1 February 4, 2008

1.1 Quotient C^* algebras

Let A be a C^* algebra, I a closed two-sided ideal. We saw last time that I is a $*$-subalgebra, hence (closed) a C^* subalgebra, with an approximate identity. We form the quotient A/I, $a \mapsto \dot{a}$, an algebra with quotient norm

$$\|\dot{a}\| \stackrel{\text{def}}{=} \inf\{\|a-d\| \text{ s.t. } d \in I\}$$

Then A/I is complete. (It is clearly a $*$-algebra.) Hence A/I is a Banach $*$-algebra. (This is true for any $*$- two-sided ideal in a Banach $*$-algebra. To show that it is a C^*-algebra required verifying the inequality.

Key Lemma: Let $\{e_\lambda\}$ be a positive norm-1 approximate identity for I. Then for any $a \in A$,

$$\|\dot{a}\| = \lim_\lambda \|a - ae_\lambda\|$$

Proof:

We can assume that A has an identity element. We can be more careful, and avoid this, but anyway. . . .

Key C^* fact: Look at $\|1 - e_\lambda\|$ for a given λ. e_λ is self-adjoint, and look at $C^*(1, e_\lambda)$, so clearly $\|1 - e_\lambda\| \le 1$ (not true in a general Banach algebra), using positivity, norm ≤ 1, and that we are in C^*-land.

Well, $ae_\lambda \in I$, so certainly $\le$ is clear in the Lemma. For $\ge$, let $\epsilon > 0$ be given. Then we can find $d \in I$ so that $\|\dot{a}\| + \epsilon \ge \|a - d\|$. Then

$$\|a - ae_\lambda\| = \|a(1-e_\lambda)\| \le \|(a-d)\underbrace{(1-e_\lambda)}_{\le 1}\| + \|d(1-e_\lambda)\| \le \|a-d\| + \underbrace{\|d - de_\lambda\|}_{\to 0}$$

□

Theorem: (Segal, 1949)

A/I is a C^*-algebra.

Proof:

From Banach-land, we have $\|\dot{a}^*\dot{a}\| \le \|\dot{a}\|^2$. We also have

$$\begin{aligned} \|\dot{a}\|^2 &= \lim \|a - ae_\lambda\|^2 = \lim \|a(1-e_\lambda)\|^2 \\ &= \lim \|(1-e_\lambda)a^*a(1-e_\lambda)\| \\ &\le \lim \|a^*a(1-e_\lambda)\| \\ &= \|\dot{\overbrace{a^*a}}\| = \|\dot{a}^*\dot{a}\| \end{aligned}$$

by general Banach algebra. □

1.2 Beginnings of non-commutative measure theory

This concludes our mining of results that follow directly from the fact that commutative C^* algebras are functions on spaces. We move to proving that general C^* algebras are algebras of bounded operators on Hilbert spaces. What we have been doing depended importantly on completeness; our new topic will not. In the wild, we often find $*$-algebras satisfying the norm identity, but that are not complete. We complete for the nice framework, but the things you add in the completion are often weird, so it's better to work with just $*$-normed algebras.

Definition: For a $*$-algebra A over $\mathbb{C}$, a linear functional μ on A is *positive* if $\mu(a^*a) \geq 0$ for all $a \in A$.

E.g. the 0-functional.

E.g. $\mathbb{C}^2$, with $(\alpha, \beta)^* \stackrel{\text{def}}{=} (\bar{\beta}, \bar{\alpha})$, then there are no non-zero positive linear functionals.

Definition: For a $*$-normed-algebra A, we say that a positive linear functional μ is a *state* if $\|\mu\| = 1$.

This is the analog of a probability measure.

Let A be a $*$-algebra and μ a positive functional. Define a sesquilinear form on A by

$$\langle a, b \rangle_\mu = \mu(b^*a)$$

****Ew. We've made the order all backwards.**** You can go in any order, but this is what is most commonly done. Called the "GNS construction" (Gelfand, Naimar ****sp?****, and Segal).

We factor by vectors $n \in N$ of length 0 to get a (positive) inner-product on A/N. Then complete, and call this $\mathcal{L}^2(A, \mu)$. We would now like to get the operators.

For $a \in A$, we let $L_a \stackrel{\text{def}}{=} b \mapsto ab$. This is a left-regular representation, and it tries to be faithful. Then

$$\langle L_a b, c \rangle = \mu(c^*ab) = \langle b, L_{a^*}c \rangle$$

Thus, $a \mapsto L_a$ is a "*-representation". (We've swept under the rug various issues of completeness, etc.)

There are issues here: E.g. Let A be all $\mathbb{C}$-valued polynomials on $\mathbb{R}$. Let $\mu(p) \stackrel{\text{def}}{=} \int_{\mathbb{R}} p(t)e^{-t^2}dt$. (The Gaussian goes to 0 at both ends so fast that this is finite for every polynomial.) Moreover, $\mu(p^*p) = \int |p(t)|^2 e^{-t^2}dt \geq 0$, so we have a genuine inner product on polynomials: $\langle p, q \rangle = \int \bar{q(t)} p(t) e^{-t^2} dt$, and we can complete with respect to this, and we get the usual $\mathcal{L}^2(\mathbb{R}, e^{-t^2}dt)$.

Now, we have the left-regular representation $p \mapsto L_p$, but L_p is not a bounded operator! Indeed, on the algebra of polynomials, there is no algebra norm that makes sense within this framework. If we work in a compact subset, we can take the supremum norm, but $e^{-t^2}dt$ lives on the whole line.

Question from the audience: What about other notions of positive, e.g. anything of the form a^*a? **Answer:** then we don't know that the sum of positive elements is positive, so not a terribly useful notion. For instance, for normed $*$-algebras, it can fail that $a^*a + b^*b \neq c^*c$, even though the notion of positive linear functionals will succeed. We can take the norm form the left-regular representation, and then complete, but this will have little to do with the original norm. E.g. G a discrete group, and look at $\ell^1(G)$, which is a fine $*$-algebra with convolution. We also have an action of ℓ^1 on $\ell^2(G)$, with a good notion of operator norm (so can complete to a C^* algebra), but has little to do with the ℓ^1 norm. **Question from the audience:** is this like Gelfand transform on an abelian group? **Answer:** of course.