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1.1 We go through a careful computation

Recall, we're interested in $\mathbb{Z}^d$ and a cocycle c_θ, and we want projective modules.

The strategy: let M be a locally compact abelian group: $M = \mathbb{R}^m \oplus \mathbb{Z}^n \oplus F$ for F finite abelian. ****If F is torsion but not finite, what do we get?**** Then let $G = M \times \hat{M}$, where $\hat{M}$ is the dual group, $\hat{M} \cong \mathbb{R}^m \oplus T^m \oplus \hat{F}$, where $\hat{F} \cong F$, but not in a canonical way — none of these equalities is canonical. Hence $G \cong \mathbb{R}^{2m} \times \mathbb{Z}^n \times T^n \times F \times \hat{F}$.

Then G has a projective "Schrodinger" unitary rep on $L^2(M)$ with cocycle β. Then what we will attempt is to find embeddings (as closed subgroup) $\mathbb{Z}^d \hookrightarrow M \times \hat{M}$, such that $\beta|_{\mathbb{Z}^d} = c_\theta$. Then we can hope that $C_c(M) \subseteq L^2(M)$ gives a projective module. The condition will be that if $\mathbb{Z}^d$ is *cocompact* in G — i.e. $G/\mathbb{Z}^d$ is compact — then we do get a projective module. A necessary condition is that $2m + n = d$. This all, at least in the abelian case, defines a *lattice* in G.

This generates a whole bunch of projective modules. It's hard to tell, and we will not in this class, whether these modules are isomorophic; to sort them out requires a non-commutative Chern class. You can prove: if θ has an irrational entry, then every projective module is a direct sum of things like this. But if θ is entirely rational, then the situation is Morita-equivalent to the commutative ($\theta = 0$) case, and you do not get all of the modules in this way, just a lot of modules.

Ok, so we begin the calculation. Let $\left(\pi_{(x,s)}\xi\right)(y) \overset{\text{def}}{=} \langle y, s\rangle\, \xi(y-x)$ for $x, y \in M$, $s \in \hat{M}$. We find the cocycle:

$$\begin{aligned}
\left(\pi_{(x,s)}\pi_{(y,t)}\xi\right)(z) &= \langle z, s\rangle \left(\pi_{(y,t)}\xi\right)(z-x) \\
&= \langle z, s\rangle \langle z-x, t\rangle \xi(z-x-y) \\
\left(\pi_{(x+y,s+t)}\xi\right)(z) &= \langle z, s+t\rangle\, \xi(z-(x+y)) \\
\left(\pi_{(x,s)}\pi_{(y,t)}\xi\right)(z) &= \overline{\langle x, t\rangle} \left(\left(\pi_{(x+y,s+t)}\xi\right)(z)\right)
\end{aligned}$$

Hence we define

$$\beta((x,s),(y,t)) = \overline{\langle x, t\rangle}$$

which is not skew-symmetric. Using u, v for letters in $G = M \times \hat{M}$, we set

$$\pi_u^* \overset{\text{def}}{=} \beta(u,u)\pi_{-u}$$

Now, let D (e.g. $D \cong \mathbb{Z}^d$) be a discrete subgroup of $M \times \hat{M} = G$. Much of what we do works with any closed subgroup, which is fine, but everywhere where we'll have a sum, you'll need an integral.

We don't need that generality, so we skip it. In any case, let's restrict π to D, and we're not going to worry about how to match up β with c_θ. In any case, restrict β to D, and then π in D is a β-projective representation of D on $C_c(M) \subseteq L^2(M)$.

Another bookkeeping: we will use right-modules, since we need to consider endomorphisms (for us, acting from the left) in order to show projectivity. So let's make $C_c(M)$ into a right $C_c(D)$-module (a certain amount carries over to L^2; of course, $C_c(D)$ for D discrete is just the functions of finite support): for $\xi \in L^2(M)$ and $f \in C_c(D)$, we set

$$\xi \cdot f \stackrel{\text{def}}{=} \sum_{u \in D} (\pi_u^* \xi)\, f(u)$$

the $*$ makes it a right-action. Check that this works out:

$$\begin{aligned}
((\xi \cdot f) \cdot g) &= \sum_u \pi_u^* (\xi \cdot f)\, g(u) \\
&= \sum_u \pi_u^* \left(\sum_v (\pi_v^* \xi)\, f(v) \right) g(u) \\
&= \sum_{u,v} \pi_u^* \pi_v^* \xi\, f(v)\, g(u) \\
&= \sum_{u,v} (\pi_v \pi_u)^* \xi\, f(v)\, g(u) \\
&= \sum_{u,v} (\beta(v,u) \pi_{v+u})^*\, \xi\, f(v)\, g(u) \\
&= \sum \bar{\beta}(v,u) \pi_{v+u}^*\, f(v)\, g(u) \\
&= \sum \bar{\beta}(v,u-v) \pi_u^* \xi\, f(v)\, g(u-v) \\
&= \sum_u (\pi_u^* \xi) \underbrace{\sum_v f(v)\, g(u-v)\, \bar{\beta}(v,u-v)}_{f \star_{\bar{\beta}} g \text{ restricted to } D}
\end{aligned}$$

Ok, so this works for $\xi \in L^2$, but let's move in the C_c direction. So we let $A = (C_c(D), \star_{\bar{\beta}})$, and later complete to a C^* algebra. We leave Hilbert space: let $\Xi = C_c(M)$, later on completed. Let's pick the ordinary inner product $\langle , \rangle_{L^2}$ on $L^2(M)$ to be linear in the first variable. Define a "bundle metric", i.e. an A-valued inner product on Ξ, by:

$$\langle \xi, \eta \rangle_A (\underset{\in D}{u}) \stackrel{\text{def}}{=} \overline{\langle \xi, \pi_u^* \eta \rangle_{L^2(M)}} = \overline{\langle \pi_u \xi, \eta \rangle_{L^2(M)}} = \int_{y \in M} \overline{(\pi_u \xi)(y)}\, \eta(y)\, dy$$

If $u = (x,s)$, then

$$\int_{y \in M} \overline{(\pi_u \xi)(y)}\, \eta(y)\, dy = \int_M \overline{\langle y, s \rangle\, \xi(y-x)}\, \eta(y)\, dy$$

Now, if η and ξ are each of compact support on M, then for each s this is certainly of compact support in x. This is more interesting; we still have this s to deal with, and there's no reason why this should be of compact support in s. Ok, so the solution is that we need to take a bigger space: when $M = \mathbb{R}^m \times \mathbb{Z}^n \times F$, we need the *Schwartz space* $\mathcal{S}(M)$: all derivatives in the $\mathbb{R}^m$ direction should exist, and everything (including all derivatives) in the $\mathbb{R}^m$ and $\mathbb{Z}^n$ directions should vanish at infinity faster than any polynomial.

Lemma: If $\xi, \eta \in \mathcal{S}(M)$, then $\langle \xi, \eta \rangle_A \in \mathcal{S}(D)$.

We'll skip this proof. Anyway, so then this really does work the way you want:

$$\begin{aligned}
\langle \xi, \eta \cdot f \rangle_A(v) &= \sum_u \langle \xi, \eta \cdot \delta_u \rangle_A(v)\, f(u) \\
\langle \xi, \eta \cdot \delta_u \rangle_A(v) &= \langle \xi, \pi_u^* \eta \rangle_A(v) \\
&=
\end{aligned}$$