1 February 8, 2008

****I was not in class. These are notes by Vinicius Ramos, TEXed much later by me — any errors are undoubtedly mine.****

1.1 GNS Construction

Let A be a $*$-algebra. Let μ be a positive linear functional on A, i.e. $\mu(a^*a) \geq 0$. Define $\langle a, b\rangle_\mu = \mu(b^*a)$.

Let $\mathcal{N}_\mu = \{a : \langle a, a\rangle_\mu = 0\}$. $\mathcal{N}_\mu$ is a linear subspace. Indeed, if $a \in \mathcal{N}_\mu$, then $|\langle a, b\rangle_\mu|^2 \leq \langle a, a\rangle_\mu \langle b, b\rangle_\mu = 0$. So $\mathcal{N}_\mu = \{a : \langle a, b\rangle_\mu = 0\, \forall b \in A\}$.

Then $\langle \cdot, \cdot\rangle_\mu$ induces a definite inner product on $A/\mathcal{N}_\mu$. Complete this, and call the completion $L^2(A, \mu)$.

The left-regular representation of A on A is a "$*$-rep": $\langle ab, c\rangle_\mu = \langle b, a^*c\rangle_\mu$.

Fact: $\mathcal{N}_\mu$ is a left ideal in A. Indeed, if $b \in \mathcal{N}_\mu$ then

$$|\langle ab, ab\rangle_\mu|^2 = |\langle b, a^*ab\rangle|^2 \leq \langle b, b\rangle_\mu^2 \cdots = 0$$

So the left regular $*$-representation induces a representation of A on $A/\mathcal{N}_\mu$ which is still a $*$-representation.

Problem: This need not be a representation by bounded operators.

Theorem: Let A be a unital $*$-normed algebra and let μ be a continuous positive linear functional. Then the "GNS"-representation for μ is by bounded operators.

Proof:

We need only show that $\|\mu\| = \mu(1)$.

Let $a, b \in A$ so that $\|L_a b\|_\mu \leq c_a \|b\|_\mu$. ****Vinicius leaves a question mark (?) in the margin; I've got nothing.**** Then

$$\|L_a b\|^2 = \langle ab, ab\rangle_\mu = \langle a^*ab, b\rangle_\mu = \mu(b^*a^*ab).$$

Define ν on A by $\nu(c) = \mu(b^*cb)$. Then ν is continuous. Thus $\|\nu\| = \nu(1)$.

$$\nu(c^*c) = \mu(b^*c^*cb) = \mu\left((cb)^*(cb)\right) \geq 0$$

$$\nu(a^*a) \leq \|a^*a\|\|\nu\| = \|a^*a\|\nu(1) = \|a^*a\|\mu(b^*b) = \|a^*a\|\|b\|_\mu^2.$$

So $\|L_a\|^2 \leq \|a^*a\| \leq \|a\|^2$. Hence $\|L_a\|_{\mathcal{B}(L^2(A,\mu))} \leq \|a\|_A$. $\square$

We have $L_1 = \mathbb{1}_{L^2(A,\mu)}$. ****On the LHS, $1 \in A$ is the unit element; on the right is $\mathbb{1}$ the identity operator on L^2.****

Definition: A continuous $*$-*representation* of a $*$-normed algebra is a continuous $*$-homomorphisms of A into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. $\mathcal{H}$ is *non-degenerate* if $\operatorname{span}\{\pi(a)\xi : a \in A, \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$.

Let $\xi_0 = 1$ viewed as an element of $L^2(A,\mu)$. Then $L_a\xi_0 = a$ viewed as an element of $L^2(A,\mu)$. Thus $\{L_a\xi_0 : a \in A\}$ is dense in $L^2(A,\mu)$.

Definition: A continuous $*$-representation $(\pi, \mathcal{H})$ of a $*$-normed algebra is *cyclic* if there is a vector ξ_0 such that $\{\pi(a)\xi_0 : a \in A\}$ is dense in $\mathcal{H}$. Such a vector is called a *cyclic vector*.

Let $(\pi, \mathcal{H})$ be a continuous $*$-representation, and let $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Define μ on A by $\mu(a) = \langle \pi(a)\xi, \xi\rangle$. Then $\mu(a^*a) = \langle \pi(a^*a)\xi, \xi\rangle = \langle \pi(a)\xi, \pi(a)\xi\rangle \geq 0$. So μ is a positive linear functional. If A has unit and $(\pi, \mathcal{H})$ is non-degenerate then $\mu(1) = \|\xi\|^2 = 1$. We need to show that $\|\pi(a)\| \leq \|a\|$, so μ is a state, called a *vector state*.

For GNS from μ, what is the vector state for ξ_0? $\langle a\xi_0, \xi_0\rangle_\mu = \langle a, 1\rangle_\mu = \mu(1^*a) = \mu(a)$. Thus μ is the vector state for the cyclic vector ξ_0.

$\mu(1_A) = 1$.

Every state on a unital $*$-normed algebra is a vector state for a representation, namely its GNS representation.