1 February 25, 2008

Theorem from last time: $\mathcal{B}_0(\mathcal{H})$ has one (up to equivalence) irreducible representation, namely $\mathcal{B}_0 : \mathcal{H} \to \mathcal{H}$. And every (non-degenerate) representation is a direct sum of these.

Proof, continued from last time:

Last time, we defined some rank-one operators $\langle \xi, \eta \rangle_0$, which we view as a $\mathcal{B}_0$-valued inner product. For bookkeeping, we let $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ be $\mathbb{C}$-linear in the second variable.

Let (π, V) be a non-degenerate representation of $\mathcal{B}_0(\mathcal{H})$. For vectors in $\mathcal{H}$ we will use $\xi, \eta, \dots$, whereas for vectors in V we will use $v, w, \dots$; we will not carry the "π" around, preferring "module notation": $Tv \overset{\text{def}}{=} \pi(T)v$ for $v \in V$ and $T \in \mathcal{B}_0(\mathcal{H})$. We take $\langle \cdot, \cdot \rangle_V$ to be linear in the second variable.)

Of course, $\mathcal{B}_0(\mathcal{H})$ is a C^*-algebra, and in particular it is complete. Thus, the representation $\pi : \mathcal{B}_0(\mathcal{H}) \to \mathcal{B}(V)$ is a $*$-homomorphism of C^*-algebras, and hence is continuous. Furthermore, $\mathcal{B}_0(\mathcal{H})$ is topologically simple: there are no *closed* two-sided ideals. Thus π is injective (its kernel is closed, since it is continuous).

So choose $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Then $\langle \xi, \xi \rangle_0$ is the rank-1 self-adjoint (and hence orthogonal) projection onto $\xi\mathbb{C}$. So $\pi(\langle \xi, \xi \rangle_0)$ is also a (self-adjoint orthogonal) projection on V (being a projection is an algebraic property), and it is not the 0 projection. So choose v_0 with $\|v_0\| = 1$ in the range of this projection.

Thus, define $Q : \mathcal{H} \to V$ by

$$Q : \eta \mapsto \langle \eta, \xi \rangle_0 v_0$$

This is obviously continuous.

$$\begin{aligned}
\langle Q\eta, Q\zeta \rangle_V &= \langle \langle \eta, \xi \rangle_0 v_0, \langle \zeta, \xi \rangle_0 v_0 \rangle_V \\
&= \langle v_0, \langle \xi, \eta \rangle_0 \langle \zeta, \xi \rangle_0 v_0 \rangle_V \text{ since } \pi \text{ is a } *\text{-representation} \\
&= \left\langle v_0, \left\langle \underbrace{\langle \xi, \eta \rangle_0 \zeta}_{=\xi \langle \eta, \zeta \rangle_{\mathcal{H}}}, \xi \right\rangle_0 v_0 \right\rangle_V \\
&= \langle \eta, \zeta \rangle \, \langle v_0, \langle \xi, \xi \rangle_0 v_0 \rangle \\
&= \langle \eta, \zeta \rangle
\end{aligned}$$

So Q is isometric. Moreover, for $T \in \mathcal{B}_0(\mathcal{H})$,

$$\begin{aligned}
Q(T\eta) &= \langle T\eta, \xi \rangle_0 v_0 \\
&= T \langle \eta, \xi \rangle_0 v_0 \\
&= T\, Q(\eta)
\end{aligned}$$

So $Q : \mathcal{H} \to V$ intertwines π with the representation of $\mathcal{B}_0(\mathcal{H})$ on $\mathcal{H}$. Thus, the range of Q is a closed subspace of V carried into itself by the action $\pi : \mathcal{B}_0(\mathcal{H}) \to \mathcal{B}(V)$. The representation

π restricted to this subspace is unitarily equivalent to the representation of $\mathcal{B}_0(\mathcal{H})$ on $\mathcal{H}$. If π itself is irreducible, then range of Q must be all of V, establishing the first part of the theorem. If the representation is not irreducible, take $Q(\mathcal{H})^\perp$, which is still a representation, and rinse and repeat (with Zorn ****the best brand of conditioning shampoo****). $\square$

****This paragraph was said after the next few, but belongs here.**** $\mathcal{B}_0(\mathcal{H})$ has only one irreducible representation, but any pure state gives an irreducible representation a la GNS. So take any pure state, do the GNS, and you'll get the representation on $\mathcal{H}$, with a cyclic vector. So every pure state of $\mathcal{B}_0(\mathcal{H})$ is represented by a vector $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, i.e. $\mu(T) = \langle T\xi, \xi\rangle$. Multiplying this vector by something in $S^1 \subset \mathbb{C}$ does not change the state; the pure states are represented by rank-1 projections p, via $\mu(T) = \operatorname{tr}(pT)$. We in fact have a bijection $\{\text{pure states of } \mathcal{B}_0(\mathcal{H})\} \leftrightarrow \{\text{rank-1 projections}\} = \mathcal{PH}$ the "projective Hilbert space". This is the setting for the quantum physics of finitely many particles. Moreover, the convex hull $S(\mathcal{B}_0(\mathcal{H})) = \{\text{"mixed states"}\} = \{\text{density operators}\} = \{D \in \mathcal{B}(\mathcal{H}) : 0 \le D, \operatorname{tr}(D) = 1\}$ where the corresponding state has $\mu_D(T) = \operatorname{tr}(DT)$. **Question from the audience:** Does $\le \mathbb{1}$ follow from that? **Answer:** Yes, but $\operatorname{tr} = 1$ is the interesting part So, with a quantum-mechanical system, it is the state that evolves, not the vector; i.e. it is the point in $\mathcal{PH}$. This often confuses people. Important questions include "What are the automorphisms of $\mathcal{PH}$?" Well, these include automorphisms of $\mathcal{H}$ and also anti-automorphisms. These anti-linear operators do occur, e.g. Time and Parity reversals.

A theorem of Burnside says that if a subalgebra of the algebra of operators on a finite-dimensional vector space acts irreducibly, then the subalgebra is the whole algebra. For example, at the purely algebraic level, then $\mathbb{C}$ and $M_n(\mathbb{C})$ have only one irreducible representation, and more generally, for any ring R, $R - \textsc{Mod}$ and $M_n(R) - \textsc{Mod}$ are equivalent as categories, under

$$_RV \mapsto _{M_n(R)} R^n_R \otimes_R {}_RV$$

This is the notion of *Morita equivalence.*

The general picture for C^*-algebras is similar. For our example, we have $\langle\cdot,\cdot\rangle_{\mathcal{B}_0(\mathcal{H})}$, ${}_{\mathcal{B}_0(\mathcal{H})}\mathcal{H}_{\mathbb{C}}$, and $\langle\cdot,\cdot\rangle_{\mathbb{C}}$. And in general, we will have $\langle x, y\rangle_A z = x\langle y, z\rangle_B$ and a module ${}_AX_B$, plus some nondegeneracy axioms, and this will give "strong Morita equivalence." Two non-commutative spaces that are strong Morita equivalent will have the same homology and cohomology.

Theorem: (analogous to a theorem of Burnside, moving towards a Stone-Weierstrass theorem)

Let A be a sub-C^*-algebra of $\mathcal{B}_0(\mathcal{H})$ and suppose that $\mathcal{H}$ is an irreducible module of the action of A. Then $A = \mathcal{B}_0(\mathcal{H})$.