**This document was last updated on April 30, 2008. A more recent version may be available as part of http://math.berkeley.edu/∼theojf/CstarAlgebras.pdf.**

# 1: April 30, 2008

Last time we defined a *standard module frame*:

**Theorem:** $A$ is a unital $C^*$-algebra (or smooth subalgebra), and $\Xi$ a right $A$-module equipped with $\langle , \rangle_A$ an $A$-valued inner product. If $\Xi$ has a (finite) standard module frame $\{\eta_j\}_{j=1}^n$ — i.e. the sum of the corresponding rank-one operators $\sum \langle \eta_j, \eta_j \rangle_0 = \mathbb{1}_\Xi$, or equivalently there's a reconstruction formula $\xi = \sum \eta_j \langle \eta_j, \xi \rangle_A$ — then $\Xi$ is projective, and in fact "isometric" to a direct summand of $A^n$ (viewed as a right module), and $\Xi$ is self-adjoint for $\langle , \rangle_A$.

**Proof:**

(Because of our conventions with left and right, everything is simple; using other conventions makes for painful bookkeeping.) Define $\Phi : \Xi \to A^n$ by $\xi \mapsto (\langle \eta_j, \xi \rangle)_{j=1}^n$. It's clear that $\Phi$ is an $A$-module homomorphism. Furthermore, by the reconstruction formula, this is injective. So $\Xi$ is (equivalent to) a submodule of $A^n$, and to show it's projective, we need to show it's a summand. This consists of displaying a projection $A^n \twoheadrightarrow \Xi$. Let $P \in M_n(A)$ the matrix algebra, acting on $A^n$ from the left. Let $P$ be given by $P_{jk} = \langle \eta_j, \eta_k \rangle_A$. Then

$$\begin{aligned}
(P^2)_{ik} &= \sum_j P_{ij} P_{jk} \\
&= \sum_j \langle \eta_i, \eta_j \rangle_A \langle \eta_j, \eta_k \rangle_A \\
&= \left\langle \eta_i, \sum_j \eta_j \langle \eta_j, \eta_k \rangle_A \right\rangle_A \\
&= \langle \eta_i, \eta_k \rangle = P_{ik}
\end{aligned}$$

Moreover, $(P^*)_{ij} = (P_{ji})^* = \langle \eta_j, \eta_i \rangle^* = \langle \eta_i, \eta_j \rangle = P_{ij}$. So we have a self-adjoint projection. But

$$\begin{aligned}
\left(P(\Phi\xi)\right)_j &= \sum_k P_{jk} (\Phi\xi)_k \\
&= \sum \langle \eta_j, \eta_k \rangle_A \langle \eta_k, \xi \rangle_A \\
&= \left\langle \eta_j, \sum_k \eta_k \langle \eta_k, \xi \rangle \right\rangle \\
&= \langle \eta_j, \xi \rangle = (\Phi\xi)_j
\end{aligned}$$

So $P\Phi = \Phi$, so $\Phi(\Xi)$ is contained in the range of $P$. On the other hand, if $v \in A^n$ and if $v \in \text{range}(P)$, so $Pv = v$, then

$$\begin{aligned} v_j &= (Pv)_j \\ &= \sum_k \langle \eta_j, \eta_k \rangle v_k \\ &= \left\langle \eta_j, \underbrace{\sum_k \eta_k v_k}_{\xi} \right\rangle \\ &= (\Phi\xi)_j \end{aligned}$$

So $P$ is the self-adjoint projection onto $\Phi(\Xi)$. And

$$A^n = \underbrace{P(A^n)}_{\cong \Xi} \oplus (1-P)(A^n)$$

For isometricity, we use the standard inner-product $\langle a, b \rangle_A = \sum_j a_j^* b_j$ on $A^n$. Then

$$\begin{aligned} \langle \Phi\xi, \Phi, \zeta \rangle_A &= \sum (\Phi\Xi)_j^* (\Phi\zeta)_j^* \\ &= \sum \langle \xi, \eta_j \rangle \langle \eta_j, \zeta \rangle \\ &= \left\langle \xi, \sum \eta_j \langle \eta_j, \zeta \rangle \right\rangle \\ &= \langle \xi, \zeta \rangle \end{aligned}$$

And last to prove is self-adjointness: Let $\phi \in \text{Hom}_A(\Xi, A_A)$. Then for $\xi \in \Xi$, we have

$$\begin{aligned} \phi(\xi) &= \phi\left(\sum \eta_j \langle \eta_j, \xi \rangle\right) \\ &= \sum \phi(\eta_j) \langle \eta_j, \xi \rangle \\ &= \left\langle \underbrace{\sum \eta_j \phi(\eta_j)^*}_{\in \Xi}, \xi \right\rangle \quad \square \end{aligned}$$

**Question from the audience:** This $P(A)$ is closed, because it's continuous. So any pre-Hilbert module is a Hilbert module? **Answer:** Certainly if $A$ is $C^*$, yes. But all of this works for any $*$-subalgebra of a $C^*$-algebra.

The smooth algebra are spectrally invariant: if $A^\infty \subseteq A$ a $C^*$-algebra, and if $a \in A^\infty$ and $a$ is invertible in $A$, then $a$ is invertible in $A^\infty$. This implies that the spectrum of $a$ in $A$ agrees with that in $A^\infty$. Hence, we have a good notion of positivity. **E.g.** $C(T) \supseteq C^\infty(T) \supseteq$ trigonometric polynomials; the MHS is spectrally invariant in the LHS, but the RHS is not, even though it's dense.

Let $Q$ be any self-adjoint projection in $A^n$, and let $\Xi = Q(A^n)$. Let $\{e_j\}$ be the standard basis for $A^n$, and $\eta_j = Qe_j$. Then $\{\eta_j\}$ is a standard module frame for $\Xi$. Some cultural remarks: Let $\mathcal{H}$ be an $\infty$-dim Hilbert space and $Q \in \mathcal{B}(\mathcal{H})$ be a self-adjoint projection, and let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Set $\eta_j = Qe_j$; then $\{\eta_j\}$ is a "normalized frame" for $Q\mathcal{H}$, in the sense that we have a (convergent) reconstruction formula:

$$\xi = \sum^{\infty} \eta_j \langle \eta_j, \xi \rangle_{\mathcal{H}}$$

Conversely, given a Hilbert space and a normalized frame, then there exists a bigger Hilbert space so that the frame is the projection of an orthonormal basis.

Let $A = C(T)$ be the continuous functions on the circle $T = \mathbb{R}/\mathbb{Z}$; so $A$ is the 1-periodic functions on $\mathbb{R}$. Then simplest non-trivial vector bundle is the Möbius strip:

$$\{\xi \in C_{\mathbb{R}}(\mathbb{R}) \text{ s.t. } \xi(t-1) = -\xi(t)\}$$

This is not a free module. Define the inner product in the obvious way: $\langle \xi, \eta \rangle_A(t) = \xi(t)\eta(t)$. Then find a standard module frame. More generally, we can write

$$\Xi_p^- = \{\xi \in C_{\mathbb{R}}(\mathbb{R}) \text{ s.t. } \xi(t-p) = -\xi(t)\}$$

$$\Xi_p^+ = \{\xi \in C_{\mathbb{R}}(\mathbb{R}) \text{ s.t. } \xi(t-p) = +\xi(t)\}$$

Don't turn these in — there are no more problem sets — but try them anyway.

**Question from the audience:** Do you want a bar somewhere? **Answer:** No, over real numbers. Over the complex numbers, the Möbius bundle is trivial. Do that example too.