen, and can be found along with many other developments in Paulsen's book [Pau02].

Theorem 5.2. *Let $\mathcal{S} \subseteq \mathcal{B}(H)$ be an operator space and let $\phi : \mathcal{S} \to \mathcal{B}(K)$ be a linear map satisfying $\|\phi\|_{cb} < \infty$. Then ϕ can be extended to a linear map $\tilde{\phi} : \mathcal{B}(H) \to \mathcal{B}(K)$ in such a way that $\|\tilde{\phi}\|_{cp} = \|\phi\|_{cb}$.*

References

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