1 March 14, 2008

We have a problem set due today. Many have asked for more time; that is fine. We'd prefer a more complete paper on Monday over a less complete paper today.

1.1 Twisted convolution, approximate identities, etc.

We were sketching what happens when G is locally compact but not discrete. We looked at $C_c(G, A)$ the continuous functions (compact support) with values in A. We have (A, G, α) with α strongly continuous, and we look for covariant representations. For a covariant representation $\{\pi, U, \mathcal{H}\}$ and $f \in C_c(G, A)$, we set

$$\sigma_f \xi = \int \pi(f(x))\, U_x \xi \, dx$$

where dx is the left Haar measure. Then we define *twisted convolution*:

$$(f \star_\alpha g)(x) \stackrel{\text{def}}{=} \int f(y)\, \alpha_y\big(g(y^{-1}x)\big)\, dx$$

Then $\sigma_f \sigma_g = \sigma_{f \star_\alpha g}$ and $\|\sigma_f\| \leq \|f\|_1 \stackrel{\text{def}}{=} \int \|f(x)\|_A \, dx$.

Now we look at G and $C_c(G)$. In the discrete case, if A has an identity (and we're using $A = \mathbb{C}$), then $C_c(G)$ has an identity element, given by the δ function at the identity. But in the non-discrete ****indiscrete?**** case, any neighborhood has infinitely many points, so the Haar measure cannot give any point positive measure. In particular, we do not have an identity in $C_c(G), \star$. All this extends to $L^1(G, A)$ by uniform continuity.

We do have an approximate identity: Let $\mathcal{N}$ be a neighborhood base of 1_G. For $U \in \mathcal{N}$, choose (Uryssohn) $f_U \in C_c(G)$ with support in U, $f_U \geq 0$, and $\|f_U\|_1 = \int f_U = 1$. By strong continuity of α, $f_U \star_\alpha g$ is very close to g. Then this is an approximate identity of norm 1 for $L^1(G)$.

In the more general case, if e_λ is an approximate identity of norm 1 for A, then $\{e_\lambda f_u\}_{\lambda, U}$ is an approximate identity of norm 1 for $L^1(G, A)$.

We've been ducking an issue here.

$$\begin{aligned}
\sigma_f^* \xi &= \int (\pi(f(x)) U_x)^* \, \xi \, dx \\
&= \int U_x^* \pi(f(x)^*) \xi \, dx \\
&= \int U_{x^{-1}} \pi(f(x)^*) \xi \, dx \\
&= \int \alpha_{x^{-1}}(\pi(f(x))^*)\, U_{x^{-1}} \xi \, dx \\
&= \int \alpha_x\big(\pi(f(x^{-1})^*)\big)\, U_x \xi \, d(x^{-1})
\end{aligned}$$

But this isn't quite right. $f \mapsto \int f(x^{-1})\,dx$ is right-translation-invariant, not left-translation-invariant. The problem is that the left Haar measure need not be right-invariant.

Definition: G is *unimodular* if left Haar = right Haar.

E.g. Abelian groups, compact groups (not immediately obvious), discrete groups, semi-simple Lie groups, nilpotent Lie groups.

But there are many solvable Lie groups that are not unimodular. **E.g.** "$ax + b$" group of affine transformations of $\mathbb{R}$, for $a \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}$. (For more details, take Math 260.) This is the simplest nonabelian solvable Lie group, and it is not unimodular. Exercise: explicitly compute the left and right Haar measures; this gets into "What is Haar on $\mathbb{R}_{>0}$ with respect to $\times$?" — this expression, with respect to Lebesgue measure, pops up all over.

For a non-unimodular group G, we have $d(x^{-1}) = \Delta(x)dx$, where $\Delta(x)$ is the "modular function" of G. It's nice: it sends $\Delta : G \to \mathbb{R}_{>0}$ under a continuous group homomorphism. It's not hard to show this, but we will not. One has to make a convention, which is not always agreed upon; some people would use $\Delta(x^{-1})$. Well, if G happens to be compact, then there are very few continuous homomorphisms into the positive reals, because there are very few compact subgroups of $\mathbb{R}_{>0}$. Hence compact groups are unimodular.

So, in the non-unimodular case, we must define the involution as:

$$f^*(x) = \alpha_x\big(f(x^{-1})^*\big)\,\Delta(x^{-1})$$

At various cases, this complicates the bookkeeping, and even worse, there are some theorems that work for unimodular and do not work for non-unimodular groups (without becoming substantially more complicated). Whenever someone thinks they have a theorem for locally compact groups, they prove it for unimodular groups and then have to go back and check with the modular functions.

Question from the audience: What is a solvable Lie group? **Answer:** Up to discrete subgroups of the center, they are of the form:

$$\begin{pmatrix} * & & & * \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}$$

Nilpotent has 1s on the diagonal. You can always embed a nonunimodular group into a unimodular one by extending by a copy of $\mathbb{R}$: you let the real line act as modular automorphisms, and get a "Type II" algebra (meaning it has traces).

Later on, one very much wants to look at homogeneous spaces G/H, which is an extremely rich collection of manifolds. Since G acts on G/H, we can ask if there is a measure on G/H that is preserved by the G action. This wraps up the modular functions on G and on H; ultimately, the answer is nice, if a bit complicated.

So anyway, we have a $*$ algebra with approximate identity of norm 1. Are there covariant representations of (G, A, α). The operations are all arranged to that covariant representations give us $*$-representations of $C_c(G, A)$. We have the "induced representations" from representations of A. We did that for discrete groups; just replace sums by integrals with respect to Haar measure. This gives a nice class of representations, which are faithful on the algebra. We can define the *reduced* C^* algebra $C_r^*(A, G, \alpha) \stackrel{\text{def}}{=} A \times_\alpha^r G$, where the norm comes from just the induced representations. And we have the full algebra $C^*(A, G, \alpha) = A \times_\alpha G$, which can be different. Even if $A = \mathbb{C}$, we can have $C_r^*(G) \neq C^*(G)$; G is ammenable iff these are equal. **E.g.** $SL(n, \mathbb{R})$ is not ammenable. We always have a quotient map $C^*(G) \to C_r^*(G)$, so representations of C_r^* give representations of C^*. We call the ones that come this way *tempered*, but this is still a very active field of investigation. It even got into the newspapers: a huge calculation that made progress into finding the representations of E_8. We have essentially a complete list of the semisimple Lie algebras, or at least the real forms of them, but sorting out the representations is hard: we get into representations that are not on Hilbert spaces, or that are not unitary. So be warned: sometimes the word "tempered" is used for non-unitarizable representations.

Theorem: There exists a bijection between covariant representations of (A, G, α) (nondegenerate as representations of A) and non-degenerate representations of $C^*(A, G, \alpha)$.

Proof:

σ is nondegenerate iff π is.

We have the mapping on one direction.

A does not need to have an identity element, but think about the multiplier algebra $M(C^*(A, G, \alpha))$. Then G and A both sit inside: $G, A \hookrightarrow M$. So $C^*(A, G, \alpha)$, which sits inside as an essential ideal (from the problem set), so any representation of C^* extends to a representation uniquely of M (by problem set), and compose with $G, A \hookrightarrow M$, giving a strongly continuous and non-degenerate covariant pair. If we take its integral form, that's actually equal to the original representation. So we really get a bijection. $\square$