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1.1 Some K theory

It's been asked that we define K_1.

When $\phi : A \to B$, we get a map $K_0(A) \xrightarrow{\phi} K_0(B)$, because if $[\Xi_1] - [\Xi_2] \in \ker(\phi)$, then $[\phi(\Xi_1)] \sim [\phi(\Xi_2)]$ in $K_0(B)$.

On the other hand, given an isomorphism $\phi(\Xi_1) \cong \phi(\Xi_2)$ over B, one can ask whether we can lift this to an isomorphism over A between Ξ_1 and Ξ_2. What this comes down to is whether given an invertible element S of $M_n(B)$, is there an invertible element T of $M_n(A)$ so that $\phi(T) = S$. I.e. "can you lift invertible elements?" We're asking to what extent the map $GL_n(A) \xrightarrow{\phi} B$ is onto. More or less, vaguely, K_1 measures the invertible elements that cannot be lifted. This is a very vague statement.

Let's make it more precise. We look for universally liftable elements of $GL_n(A)$ (which was the B up above). We want $\phi : A \to B$ to be onto, and for the moment these are unital algebras without topology. Let's give some examples:

$$\begin{pmatrix} 1 & & 0 & \\ & 1 & & r_{ij} \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

These clearly can all be lifted, since $A \to B$ is onto, and is invertible for any single value r_{ij}. Call the (normal) subgroup generated by such things $El_n(A)$: then

$$GL_n(A)/El_n(A) \to GL_{n+1}(A)/El_{n+1}(A) \to \cdots \to \text{limit} = GL_\infty(A)/El_\infty(A)$$

under

$$T \mapsto \begin{pmatrix} T & \\ & 1 \end{pmatrix}$$

and

$$GL_\infty(A) = \begin{pmatrix} \boxed{\text{invertible}} & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

and we can look at the image of $El_n(A)$ in $GL_{2n}(A)$, which sits in $[GL_{2n}(A), GL_{2n}(A)] \subseteq El_{4n}(A)$. Then we define

$$K_1^{\text{alg}}(A) \stackrel{\text{def}}{=} GL_\infty(A)/[GL_\infty(A), GL_\infty(A)]$$

The denominator is the commutator subgroup, so this is abelian.

There is no good algebraic definition of K_1 for non-unital algebras. One way to do it is to use an ideal $J \leq A$, and then writing down the sequence

$$K_1^{\text{alg}}(J, A) \to K_1^{\text{alg}}(A) \to K_1^{\text{alg}}(J)$$

but it becomes even harder to get K_2, etc., indeed someone won a Fields Medal for such stuff.

For unital Banach algebras, again we look for universally liftable elements of $M_n(A)$. If $T \in M_n(A)$ with an appropriate Banach norm on $M_n(A)$, and if $\|T - \mathbb{1}\| < 1$, then we can use holomorphic functional calculus to define $\log(T) = S$. Then $T = e^S$, and any e^S is liftable, because S is just some matrix and we have a Banach homomorphism that's onto. So everything close to $\mathbb{1}$ is universally liftable; this is an open neighborhood of $\mathbb{1}$ in the group of invertible elements. And the point is that the connected component of $\mathbb{1}$ in $GL_n(A)$ is algebraically generated by any open neighborhood of the identity. Thus everything in $GL_n^0(A)$ is universally liftable. So in this context we define the topological K_1 by the sequence of discrete groups:

$$GL_n(A)/GL_n^0(A) \to GL_{n+1}(A)/GL_{n+1}^0(A) \to \cdots \to GL_\infty(A)/GL_\infty^0(A)$$

and, of course, $[GL_\infty, GL_\infty] \subseteq GL_\infty^0$. What happens is that we're deviding out by more: $K_1^{\text{alg}} \twoheadrightarrow K_1^{\text{top}}$. And

$$K_1(\text{non-unital } A) = \ker\Big(K_1(\tilde{A}) \to K_1(\text{field})\Big)$$

Then we have the famous

Bott periodicity theorem: If we are over $\mathbb{C}$, then $K_2(A) \cong K_0(A)$.

So we don't have to worry about K_2 and higher. The surprise is that the following six-term sequence is exact everywhere:

$$\begin{tikzcd}
 & J & A & A/J \\
K_1 & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[d] \\
K_0 & \bullet \arrow[u] & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}$$

Over $\mathbb{R}$, the iso is $K_8(A) \cong K_0(A)$, because you get tied up in quaternions and Clifford algebras.

Good questions: if G is discrete and we take $C^*(G)$ or $C_r^*(G)$, what are the K-groups of these? By now there is a large literature using non-commutative geometry in intense ways, e.g. Dirac operators, to answer those questions at least for large classes of groups in a way that you could imagine you might be able to actually compute these. Part of the difficulty is figuring out what all the projective modules over these, e.g. $\mathbb{Z}^10$ no one has in an effective way shown how to list all of the projective modules over the commutative 10-torus.

1.2 Return to tori and projective modules

For $\widehat{\mathbb{Z}^2} = T^2$, we have a commutative C^*-algebra $A = C(T^2)$, where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then we skip the proofs, and have:

$$\Xi(q,a) \stackrel{\text{def}}{=} \{\xi \in C(\mathbb{R}^2 \to \mathbb{C}) : \xi(s+q,t) = \xi(s,t),\, \xi(s,t+1) = e^{2\pi i a s}\xi(s,t)\}$$

Theorem: Every projective module over $C(T^2)$ is a free module or isomorphic to a $\Xi(q,a)$. And when $a \geq 0$, the $\Xi(q,a)$ are all isomorphic.

This does not give a particularly good clue how to deal with non-commutative tori. We have $\mathbb{Z}^d$ and a matrix $\theta \in M_d(\mathbb{R})$, and we form A_θ as before in terms of the bicharacter c_θ.

In any case, $\mathbb{Z}^d$ fits inside A_θ, not comfortably as a subgroup, because of the twisting, but as a subgroup. And precisely this means that a projective module will give a "c_θ-projective representation of $\mathbb{Z}^d$", although we don't have a Hilbert space. (This is a way of thinking of this stuff in hindsight.) We can look for c_θ-projective representations, and there aren't a lot of ways to construct these:

Let M be a locally compact Abelian group, and $\hat{M}$ its dual group. Let $G = M \times \hat{M}$; then on $L^2(M)$ we have

$$(\pi_{(x,s)}\xi)(y) \stackrel{\text{def}}{=} \langle y, s\rangle\, \xi(y-x)$$

the "Schrodinger representation." Then π is a projective representation of G on this Hilbert space, with bicharacter β (easily enough computed).

Strategy:

- Find embeddings of $\mathbb{Z}^d$ into $M \times \hat{M}$ such that $\beta|_{\mathbb{Z}^d} = c_\theta$.
- If $\mathbb{Z}^d$ is a lattice in $M \times \hat{M}$, restrict attention to $C_c(M)$; then this leads to a projective module.

The difficulty: this gives zillions of projective modules, and it's hard to figure out when two such things are isomorphic.