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1 April 4, 2008

We begin with some homological algebra. We have $C_2(G, A) = \{c : G \times G \to A\}$ and we define $\partial c(x, y, z) = c(xy, z)c(x, y) - c(y, z)c(x, yz)$. For $c \in C_1 = \{c : G \to A\}$, we define $\partial c(x, y) = c(x)c(y)c(xy)$. Then the second homology $H_2 = Z_2/B_2 = \ker \partial / \operatorname{im} \partial$ classify extensions $0 \to A \to E_c \to G \to 0$, at least at the algebraic level. Since B_n may not be closed in Z_n, the quotient can get a non-Hausdorff topology; this adds difficulty to the theory.

1.1 A specific class of examples

We have matrices $\theta \in M_d(\mathbb{R})$ and $\mathbb{Z}^2$, and we define a cocycle $c_\theta(m, n) = e^{2\pi i m \cdot \theta n}$. ****We should call θ a $0, 2$-tensor.**** We look at projective unitary reps of $\mathbb{Z}^d$ for bicharacter c_θ:

$$U_m U_n = c_\theta(m, n) U_{m+n} \tag{1}$$

We look at the universal C^*-algebra generated by the unitary symbols U_m ($U_0 = \mathbb{1}$) and the the relation (1).

We can construct this. $\mathbb{Z}^d \hookrightarrow C_c(\mathbb{Z}^d)$ with the twisted convolution

$$(f \star_\theta g)(m) \stackrel{\text{def}}{=} \sum_n f(n) g(m - n) c_\theta(n, m - n)$$

We need a $*$-operation. The relation (1) gives $U_m U_{-m} = c_\theta(m, -m) = \overline{c_\theta(m, m)}$, so $(U_m)^* = (U_m)^{-1} = c_\theta(m, m) U_{-m}$. Saying this again, for C_c:

$$f^*(m) = c_\theta(m, m) f(-m)$$

Then on this algebra the universal C^*-norm $\|f\|_{C^*}$ is well-defined. Indeed, looking at (the integrated form of) a representation $f \mapsto \sum f(m) U_m$, we see that $\|U_f\| \leq \|f\|_{\ell^1}$.

Question from the audience: Do you have a specific representation in mind? **Answer:** No. This is for any rep. **Question from the audience:** And the cocycle condition is equivalent to associativity? **Answer:** Yes. This is a bicharacter, so certainly a cocycle.

So, we can complete $C_c(\mathbb{Z}^d)$ — by the way, the "c" here means "of finite support", it doesn't have anything to do with the cocycle — to get C^*-algebra A_θ, and this is the universal C^*-algebra as above. Are there any projective representations? Look at the left-regular representation, and see what you can do. Let $C_c(G)$ act on $\ell^2(\mathbb{Z}^d)$, a fine Hilbert space by left convolution:

$$(f, \xi) \mapsto f \star_\theta \xi$$

for $\xi \in \ell^2(\mathbb{Z}^d)$

Question from the audience: I'm still confused. Why didn't we just define A_c as the universal algebra? **Answer:** We did. This is a description, using (1). If you take any juxtaposition of these symbols, we just get a scalar times another symbol. So any combination is a linear combination.

In any case, the norm you would get on this space is the "reduced" norm. We will see later that the full norm is the reduced norm.

We'd like to understand better the structure of the algebra A_θ. Certainly this will depend on θ — when θ is 0 ****mod $\mathbb{Z}$**** we get the commutative algebra with the usual convolution; when θ is not zero, we do not expect a commutative answer.

The dual group $\widehat{\mathbb{Z}^d} \cong T^d = \mathbb{R}^d/\mathbb{Z}^d$, where we identify the character $e_t(n) = e(n \cdot t) \stackrel{\text{def}}{=} \langle n, t\rangle$ ****bah, dot products****, and have adopted the notation $e(\tau) \stackrel{\text{def}}{=} e^{2\pi i \tau}$. There is an action α of T^d on A_θ via

$$(\alpha_t(f))(m) = \langle m, t\rangle f(m)$$

This action is independent of θ. We can check

$$\begin{aligned}
(\alpha_t(f \star_\theta g))(m) &= \langle m, t\rangle \sum f(n) g(m-n) c_\theta(n, m-n) \\
&= \sum \langle n, t\rangle f(n)\, \langle m-n, t\rangle g(m-n)\, c_\theta(n, m-n) \\
&= (\alpha_t(f) \star_\theta \alpha_t(g))(m)
\end{aligned}$$

In general, for any G (discrete, or even non-discrete) abelian, and cocycle c, then $\hat{G}$ acts on $C^*(G, c)$ exactly by the analog of this pairing. This is because if G abelian acts on a C^*-algebra A by an action β, then $\hat{G}$ acts on the cross-product algebra $A \times_\beta G$. The formula is the same — we get the "dual action".

Question from the audience: What can we say about $A \times_\beta G \times_\alpha \hat{G}$? **Answer:** Quite a lot. More generally, we can consider $G \rightsquigarrow \mathbb{C}[G]$ a Hopf algebra, and $\hat{G}$ the dual Hopff algebra. There's a lot to be said about C^*-Hopff algebras. Compact case is more-or-less understood, but locally compact quantum groups are hard even to define. E.g. no one knows how to prove the existence of a Haar measure in the non-compact case.

In any case, the action $(\alpha_t(f))(m) = \langle m, t\rangle f(m)$ is strongly continuous: $t \mapsto \alpha_t(f)$ is continuous for the norm $\|\cdot\|_{A_\theta}$. It's worth generalizing. Let G be a compact Abelian group with an action α on some C^*-algebra A. We can try to do "Fourier analysis" We have $\hat{G}$ discrete. For $a \in A$ and $m \in \hat{G}$, the mth Fourier coef of a is

$$a_m \stackrel{\text{def}}{=} \int_G \alpha_t(a)\, \overline{\langle t, m\rangle}\, dt \in A$$