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****I arrived late.**** When G is compact, $\hat{G}$ is discrete, (if G is not abelian, $\hat{G}$ are the equivalence classes of irreps of G) and given the right set-up ****an action of G on Banach space B?****, we can average and define an "isotypic subspace": $B_m = \{\xi : \alpha_x(\xi) = \langle x, m\rangle \xi\}$.

Another view: Let $e_m(x) = \overline{\langle x, m\rangle}$. Then $e_m \in L^1(G)$, and $(e_m \star e_n)(x) = \int_G e_m(y)\, e_n(x-y)\, dy = \int_G \overline{\langle y, m\rangle \langle x-y, n\rangle} dy = \overline{\langle x, n\rangle} \int \overline{\langle y, m-n\rangle} dy$. Based on experience with these things over, e.g. the torus, we can show that this integral is $\int \overline{\langle y, m-n\rangle} dy = \delta_{m,n}$. So e_m is an idempotent in $L^1(G)$ and $e_m \star e_n = 0$ if $m \neq n$. So set $\xi_m = \alpha_{e_m}(\xi)$; then α_{e_m} is a projection of B onto B_m.

Prop: If $\xi_m = 0$ for all m, then $\xi = 0$.

Proof:

> $\alpha_{e_m}(\xi) = 0$. The finite linear combinations of the e_ms form a subalgebra under pointwise multiplication — $e_m e_n = e_{m+n}$ — and under complex conjugation. There are lots of characters of a compact group: we even have the machinery to show this ****and sketched the proof verbally, but I didn't catch it****. This algebra separates points, hence is dense in $C(G)$ with ∞-norm, so dense in $L^1(G)$. Thus $\alpha_f(\xi) = 0$ for any $f \in L^1(G)$. But let f_λ be an approximate identity for $L^1(G)$, so $0 = \alpha_{f_\lambda}(\xi) \to \xi$. $\square$

Question from the audience: So this is saying that if all the Fourier coefficients of a function are zero, then it's zero? **Answer:** Precisely. And more generally.

Cor: If $a \in A_\theta = C^*(\mathbb{Z}^d, c_\theta)$, and if $a_n = 0$ for all n, then $a = 0$.

If G compact Abelian, α and action on C^*-algebra A, we can define A_m for each $m \in \hat{G}$. We pick $a \in A_m$ and $b \in A_n$; then $\alpha_x(ab) = \alpha_x(a)\alpha_x(b) = \langle x, m\rangle a \,\langle x, n\rangle b = \langle x, m+n\rangle ab$. So $ab \in A_{m+n}$. And $a^* \in A_{-m}$. So then $\bigoplus A_m$ is a dense subalgebra fibered over $\hat{G}$. ****I would call this "graded".**** We can even do this in the nonabelian case. These are often called "Fell bundles".

So, back in our torus case, let U_m be a unitary generator for A_θ (corresponds to δ_m).

$$\begin{aligned}
(U_m)_n &= \int \overline{\langle x, m\rangle}\, \alpha_x(U_m)\, dx \\
&= \int \overline{\langle x, b\rangle} \langle x, m\rangle U_m \, dx \\
&= \int \langle x, m-n\rangle dx\, U_m \\
&= \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}
\end{aligned}$$

So $(A_\theta)_m = \text{span}(U_m)$. We can try to ask at a convergence level whether $a \sim \sum a_m U_m$. This doesn't have a good answer, even in the continuous case: Which collections of Fourier coefficients come from continuous functions?

For G acting on a C^*-algebra A, the fiber over 0 is a C^*-subalgebra. $A_0 = \{a : \alpha_x(a) = a \,\forall x\} \overset{\text{def}}{=} A^G$. If $P = \alpha_{e_0}$, then $P(a) = \int_G \alpha_x(a)\,dx$.

Prop: P is a *conditional expectation* from A onto A^G:

(a) If $a > 0$, then $P(a) > 0$.

(b) If $a \in A$ and $b \in A^G$, then $P(ba) = bP(a)$ and $P(ab) = P(a)b$.

(c) $P(\alpha_x(a)) = P(a) \,\forall x \in G$.

For A_θ, $P(a) = \int_{T^d} \alpha_x(a)\,dx = a_0 U_0 = a_0 \mathbb{1}$. So we can view P as a linear functional $\tau : a \mapsto a_0 \in \mathbb{C}$. It's positive, from what we've seen, and $P(\mathbb{1}) = P(U_0) = 1$, so it's a state, but also *tracial*: $\tau ab = \tau(ba)$. It's enough to check this on generators:

$$\tau(U_m U_n) = \int \langle m, x\rangle U_m \langle n, x\rangle U_n \,dx = \begin{cases} 0, & m \neq -n \\ U_m U_{-m} = c_\theta(m, -m), & m = -n \end{cases}$$

Cor: A_θ contains no proper α-invariant ideal.

Proof:

If I is an ideal, $a \in I$, $a \neq 0$, then $a^*a \in I$ and $a^*a \neq 0$. So $P(a^*a) = \int \alpha_x(a^*a)\,dx > 0$, but it is in $\mathbb{C}\mathbb{1}$, so $1 \in I$, so $I \in A_\theta$. $\square$

Cor: The rep of A_θ on $L^2(\mathbb{Z}^d)$ is faithful.

Next time.

Cor: τ is the only α-invariant tracial state.

Proof:

If τ_0 is another one, then $\tau_0(a) = \tau_0(\alpha_x(a)) = \int_G \tau_0(\alpha_x(a))dx = \tau_0\left(\int \alpha_x(a)dx\right) = \tau_0(a_0) = \tau_0(\tau(a)1) = \tau(a)$. $\square$