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We review classical dynamical systems: a group G acts as diffeomorphisms on a locally compact space M, thought of as the phase space of the system. Then we get an action on $A = C_\infty(M)$, since $C_\infty(-)$ is contravariant: if $\alpha : G \to \mathrm{Homeo}(M)$, then G acts on A via $\alpha_x(f)(m) = f(\alpha_{x^{-1}}(m))$. So we can form $A \times_\alpha G$. If the action on M is sufficiently continuous, then α is strongly continuous on A.

Theorem: For (M, G, α), with M *second countable* (i.e. a countable base for its topology): let $(\sigma, \mathcal{H})$ be an irreducible representation of $A \times_\alpha G$, where $A = C_\infty(M)$; let σ be the integrated form of $(\pi, U, \mathcal{H})$. Let $I = \ker(\pi)$. (**Question from the audience:** Is π irreducible? **Answer:** Absolutely not.) Then I is a closed ideal of A; let $Z_I = \mathrm{hull}(I)$ (i.e. maximal ideals that contain I — maximal ideals of A correspond to points in M), so $I = \{f \in A : f|_{Z_I} = 0\}$.

Then Z_I is the closure of an orbit in M, i.e. $\exists\, m_0 \in M$ s.t. $\overline{\{\alpha_x(m) : x \in G\}} = Z_I$. (There's no reason the orbit ought to be closed, e.g. an action of $\mathbb{Z}$ on a compact space.)

Proof:

Note: $\alpha_x(I) \subseteq I$ for all X. ($d \in I$, then $\pi(\alpha_x(d)) = U_x \pi(d) U_{x^{-1}} = 0$; this uses only the covariance relation.) We say that I is "α-invariant". We say $\subseteq$, but it's true for x^{-1}, so we get equality. This implies that $\alpha_x(Z_I) = Z_I$.

Choose a countable base for the topology of M; let $\{B_n\}$ be (an enumeration of) those elements of the base that meet Z_I. (Thus $\{B_n \cap Z_I\}$ is a base for the relative topology of Z_I.) For each n, let $O_n = \bigcup_{x \in G} \alpha_x(B_n) = \alpha_G(B_n)$. Since each B_n is open and α is homeo, this is open; it's also clear that O_n is α-invariant, in that it's carried into itself by the G-action. Let $J_n = C_\infty(O_n)$. We view these as continuous functions on M that vanish outside O_n; J_n is exactly those functions that vanish on the closed set $M \smallsetminus O_n$. So J_n is an ideal of A.

Furthermore, because $B_n \cap Z_I \neq \emptyset$, we can find $f \in C_\infty(B_n)$ so that $f|_{Z_I} \neq 0$. Thus $J_n \not\subseteq I$. So $J_n \times_\alpha G$ is an ideal in $A \times_\alpha G$, and it is not a subideal of $I \times_\alpha G = \ker(\sigma)$. Since σ is irreducible, $\sigma|_{J_n \times_\alpha G}$ is non-degenerate. Thus $\pi|_{J_n}$ is non-degenerate.

Choose $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Define $\mu \in S(A)$ to be the vector state: $\mu(f) = \langle \pi(f)\xi, \xi \rangle$. I.e. μ is a probability Radon measure on M. Since $\pi|_{J_n}$ is non-degenerate, choose $\{e_\lambda\}$ a positive approximate identity of norm 1. Then $\mu(e_\lambda) = \langle \pi(e_\lambda)\xi, \xi \rangle \xrightarrow{\lambda} \langle \xi, \xi \rangle = 1$. So $\left\| \mu|_{J_n} \right\| = 1$. Let μ also be the corresponding Borel measure. ****huh?**** I.e. we view μ as giving sizes of sets: $\mu(O_n) = 1$. Then $\mu(M \smallsetminus O_n) = 0$. So $\mu\left(\bigcup_n (M \smallsetminus O_n)\right) = 0$ — this is where we use the separability hypothesis —, so $\mu(\bigcap_n O_n) = 1$, so $\bigcap O_n \neq \emptyset$. Pick any $m_0 \in \bigcap O_n$.

If $f \in I$, then $\pi(f) = 0$, so $\mu(f) = 0$. Thus, $\mu(M \smallsetminus Z_I) = 0$, so $\mu(Z_I) = 1$, and we should have intersected all our O_n in the previous paragraph with Z_I. So we have $m_0 \in \bigcap (O_n \cap Z_I)$. So $\alpha_G(m_0) \subseteq \bigcap (O_n \cap Z_I) = \bigcap \left(\alpha_G(B_n \cap Z_I)\right)$. So for each n, $\alpha_G(m_0) \cap B_n \neq \emptyset$. So $\alpha_G(m_0)$ meets each elements of a base for the topology of Z_I, and so is dense in Z_I. $\square$

(For the record, this argument works for "factor representations" of von-Neuman algebras.)