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Recall we have A_θ and the "dual" action α of T^d. A_θ has no proper α-invariant ideals.

Theorem: The rep π of A_θ on $\ell^2(\mathbb{Z}^d)$ (the GNS rep of the (unique) tracial state)

$$\pi(f)\xi \stackrel{\text{def}}{=} f \star_{c_\theta} \xi$$

is faithful, i.e. kernel $= 0$.

Proof:

Slogan: "α is unitarily implemented on $\ell^2(\mathbb{Z}^d)$." I.e. there is a unitary representation W of T^d on $\ell^2(\mathbb{Z}^d)$ such that, for $x \in T^d$:

$$\pi(\alpha_x(a)) = W_x \pi(a) W_x^* \tag{1}$$

What this is saying is that $(\pi, W, \ell^2(\mathbb{Z}^d)$ is a covariant rep for (A_θ, T^d, α).

If so (we haven't justified the above yet), then if $a \in \ker \pi$, then $\alpha_x(a) \in \ker$ for all $x \in T^d$, so kernel is α-invariant. But the kernel is not the whole algebra — there are non-zero operators — then by last time, $\ker = 0$.

Ok, so for unitary equivalence, set:

$$\left(W_x \xi\right)(m) \stackrel{\text{def}}{=} \langle m, x\rangle \xi(m)$$

This is almost the same formula as for α: $\left(\alpha_x(f)\right)(m) \stackrel{\text{def}}{=} \langle m, x\rangle f(m)$. From these, it's an easy exercise to sort out the slogan (1). $\square$

Well, so, from before, $U_m U_n = c_\theta(m,n) U_{m+n}$, and $U_m^* = c_\theta(m,n) U_{-m}$. These are unitary generators of the algebra; we can multiply each by a complex number of modulus 1, and we'll still have unitary generators. So, set

$$V_m \stackrel{\text{def}}{=} c_{\theta/2}(m,m) U_m = e^{2\pi i m \cdot \frac{\theta}{2} n} U_m$$

Claim:

- $V_m^* = \left(c_{\theta/2}(m,m) U_m\right)^* = \overline{c_{\theta/2}}(m,m)\, c_\theta(m,m)\, U_{-m} = c_{\theta/2}(-m,-m)\, U_{-m} = V_{-m}$
- $V_m V_n = c_\zeta(m,n) V_{m+n}$, where $\zeta = \left(\theta - \theta^t\right)/2$ is the skew-symmetric. What's going on is that the cocycle c_θ is homologous to c_ζ.

We're using the fact that in T^d everything has a square root. This is not always the case for dual groups, e.g. of finite groups. We won't check the second fact in the claim on the board. In any case, when convenient, we can always insist that θ be skew-symmetric. (Recall that θ is a real $d \times d$ matrix, and we can take it up to mod $\mathbb{Z}$.)

For a given n, consider the conjugation of A_θ by U_n.

$$\begin{aligned} U_n U_m U_n^* &= c_\theta(n,m) U_{m+n} c_\theta(n,n) U_{-n} \\ &= c_\theta(n,m) c_\theta(n,n) c_\theta(m+n,-n) U_m \\ &= e^{2\pi i(n \cdot \theta m - m \cdot \theta n)} U_m \\ &= \overline{c_{\theta-\theta^t}}(m,n) U_m \\ &\overset{\text{def}}{=} \rho_\theta(m,n) U_m \end{aligned}$$

Writing "$\sim$" for "homologous", we see that $\overline{\rho_\theta} = \left(c_{(\theta-\theta^t)/2}\right)^2 \sim c_\theta^2$. Then $\rho_\theta(\cdot,n) \in \widehat{\mathbb{Z}^d} \cong T^d$ is a character. And indeed

$$U_n U_m U_n^* = \alpha_{\rho_\theta(\cdot,n)}(U_m)$$

(We can turn things around and get rid of the complex conjugate sign.)

So, view ρ_θ as a map $\mathbb{Z}^d \to T^d$ by $n \mapsto \rho_\theta(\cdot,n)$. There's no reason whatsoever why the image should be a closed subset. Let $H_\theta = \overline{\{\rho_\theta(\cdot,n) : n \in \mathbb{Z}^d\}}$ be the closure of the image in T^d. So H_θ is a closed subgroup of T^d. Then there's a little taking duals: any closed subgroup has a connected component of the identity, and any connected closed subgroup is another torus stuck in skew-wise. So

$$H_\theta \cong T^e \times F$$

where $e \leq d$ and F is finite abelian. We like this version, because we have a compact group and we'd like to average over it, and we know how to do so on each piece.

Let J be any closed 2-sided ideal in A_θ. Then $\alpha_{\rho_\theta(\cdot,n)}(J) = U_n J U_n^* = J$. But α is continuous, and the $\rho_\theta(\cdot,n)$ are dense in H_θ, so $\alpha_x(J) = J$ for all $x \in H_\theta$. On $a \in A_\theta$, define the average

$$Q(a) = \int_{H_\theta} \alpha_x(a)\, dx$$

This is a conditional expectation, and $Q \geq 0$ and Q is faithful. Then $Q(J) \subseteq J$.

If $H_\theta = T^d$ (big if), then $Q(A_\theta) = P(A_\theta) = \mathbb{C}1$ from last time. So if J is not the zero ideal, still in the $_\theta = T^d$ case, then $1 \in J$. In sum:

Theorem: If $\{(\theta - \theta^t)(n) : n \in \mathbb{Z}^d\}$ is dense in $\mathbb{R}^d/\mathbb{Z}^d$, then A_θ has no proper ideals, i.e. is a simple C^* algebra. It's certainly unital and ∞-dimensional, but definitely not GCR. Nevertheless, we can write down many irreducible representations.