1 January 30, 2008

Question from the audience: Why are C^* algebras called that? **Answer:** They are $*$ algebras, and they are "closed". Von Neuman algebras are "W^*", because they're "weak-closed". C^* is also called "spectral". Much less poetic than "sea-star".

1.1 The positive cone

Last time we stated a big theorem:

Theorem: (Fukamya, 1952. Independently in 1953, simpler proof by Kelly and Vaight ****spell?****.)

Let A be a C^* algebra, $A^+ = \{a \in A \text{ s.t. } a = a^* \text{ and } \sigma(a) \in \mathbb{R}^+\}$. Then A^+ is a cone. I.e. if $a, b \in A^+$ then $a + b \in A^+$.

Key Lemma: For any commutative $C(M)$, M compact, $f \in C(M)$, $f = \bar{f}$ the following are equivalent:

(a) $f \geq 0$ (i.e. $\sigma(f) \in \mathbb{R}^+$).

(b) For some $t \geq \|f\|_\infty$, $\|f - t1\| \leq t$.

(c) For all $t \geq \|f\|_\infty$, $\|f - t1\| \leq t$.

Proof of Lemma:

(a $\Rightarrow$ c) Given $t \geq \|f\|_\infty$ and $m \in M$, $0 \geq f(m) - t \geq t$, so $|f(m) - t| \leq t$.

(c $\Rightarrow$ b) Obvious.

(b $\Rightarrow$ a) For each $m \in M$, $|f(m) - t| \leq t$, and $t \geq |f(m)|$. $\square$

Proof of Theorem:

Can assure $1 \in A$. Given $a, b \in A^+$, let $s = \|a\|$, $t = \|b\|$. So $\|a - s1\| \leq s$, $\|b - t1\| \leq t$. Then $\|a + b\| \leq \|a\| + \|b\| = s + t$. Then $\|a + b - (s + t)1\| \leq \|a - s1\| + \|b - t1\| \leq s + t$. So by Key Lemma, $\sigma(a + b) \in \mathbb{R}^+$. $\square$

Also, A^+ is norm-closed: Say we have $a_n \to a$, $a_n \in A^+$. Choose a $t \geq \|a_n\| \forall n$. so $\|a_n - t1\| \leq t$ for all n, and the expression is continuous in the norm, so $\|a - t1\| \leq t$. Moreover, $a_n = a_n^*$, so a is also self-adjoint, and hence $a \in A^+$.

Theorem: (Kaplansky)

If $c \in A$, then $c^* c \geq 0$.

Proof:

If not, we can find $b = b^*$ so that $bc^*c = c^*cb$ and $b^2c^*c \leq 0$, by taking a bump function. Then set $d = cb \neq 0$. Then $d^*d = b^*c^*cb = bc^*cb = b^2c^*c \leq 0$. I.e. we have $d \in A$ with $d^*d \leq 0, d \neq 0$.

Now we take real and imaginary parts: $d = h + ik$, $h, k \in A$ $h^* = h$, $k^* = k$. Then

$$d^*d + dd^* = (h + ik)(h - ik) + (h - ik)(h + ik) = 2(h^2 + k^2)$$

This is a sum of squares of self-adjoint elements, so is positive.

Ok, but $-d^*d \geq 0$, so $dd^* \geq 0$ is the sum of positive things. So $\sigma(dd^*) \in \mathbb{R}^+$, whereas $\sigma(d^*d) \in \mathbb{R}^-$.

Prop: Let A be an algebra over field F, and $a, b \in A$. Then $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.

E.g. Let $\mathcal{H}$ be an ∞-dim separable Hilbert space, and $\mathcal{K}$ and ∞-dim subspace, and let S be an isometry of $\mathcal{H}$ onto $\mathcal{K}$, then $S^*S = \mathbb{1}_{\mathcal{H}}$, so $\sigma S^*S = \{1\}$. But SS^* = orthogonal projection onto $\mathcal{K}$, so $\sigma(SS^*) = \{1, 0\}$.

Proof of Prop:

If $\lambda \in \sigma(ab)$, $\lambda \neq 0$, i.e. $(ab - \lambda 1)$ is not invertible, so $(\frac{a}{\lambda}b - 1)$ is not invertible. So it suffices to show: if $(ab - 1)$ is invertible, so is $(ba - 1)$. But formally

$$\begin{aligned}
(1 - ab)^{-1} &= 1 + ab + (ab)^2 + (ab)^3 \ldots \\
&= 1 + a\left(1 + ba + (ba)^2 + \ldots\right) b \\
&= 1 + a(1 - ba)^{-1} b
\end{aligned}$$

and we can check directly that these are inverses. $\square$

This completes the proof of the theorem. $\square$

A few final facts: the self-adjoint elements are differences of positive elements, and $A^+ \cap -A^+ = \{0\}$.