# Problem Set 2
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**A. Fields of $C^*$-algebras.**

**B. An important extension theorem.** *Prove that if $I$ is a $*$-ideal of a $*$-normed algebra $A$, and if $I$ has an approximate identity of norm one for itself, then every non-degenerate $*$-representation of $I$ extends uniquely to a non-degenerate representation of $A$.*

We begin with uniqueness, as that usually tells us how to do a construction. Recall that $(\phi, \mathcal{H})$ a representation on $I$ is non-degenerate if $\operatorname{span}\{\phi(j)\xi : j \in I, \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$. But then if continuous $T, S$ satisfy $T\phi(j)\xi = S\phi(j)\xi$ for every $j$ and $\xi$, then $T = S$; we can clearly strip the $\xi$ from this. So let $\sigma, \tau : A \to \operatorname{End}(\mathcal{H})$ be representations, with $\sigma|_I = \tau|_I$ non-degenerate. Then $\sigma(a)\sigma(j) = \sigma(aj) = \sigma(aj) = \tau(a)\tau(j) = \tau(a)\sigma(j)$ for every $a \in A$, $j \in I$, so $\sigma = \tau$ as functions on $A$.

Conversely, if $\phi : I \to \operatorname{End}(\mathcal{H})$ is non-degenerate (and continuous), and $I$ has an approximate identity $e_\lambda$, then $\phi(e_\lambda)\phi(j) = \phi(e_\lambda j) \underset{\lambda}{\longrightarrow} \phi(j)$ for every $j \in I$, so $\phi(e_\lambda) \underset{\lambda}{\longrightarrow} \mathbb{1}_{\mathcal{H}}$. We extend $\phi$ to $A$ by $\phi(a) \overset{\text{def}}{=} \lim_\lambda \phi(ae_\lambda) = \lim_\lambda \phi(e_\lambda a)$ (the last equality follows from some expeditious limit-swapping). The check that this is a homomorphism is immediate, and non-degeneracy is trivial since it is there restricted to $I$.

**C. The non-commutative Stone-Čech compactification.** *Let $M(A)$ denote the set of double centralizers of $A$, i.e. pairs $(S, T)$ of operators on $A$ so that for all $a, c \in A$, we have $S(ac) = S(a)c$, $T(ac) = aT(c)$, and $aS(c) = T(a)c$.*

1. *Using the example of $A$ as ideal in $B$ as motivation, define operations on $M(A)$ making it into an algebra, with a homomorphism of $A$ onto an ideal of $M(A)$.*

   If $(S, T)$ and $(S', T')$ are double centralizers, then by linearity $(S + S', T + T')$ is also a double centralizer, as is any product of $(S, T)$ by an element of the center $Z(A)$ of $A$. We can multiply double centralizers by $(S, T)(S', T') = (SS', T'T)$, where the inner multiplication is composition as operators. (This ordering agrees with the ordering in the example of $B$ acting on $A \subseteq B$ as an ideal.)

   For any $a \in A$, the pair of left- and right-multipliers $(L_a, R_a) : b \mapsto (ab, ba)$ is a double centralizer, and the addition and multiplication in $M(A)$ is such that we have a homomorphism $A \to M(A)$ by $a \mapsto (L_a, R_a)$. This is an ideal:

$$\begin{aligned}
(S,T)(L_a,R_a)(b) &= (SL_a, R_aT)(b) \\
&= (S(L_a(b)), R_a(T(b))) \\
&= (S(ab), T(b)a) \\
&= (S(a)b, aS(b)) \\
&= (L_{S(a)}, R_{S(a)})(b) \\
\text{hence } (S,T)(L_a,R_a) &= (L_{S(a)}, R_{S(a)}) \\
(L_a,R_a)(S,T)(b) &= (L_a(S(b)), T(R_a(b))) \\
&= (aS(b), T(ba)) \\
&= (T(a)b, bT(a)) \\
\text{hence } (L_a,R_a)(S,T) &= (L_{T(a)}, R_{T(a)})
\end{aligned}$$

Unless $A$ has elements $a \neq 0$ so that for every $b \in A$, $ab = ba = 0$, this map is injective.

2. *Show that if $A$ is a Banach algebra with approximate identity of norm one, and if we require $S$ and $T$ to be continuous (which actually is automatic), then $M(A)$ can be made into a Banach algebra in which $A$ sits isometrically as an essential ideal. Show that if $A$ is a $*$-Banach algebra, then its involution extends uniquely to make $M(A)$ a $*$-algebra. Note then that the theorem of problem B. above says that every nondegenerate $*$-representation of $A$ extends to $M(A)$.*

   If $A$ is Banach with two-sided approximate identity $\{e_\lambda\}$ of norm 1, then we take

$$\|(S,T)\| \stackrel{\text{def}}{=} \lim_\lambda \|S(e_\lambda)\|$$

   Let $\mu$ and $\lambda$ be two variables ranging over the index of the approximate identity. Then, assuming $S$ and $T$ commutative:

$$\begin{array}{ccccc}
\|e_\mu S(e_\lambda)\|_A & \xrightarrow[\mu]{} & \|S(e_\lambda)\|_A & \xrightarrow[\lambda]{} & \|S\|_{M(A)} \\
\| & & & & \\
\|T(e_\mu)e_\lambda\|_A & \xrightarrow[\lambda]{} & \|T(e_\mu)\|_A & \xrightarrow[\mu]{} & \|T\|_{M(A)}
\end{array}$$

   This justifies the choice in the definition of the norm on $M(A)$. We also have the Banach condition:

$$\begin{aligned}
\|SS'\| &= \lim_\lambda \|S(S'(e_\lambda))\|_A = \lim_\lambda \lim_\mu \|S(e_\mu S'(e_\lambda))\|_A = \lim_\lambda \lim_\mu \|S(e_\mu)S'(e_\lambda)\|_A \\
&\leq \lim_\lambda \lim_\mu \|S(e_\mu)\|_A \|S(e_\lambda)\|_A = \|S\|\|S'\|
\end{aligned}$$

   Moreover, under $a \mapsto (L_a, R_a)$, we have $\|L_a\| = \lim_\lambda \|ae_\lambda\|_A = \|a\|_A$, so $A \hookrightarrow M(A)$ isometrically. We observe that $S(e_\lambda)a \xrightarrow[\lambda]{} S(a)$ and $aT(e_\lambda) \xrightarrow[\lambda]{} T(a)$.

We must check that $M(A)$ is closed under this norm. But if $(S_n, T_n)$ is a Cauchy sequence, then for any $a$ for large enough $\lambda$, $\|(S_m - S_n)a\| \approx \|(S_m(e_\lambda) - S_n(e_\lambda))a\| \leq \|S_m(e_\lambda) - S_n(e_\lambda)\|\|a\| \approx \|S_m - S_n\|\|a\|$. Hence $S_n(a)$ is a Cauchy sequence, as is $T_n(a)$, and it is easy to check that $(S_\infty, T_\infty) : a \mapsto (\lim_n S_n(a), \lim_n T_n(a))$ is a double centralizer.

When $A$ is a $*$-algebra, we define $S^* : b \mapsto (S(b^*))^*$; then we have an involution on $M(A)$ given by $(S,T)^* \overset{\text{def}}{=} (T^*, S^*)$. It's trivial to check that this is also a double centralizer, and that, if $A$ is $*$-Banach, then $\|S\| = \|S^*\|$.

3. *Show that if $A$ is a $C^*$-algebra, then so is $M(A)$.*

   We have only to check the $C^*$ identity that $\|x^*x\| = \|x\|^2$. But

$$\begin{aligned}\|(S,T)^*(S,T)\| = \|(T^*S, TS^*)\| &= \lim_\lambda \|T^*(S(e_\lambda))\| = \lim_\lambda \lim_\mu \|T^*(e_\mu S(e_\lambda))\| \\ &= \lim_\lambda \lim_\mu \|T^*(e_\mu) S(e_\lambda)\|\end{aligned}$$

   Now, $T^*(e_\mu) = (T(e_\mu^*))^*$, and $e_\nu^*$ is a two-sided approximate identity, so $S(e_\lambda) = \lim_\nu e_\nu^* S(e_\lambda) = T(e_\nu^*)e_\lambda$. So, using the $C^*$ identity in $A$, and taking a limit along $\mu = \nu$:

$$\begin{aligned}\|(S,T)^*(S,T)\| &= \lim_\lambda \lim_\mu \|T^*(e_\mu)S(e_\lambda)\| = \lim_{\lambda,\mu,\nu} \|(T(e_\mu^*))^*\, T(e_\nu^*)e_\lambda\| \\ &= \lim_\mu \|(T(e_\mu^*))^*\|\|T(e_\mu^*)\| = \|T^*\|\|T\|\end{aligned}$$

4. *Let $A$ be a $C^*$-algebra, and let $X = A_A$ as a right $A$-module, with $A$-valued inner product as defined in class. Let $B_A(X)$ be the algebra of all continuous (which actually is automatic) $A$-module endomorphisms of $X$ that have a continuous enomorphism as adjoint for the $A$-valued inner product (which is not automatic). Show that in a very natural way $M(A) = B_A(X)$.*

   Ignoring issues of continuity, this is exactly an unpacking of definitions. The inner product on $X$ is that $\langle a, b\rangle \overset{\text{def}}{=} ab \in A$; then $S \in B_A(X)$ if it is (1) a module map, i.e. $S(ab) = S(a)b$, and (2) if there is a left-adjoint $T$, i.e. $aS(b) = \langle a, S(b)\rangle = \langle T(a), b\rangle = T(a)b$. Thus, $S \in B_A(X)$ with adjoint $T$ if and only if $(S,T) \in M(A)$, and the multiplication agrees in the two settings.

5. *For $A$ a $C^*$-algebra, show that if $B$ is any $C^*$-algebra in which $A$ sits as an essential ideal, then $B$ can be identified as a subalgebra of $M(A)$, so $M(A)$ is maximal in this sense.*

   If $A \hookrightarrow B$ as an ideal, then $b \in B$ acts from the right and left as endomorphisms of $A$: i.e. $b \mapsto (L_b, R_b)$ is a homomorphism $B \to M(A)$. If $A$ is an essential ideal of $B$, then there is no $b \in B$ so that $L_b$ and $R_b$ both act as the zero map on $A$, and so this homomorphism $B \to M(A)$ is an injection.

6. *Determine $M(A)$ when $A = C_\infty(X)$, and when $A = \mathcal{B}_0(\mathcal{H})$, the algebra of compact operators on a Hilbert space $\mathcal{H}$.*

   When $A = C_\infty(X)$, we expect $M(A) = C(\beta X)$, where $\beta X$ is the Stone-Čech compactification of $X$. Indeed, since $M(A)$ is unital (we take $1_{M(A)} = (\mathbb{1}_A, \mathbb{1}_A)$ to be the pair of identity operators on $A$). If $A$ is commutative (and, say, Banach with approximate identities) then, since $S(ab) = S(a)b = bS(a) = T(b)a = aT(b) = T(ab)$, so $S = T$ and $(S,S)(S',S') = (SS', S'S)$, so $SS' = S'S$ and $M(A)$ is also commutative. Thus, if $A = C_\infty(X)$, then $M(A)$ is commutative with unit, so it's $C(Y)$ for some compact $Y$, and the universal property in the previous subproblems translates to the universal property of $\beta X$. We can describe this algebra another way: $M(C_\infty(X)) = C(\beta(X)) = C_b(X)$ is the algebra of bounded functions on $X$.

   As for the $A = \mathcal{B}_0(\mathcal{H})$ case: $\mathcal{B}_0(\mathcal{H}) \hookrightarrow \mathcal{B}(H)$ (all bounded operators) as an essential ideal, and this later algebra has a unit. Thus $\mathcal{B}(\mathcal{H}) \subseteq M(\mathcal{B}_0(\mathcal{H}))$. Conversely, a continuous centralizing pair on $\mathcal{B}_0(\mathcal{H})$ is determined by its action on the rank-1 operators, which are in nice correspondence with $\mathcal{H}$, so $M(\mathcal{B}_0(\mathcal{H})) \subseteq \mathcal{B}(\mathcal{H})$.

## D. Morphisms.

1. *Give a characterization of those homomorphisms from $C_\infty(Y)$ to $C_b(X)$ which arise from maps from $X$ to $Y$ ("morphisms"). Your characterization should be phrased so that it makes sense for non-commutative $C^*$-algebras. (Hint: recall the definition of a representation being non-degenerate.)*

   Given $Y \xleftarrow{\phi} X$, the usual pull-back map $f \overset{\phi}{\longmapsto} f \circ \phi$ on functions takes bounded functions to bounded functions. Thus, the map in the exercise is in fact $C_\infty(Y) \hookrightarrow C_b(Y) \to C_b(X)$. (Of course, the example of $[0,1] \hookleftarrow (0,1)$ shows that we cannot achieve $C_\infty(X)$ as the range of any such pull-back, except if $X$ is compact.) From the previous exercise, we recognize $C_b(X) = C(\beta X) = M(C_\infty(X))$ is the algebra of functions on the Stone-Čech compactification of $X$. Of course, by the universal property, any map $Y \leftarrow X$ extends uniquely to a map $\beta Y \leftarrow \beta X$. In any case, to understand homomorphisms $C_\infty(Y) \to C_b(X)$ that arise from continuous maps, we need only to understand them in the compact case.

   When $X$ is compact, we have the following dictionary, via Algebraic Geometry, between $X$ and $C(X)$:

   | Topology | Algebra |
   |---|---|
   | closed set $S \subseteq X$ | primary ideal $\mathcal{J}_S \overset{\text{def}}{=} \{f \text{ s.t. } f\|_S = 0\} \subseteq C(X)$ |
   | point $x \in X$ | maximal ideal $\mathcal{J}_x \subseteq C(X)$ |

   Then $C(Y) \to C(X)$ is a map $Y \leftarrow X$ of points only if the inverse image of a maximal ideal is maximal. In this case, a partition-of-unity argument shows that any function of points $Y \leftarrow X$ that takes all continuous functions to continuous functions $C(Y) \to C(X)$ is automatically continuous.

In the non-commutative case, the correct generalization of "maximal" (i.e. the kernel of an irreducible one-dimensional representation) is "primitive" (the kernel of an irreducible representation), so we define a *morphism* from $A$ ($C^*$ with unit) to $B$ ($C^*$, probably non-unital) to be a homomorphism $B \to A$ that extends to a homomorphism $M(B) \to A$, for which the inverse image of any primitive ideal is primitive. If $A$ is not unital, then by passing to $M(A)$ we've described maps into the Stone-Čech compactification. Pushing functions forward is hard, so I'm not likely to do better.

2. *For the non-commutative case explain how to compose morphisms.*

   The condition that "the pull-back of an orange ideal is orange" is preserved under composition of homomorphisms, for any value of "orange".