This document was last updated on April 28, 2008. A more recent version may be available as part of http://math.berkeley.edu/~theojf/CstarAlgebras.pdf.

1 April 28, 2008

1.1 More on vector bundles

We had been talking about vector bundles, in preparation of the non-commutative case. It will be more convenient to use right modules.

Definition: Let A be a unital C^*-algebra (or a "nice" $*$-subalgebra thereof), and Ξ a right A-module. An A-valued *inner product* on Ξ is a map

$$\langle \cdot, \cdot \rangle_A : \Xi \times \Xi \to A$$

satisfying "A-sesquilinearity" and "positivity":

(a) Bi-additivity **bilinearity over $\mathbb{Z}$**

(b) $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$

(c) $\langle \xi, \eta \rangle_A^* = \langle \eta, \xi \rangle_A$ (Hence $\langle \xi a, \eta \rangle_A = a^* \langle \xi, \eta \rangle_A$)

(d) $\langle \xi, \xi \rangle_A \geq 0$ (the notion of positivity requires something about C^*-algebras)

(e) Sometimes: $\langle \xi, \xi \rangle_A = 0$ implies $\xi = 0$

E.g. E a vector bundle over M compact, and $\Xi = \Gamma(E)$, $A = C(M)$. Then take the inner-product that's $\mathbb{C}$-linear in the second variable.

We say that Ξ is a "Hilbert C^*-module over A". **Question from the audience:** In order to use the name "Hilbert", shouldn't there be some sort of completeness? **Answer:** Yes. So perhaps above is a "pre-Hilbert module". We can set a norm

$$\|\xi\|_\Xi \stackrel{\text{def}}{=} \|\langle \xi, \xi \rangle_A\|_A^{1/2}$$

We can show this is a norm, and for Hilbert we need some sort of Cauchy-Schwarts condition. Our above example will be complete once you do all that.

Well, if you have a Hilbert space, it's common to discuss rank-one operators. Here we can do the analogous thing: Given $\xi, \eta \in \Xi$, we set $\langle \xi, \eta \rangle_0$ ($= \langle \xi, \eta \rangle_E$, for "endomorphism", but a different "E" than in the above example) to be the element of $\mathrm{End}_A(\Xi)$ defined by

$$\langle \xi, \eta \rangle_0 \zeta = \xi \langle \eta, \zeta \rangle_A$$

We write A-things on the right so that we can put endomorphisms on the left; then there is no crossing. The formalism works just like with rank-one operators.

We write $\mathcal{B}(\Xi)$ for the bounded operators for the above norm, except that sometimes operators don't have adjoints, and this is sad. Hence, we use:

- The *adjoint* (with respect to the norm $\langle,\rangle_A$) of an operator $T \in \mathrm{End}_A(\Xi)$ is an operator $S \in \mathrm{End}_A(\Xi)$ such that
$$\langle T\xi, \eta\rangle_A = \langle \xi, S\eta\rangle_A$$
for every $\xi, \eta \in \Xi$. If $\langle,\rangle$ is definite, then S is unique, and we write $S = T^*$.

- Then
$$\mathcal{B}(\Xi) \stackrel{\text{def}}{=} \{T \in \mathrm{End}_A(\Xi) : \|T\|, \|T^*\| < \infty\}$$

In any case, we see that $\langle \xi, \eta\rangle_0^* = \langle \eta, \xi\rangle_0$. If $T \in \mathcal{B}(\Xi)$, then $T\langle \xi, \eta\rangle_0 = \langle T\xi, \eta\rangle_0$. So "$\langle \cdot, \cdot\rangle_0$ is a $\mathcal{B}(\Xi)$-valued inner-product, where we consider Ξ as a left-module over $\mathcal{B}(\Xi)$."

E.g. In the above vector-bundle example, this works, and is appropriately continuous (the notion of continuity can be derived from a suitable open cover).

Question from the audience: Most of these notions are in your paper? **Answer:** Various papers, yes.**Question from the audience:** Do the rank-one operators form an ideal? **Answer:** No, you have to take sums. The rank-one operators span an ideal; denote its closure $\mathcal{K}(\Xi)$ for "compact": these are not compact in the usual range sense, but it's an extremely useful ideal. If $\mathrm{span}\langle \xi, \eta\rangle_A$ is dense in A (we never said how big a module we had; this means it's not tiny), then
$${}_{\mathcal{K}(A)}\Xi_A$$
is a Morita equivalence. ****Perhaps the left subscript should be $\mathcal{K}(\Xi)$? The board says $\mathcal{K}(A)$, which is a natural notion, as in the subsequent question.****

Question from the audience: Is this a simple ideal, topologically? **Answer:** No. For instance, take A, with the obvious right-action and inner product. If A is unital, then $\mathcal{K}(A) = A$, so you can't say much.

In our vector-bundle $E \xrightarrow{\pi} M$ example, we pick a cover $\mathcal{O}_j$ with partition-of-unity ϕ_j and trivialization $\pi^{-1}(\mathcal{O}_j) \cong \mathcal{O}_j \times \mathbb{C}^n$. Then pick unit vectors e_k of $\mathbb{C}^n$, and set $\zeta_k^j = \phi_j(x)\, e_k$. Then
$$T_j \stackrel{\text{def}}{=} \sum_k \left\langle \zeta_k^j(x), \zeta_k^j(x)\right\rangle_0 \geq c(x)\mathbb{1}$$
for $c(x) \neq 0$ if $\phi_j(X) \neq 0$. Then $T \stackrel{\text{def}}{=} \sum T_j$ has
$$T(x) = \sum T_j(x) \geq c(x)\mathbb{1} \geq \epsilon\mathbb{1}$$
where $c(x)$ is some always-positive function, and M is compact, hence $c(x) \geq \epsilon > 0$.

Now set $S(x) = T(x)^{-1/2}$, and $\eta_k^j = S\zeta_k^j$. Then
$$\sum_{j,k} \left\langle \eta_k^j, \eta_k^j\right\rangle_0 = \sum_{j,k} \left\langle S\zeta_k^j, S\zeta_k^j\right\rangle_0 = S\sum_{j,k} \langle \zeta_k^j, \zeta_k^j\rangle_0 S = STS = \mathbb{1}$$

so $\mathcal{K}(E)$ includes the identity operator.

Definition: Given unital C^*-algebra A and a right-module Ξ with $\langle,\rangle_A$. By a "standard module frame" for Ξ we mean a finite set $\{\eta_j\}$ of elements of Ξ such that $\mathbb{1}_\Xi = \sum_j \langle \eta_j, \eta_j \rangle_0$.

This is not entirely standard language, but is catching on. Some people think about infinite sums, with all their subsequent convergence questions. We've seen that any vector bundle over a compact space can receive an inner product with a standard module frame. In general, a frame has many more vectors than the dimension; nevertheless, frames are like bases, and are increasingly used in simple old Hilbert land.

Equivalent Definition: For any $\xi \in \Xi$,

$$\xi = \mathbb{1}_\Xi \xi = \sum \langle \eta_j, \eta_j \rangle_0 \xi = \sum \eta_j \langle \eta_j, \xi \rangle_A$$

which looks just like the reconstruction formula for a basis in finite-dimensional-land. This stuff is useful for, e.g., error-correction and signal processing.

Definition: Let R be a unital ring (possibly non-commutative). We will always deal with finitely-generated modules. A *free module* (right or left) over R is a (right- or left-) module isomorphic to R^n (as a right- or left-) module, for some n.

Question from the audience: Does finitely-generated assure a unique n? **Answer:** Absolutely not. E.g. $R = \mathcal{B}(\mathcal{H})$.

Definition: A *projective module* is a direct summand of a free module.

Next time:

Theorem: Let A be a unital C^*-algebra (or nice subalgebra), and Ξ, $\langle,\rangle$ a (Hilbert, but we don't so much need this, by finite-generated-ness) C^*-module over A. If Ξ has a standard module frame, then Ξ is a projective A-module and is self-dual for $\langle,\rangle_A$.

Corollary: (Swan's theorem)

For M a compact space and E a vector bundle, $\Gamma(E)$ is a projective $C(M)$-module (and conversely).