****This document was last updated on April 20, 2008. A more recent version may be available as part of http://math.berkeley.edu/~theojf/CstarAlgebras.pdf.****

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1.1 Examples

We begin with some rather abstract examples, but we won't stay for very long. It's good to have a framework for examples: generators and relations. We can have a (possibly very infinite) collection $\{a_1, a_2, \dots\}$ of *generators*, and we should also have $a_1^*, a_2^*, \dots$. The *relations* are non-commutative polynomials in the generators. Then form the *free algebra* $\mathcal{F}$ on the generators, which is a $*$-algebra: $* : a_i \mapsto a_i^*$ and $*$ reverses the order of words and is anti-linear over $\mathbb{C}$. Let I be the $*$-ideal generated by the relations. Then form $A = \mathcal{F}/I$, which is a $*$-algebra: it is *the universal $*$-algebra for the given generators and relations.*

We can look for $*$-representations of A, i.e. $*$-homomorphisms of A into $\mathcal{B}(\mathcal{H})$ for various Hilbert spaces $\mathcal{H}$. For $a \in A$, set

$$\|a\|_{C^*} \overset{\text{def}}{=} \sup\{\|\pi(a)\| : \ \pi \text{ is a } *\text{-rep of } A\}$$

This might be $+\infty$. **E.g.** one generator (and its adjoint) and no relations. We might also have $\|a\|_{C^*} = 0$ forall a. **E.g.** one generator, with relation $a^*a = 0$.

So the issues are:

1. Do the relations force $\|a\|_{C^*} < \infty$ on the generators (if it's finite on the generators, then it's finite on any polynomial).

   E.g. $u^*u = 1 = uu^*$ (we have a generators called "1", satisfying all the relations 1 should have). Then u is unitary in any representations, so $\|u\| = 1$.

2. Are there non-zero representations?

   It may happen that $\|a\|_{C^*} = 0$ for certain $a \in A$, and such a form an ideal, by which we can factor out.

If 1. holds, then the quotient in 2. will have a norm satisfying the C^* relation, and factoring out gives a C^*-norm, so we can complete. This gives *the C^*-algebra for the generators and relations.*

3. There may be a natural class of $*$-representations which give a C^*-norm $\|\cdot\|'_{C^*}$, but this $\|\cdot\|'_{C^*}$ does not give the full $\|\cdot\|_{C^*}$. It might be smaller.

Question from the audience: Like the atomic norm, taking just irreducible representations? **Answer:** No, that will just give us back the same thing. Indeed, doing this construction to a C^*-algebra will leave it intact.

E.g. If we just have the one relation $S^*S = 1$ and not on the other side, then we get the C^*-algebra for the unilateral shift on $\ell^2(\mathbb{N})$.

E.g. Let G be a discrete group, and take the elements of G as generators, with relations as in G. Also demand that $x^* = x^{-1}$. Then the representations of this set of generators and relations is the same as unitary representations — well, we never demand that an algebra's identity element go to $\mathbb{1} \in \mathcal{B}(\mathcal{H})$, but it will go to an idempotent, i.e. a projection operator, so we can cut down — on subspaces. All words in A in the the generators are just given by elements of G. So A (purely at the algebraic level; we haven't completed) consists of finite linear combinations of elements of G. I.e. given by functions $f \in C_c(G)$ (continuous of compact support). ****I would call this $\mathbb{C}[G]$ instead. This construction is covariant in G, but $C_c(-)$ is by rights contravariant.**** I.e. an elements of A is $\sum_{x\in G} f(x)x$. **Question from the audience:** Compact support for the discrete group is just ... **Answer:** Finite support. **Question from the audience:** So this is exactly the group ring. **Answer:** Yes.

E.g. $G = SL(3, \mathbb{Z})$. Where do we find irreducible unitary representations of this?

In fact, for this setting, there always exist two unitary representations:

(a) The trivial representation, 1-dimensional on $\mathcal{H} = \mathbb{C}$.

(b) The left-regular representation of G on $\ell^2(G)$:

$$(L_x\phi)(y) = \phi(x^{-1}y)$$

We need the inverse to preserve $L_xL_z = L_{xz}$. This really is a unitary operator, satisfying all the necessary relations.

Some group-ring calculations:

$$\left(\sum f(x)x\right)\left(\sum f(y)y\right) = \sum_{x,y} f(x)g(y)xy = \sum_z \left(\sum_{xy=z} f(x)g(y)\right) z = \sum_y \underbrace{\left(\sum_x f(x)g(x^{-1}y)\right)}_{\in C_c(G)} y$$

I.e. we define *convolution* on $C_c(G)$ by

$$(f \star g)(y) \stackrel{\text{def}}{=} \sum_x f(x)g(x^{-1}y)$$

Then $\left(\sum f(x)x\right)\left(\sum f(y)y\right) = \sum (f \star g)(z)z$.

What about the $*$?

$$\left(\sum f(x)x\right)^* = \sum \overline{f(x)}x^* = \sum \bar{f}(x)x^{-1} = \sum \bar{f}(x^{-1})x$$

So on $C_c(G)$ we set

$$f^*(x) \stackrel{\text{def}}{=} \overline{f(x^{-1})}$$

Anyway, for a representation $(\pi, \mathcal{H})$ of G, we have $\pi\left(\sum f(x)x\right) \stackrel{\text{def}}{=} \sum f(x)\pi(x)$. So, for $f \in C_c(G)$, set $\pi_f \stackrel{\text{def}}{=} \sum f(x)\pi(x)$. Then $f \to \pi_f$ is a $*$-representation of $C_c(G)$. Conversely, a $*$-representation of $C_c(G)$ must restrict to a unitary representation of G, since we can view $G \hookrightarrow C_c(G)$, by $x \mapsto \delta_x$. ****Gah! If G is not discrete, then the δ functions are not in $C_c(G)$, although they are in $\mathbb{C}[G]$ the group ring.****

Let L be the left-regular representation of G on $\ell^2(G)$, and look at $\delta_e \in \ell^2(G)$ be the vector at the identity. Then

$$L_f\delta_e = \left(\sum f(x)L_x\right)\delta_e = \sum f(x)\delta_x \in \ell^2(G)$$

So if $L_f = 0$, then $f = 0$. So L is a faithful $*$-representation of $C_c(G)$. So left $\|f\|^r_{C^*} \stackrel{\text{def}}{=} \|L_f\|$; this is a legitimate C^*-norm on $C_c(G)$. This is an example of 3. above. (r for "reduced")

Theorem: $\|\cdot\|^r_{C^*(G)} = \|\cdot\|_{C^*(G)}$ if and only if G is *amenable*.

There are twenty different equivalent definitions of "amenable". Where does the name come from? G is *amenable* if on $\ell^\infty(G)$ there is a *state* ("mean") μ which is invariant under left translation, e.g. $\mu(L_x\phi) = \mu(\phi)$ forall $\phi \in \ell^\infty(G)$. All abelian groups are amenable. **Exercise:** Why is $\mathbb{Z}$ amenable?

Question from the audience: You get the left-invariant representation by looking at .... Where is the other one? What is the norm on $C^*(G)$? **Answer:** We defined $\|L_f\| \stackrel{\text{def}}{=} \|L_f\|_{\mathcal{B}(\ell^2(G))}$.

Question from the audience: What is the topology on G? **Answer:** Discrete. We will eventually imitate this on locally compact groups.

Question from the audience: How is this μ related to the Haar measure? **Answer:** Every finite group is amenable. Just use the Haar measure.