1 February 15, 2008

****I was not in class. These are notes by Vinicius Ramos, TEXed much later by me — any errors are undoubtedly mine.****

Prop: Let A be a $*$-normed algebra with approximate identity $\{e_\lambda\}$ of norm 1. Let (π, B) be a continuous representation of A on a normed space. If π is *nondegenerate* (i.e. $\text{span}(\pi(A)B)$ is dense in B), then $\forall \xi \in B$ we have $\pi(e_\lambda)\xi \to \xi$.

Proof:

If $\xi = \pi(a)\eta$, then $\pi(e_\lambda)\xi = \pi(ae_\lambda)\eta \to \pi(a)\eta = \xi$. So this is also true for $\pi \in \text{span}(\pi(A)B)$. Use continuity to show it holds for B. $\square$

Cor: For A a $*$-normed algebra with approximate identity $\{e_\lambda\}$ of norm 1, for μ a continuous positive linear functional, we have $\mu(e_\lambda) \to \|\mu\|$.

Proof:

Form $(\pi_\lambda, \mathcal{H}_\lambda, \xi_\lambda)$;

$$\mu(e_\lambda) = \langle \pi(e_\lambda)\xi_\lambda, \lambda\rangle \to \langle \xi_\lambda, \xi\lambda\rangle = \tilde{\mu}(1_{\tilde{A}}) = \|\mu\|$$

$\square$

Let A be a $*$-normed algebra (with approximate identity e_λ); μ is a *state* of A if $\mu \geq 0$ and $\|\mu\| = 1$. Let $S(A) =$ set of states of A.

If $1_A \in A$, then $S(A)$ is a w^*-closed bounded convex subset of A and so is w^*-compact.

Proof:

$\mu(a^*a) \geq 0$ and $\mu(1) = \|\mu\| = 1$. $\square$

$S(A)$ can be viewed as "the set of non-commutative probability measures on A."

May be false in the non-unital case: $A = C_\infty(\mathbb{R})$, $\mu_n = \delta_n \overset{n\to\infty}{\underset{w^*}{\longrightarrow}} 0$; that is not a state.

If $1 \notin A$, let $QS(A) = \{\mu : \mu \geq 0, \|\mu\| \leq 1\}$ be the set of "quasi-states". This is the w^*-closed convex hull of $S(A) \cup \{0\}$. Then $QS(A)$ is w^*-closed convex ****illegible****, so compact.

Definition: A representation (π, B) of A on a Banach space is *irreducible* if there is no proper closed subspace that is carried into itself by the representation, i.e. no non-zero $C \subsetneq B$ so that $\pi(A)C \subseteq C$.

Schur's Lemma: Let $(\pi, \mathcal{H})$ be a $*$-representation of a $*$-normed algebra A. Then π is irreducible if and only if $\text{End}_\pi(\mathcal{H}) = \{\mathbb{C}\mathbb{1}_\mathcal{H}\}$. (Recall, the left hand side is the intertwiners $\{T \in \mathcal{B}(\mathcal{H}) : \pi(a)T = T\pi(a)\, \forall a\}$.)

Proof:

($\Leftarrow$) If not irreducible, let P be the orthogonal projection onto an invariant proper closed subspace. Then $P \in \mathrm{End}_\pi(\mathcal{H})$.

($\Rightarrow$) Suppose π is irreducible and let $T \in \mathrm{End}_\pi(\mathcal{H})$. Suppose $T \notin \mathbb{C}\mathbb{1}_{\mathcal{H}}$. Then either $\Re T = \frac{T+T^*}{2}$ or $\Im T = \frac{T-T^*}{2i} \notin \mathbb{C}\mathbb{1}_{\mathcal{H}}$. So we can assume $T^* = T$. Then $C^*(T, \mathbb{1}) \cong \mathbb{C}(\sigma(T))$. A proper spectral projection for T will be in $\mathrm{End}_\pi(\mathcal{H})$ and its range will be a proper closed invariant subspace. Contradicts irreducibility of π. $\square$

For μ, ν positive linear functionals, we write $\mu \geq \nu$ if $\mu - \nu \geq 0$. For $(\pi_\mu, \mathcal{H}_\mu, \xi_\mu)$, let $T \in \mathrm{End}_\pi(\mathcal{H}_\mu)$ with $0 \leq T \leq \mathbb{1}$. Set $\nu(a) = \langle T\pi_\mu(a)\xi_\mu, \xi_\mu\rangle$, then $\nu \geq 0$.

$$(\mu - \nu)(a) = \langle \pi_\mu(a)\xi_\mu, \xi_\mu\rangle - \langle T\pi_\mu(a)\xi_\mu, \xi_\mu\rangle = \langle (\mathbb{1} - T)\pi_\mu(a)\xi_\mu, \xi_\mu\rangle$$

So $\mu - \nu \geq 0$ since $\mathbb{1} - T \geq 0$.

Theorem: Given $\mu \geq 0$, the $\nu \geq 0$ such that $\mu \geq \nu$ are exactly of the form $\nu(a) = \langle T\pi_\mu(a)\xi_\mu, \xi_\mu\rangle$.