# $C^*$ Algebras

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## Contents

## 1: Introduction

These notes are from the class on $C^*$-algebras, taught by Prof. Marc Rieffel at UC-Berkeley in the Spring of 2008. The class meets three times a week — Mondays, Wednesdays, and Fridays — from 10am to 11am.

I typed these notes mostly for my own benefit — I wanted to try live-TEXing a class, and this one had fewer diagrams and more linear thoughts than my other classes. I do hope that they will be of use to other readers; I apologize in advance for any errors or omissions. Places where I did not understand what was written or think that I in fact have an error will be marked ****like this****. Please e-mail me (theojf@math.berkeley.edu) with corrections. For the foreseeable future, these notes are available at http://math.berkeley.edu/∼theojf/CstarAlgebras.pdf.

These notes are typeset using TEXShop Pro on a MacBook running OS 10.5. The raw TEX sources are available at http://math.berkeley.edu/∼theojf/Cstar.tar.gz. These notes were last updated May 12, 2008.

## 2: January 23–28, 2008

****I had not started taking notes until January 30, 2008. We began with some historical background, then set in on definitions. I may eventually find and type up notes from this first week.****

## 3: January 30, 2008

**Question from the audience:** Why are $C^*$ algebras called that? **Answer:** They are $*$ algebras, and they are "closed". Von Neuman algebras are "$W^*$", because they're "weak-closed". $C^*$ is also called "spectral". Much less poetic than "sea-star".

## 3.1 The positive cone

Last time we stated a big theorem:

**Theorem:** (Fukamya, 1952. Independently in 1953, simpler proof by Kelly and Vaight ****spell?****.)

Let $A$ be a $C^*$ algebra, $A^+ = \{a \in A \text{ s.t. } a = a^* \text{ and } \sigma(a) \in \mathbb{R}^+\}$. Then $A^+$ is a cone. I.e. if $a, b \in A^+$ then $a + b \in A^+$.

**Key Lemma:** For any commutative $C(M)$, $M$ compact, $f \in C(M)$, $f = \bar{f}$ the following are equivalent:

(a) $f \geq 0$ (i.e. $\sigma(f) \in \mathbb{R}^+$).

(b) For some $t \geq \|f\|_\infty$, $\|f - t1\| \leq t$.

(c) For all $t \geq \|f\|_\infty$, $\|f - t1\| \leq t$.

**Proof of Lemma:**

(a $\Rightarrow$ c) Given $t \geq \|f\|_\infty$ and $m \in M$, $0 \geq f(m) - t \geq t$, so $|f(m) - t| \leq t$.

(c $\Rightarrow$ b) Obvious.

(b $\Rightarrow$ a) For each $m \in M$, $|f(m) - t| \leq t$, and $t \geq |f(m)|$. $\square$

**Proof of Theorem:**

Can assure $1 \in A$. Given $a, b \in A^+$, let $s = \|a\|$, $t = \|b\|$. So $\|a - s1\| \leq s$, $\|b - t1\| \leq t$. Then $\|a + b\| \leq \|a\| + \|b\| = s + t$. Then $\|a + b - (s + t)1\| \leq \|a - s1\| + \|b - t1\| \leq s + t$. So by Key Lemma, $\sigma(a + b) \in \mathbb{R}^+$. $\square$

Also, $A^+$ is norm-closed: Say we have $a_n \to a$, $a_n \in A^+$. Choose a $t \geq \|a_n\| \forall n$. so $\|a_n - t1\| \leq t$ for all $n$, and the expression is continuous in the norm, so $\|a - t1\| \leq t$. Moreover, $a_n = a_n^*$, so $a$ is also self-adjoint, and hence $a \in A^+$.

**Theorem:** (Kaplansky)

If $c \in A$, then $c^*c \geq 0$.

**Proof:**

If not, we can find $b = b^*$ so that $bc^*c = c^*cb$ and $b^2c^*c \leq 0$, by taking a bump function. Then set $d = cb \neq 0$. Then $d^*d = b^*c^*cb = bc^*cb = b^2c^*c \leq 0$. I.e. we have $d \in A$ with $d^*d \leq 0, d \neq 0$.

Now we take real and imaginary parts: $d = h + ik$, $h, k \in A$ $h^* = h$, $k^* = k$. Then

$$d^*d + dd^* = (h + ik)(h - ik) + (h - ik)(h + ik) = 2(h^2 + k^2)$$

This is a sum of squares of self-adjoint elements, so is positive.

Ok, but $-d^*d \geq 0$, so $dd^* \geq 0$ is the sum of positive things. So $\sigma(dd^*) \in \mathbb{R}^+$, whereas $\sigma(d^*d) \in \mathbb{R}^-$.

**Prop:** Let $A$ be an algebra over field $F$, and $a, b \in A$. Then $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.

**E.g.** Let $\mathcal{H}$ be an $\infty$-dim separable Hilbert space, and $\mathcal{K}$ and $\infty$-dim subspace, and let $S$ be an isometry of $\mathcal{H}$ onto $\mathcal{K}$, then $S^*S = \mathbb{1}_{\mathcal{H}}$, so $\sigma S^*S = \{1\}$. But $SS^*$ = orthogonal projection onto $\mathcal{K}$, so $\sigma(SS^*) = \{1, 0\}$.

**Proof of Prop:**

If $\lambda \in \sigma(ab)$, $\lambda \neq 0$, i.e. $(ab - \lambda 1)$ is not invertible, so $(\frac{a}{\lambda}b - 1)$ is not invertible. So it suffices to show: if $(ab - 1)$ is invertible, so is $(ba - 1)$. But formally

$$\begin{aligned}(1-ab)^{-1} &= 1 + ab + (ab)^2 + (ab)^3 \dots \\ &= 1 + a\left(1 + ba + (ba)^2 + \dots\right)b \\ &= 1 + a(1-ba)^{-1}b\end{aligned}$$

and we can check directly that these are inverses. $\square$

This completes the proof of the theorem. $\square$

A few final facts: the self-adjoint elements are differences of positive elements, and $A^+ \cap -A^+ = \{0\}$.

## 4: February 1, 2008

Today we present a few simple consequences of this theorem of positivity. Notation: we say $a \leq b$ is $b - a \in A^+$ and $a$ and $b$ are self-adjoint.

**Prop:** If $0 \leq a \leq b$, then $\|a\| \leq \|b\|$.

**Proof:**

We can assume $1 \in A$. Then $b \leq \|b\|1$, by looking at $C^*(b, 1)$. I.e. $\|b\|1 - b, b - a \in A^+$, and add, so $\|b\|1 - a \in A^+$. Then $\|a\| \leq \|b\|$ by looking at $C^*(a, 1)$. $\square$

**Prop:** If $a \leq b$, then for any $c \in A$, we have

$$c^*ac \leq c^*bc$$

**Proof:**

$b - a \in A^+$ so $b - a = d^2$ for $d \geq 0$. Then $c^*bc - c^*ac = c^*(b-a)c = c^*d^2c = (dc)^*(dc) \geq 0$. $\square$

## 4.1 Ideals

We now turn our attention to two-sided ideals of $C^*$ algebras. These generally do not have identity elements; we will talk about approximate identities. Recall: for a normed algebra $A$ without 1, a *left approximate identity* for $A$ is a net $\{a_\lambda\}$ in $A$ such that $e_\lambda a \to a$ for all $a \in A$. These don't always exist — stupid counterexample is a Banach space with multiplication always 0. The notions of "right approximate identity" and "two-sided approximate identity" are obvious. We can also ask if our approximate identity is "bounded": does there exist a $k$ so that $\|e_\lambda\| \leq k$ for every $\lambda$. Similarly for "approximate identities of norm 1": $\|e_\lambda\| \leq 1 \forall \lambda$.

In the commutative case, $C_c(X)$ of fns with compact support is a dense ideal in $C_\infty(X)$. In the non-commutative case, e.g. $B_0(\mathcal{H})$ of compact operators, there are trace-class operators $\mathcal{L}^1$, Hilbert-Schmidt $\mathcal{L}^2$, etc. Alain Connes has advocated viewing compact operators, and specifically elements of these ideals, as infinitesimals. In our department, we have a leading expert: Voidjitski ****spell?****. These are non-closed ideals, which are important, but in this course, we will mostly focus on closed ideals.

**Theorem:** Let $L$ be a left ideal (not necessarily closed) in a $C^*$-algebra $A$. Then $L$ has a right approximate identity $\{e_\lambda\}$ with $e_\lambda \geq 0$ and $\|e_\lambda\| \leq 1$.

We can even arrange that if $\lambda \geq \mu$, then $e_\lambda \geq e_\mu$, but we will not take the time to show that. Davidson does this for closed ideals. Indeed, what he shows is that $\{a \in A^+, \|a\| < 1\}$ with the usual ordering really is a net, and that for $A$ a complete $C^*$ algebra without identity, then this is an approximate identity — closed ideals are complete $C^*$ algebras without identity.

If $L$ is separable, then we can have an approximate identity given by a sequence (no need for nets), although for us, we allow any cardinality.

**Proof of Theorem:**

We can assume that $A$ has an identity element.

Choose a dense subset $S$ of $L$ (e.g. $S = L$, or if $L$ is separable, we can take $S$ countable). Set $\Lambda =$ finite subsets of $S$ ordered by inclusion. For $\lambda = \{a_1, \dots, a_n\} \in \Lambda$, set $b_\lambda = \sum a_j^* a_j$. Well, $L$ is a left ideal, so this is in $L$, and is positive.

Well, look at $\left(\frac{1}{n}1 + b_\lambda\right)$, which is certainly a strictly-positive function, and hence invertible, so we define

$$e_\lambda = \left(\frac{1}{n}1 + b_\lambda\right)^{-1} b_\lambda \in L \cap A^+$$

Then checking norms of both multiplicands, we see that $\|e_\lambda\| \leq 1$.

Claim: this is a right approximate identity. First we show for $a \in S$, and then by boundedness and density, we will have the desired result. We examine

$$\begin{aligned} \|a - ae_\lambda\|^2 &= \|(a-ae_\lambda)^*(a-ae_\lambda)\| \\ (a-ae_\lambda)^*(a-ae_\lambda) &= (1-e_\lambda)^* a^* a (1-e_\lambda) \\ &\leq (1-e_\lambda)^* b_\lambda (1-e_\lambda) \\ &= \left(1 - \left(\frac{1}{n} + b_\lambda\right)^{-1} b_\lambda\right)^2 b_\lambda \\ &= \frac{1}{n} \frac{\frac{1}{n}}{\frac{1}{n}+b_\lambda} \frac{b_\lambda}{\frac{1}{n}+b_\lambda} \\ \|\text{RHS}\| &\leq \frac{1}{n} \end{aligned}$$

Where we have assumed that $a \in \lambda$, and hence $b_\lambda \geq a^* a$. $\square$

**Cor:** And $C^*$-algebra $A$ has a two-sided approximate identity, self-adjoint with norm 1.

**Proof:**

Let $\{e_\lambda\}$ be a right approximate identity. But it is self-adjoint, so it's a left approximate identity for $a^*$. $\square$

**Cor:** Let $I$ be a closed two-sided ideal in $A$. Then $I$ is closed under $*$, so it is a $C^*$ algebra.

**Proof:**

Let $\{e_\lambda\}$ be a right approximate identity, and look at $e_\lambda a^* - a^*$. Because it's a two-sided ideal, $e_\lambda a^* \in I$. But $\|e_\lambda a^* - a^*\| = \|ae_\lambda - a\| \to 0$, so since $I$ is closed, $a^* \in I$. $\square$

**Cor:** If $I$ is a closed two-sided ideal in $A$, and $J$ is a closed two-sided ideal in $I$, then $J$ is a closed two-sided ideal in $A$.

**Proof:**

Let $d \in J$ and $a \in A$; we want to show $ad \in J$. Well, take $\{e_\lambda\}$ an approximate identity for $I$. Then $\underbrace{\underbrace{ae_\lambda}_{\in I} d}_{\in J} \to ad$. $\square$

**Cor:** If $I, J$ are closed two-sided ideals in $A$, then $I \cap J = IJ \overset{\text{def}}{=}$ closed linear span of products.

**Proof:**

$\supseteq$ is clear. On the other hand, let $d \in I \cap J$. Consider $de_\lambda \in IJ$, where $\{e_\lambda\}$ is an approximate ideal for $J$. But $IJ$ is closed, so $de_\lambda \to d$ must be in $IJ$. $\square$

Next time: quotient spaces.

# 5: February 4, 2008

## 5.1 Quotient $C^*$ algebras

Let $A$ be a $C^*$ algebra, $I$ a closed two-sided ideal. We saw last time that $I$ is a $*$-subalgebra, hence (closed) a $C^*$ subalgebra, with an approximate identity. We form the quotient $A/I$, $a \mapsto \dot{a}$, an algebra with quotient norm

$$\|\dot{a}\| \stackrel{\text{def}}{=} \inf\{\|a - d\| \text{ s.t. } d \in I\}$$

Then $A/I$ is complete. (It is clearly a $*$-algebra.) Hence $A/I$ is a Banach $*$-algebra. (This is true for any $*$- two-sided ideal in a Banach $*$-algebra. To show that it is a $C^*$-algebra required verifying the inequality.

**Key Lemma:** Let $\{e_\lambda\}$ be a positive norm-1 approximate identity for $I$. Then for any $a \in A$,

$$\|\dot{a}\| = \lim_\lambda \|a - ae_\lambda\|$$

**Proof:**

We can assume that $A$ has an identity element. We can be more careful, and avoid this, but anyway....

Key $C^*$ fact: Look at $\|1 - e_\lambda\|$ for a given $\lambda$. $e_\lambda$ is self-adjoint, and look at $C^*(1, e_\lambda)$, so clearly $\|1 - e_\lambda\| \leq 1$ (not true in a general Banach algebra), using positivity, norm $\leq 1$, and that we are in $C^*$-land.

Well, $ae_\lambda \in I$, so certainly $\leq$ is clear in the Lemma. For $\geq$, let $\epsilon > 0$ be given. Then we can find $d \in I$ so that $\|\dot{a}\| + \epsilon \geq \|a - d\|$. Then

$$\|a - ae_\lambda\| = \|a(1 - e_\lambda)\| \leq \|(a-d)\underbrace{(1-e_\lambda)}_{\leq 1}\| + \|d(1-e_\lambda)\| \leq \|a - d\| + \underbrace{\|d - de_\lambda\|}_{\to 0}$$

$\square$

**Theorem:** (Segal, 1949)

$A/I$ is a $C^*$-algebra.

**Proof:**

From Banach-land, we have $\|\dot{a}^*\dot{a}\| \leq \|\dot{a}\|^2$. We also have

$$\begin{aligned}
\|\dot{a}\|^2 &= \lim \|a - ae_\lambda\|^2 = \lim \|a(1-e_\lambda)\|^2 \\
&= \lim \|(1-e_\lambda)a^*a(1-e_\lambda)\| \\
&\leq \lim \|a^*a(1-e_\lambda)\| \\
&= \|\dot{\overbrace{a^*a}}\| = \|\dot{a}^*\dot{a}\|
\end{aligned}$$

by general Banach algebra. $\square$

### 5.2 Beginnings of non-commutative measure theory

This concludes our mining of results that follow directly from the fact that commutative $C^*$ algebras are functions on spaces. We move to proving that general $C^*$ algebras are algebras of bounded operators on Hilbert spaces. What we have been doing depended importantly on completeness; our new topic will not. In the wild, we often find $*$-algebras satisfying the norm identity, but that are not complete. We complete for the nice framework, but the things you add in the completion are often weird, so it's better to work with just $*$-normed algebras.

**Definition:** For a $*$-algebra $A$ over $\mathcal{C}$, a linear functional $\mu$ on $A$ is *positive* if $\mu(a^*a) \geq 0$ for all $a \in A$.

**E.g.** the 0-functional.

**E.g.** $\mathcal{C}^2$, with $(\alpha, \beta)^* \stackrel{\text{def}}{=} (\bar{\beta}, \bar{\alpha})$, then there are no non-zero positive linear functionals.

**Definition:** For a $*$-normed-algebra $A$, we say that a positive linear functional $\mu$ is a *state* if $\|\mu\| = 1$.

This is the analog of a probability measure.

Let $A$ be a $*$-algebra and $\mu$ a positive functional. Define a sesquilinear form on $A$ by

$$\langle a, b \rangle_\mu = \mu(b^*a)$$

****Ew. We've made the order all backwards.**** You can go in any order, but this is what is most commonly done. Called the "GNS construction" (Gelfand, Naimar ****sp?****, and Segal).

We factor by vectors $n \in N$ of length 0 to get a (positive) inner-product on $A/N$. Then complete, and call this $\mathcal{L}^2(A, \mu)$. We would now like to get the operators.

For $a \in A$, we let $L_a \stackrel{\text{def}}{=} b \mapsto ab$. This is a left-regular representation, and it tries to be faithful. Then

$$\langle L_a b, c \rangle = \mu(c^*ab) = \langle b, L_{a^*} c \rangle$$

Thus, $a \mapsto L_a$ is a "*-representation". (We've swept under the rug various issues of completeness, etc.)

There are issues here: E.g. Let $A$ be all $\mathcal{C}$-valued polynomials on $\mathcal{R}$. Let $\mu(p) \stackrel{\text{def}}{=} \int_{\mathcal{R}} p(t) e^{-t^2} dt$. (The Gaussian goes to 0 at both ends so fast that this is finite for every polynomial.) Moreover, $\mu(p^*p) = \int |p(t)|^2 e^{-t^2} dt \geq 0$, so we have a genuine inner product on polynomials: $\langle p, q \rangle = \int \bar{q(t)} p(t) e^{-t^2} dt$, and we can complete with respect to this, and we get the usual $\mathcal{L}^2(\mathcal{R}, e^{-t^2} dt)$.

Now, we have the left-regular representation $p \mapsto L_p$, but $L_p$ is not a bounded operator! Indeed, on the algebra of polynomials, there is no algebra norm that makes sense within this framework. If we work in a compact subset, we can take the supremum norm, but $e^{-t^2} dt$ lives on the whole line.

**Question from the audience:** What about other notions of positive, e.g. anything of the form $a^*a$? **Answer:** then we don't know that the sum of positive elements is positive, so not a terribly useful notion. For instance, for normed $*$-algebras, it can fail that $a^*a + b^*b \neq c^*c$, even though the notion of positive linear functionals will succeed. We can take the norm form the left-regular representation, and then complete, but this will have little to do with the original norm. E.g. $G$ a discrete group, and look at $\ell^1(G)$, which is a fine $*$-algebra with . We also have an action of $\ell^1$ on $\ell^2(G)$, with a good notion of operator norm (so can complete to a $C^*$ algebra), but has little to do with the $\ell^1$ norm. **Question from the audience:** is this like Gelfand transform on an abelian group? **Answer:** of course.

## 6: February 6, 2008

Any questions? No. Today we will look seriously at positive linear functionals.

### 6.1 Positive Linear Functionals

If you don't have an identity elements, things are more complicated, and we will have to deal with that; as always, we begin with the unital situation.

**Prop:** Let $A$ be a unital $*$-normed algebra. Let $\mu$ be a positive linear functional on $A$. If either

(a) $\mu$ is continuous (with respect to norm). This is situation in many examples.

(b) $A$ is complete (i.e. a Banach algebra). (If (a) holds, then $\mu$ extended to the completion of $A$, so reduces to (b), whereas if (b) holds, then $\mu$ is automatically continuous, as we will show.)

Then $\|\mu\| = \mu(1)$. (In particular, it is continuous.)

**Proof:**

We assume, by parenthetical remark above, that we are in case 2. above. We do not assume continuity. Certainly $1^* = 1 = 1^* \times 1$, so $\mu(1)$ is a nonnegative real number.

Consider first $a$ with $\|a\| < 1$ and $a = a^*$. Claim: then $1 - a = b^*b$ for some $b$ in $A$. Why? Consider $\sqrt{1-z} : \mathbb{C} \to \mathbb{C}$ which is holomorphic near 0, so has power series $\sum r_n z^n$, converging absolutely and uniformly on any disk about 0 with radius less than 1. Thus $b = \sum r_n a^n$ converges in $A$, by completeness. The $r_n \in \mathbb{R}$, so $b = b^*$, and $b^2 = 1 - a$.

Then $\mu(1) - \mu(a) = \mu(b^*b) \geq 0$, so $\mu(a) \in \mathbb{R}$, so $\mu(1) \geq \mu(a)$, and also $\mu(1) \geq \mu(-a)$, and hence $\mu(1) \geq |\mu(a)|$.

For general $a$ with $\|a\| < 1$ (but no longer considering $a = a^*$, we consider, using Cauchy-

Schwartz ($\mu$ is positive, and C.S. does not require definiteness), that

$$|\mu(a)|^2 = |\mu(1a)|^2 = |\langle 1, a\rangle_\mu|^2 \overset{\text{C.S.}}{\leq} |\langle 1, 1\rangle_\mu||\langle a, a\rangle_\mu| = \mu(1)\mu(a^*a) \leq \mu(1)^2$$

where the last inequality follows from the previous paragraph, since $\|a^*a\| \leq \|a^*\|\|a\|$, and we assume that $*$ is isometric. (This is part of the word "$*$-normed".) This completes the proof ($\|\mu\|$ is the sup of $|\mu(a)|$ for $\|a\| \leq 1$). $\square$

**Prop:** Let $A$ be a $C^*$-algebra with 1, and let $\mu$ be a continuous linear functional on $A$. If $\mu(1) = \|\mu\|$, then $\mu$ is positive. I.e., this condition characterizes positivity.

**Proof:**

Let $a \in A$. We must show that $\mu(a^*a) \geq 0$. We can write $a^*a = b^2$ for some $b = b^*$, so suffice to show that $\mu(b^2) \geq 0$. Let $B = C^*(b, 1) = C(\sigma(b))$, and restrict $\mu$ to $B$. Thus, we can verify the result in the commutative case.

So, we need to show: if $A = C(M)$ with $M$ compact, and $\mu$ is a linear functional on $A$ with $\mu(1) = \|\mu\|$, then $\mu$ is positive. By dividing, we can assume that $\mu(1) = \|\mu\| = 1$.

If $f^* = f$, then $\mu(f) \in \mathbb{R}$. Why? Let $\mu(f) = \alpha + i\beta$. Then $|\mu(f + it1)|^2 = |\alpha + i(\beta + t)|^2 = \alpha^2 + \beta^2 + 2\beta t + t^2$. On the other hand, $|\mu(f + it)|^2 \leq \|f + it\|^2 \leq \|f\|^2 + t^2$ since $f$ is $\mathbb{R}$-valued. Then for every $t$, $\underbrace{\alpha^2 + \beta^2}_{|\mu(f)|} + 2\beta t \leq \|f\|^2$. Thus, $\beta = 0$, and we see that $f \geq 0$ implies that $\|f - \|f\|1\| \leq \|f\|$, so $|\mu(f) - \|f\|| = |\mu(f - \|f\|1)| \leq \|f\|$, so $\mu(f) \geq 0$. $\square$

**Theorem:** Let $A$ be a $C^*$-algebra with 1. For any $a \in A$ with $a = a^*$, and for any $\lambda \in \sigma(a) \subseteq \mathbb{R}$, there is a state $\mu$ n $A$ such that $\mu(a) = \lambda$.

**Proof:**

Let $B = C^*(a, 1) = C^*(\sigma(a))$, and let $\mu_0$ on $B$ be the $\delta$-function at $\lambda$. Then $\mu_0(a) = \lambda$. Then $\|\mu_0\| = 1 = \mu_0(1)$.

We invoke the Hahn-Banach theorem (big, mysterious, uses Choice). This extended $\mu_0$ to $\mu$ on $A$, with $\|\mu\| = \|\mu_0\|$. But $\|\mu_0\| = 1 = \mu_0(1) = \mu(1)$. So by the previous proposition, $\mu \geq 0$, and hence a state (positive linear functional of norm 1). $\square$

In the commutative case, we care about this kind of thing because we want, e.g., $\ell^\infty(\mathbb{Z}) = C(?)$, where ? = maximal ideals, or something. In separable case, we can get states in a more hands-on way. In normed Banach spaces, we don't know that there are states, but, e.g., for $\ell^1(G)$ or $\ell^2(G)$, we can see there are some.

Next time, we will use this result, and dig into the GNS construction.

## 7: February 8, 2008

****I was not in class. These are notes by Vinicius Ramos, TEXed much later by me — any errors are undoubtedly mine.****

### 7.1 GNS Construction

Let $A$ be a $*$-algebra. Let $\mu$ be a positive linear functional on $A$, i.e. $\mu(a^*a) \geq 0$. Define $\langle a, b\rangle_\mu = \mu(b^*a)$.

Let $\mathcal{N}_\mu = \{a : \langle a, a\rangle_\mu = 0\}$. $\mathcal{N}_\mu$ is a linear subspace. Indeed, if $a \in \mathcal{N}_\mu$, then $|\langle a, b\rangle_\mu|^2 \leq \langle a, a\rangle_\mu \langle b, b\rangle_\mu = 0$. So $\mathcal{N}_\mu = \{a : \langle a, b\rangle_\mu = 0 \,\forall b \in A\}$.

Then $\langle \cdot, \cdot\rangle_\mu$ induces a definite inner product on $A/\mathcal{N}_\mu$. Complete this, and call the completion $L^2(A, \mu)$.

The left-regular representation of $A$ on $A$ is a "$*$-rep": $\langle ab, c\rangle_\mu = \langle b, a^*c\rangle_\mu$.

**Fact:** $\mathcal{N}_\mu$ is a left ideal in $A$. Indeed, if $b \in \mathcal{N}_\mu$ then

$$|\langle ab, ab\rangle_\mu|^2 = |\langle b, a^*ab\rangle|^2 \leq \langle b, b\rangle_\mu^2 \cdots = 0$$

So the left regular $*$-representation induces a representation of $A$ on $A/\mathcal{N}_\mu$ which is still a $*$-representation.

**Problem:** This need not be a representation by bounded operators.

**Theorem:** Let $A$ be a unital $*$-normed algebra and let $\mu$ be a continuous positive linear functional. Then the "GNS"-representation for $\mu$ is by bounded operators.

**Proof:**

We need only show that $\|\mu\| = \mu(1)$.

Let $a, b \in A$ so that $\|L_a b\|_\mu \leq c_a \|b\|_\mu$. ****Vinicius leaves a question mark (?) in the margin; I've got nothing.**** Then

$$\|L_a b\|^2 = \langle ab, ab\rangle_\mu = \langle a^*ab, b\rangle_\mu = \mu(b^*a^*ab).$$

Define $\nu$ on $A$ by $\nu(c) = \mu(b^*cb)$. Then $\nu$ is continuous. Thus $\|\nu\| = \nu(1)$.

$$\nu(c^*c) = \mu(b^*c^*cb) = \mu\left((cb)^*(cb)\right) \geq 0$$

$$\nu(a^*a) \leq \|a^*a\|\|\nu\| = \|a^*a\|\nu(1) = \|a^*a\|\mu(b^*b) = \|a^*a\|\|b\|_\mu^2.$$

So $\|L_a\|^2 \leq \|a^*a\| \leq \|a\|^2$. Hence $\|L_a\|_{\mathcal{B}(L^2(A,\mu))} \leq \|a\|_A$. $\square$

We have $L_1 = \mathbb{1}_{L^2(A,\mu)}$. ****On the LHS, $1 \in A$ is the unit element; on the right is $\mathbb{1}$ the identity operator on $L^2$.****

**Definition:** A continuous $*$-*representation* of a $*$-normed algebra is a continuous $*$-homomorphisms of $A$ into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. $(\pi, \mathcal{H})$ is *non-degenerate* if $\text{span}\{\pi(a)\xi : a \in A, \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$.

Let $\xi_0 = 1$ viewed as an element of $L^2(A,\mu)$. Then $L_a\xi_0 = a$ viewed as an element of $L^2(A,\mu)$. Thus $\{L_a\xi_0 : a \in A\}$ is dense in $L^2(A,\mu)$.

**Definition:** A continuous $*$-representation $(\pi, \mathcal{H})$ of a $*$-normed algebra is *cyclic* if there is a vector $\xi_0$ such that $\{\pi(a)\xi_0 : a \in A\}$ is dense in $\mathcal{H}$. Such a vector is called a *cyclic vector*.

Let $(\pi, \mathcal{H})$ be a continuous $*$-representation, and let $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Define $\mu$ on $A$ by $\mu(a) = \langle \pi(a)\xi, \xi\rangle$. Then $\mu(a^*a) = \langle \pi(a^*a)\xi, \xi\rangle = \langle \pi(a)\xi, \pi(a)\xi\rangle \geq 0$. So $\mu$ is a positive linear functional. If $A$ has unit and $(\pi, \mathcal{H})$ is non-degenerate then $\mu(1) = \|\xi\|^2 = 1$. We need to show that $\|\pi(a)\| \leq \|a\|$, so $\mu$ is a state, called a *vector state*.

For GNS from $\mu$, what is the vector state for $\xi_0$? $\langle a\xi_0, \xi_0\rangle_\mu = \langle a, 1\rangle_\mu = \mu(1^*a) = \mu(a)$. Thus $\mu$ is the vector state for the cyclic vector $\xi_0$.

$\mu(1_A) = 1$.

Every state on a unital $*$-normed algebra is a vector state for a representation, namely its GNS representation.

## 8: February 11, 2008

We were speaking in generalities about representations, and were in the midst of contemplating

$$\mathcal{H} = \bigoplus \mathcal{H}_\lambda$$

indexed by a vast set $\lambda \in \Lambda$. Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a family of $*$-representations of a $*$-algebra $A$. We want to define the direct sum $\bigoplus \pi_\lambda$ on $\bigoplus \mathcal{H}_\lambda$: the obvious answer is

$$\pi_\lambda(a)\xi = (\pi_\lambda(a)\xi_\lambda)_\lambda$$

How do we know that the RHS is square-integrable? This construction works if there is a constant $K$ such that

$$\|\pi_\lambda(a)\| \leq K\|a\| \,\forall\lambda$$

**Prop:** Let $A$ be a Banach $*$-algebra and $\pi$ a $*$-homomorphism into a $C^*$-algebra (e.g. $\mathcal{B}(\mathcal{H})$) (or $A$ is a $*$-normed algebra with each $\pi$ continuous, so that $\pi$ extends to the Banach completion). Then $\|\pi(a)\| \leq \|a\|$ for each $a \in A$.

**Proof:**

Adjoin identity elements. This is a little bit funny: $A$ may well have an identity element, but the homomorphism need not be identity-preserving. Even if $A$ has an identity elements, you can still adjoin another, in such a way as to make the homomorphism unital.

For each $a = a^*$, we have $\|\pi(a)\| = \rho(\pi(a))$ = the spectral radius (since $\pi(a)$ is in a $C^*$ algebra). But if $A \overset{\text{unital}}{\longrightarrow} B$, then we have only introduced more inverses, so $\rho(\pi(a)) \leq \rho(a) \leq \|a\|$.

For general $a$, do the usual thing: $\|\pi(a)\|^2 = \|\pi(a^*a)\| \leq \|a^*a\| \leq \|a\|^2$. $\square$

**Question from the audience:** When you add a unit to a unital algebra, I think of this as compactifying an already compact space? **Answer:** yes; the original unit is an idempotent, so you are just adding a point that has nothing to do with the rest of the space.

**Question from the audience:** Are we assuming, in the continuous non-Banach case, that $\pi$ has unit norm? **Answer:** no, that's a corollary.

**Theorem:** Any abstract $C^*$-algebra is isomorphic to a concrete $C^*$-algebra.

**Proof:**

Namely, let $A$ be a $C^*$-algebra, and adjoin 1 if necessary. Let $S(A)$ be the state space. For $\mu(S(A))$, let $(\pi_\mu, \mathcal{H}_\mu)$ be the GNS representation. Let

$$(\pi, \mathcal{H}) = \bigoplus_{\mu \in S(A)} (\pi_\mu, \mathcal{H}_\mu)$$

This is a faithful representation of $A$.

Given self-adjoint $a$, there exists $\mu$ with $|\mu(a)| = \|a\|$. On the other hand, $|\mu(a)| = \langle \pi_\mu(a)\xi_\mu, \xi_\mu \rangle$ where $\xi_\mu$ is a cyclic element from GNS. This implies that $\|\pi(a)\| \geq \|a\|$. For general $a$, do the usual squaring. By the previous Prop, we have $\|\pi(a)\| = \|a\|$. $\square$

If $A$ is separable, then we can use a countable number of states, so we can get $\mathcal{H}$ separable.

Lurking in the background, we used Choice to get all these states.

**Question from the audience:** What is the advantage of finding pure states, and working just with those? Davidson does this. **Answer:** We haven't talked about pure states yet. They are exactly the ones that give irreducible representations. We can sometimes get a smaller Hilbert space by working just with pure states.

**Prop:** Let $A$ be a $*$-algebra, and let $(\pi_j, \mathcal{H}_j, \xi_j)$ for $j = 1, 2$ be two cyclic representations of $A$. Let $\mu_j = \langle \pi_j(a)\xi_j, \xi_j \rangle$ be the corresponding positive linear functionals on $A$. If $\mu_1 = \mu_2$, then there is a unique unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ with $\xi_1 \mapsto \xi_2$ and intertwining the $A$-action (i.e. $\pi_2(a)U = U\pi_1(a)$, i.e. $U$ is an $A$-module homomorphism).

**Proof:**

Try to define $U$ by

$$U(\pi_1(a)\xi_1) = \pi_2(a)\xi(2)$$

since the $\pi_j(a)\xi_j$ are dense. It's not clear that this is well-defined. Ducking this for a moment,

$$\begin{aligned}
\langle U(\pi_1(a)\xi_1), U(\pi_1(b)\xi_1)\rangle_{\mathcal{H}_2} &= \langle \pi_2(a)\xi_2, \pi_2(b)\xi_2\rangle \\
&= \langle \pi_2(b^*a)\xi_2, \xi_2\rangle \\
&= \mu_2(b^*a) \\
&= \mu_1(b^*a) \\
&= \dots \\
&= \langle \pi_1(a)\xi_1, \pi_1(b)\xi_1\rangle_{\mathcal{H}_1}
\end{aligned}$$

So $U$ is certainly length-preserving, so extends to all of $\mathcal{H}_1$.

But if the RHS is 0, so must be the LHS, so $U$ is well-defined and isometric and unitary. $\square$

Thus, for $*$-normed algebras, there is a bijection between {continuous positive linear functionals} and {pointed cyclic representations} (i.e. has a specific choice of cyclic vector). ****The board says "isomorphism classes", but if we have a unique isomorphism as thingies between two thingies, then I say that as thingies they are the same thingy.****

If $\mu$ is a positive linear functional on a $*$-algebra $A$, do we have $\mu(a^*) = \overline{\mu(a)}$? No, e.g. $A =$ polynomials vanishing at 0 on $[0,1]$. Then let $\mu(p) = ip'(0)$.

On the other hand, sesquilinearity and positivity guarantee that $\langle a, b\rangle_\mu = \overline{\langle b, a\rangle_\mu}$ for any positive linear functional. If $A$ is unital, we can let $b = 1_A$, so $\mu(1) = \langle a, 1\rangle_\mu = \overline{\langle 1, a\rangle_\mu} = \overline{\mu(a^*)}$.

**Remark:** "An approximate identity is enough."

Let $A$ be a $*$-normed algebra with $\{e_\lambda\}$ a two-sided approximate identity of norm 1. (We do not require these to be self-adjoint; if it is two-sided, then $\{e_\lambda^*\}$ is also a two-sided approximate identity.) And let $\mu$ be a continuous positive linear funcitonal on $A$. Then

(a) $\mu(a^*) = \overline{\mu(a)}$

(b) $|\mu(a)|^2 \leq \|\mu\|\mu(a^*a)$

## 9: February 13, 2008

The first problem set is given out today. It's elementary, but explores positivity to prevent against wrong intuitions. Due next week.

For the homework, some notation: e.g. $a^+$ for $a$ in a $C^*$-algebra is just the positive part of the function $a \in C^*(a, 1) = C(M)$.

## 9.1 We continue the discussion from last time

**Prop:** Let $A$ be a $*$-normed algebra with two-sided approximate identity $e_\lambda$ of norm 1. Let $\mu$ be a continuous positive linear funcitonal on $A$. Then

(a) $\mu(a^*) = \overline{\mu(a)}$

(b) $|\mu(a)|^2 \leq \|\mu\||\mu(a^*a)|$

**Proof:**

(a) $\mu(a^*) = \lim \mu(a^* e_\lambda) = \lim\langle e_\lambda, a\rangle_\mu = \lim \overline{\langle a, e_\lambda\rangle} = \lim \overline{\mu(e_\lambda^* a)} = \overline{\mu(a)}$.

(b) $\underbrace{|\mu(e^*a)|^2}_{\to|\mu(a)|^2} = |\langle a, e_\lambda\rangle_\mu|^2 \overset{C.S.}{\leq} \langle a, a\rangle_\mu \langle e_\lambda, e_\lambda\rangle_\mu = \mu(a^*a)\mu(e_\lambda^* e_\lambda) \leq \mu(a^*a)\|\mu\|$. $\square$

**Question from the audience:** Do we assume positivity of $e_\lambda$? Self-adjointness? **Answer:** Positivity only makes sense in a $C^*$-algebra. We can assume self-adjointness: if $e_\lambda$ is two-sided, then $e_\lambda^*$ is also a two-sided approximate identity, so $(e_\lambda + e_\lambda^*)/2$ is a self-adjoint approximate identity.

**Prop:** Let $A$ be a $*$-normed algebra (without identity). Let $\mu$ be a positive linear functional on $A$. Suppose that $\mu$ satisfies the two results of the previous proposition. Let $\tilde{A}$ be $A$ with 1 adjoined, and define $\tilde{\mu}$ on $\tilde{A}$ by $\tilde{\mu}(a + z1) = \mu(a) + z\|\mu\|$. Then $\tilde{\mu}$ is positive.

**Proof:**

$$\begin{aligned}
\tilde{\mu}\left((a+z1)^*(a+z1)\right) &= \mu(a^*a) + \mu(a^*z) + \mu(\bar{z}a) + \mu(\bar{z}z) \\
&= \mu(a^*a) + \underbrace{z \underbrace{\mu(a^*)}_{=\overline{\mu(a)}} + \bar{z}\mu(a)}_{2\Re(\bar{z}\mu(a))} + |z|^2\|\mu\| \\
&\geq \mu(a^*a) - 2|z||\mu(a)| + |z|^2\|\mu\| \\
&\geq \mu(a^*a) - 2|z|\|\mu\|^{1/2}\mu(a^*a)^{1/2} + |z|^2\|\mu\| \\
&= \left(\mu(a^*a)^{1/2} - |z|\|\mu\|^{1/2}\right)^2 \\
&\geq 0 \square
\end{aligned}$$

**Cor:** Let $A$ be a $*$-normed algebra with approximate identity of norm 1. Let $\mu$ be a continuous positive linear functional on $A$. Define $\tilde{\mu}$ on $\tilde{A}$ by $\tilde{\mu}(a + z1) = \mu(a) + z\|\mu\|$. Then $\tilde{\mu} \geq 0$.

**Theorem:** Let $(\pi, \mathcal{H}, \xi_0)$ be the GNS representation for $\tilde{A}, \tilde{\mu}$. Then when $\pi$ is restricted to $A$, we might worry that it is degenerate. But in fact it is non-degenerate. In particular, $\xi_0 \in \overline{\text{span}\{\pi(A)\mathcal{H}\}}$. Also, $\mu(a) = \langle\pi(a)\xi_*, \xi_*\rangle$.

**Proof:**

Let $\{a_j\}$ be a sequence of states of $A$ such that $\|a_j\| \leq 1$ and $\mu(a_j) \to \|\mu\|$. Then view these in the GNS Hilbert space.

$$\begin{aligned} \| \underbrace{\xi_{a_j}}_{=a_j \text{ in } \mathcal{H}} - \xi_* \|^2 = \left\langle \xi_{a_j} - \xi_*, \xi_{a_j} - \xi_* \right\rangle_{\tilde{\mu}} &= \mu(a_j^* a_j) - \mu(a_j^*) - \mu(a_j) + \tilde{\mu}(1) \\ &\leq \|\mu\| - \overline{\mu(a_j)} - \mu(a_j) + \|\mu\| \\ &\to 0\Box \end{aligned}$$

This completes the GNS business.

### 9.2 Irreducible representations

**Prop (do earlier):** Let $A$ be a $*$-algebra with 1 (or really a $*$-set, e.g. a group where $x^* \stackrel{\text{def}}{=} x^{-1}$). Let $(\pi, \mathcal{H})$ be a non-degenerate representation of $A$ (i.e. it assignes $* \mapsto$ adjoint). Then $(\pi, \mathcal{H})$ is the direct sum (possibly vast) of cyclic representations.

**Definition:** A subspace $\mathcal{K}$ of $\mathcal{H}$ is $\pi$-*invariant* if $\pi(A)\mathcal{K} \subseteq \mathcal{K}$.

**Prop:** If $\mathcal{K}$ is $\pi$-invariant, then so is $\mathcal{K}^\perp$. So $\pi = \pi|_{\mathcal{K}} \oplus \pi|_{\mathcal{K}^\perp}$.

**Proof:**

Let $\xi \in \mathcal{K}^\perp$, $a \in A$, with $\pi(a)\xi \in \mathcal{K}^\perp$. Let $\eta \in \mathcal{K}$. Then $\langle \pi(a)\xi, \eta \rangle = \langle \xi, \underbrace{\pi(a^*)\eta}_{\in \mathcal{K}} \rangle = 0$. $\Box$

So we have a sort of "semi-simplicity".

**Proof of "do earlier" Prop:**

Choose $\xi \in \mathcal{H}$, $\xi \neq 0$. Let $\mathcal{H}_1$ be $\overline{\pi(A)\xi}$. This is obviously $\pi$-invariant (by continuity). So $\mathcal{H}_1^\perp$ is $\pi$-invariant. Choose $\xi_2 \in \mathcal{H}_1^\perp$, $\xi_2 \neq 0$, so $\mathcal{H}_2 \stackrel{\text{def}}{=} \overline{\pi(A)\xi_2}$. Then $\mathcal{H}_2 \perp \mathcal{H}_1$. Then $(\mathcal{H}_1 \oplus \mathcal{H}_2)^\perp$ is $\pi$-invariant. Choose $\xi_3, \ldots$. Really, use Zorn. $\Box$

In the separable case, we choose a countable dense "basis", and work with that sequence, eschewing Choice.

## 10: Problem Set 1: "Preventive Medicine" Due February 20, 2008

****For posterity's sake, I will also type up the handed-out problem sets.****

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$, for $s, t \geq 0$.

1. Determine for which $s, t$ we have $B \geq A$.
2. Determine for which $s, t$ we have $B \geq A^+$. ****Recall from lecture: $A^+$ is the positive part of $A$, where we consider $A$ as a function on its spectrum.****
3. Find values of $s, t$ for which $B \geq A$, $B \geq 0$, and yet $B \not\geq A^+$. (So be careful about false proofs.)
4. Find values of $s, t$ such that $B \geq A^+ \geq 0$ and yet $B^2 \not\geq (A^+)^2$. (So again be careful.)
5. Can you find values of $s, t$ such that $B \geq A^+$ and yet $B^{1/2} \not\geq (A^+)^{1/2}$?
6. For $2 \times 2$ matrices $T$ and $P$ such that $T \geq 0$ and $P$ is an orthogonal projection, is it always true that $PTP \leq T$?

## 11: February 15, 2008

****I was not in class. These are notes by Vinicius Ramos, TEXed much later by me — any errors are undoubtedly mine.****

**Prop:** Let $A$ be a $*$-normed algebra with approximate identity $\{e_\lambda\}$ of norm 1. Let $(\pi, B)$ be a continuous representation of $A$ on a normed space. If $\pi$ is *nondegenerate* (i.e. $\mathrm{span}(\pi(A)B)$ is dense in $B$), then $\forall \xi \in B$ we have $\pi(e_\lambda)\xi \to \xi$.

**Proof:**

If $\xi = \pi(a)\eta$, then $\pi(e_\lambda)\xi = \pi(ae_\lambda)\eta \to \pi(a)\eta = \xi$. So this is also true for $\pi \in \mathrm{span}(\pi(A)B)$. Use continuity to show it holds for $B$. $\square$

**Cor:** For $A$ a $*$-normed algebra with approximate identity $\{e_\lambda\}$ of norm 1, for $\mu$ a continuous positive linear functional, we have $\mu(e_\lambda) \to \|\mu\|$.

**Proof:**

Form $(\pi_\lambda, \mathcal{H}_\lambda, \xi_\lambda)$;

$$\mu(e_\lambda) = \langle \pi(e_\lambda)\xi_\lambda, \lambda \rangle \to \langle \xi_\lambda, \xi\lambda \rangle = \tilde{\mu}(1_{\tilde{A}}) = \|\mu\|$$

$\square$

Let $A$ be a $*$-normed algebra (with approximate identity $e_\lambda$); $\mu$ is a *state* of $A$ if $\mu \geq 0$ and $\|\mu\| = 1$. Let $S(A)$ = set of states of $A$.

If $1_A \in A$, then $S(A)$ is a $w^*$-closed bounded convex subset of $A$ and so is $w^*$-compact.

**Proof:**

$\mu(a^*a) \geq 0$ and $\mu(1) = \|\mu\| = 1$. $\square$

$S(A)$ can be viewed as "the set of non-commutative probability measures on A."

May be false in the non-unital case: $A = C_\infty(\mathbb{R})$, $\mu_n = \delta_n \xrightarrow[w^*]{n\to\infty} 0$; that is not a state.

If $1 \notin A$, let $QS(A) = \{\mu : \mu \geq 0, \|\mu\| \leq 1\}$ be the set of "quasi-states". This is the $w^*$-closed convex hull of $S(A) \cup \{0\}$. Then $QS(A)$ is $w^*$-closed convex ****illegible****, so compact.

**Definition:** A representation $(\pi, B)$ of $A$ on a Banach space is *irreducible* if there is no proper closed subspace that is carried into itself by the representation, i.e. no non-zero $C \subsetneq B$ so that $\pi(A)C \subseteq C$.

**Schur's Lemma:** Let $(\pi, \mathcal{H})$ be a $*$-representation of a $*$-normed algebra $A$. Then $\pi$ is irreducible if and only if $\mathrm{End}_\pi(\mathcal{H}) = \{\mathbb{C}\mathbb{1}_\mathcal{H}\}$. (Recall, the left hand side is the intertwiners $\{T \in \mathcal{B}(\mathcal{H}) : \pi(a)T = T\pi(a)\, \forall a\}$.)

**Proof:**

($\Leftarrow$) If not irreducible, let $P$ be the orthogonal projection onto an invariant proper closed subspace. Then $P \in \mathrm{End}_\pi(\mathcal{H})$.

($\Rightarrow$) Suppose $\pi$ is irreducible and let $T \in \mathrm{End}_\pi(\mathcal{H})$. Suppose $T \notin \mathbb{C}\mathbb{1}_\mathcal{H}$. Then either $\Re T = \frac{T+T^*}{2}$ or $\Im T = \frac{T-T^*}{2i} \notin \mathbb{C}\mathbb{1}_\mathcal{H}$. So we can assume $T^* = T$. Then $C^*(T, \mathbb{1}) \cong \mathbb{C}(\sigma(T))$. A proper spectral projection for $T$ will be in $\mathrm{End}_\pi(\mathcal{H})$ and its range will be a proper closed invariant subspace. Contradicts irreducibility of $\pi$. $\square$

For $\mu, \nu$ positive linear functionals, we write $\mu \geq \nu$ if $\mu - \nu \geq 0$. For $(\pi_\mu, \mathcal{H}_\mu, \xi_\mu)$, let $T \in \mathrm{End}_\pi(\mathcal{H}_\mu)$ with $0 \leq T \leq \mathbb{1}$. Set $\nu(a) = \langle T\pi_\mu(a)\xi_\mu, \xi_\mu\rangle$, then $\nu \geq 0$.

$$(\mu - \nu)(a) = \langle \pi_\mu(a)\xi_\mu, \xi_\mu\rangle - \langle T\pi_\mu(a)\xi_\mu, \xi_\mu\rangle = \langle (\mathbb{1} - T)\pi_\mu(a)\xi_\mu, \xi_\mu\rangle$$

So $\mu - \nu \geq 0$ since $\mathbb{1} - T \geq 0$.

**Theorem:** Given $\mu \geq 0$, the $\nu \geq 0$ such that $\mu \geq \nu$ are exactly of the form $\nu(a) = \langle T\pi_\mu(a)\xi_\mu, \xi_\mu\rangle$.

## 12: February 20, 2008

This piece we're doing right now is the last piece of the purely theoretical bit. We will soon start edging into examples, with theory along the way.

We are trying to show that for unital $*$-normed algebras, that the state space (a compact convex subset of the dual) has extreme points which, per the GNS construction, result in irreducible representations. It has been convenient to think not in terms of states but in terms of positive linear functional.

Let $A$ be a unital $*$-normed algebra, with positive linear functionals $\mu, \nu$ with $\mu \geq \nu \geq 0$. Let $T \in \mathrm{End}_A(\mathcal{H}_\mu) = \mathcal{B}(L^2(A, \mu))$ and $0 \leq T \leq \mathbb{1}$. Then for example with can set

$$\nu : a \mapsto \langle Ta\xi_0, \xi_0\rangle_\mu. \tag{1}$$

Conversely, given $\nu$, we consider:

$$|\nu(b^*a)| = |\langle a, b\rangle_\nu| \leq \nu(a^*a)^{1/2}\nu(b^*b)^{1/2} \leq \mu(a^*a)^{1/2}\mu(b^*b)^{1/2} = \|a\xi_\mu\|_\mu \|b\xi_\mu\|_\mu \tag{2}$$

Thus, if we set $\langle a\xi_\mu, b\xi_\mu\rangle_\nu \overset{\text{def}}{=} \nu(b^*a)$, is this well-defined? Yes, by equation (2), because if the difference on is the 0-vector, then the RHS of above is 0. So then $a\xi_\mu \mapsto \langle a\xi_\mu, b\xi_\mu\rangle_\nu$ is a continuous linear functional for $\|\cdot\|_\mu$, so it extends to $\mathcal{H}_\mu$ (the vectors of the form $a\xi_\mu$ are dense in $\mathcal{H}$ = the completion). But on a complete Hilbert space, every positive linear functional comes from a vector. So there is a vector $T^*b\xi_m u$ so that $\langle a\xi_\mu, b\xi_\mu\rangle_\nu = \langle a\xi_m u, T^*b\xi_\mu\rangle_\mu$. This defines $T^*$ for vectors of the form $b\xi_m u$, but also by equation (2), we see that $b\xi_\mu \mapsto T^*b\xi_m u$ is continuous for $\|\cdot\|_\mu$ so $T^*$ extends to $\mathcal{H}_\mu$, and chasing constants gives $\|T\| \leq 1$. So we see that $\nu(b^*a) = \langle Ta\xi_\mu, b\xi_\mu\rangle_\mu$. Letting $b = 1$ gives $\nu(a) = \langle Ta\xi_\mu, \xi_\mu\rangle_\mu$.

Checking that $T$ is in fact an endomorphism over $A$:

$$\langle TL_a b\xi_\mu, c\xi_\mu\rangle = \langle T(ab)\xi_\mu, c\xi_\mu\rangle = \nu(c^*(ab)) = \nu((a^*c)^*b) = \langle Tb\xi_\mu, a^*c\xi_\mu\rangle = \langle L_a Tb\xi_\mu, c\xi_\mu\rangle$$

This checks that $T$ commutes with $L_a$ on a dense subspace, so $T$ is in fact an endomorphism. Chasing inequalities gives $0 \leq T \leq \mathbb{1}$. Thus every $0 \leq \nu \leq \mu$ is of the form (1).

Moreover, if we have two different $T$s giving the same positive linear functional, then taking their difference gives the zero positive linear functional ($T \mapsto \nu$ in (1) is linear), so $\nu \mapsto T$ is injective. Thus, there is a bijection between $\{\nu : \mu \geq \nu \geq 0\}$ and $\{T \in \mathrm{End}_A(\mathcal{H}_\mu) : 0 \leq T \leq \mathbb{1}\}$.

**Definition:** A positive linear functional $\mu$ is *pure* if whenever $\mu \geq \nu \geq 0$, then $\nu = r\mu$ for some $r \in [0, 1]$.

**Theorem:** For positive linear functional $\mu$, its GNS representation $(\pi_\mu, \mathcal{H}_\mu, \xi_\mu)$ is irreducible if and only if $\mu$ is pure.

**Proof:**

If the GNS representation is not irreducible, then there exists $P \in \mathrm{End}_A(\mathcal{H}_\mu)$ a proper projection. (So $0 \leq P \leq \mathbb{1}$.) Use $T = P$ in equation (1), and then $\nu \leq \mu$ but $\nu \notin [0, 1]\mu$, because $r\mu$ gives the same GNS representation as $\mu$ (except that $\xi_{r\mu} = r\xi_m u$), whereas $\nu$ gives a different one (shrunk by $P$). Thus $\mu$ is not pure.

Conversely, if $\mu$ is not pure, then there is a positive $\nu \leq \mu$ with $\nu \notin [0, 1]\mu$, so $T_\nu \notin \mathbb{C}\mathbb{1}$, so $\mathrm{End}_A(\mathcal{H}_\mu) \neq \mathbb{C}\mathbb{1}$. So by Schur's lemma, the GNS representation is not irreducible. I.e. $T_\nu$ will map onto a proper invariant subspace. □

Now we want to convert this statement about positive linear functionals into one about states.

**Reminder:** For a convex set $C$, a point $\mu$ is an *extreme* point if whenever $\mu = t\nu_1 + (1-t)\nu_0$ and $0 < t < 1$, then $\nu_0 = \nu_1 = \mu$.

**Theorem:** (Krein-Milman)

A compact convex set is the closed convex hull of its extreme points.

**Question from the audience:** Any topological vector space? **Answer:** Locally convex. **Question from the audience:** Not just a Banach space? **Answer:** No. We are applying it to the dual of a hairy space, so not necessarily Banach.

Even in the finite-dimensional case, the set of extreme points need not be closed.

**Theorem:** For a $*$-normed algebra $A$ with 1, and a state $\mu$, the GNS representation for $\mu$ is irreducible if and only if $\mu$ is an extreme point of $S(A) =$ state space.

Since "pure" has fewer syllables than "extreme", we refer to extreme points as pure points.

**Proof:**

Suppose $\mu$ is extreme. It is sufficient to show that it is pure.

Suppose $\mu > \nu > 0$. Then

$$\mu = \nu + (\mu - \nu) = \|\nu\| \overbrace{\frac{\nu}{\|\nu\|}}^{\in S(A)} + \|\mu - \nu\| \overbrace{\frac{\mu - \nu}{\|\mu - \nu\|}}^{\in S(A)}$$

But $\|\nu\| + \|\mu - \nu\| = \nu(1) + (\mu - \nu)(1) = \mu(1) = 1$ by positivity, and since $\mu$ is extreme, we must have $\frac{\nu}{\|\nu\|} = \mu$, so $r = \|\nu\|$ and $\mu$ is pure.

Conversely, if $\mu$ is pure, we should show that it is extreme. Consider $\mu = t\nu_1 + (1-t)\nu_0$ with $0 < t < 1$ and $\nu_0, \nu_1 \in S(A)$. Then $\mu - t\nu_1 = (1-t)\nu_0 \geq 0$, so $\mu \geq t\nu_1$, so $t\nu_1 = r\mu$, but $\nu_1(1) = 1 = \mu(1)$, so $r = t$, so $\nu_1 = \mu$ (and same argument works for $1-t$ and $\nu_0$). $\square$

**Question from the audience:** Are these the point measures? **Answer:** Yes, exactly. These are the $\delta_x$ on $C(X)$, and $L^2(X, \delta_x)$ are the irreducible representations. In the noncommutative case, things are more complicated.

We have two minutes left, and will talk about non-unital algebras.

For $A$ non-unital $*$-normed with approximate two-sided identity of norm 1, we define a quasi-state space $\mathcal{QS}$. We drew a picture ****perhaps I'll add later: a cone with $0$ at the vertex and $S(A)$ at the base****, and the extreme points of $\mathcal{QS}(A)$ are exactly the extreme points of $S(A)$ together with 0. Certainly 0 does not give an interesting representation (the 0 Hilbert space). Even in this non-unital case, the extreme points of the now non-closed $S(A)$ are almost enough; $\mathcal{QS}$ is weak closed.

## 13: February 22, 2008

(We begin by handing back the homework turned in last time.)

We begin with a few comments to wind things up.

Let $A$ be a $*$-normed algebra with approximate identity of norm 1. Then for any continuous $*$-representation $(\pi, \mathcal{H})$, of $A$, then we know that $\|\pi(a)\| \leq \|a\|$. So set

$$\|a\|_{C^*} = \sup\left\{\|\pi(a)\| : (\pi, \mathcal{H}) \text{ is a cont's } *\text{-rep of } A\right\}. \tag{3}$$

So $\|a\|_{C^*} \leq \|a\|$, and you can check that $\|a^*a\|_{C^*} = (\|a\|_{C^*})^2$, so the completion of $A$ with respect to $\|\cdot\|_{C^*}$ is a $C^*$-algebra, called the "universal $C^*$-enveloping algebra of $A$". It has the property that there is a natural bijection of

$$\{\text{continuous } *\text{-reps of } A\} \leftrightarrow \{*\text{-reps of } C^*(A)\}$$

**Comment:** If $(\pi, \mathcal{H})$ is a representation, look at

$$\underbrace{\overline{\pi A \mathcal{H}}}_{\substack{\text{rep is}\\ \text{non-degen}}} \oplus \underbrace{\left(\overline{\pi A \mathcal{H}}\right)^\perp}_{0-\text{rep}}.$$

**Question from the audience:** Is there some diagram that goes with this? **Answer:** Yes:

$$\begin{tikzcd} C^*(A) \arrow[dr, dashed] & \\ A \arrow[u] \arrow[r, "\pi"] & \mathcal{B}(\mathcal{H}) \end{tikzcd}$$

Now, we form

$$\bigoplus_{\mu \in S(A)} (\pi_\mu, \mathcal{H}_\mu)$$

which has the same norm, since we could add the word "cyclic" to eqn (3).

Instead, let's look at the "atomic" representation:

$$\begin{aligned} (\pi, \mathcal{H}) &\stackrel{\text{def}}{=} \bigoplus_{\mu \text{ a pure state of } A} (\pi_\mu, \mathcal{H}_m u) \\ \|a\|_{\text{atomic}} \stackrel{\text{def}}{=} \|\pi(a)\| &= \sup\{\|\pi(a)\| : (\pi, \mathcal{H}) \text{ is an irreducible rep of } A\} \end{aligned}$$

This is a good construction, because there is more than a *set* of irreducible representations, but only a set of pure states.

**Prop:** $\|a\|_{\text{atomic}} = \|a\|_{C^*}$. (These are both obviously $C^*$-seminorms.)

**Proof:**

$\leq$ is clear. For $\geq$, by $C^*$-identity, it suffices to show for $a^*a$, i.e. we can assume that $a \geq 0$.

If $\mu(a) \leq c$ for all pure states $\mu$ (so certainly for any convex combination of pure states, so also for the closure of the convex combinations), then by the Krein-Milman ****sp?**** theorem, we have $\mu(a) \leq c$ for any $\mu$. Take $a^*a$ and look at the commutative $C^*$-algebra it generates, and we can find a state that returns the norm, and use the Hahn-Banach theorem to extend this state to the whole algebra. So there is a state $\mu$ with $\|a^*a\|_{C^*} = \mu(a^*a)$, so $\|a^*a\|_{C^*} \leq c$. This gives us the reverse inequality. $\square$

Getting a little ahead of ourselves: if $G$ is locally compact, with a left Haar measure on $G$, we can form $L^1(G)$ under convolution. This has a faithful (i.e. injective) representation as operators on $L^2(G)$ (again by convolution). Then every vector in $L^2$ is a state in $L^1$ — it has enough states to separate the points. So $L^1$ has lots of pure states, and hence $G$ has lots of irreducible unitary representations. In 1943, Gelfand and Raikov showed this. We will go into this in more detail later; this is just foreshadow.

If you respond "Ok, it has lots, show me some", then that's hard. The Krein-Milman theorem is non-constructive — it needs Axiom of Choice — so this is all very encouraging, but it doesn't give you any real technique for finding these representations. How to find things depends on how the example is presented. There's an enormous literaturs on, say, $GL(3, \mathbb{R})$, which is understood, but $GL(3, \mathbb{Z})$ is not.

### 13.1 Compact operators

For a Hilbert space $\mathcal{H}$, let $\mathcal{B}_0(\mathcal{H})$ be the $C^*$-algebra of compact operators on $\mathcal{H}$. We should ask what definition we mean for "compact operators". In Banach land, there is a definition, and you have to look fairly far before you come across an example where the compact operators are not just the closure of finite-rank operators. Hilbert spaces are more constrained; even finite-dimensional normed spaces can be bewildering. For Hilbert spaces, compact is nice: $\mathcal{B}_0(\mathcal{H}) \stackrel{\text{def}}{=}$ the closure of finite-rank operators.

Under the operator norm, this is a $C^*$-algebra. Moreover, $\mathcal{B}_0(\mathcal{H})$ is topologically simple: there are no proper closed 2-sides ideals. It has lots of important non-closed 2-sided ideals (trace-class, Hilbert-Schmidt, etc.; these are in fact ideals of $\mathcal{B}(\mathcal{H})$). But any ideal that isn't 0, then we can compress it between two rank-1 projections, so there are rank-1 operators in the ideal, so all are, so all finite-rank operators are in the ideal.

**Theorem:** (Shows up lots of places, e.g. uniqueness of Heisenberg commutation relations.)

Up to unitary equivalence, $\mathcal{B}_0(\mathcal{H})$ has exactly one irreducible representation, namely the one on $\mathcal{H}$. Furthermore, every non-degenerate representation is a direct sum of copies of $\mathcal{H}$.

**Proof:**

Let $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ be linear in the second variable ****this is how I like it****, and put the scalars on the right. For $\xi, \eta \in \mathcal{H}$, define $\langle \xi, \eta \rangle_0 \in \mathcal{B}_0(\mathcal{H})$. This will be a familiar object: $\langle \xi, \eta \rangle_0 \stackrel{\text{def}}{=} \{\zeta \mapsto \xi \langle \eta, \zeta \rangle_{\mathcal{H}}\}$. I.e. we have a bimodule ${}_{\mathcal{B}_0(\mathcal{H})}\mathcal{H}_{\mathbb{C}}$.

For $T \in \mathcal{B}_0(\mathcal{H})$ (or in fact in $\mathcal{B}(\mathcal{H})$), we have

$$T\langle \xi, \eta \rangle_0 = \langle T\xi, \eta \rangle_0$$

and you can check that $(\langle \xi, \eta \rangle_0)^* = \langle \eta, \xi \rangle_0$. The consequence is that $\langle \xi, \eta \rangle_0 T = \langle \xi, T^* \eta \rangle_0$. (Of course, $\langle \cdot, \cdot \rangle_0$ is $\mathbb{C}$-linear in the first variable.) $\square$

(Time is up; we will continue this discussion next time.)

## 14: February 25, 2008

**Theorem from last time:** $\mathcal{B}_0(\mathcal{H})$ has one (up to equivalence) irreducible representation, namely $\mathcal{B}_0 : \mathcal{H} \to \mathcal{H}$. And every (non-degenerate) representation is a direct sum of these.

**Proof, continued from last time:**

Last time, we defined some rank-one operators $\langle \xi, \eta \rangle_0$, which we view as a $\mathcal{B}_0$-valued inner product. For bookkeeping, we let $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ be $\mathbb{C}$-linear in the second variable.

Let $(\pi, V)$ be a non-degenerate representation of $\mathcal{B}_0(\mathcal{H})$. For vectors in $\mathcal{H}$ we will use $\xi, \eta, \dots,$ whereas for vectors in $V$ we will use $v, w, \dots$; we will not carry the "$\pi$" around, preferring "module notation": $Tv \overset{\text{def}}{=} \pi(T)v$ for $v \in V$ and $T \in \mathcal{B}_0(\mathcal{H})$. We take $\langle \cdot, \cdot \rangle_V$ to be linear in the second variable.)

Of course, $\mathcal{B}_0(\mathcal{H})$ is a $C^*$-algebra, and in particular it is complete. Thus, the representation $\pi : \mathcal{B}_0(\mathcal{H}) \to \mathcal{B}(V)$ is a $*$-homomorphism of $C^*$-algebras, and hence is continuous. Furthermore, $\mathcal{B}_0(\mathcal{H})$ is topologically simple: there are no *closed* two-sided ideals. Thus $\pi$ is injective (its kernel is closed, since it is continuous).

So choose $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Then $\langle \xi, \xi \rangle_0$ is the rank-1 self-adjoint (and hence orthogonal) projection onto $\xi\mathbb{C}$. So $\pi(\langle \xi, \xi \rangle_0)$ is also a (self-adjoint orthogonal) projection on $V$ (being a projection is an algebraic property), and it is not the 0 projection. So choose $v_0$ with $\|v_0\| = 1$ in the range of this projection.

Thus, define $Q : \mathcal{H} \to V$ by

$$Q : \eta \mapsto \langle \eta, \xi \rangle_0 v_0$$

This is obviously continuous.

$$\begin{aligned}
\langle Q\eta, Q\zeta \rangle_V &= \langle \langle \eta, \xi \rangle_0 v_0, \langle \zeta, \xi \rangle_0 v_0 \rangle_V \\
&= \langle v_0, \langle \xi, \eta \rangle_0 \langle \zeta, \xi \rangle_0 v_0 \rangle_V \quad \text{since } \pi \text{ is a } *\text{-representation} \\
&= \left\langle v_0, \left\langle \underbrace{\langle \xi, \eta \rangle_0 \zeta}_{=\xi \langle \eta, \zeta \rangle_{\mathcal{H}}}, \xi \right\rangle_0 v_0 \right\rangle_V \\
&= \langle \eta, \zeta \rangle \, \langle v_0, \langle \xi, \xi \rangle_0 v_0 \rangle \\
&= \langle \eta, \zeta \rangle
\end{aligned}$$

So $Q$ is isometric. Moreover, for $T \in \mathcal{B}_0(\mathcal{H})$,

$$\begin{aligned} Q(T\eta) &= \langle T\eta, \xi\rangle_0 v_0 \\ &= T\langle \eta, \xi\rangle_0 v_0 \\ &= T\,Q(\eta) \end{aligned}$$

So $Q : \mathcal{H} \to V$ intertwines $\pi$ with the representation of $\mathcal{B}_0(\mathcal{H})$ on $\mathcal{H}$. Thus, the range of $Q$ is a closed subspace of $V$ carried into itself by the action $\pi : \mathcal{B}_0(\mathcal{H}) \to \mathcal{B}(V)$. The representation $\pi$ restricted to this subspace is unitarily equivalent to the representation of $\mathcal{B}_0(\mathcal{H})$ on $\mathcal{H}$. If $\pi$ itself is irreducible, then range of $Q$ must be all of $V$, establishing the first part of the theorem. If the representation is not irreducible, take $Q(\mathcal{H})^\perp$, which is still a representation, and rinse and repeat (with Zorn ****the best brand of conditioning shampoo****). $\square$

****This paragraph was said after the next few, but belongs here.**** $\mathcal{B}_0(\mathcal{H})$ has only one irreducible representation, but any pure state gives an irreducible representation a la GNS. So take any pure state, do the GNS, and you'll get the representation on $\mathcal{H}$, with a cyclic vector. So every pure state of $\mathcal{B}_0(\mathcal{H})$ is represented by a vector $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, i.e. $\mu(T) = \langle T\xi, \xi\rangle$. Multiplying this vector by something in $S^1 \subset \mathbb{C}$ does not change the state; the pure states are represented by rank-1 projections $p$, via $\mu(T) = \mathrm{tr}(pT)$. We in fact have a bijection $\{\text{pure states of } \mathcal{B}_0(\mathcal{H})\} \leftrightarrow \{\text{rank-1 projections}\} = \mathcal{PH}$ the "projective Hilbert space". This is the setting for the quantum physics of finitely many particles. Moreover, the convex hull $S(\mathcal{B}_0(\mathcal{H})) = \{\text{"mixed states"}\} = \{\text{density operators}\} = \{D \in \mathcal{B}(\mathcal{H}) : 0 \le D, \mathrm{tr}(D) = 1\}$ where the corresponding state has $\mu_D(T) = \mathrm{tr}(DT)$. **Question from the audience:** Does $\le \mathbb{1}$ follow from that? **Answer:** Yes, but $\mathrm{tr} = 1$ is the interesting part So, with a quantum-mechanical system, it is the state that evolves, not the vector; i.e. it is the point in $\mathcal{PH}$. This often confuses people. Important questions include "What are the automorphisms of $\mathcal{PH}$?" Well, these include automorphisms of $\mathcal{H}$ and also anti-automorphisms. These anti-linear operators do occur, e.g. Time and Parity reversals.

A theorem of Burnside says that if a subalgebra of the algebra of operators on a finite-dimensional vector space acts irreducibly, then the subalgebra is the whole algebra. For example, at the purely algebraic level, then $\mathbb{C}$ and $M_n(\mathbb{C})$ have only one irreducible representation, and more generally, for any ring $R$, $R - \textsc{Mod}$ and $M_n(R) - \textsc{Mod}$ are equivalent as categories, under

$$_RV \mapsto_{M_n(R)} R^n_R \otimes_R {}_RV$$

This is the notion of *Morita equivalence.*

The general picture for $C^*$-algebras is similar. For our example, we have $\langle\cdot,\cdot\rangle_{\mathcal{B}_0(\mathcal{H})}$, ${}_{\mathcal{B}_0(\mathcal{H})}\mathcal{H}_{\mathcal{C}}$, and $\langle\cdot,\cdot\rangle_{\mathbb{C}}$. And in general, we will have $\langle x, y\rangle_A z = x\langle y, z\rangle_B$ and a module ${}_AX_B$, plus some nondegeneracy axioms, and this will give "strong Morita equivalence." Two non-commutative spaces that are strong Morita equivalent will have the same homology and cohomology.

**Theorem:** (analogous to a theorem of Burnside, moving towards a Stone-Weierstrass theorem)

Let $A$ be a sub-$C^*$-algebra of $\mathcal{B}_0(\mathcal{H})$ and suppose that $\mathcal{H}$ is an irreducible module of the action of $A$. Then $A = \mathcal{B}_0(\mathcal{H})$.

# 15: February 27, 2008

One more comment on the theory of bounded operators:

**Naimark Conjecture:** (1940s)

> Let $A$ be a $C^*$-algebra. Suppose $A$ has the property that, up to equivalence, $A$ has only one irreducible representation. Then $A \cong \mathcal{B}_0(\mathcal{H})$ for some $\mathcal{H}$.

Rarely do we come across algebras that we don't know are $\mathcal{B}_0$, but for neatness, this would be nice to know. The answer is "Yes" if $A$ is separable. In 2003, however, there was a surprise: Chuck Akerman and Nick Weaver (both Ph.D. students of Prof. Bade, at UC-Berkeley) showed that the answer depends on the axioms of set theory. For usual axioms ****ZF definitely, and probably ZFC?****, the question is undecidable. More specifically, If we assume the Diamond Principle, we can construct a counterexample to Naimark's conjecture, and Diamond is consistent with usual axioms.

**Question from the audience:** What does separable mean? **Answer:** As a Banach space, it is separable: there is a countable norm-dense subset. **Question from the audience:** If $A$ acts on a separable Hilbert space, does that imply $A$ is separable? **Answer:** I don't know. I'd expect that that does imply the result, but I did not try to understand the paper, since I don't know set theory.

## 15.1 Continuing from last time

We stated this Burnside theorem:

**Theorem:** If $A$ is a $C^*$-subalgebra of $\mathcal{B}_0(\mathcal{H})$ that acts irreducibly on $\mathcal{H}$, then $A = \mathcal{B}_0(\mathcal{H})$.

**Proof:**

> The action must clearly be non-degenerate. $A$ is a $C^*$-subalgebra, and $\mathcal{H}$ is of non-zero dimension, so $A$ has non-zero elements, and $T^*T$ is non-zero if $T$ is. So we pick out $T \in A$ with $T \neq 0$, $T \geq 0$. Then $T$ is compact — specturm is discrete, except perhaps 0 could be an accumulation point —, by the spectral theorem of compact self-adjoint operators. So $T$ has a non-zero eigenvalue, and the projection onto the eigensubspace is in $A$. Thus $A$ contains proper projections onto finite-dimensional subspace.
>
> So we look at all projections, and find a minimal one: Let $P \in A$ be a projection of minimal positive dimension (of range of $P$). Then for any $T \in A$ with $T = T^*$, we look at $PTP$, which is clearly self-adjoint with finite dimension. So it has spectral projections, and it's obvious that the spectral projections must be smaller than $P$ (in the strongest sense: they are onto subspaces of range of $P$). These spectral projections are certainly still in $A$, since they are polynomials in $PTP$. But $P$ is minimal, so the only possible spectral projections for $PTP$ are 0 and $P$. Thus, there exists $\alpha(T) \in \mathbb{R}$ (self-adjoint implies real eigenvalues) such that

$PTP = \alpha(T)P$. By splitting operators into their real and imaginary parts, we can extend this from self-adjoint $T$ to all $T$: for any $T \in A$, we have $\alpha(T) \in \mathbb{C}$ so that $PTP = \alpha(T)P$.

Let $\xi, \eta \in$ range of $P$, with $\|\xi\| = 1$ and $\eta \perp \xi$. We'd like to show that $\eta = 0$, since we're trying to show that range of $P$ is one-dimensional. Well, for $T \in A$,

$$\begin{aligned}
\langle T\xi, \eta \rangle &= \langle TP\xi, P\eta \rangle \text{ since } P\xi = \xi, \text{ etc.} \\
&= \langle PTP\xi, \eta \rangle \text{ since } P^* = P \\
&= \langle \alpha(T)\xi, \eta \rangle \\
&= 0
\end{aligned}$$

But $\overline{\{T\xi : T \in A\}}$ is $A$-invariant, so $= \mathcal{H}$, so $\eta = 0$.

So $A$ contains a rank-1 projection $P$ on $\mathbb{C}\xi$. Then $\{T\xi\}$ are dense in $\mathcal{H}$, so $TP$ is rank-$\leq 1$ taking $\xi$ to $T\xi$. Since $A$ is norm-closed, if we take any vector t$\eta \notin \{T\xi\}$, we can approximate it by such, and then look at corresponding $TP$, which will converge to the rank-one operator on $\eta$. I.e., for any $\eta \in \mathcal{H}$, the rank-one operator $\langle \eta, \xi \rangle_0$ is in $A$. But $A$ is closed under $*$, so $\langle \xi, \zeta \rangle_0$ is also in $A$ for all $\zeta \in \mathcal{H}$. Multiplying gives $\langle \eta, \zeta \rangle_0 \in A$, so all rank-one operators in $\mathcal{B}_0(\mathcal{H})$ are in $A$, and so all of $\mathcal{B}_0(\mathcal{H})$ is in $A$. (All rank-one are in, so all finite-rank, and we defined $\mathcal{B}_0$ to be the closure of finite-rank. Remember that you have to look fairly far to find a Banach algebra where the compact operators are not the closure of finite-rank ones, but there are some examples, but in $C^*$-land they all are.) $\square$

**Question from the audience:** This is a converse of Schur's lemma. **Answer:** In some sense.

## 15.2 Relations between irreducible representations and two-sided ideal

We don't need the full strength of a $C^*$-algebra.

**Prop:** Let $A$ be a $*$-normed algebra, $I$ a two-sided ideal, and assume that $I$ has a two-sided bounded approximate identity (for $I$). Let $(\pi, \mathcal{H})$ be a continuous irreducible representation of $A$. Then either

(a) $\pi(I) = 0$, or

(b) $\pi|_I$ is irreducible.

**Proof:**

If $\pi(I) \neq 0$, then look at $\overline{\{\pi(I)\mathcal{H}\}} \neq 0$ (meaning linear span), which is clearly $A$-invariant. So it is all of $A$, and hence $\overline{\{\pi(I)\mathcal{H}\}} = \mathcal{H}$, i.e. $\pi|_I$ is non-degenerate. Let $\{e_j\}$ be an approximate identity for $I$. We showed that $\pi(e_j)\xi \to \xi$ for all $\xi \in \mathcal{H}$. Let $\mathcal{K}$ be a closed $\pi|_I$-invariant subspace. Then $\mathcal{K}$ is $A$-invariant, because: given $\xi \in \mathcal{K}$ and $a \in A$, and switching to module notation, $a\xi = \lim a(e_j\xi) = \lim(ae_j)\xi$. But $ae_j \in I$, so $ae_j\xi \in \mathcal{K}$, and since $\mathcal{K}$ is closed, $a\xi \in \mathcal{K}$. $\square$

**Prop:** Let $A$ be a $*$-normed algebra, $I$ an ideal with approximate identity. Let $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ be two irreducible representations of $A$. If $\pi(I), \rho(I) \neq 0$ (so $\pi|_I$ and $\rho|_I$ are irreducible), and if $\pi|_I$ is unitarily equivalent to $\rho|_I$, then $\pi$ and $\rho$ are unitarily equivalent.

**Proof:**

Let $U : \mathcal{H} \to \mathcal{K}$ be a unitary equivalence over $I$. I.e. $U\pi(d) = \rho(d)U$ for $d \in I$, and $U$ unitary. Then for $a \in A$, we have $U\pi(a)\xi = \lim U\pi(a)\pi(e_j)\xi = \lim U\pi(ae_j)\xi = \lim \rho(ae_j)U\xi = \lim \rho(a)\rho(e_j)U\xi = \rho(a)U\xi$. $\square$

**Theorem:** Let $A$ be a $C^*$-algebra and $(\pi, \mathcal{H})$ an irreducible representation of $A$. If $\pi(A)$ contains at least one non-zero compact operator, then $\pi(A)$ contains all compact operators. In this case, moreover, any irreducible representation of $A$ with the same kernel as $\pi$ is unitarily equivalent to $(\pi, \mathcal{H})$.

We will give the proof next time.

## 16: February 29, 2008

****I was 10 minutes late.****

We prove the theorem stated at the end of lecture last time. ****Proof omitted, since I was late. The proof follows from the results stated in last time's lecture.****

**Definition:** (Kaplansky 1950s)

For $A$ a $C^*$-algebra,

(a) $A$ is *CCR* ("completely continuous representation", not "canonical commutation relations") if for every irreducidble representation $(\pi, \mathcal{H})$ of $A$, we have $\pi(A) = \mathcal{B}_0(\mathcal{H})$. In Dixmier's ****sp?**** book, these are called *liminal*.

(b) $A$ is *GCR* ("generalized") if for every irreducible representation, $\pi(A) \subseteq \mathcal{B}_0(\mathcal{H})$. Also called *postliminal*.

(c) $A$ is *NCR* ("not") if $\pi(A) \cap \mathcal{B}_0(\mathcal{H}) = 0$ for all irreducible representations. Also "antiliminal". (These might be "NGR" rather than "NCR".)

**E.g.** Let $K = \bigcap\{\ker(\pi) : \pi(A) \cap \mathcal{B}_0(\mathcal{H}) \neq 0\}$, then $K$ is $GCR$ and $A/K$ is $NCR$.

**Theorem:** (Harish-Chandra, 1954)

For $G$ a semi-simple Lie group (e.g. $SL(n, \mathbb{R})$), then $\pi(L^1(G)) \subseteq \mathcal{B}_0(\mathcal{H})$, i.e. $C^*(G)$ is CCR.

**Theorem:** (Dixmier)

If $G$ is a nilpotent Lie group (i.e. closed connected subgroups of $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ — every Lie group satisfying ****some conditions**** is a discrete quotient of one of these), then $C^*(G)$ is CCR.

Many solvable groups — $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ — are GCR. However, Mantner showed by 1950s, there exists solvable groups that are not GCR. **E.g.** Take $\mathbb{C}^2$, and let $\alpha$ be the action of $\mathbb{R}$ on $\mathbb{C}^2$ by $\alpha_t : (z, w) \mapsto (e^{it}z, e^{it\theta}w)$ with $\theta$ irrational. This doesn't change the lengths of vectors, so e.g. if $z, w \in S^1 \subseteq \mathbb{C}$, they are preserved. I.e. we have an orbit that, since $\theta \notin \mathbb{Q}$, is dense in the torus $T^2 = S \times S$. So let $G = \mathbb{C}^2 \rtimes_\alpha \mathbb{R}$. Topologically this is $\mathbb{R}^5$, but the group is not GCR.

**Prop:** Let $A$ be a unital $\infty$-dimensional simple (no proper ideals) $C^*$-algebra. Then $A$ is NCR.

**E.g.** Canonical anti-commutation relations:

$$M_2(\mathbb{C}) \longrightarrow M_4(\mathbb{C}) \longrightarrow M_8(\mathbb{C}) \longrightarrow \cdots$$

$$T \longmapsto \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \longmapsto \cdots$$

Taking $M_3 \to M_9 \to \ldots$, or $2 \mapsto p$ for other primes, give "ultra hyperfinite" algebras. Jim Glimm studied these in his doctoral thesis, and then went off into QFT, working with Arthur Jaffey, and then wandered into PDEs and shock waves, then numerical analysis.

Also, $C_r^*(F_n)$, where $F_n$ is a free group, is an interesting example for $n \geq 2$.

**Theorem:** (Thoma, 1962)

For a discrete group $G$, if $C^*(G)$ is GCR, then $G$ has an Abelian normal subgroup $N$ such that $G/N$ is finite.

**Definition:** For a $*$-algebra $A$, an ideal is *primitive* if it is the kernel of an irreducible $*$-representation on a Hilbert space. (Or omit the $*$ and look at Banach spaces.)

**Notation:** For a $*$-normed algebra $A$, we write $\hat{A}$ for the set of unitary equivalence classes of irreducible representations on Hilbert space.

**Cor:** For a GCR $C^*$-algebra, there is a bijection between $\hat{A}$ and the set of primitive ideals of $A$.

**Theorem:** (Machey, Dixmier, Fell, and final hard part by Glimm)

Let $I$ be a primitive ideal of $A$ with irreducible representation $(\pi, \mathcal{H})$ (i.e. $\ker \pi = I$) with $\pi(A) \cap \mathcal{B}_0(\mathcal{H}) = 0$. (So-called "bad case.") Then there is an uncountable number of inequivalent irreducible representations of $A$ with kernel $I$, and they are unclassifiable (in a specific technical sense stated by Machey).

**Proof idea:**

Let $B = A/I$, and look at irreducible faithful reps. Can find a Hilbert space $\mathcal{H}$ on which each rep can be realized. So ... ****out of time**** $\square$

# 17: March 3, 2008

****Again I was late to class.****

**Theorem:** (Glimm)

> If $A$ has an irreducible representation $\pi$ such that $\pi(A) \cap \mathcal{B}_0(\mathcal{H}) = 0$, then the irreducible representations are not classifiable.

To "classify" the irreducible representations means to find a countable number of real-valued Borel functions on $\mathbb{R}$ that are constant on the equivalence classes.

Because of the analogy with von Neuman algebras, GCR is also called "type I".

## 17.1 Some topology and primitive ideals

For non-GCR algebras, don't look for all irreducible representations. Look instead at the kernels of irreducible representations.

**Definition:** For a $C^*$ algebra $A$, a two-sided ideal is *primitive* if it is the kernel of an irreducible representation.

We write $\mathrm{Prim}(A)$ for the set of primitive ideals, although this is an unfortunate notation, as it looks like "prime". ****I will use "**$\mathrm{Spec}(R)$**" for the primes of a ring** $R$**.**** (Recall, $J$ a two-sided ideal of $R$ is *prime* if whenever $K_1K_2 \subseteq J$ and $K_1$ and $K_2$ are two-sided ideals, then at least one of the $K_i$ is in $J$.) In fact, if $A$ is $C^*$, then any primitive ideal is prime. **Proof:** Let $K_1, K_2$ be subideals of primitive ideal $J$ (with associated representation $(\pi, \mathcal{H})$) with $K_1K_2 \subseteq J$. Since $J$ is closed, we can assume that $K_1$ and $K_2$ are. Assume $K_1 \not\subseteq J$, and then $\pi(K_1) \neq 0$, and so by irreducibility $\overline{\pi(K_1)\mathcal{H}} = \mathcal{H}$. Ditto for $K_2$, and everything is continuous, so $\mathcal{H} = \overline{\pi(K_2)\mathcal{H}} = \overline{\pi(K_2)\overline{\pi(K_1)\mathcal{H}}} = \overline{\pi(K_2K_1)\mathcal{H}} = 0$. $\square$

**Rhetorical Question:** Is every prime ideal primitive? **Answer:** No. Consider $A = C([0,1])$ and $J = \{f \in A : f = 0 \text{ in a nbhd of } 1/2\}$. This is not a closed ideal, but it is prime. There are more complicated examples as well. But primitive ideals are closed. **Rhetorical Question′:** Is every closed prime ideal primitive? **Answer:** If $A$ is separable, yes, using Baire category theorem. In 2001, Nik Weaver constructed a (large) counterexample using transfinite induction.

For any ring $R$ (with 1), on the set $\mathrm{Spec}(R)$ of prime ideals we have the "hull-kernel" or "Jacobson" topology (for commutative rings called "Zariski" topology). We can take its restriction to any subset $M$ of prime ideals. Misusing the word ker, given $S \subseteq M$, we set $\ker_M(S) = \bigcap\{J \in S\}$. Conversely, given $I \in \mathrm{Spec}(R)$, we set $\mathrm{hull}_M(I) = \{J \in M : J \supseteq I\}$. Then we declare our topology: For $S \subseteq M$, we define the closure $\overline{S} \overset{\text{def}}{=} \mathrm{hull}_M \ker_M(S)$.

The Kuratowski closure axioms tell us when a definition of "closure" defines a topology.

1. $\overline{\emptyset} = \emptyset$

2. $\overline{S} \supseteq S$ ****original notes said $\subseteq$, but that is surely wrong, and I probably mistranscribed from the board****
3. $\overline{\overline{S}} = \overline{S}$
4. $\overline{S \cup T} = \overline{S} \cup \overline{T}$

Only this last property requires any thinking. If $S \subseteq T$, then $\overline{S} \subseteq \overline{T}$, clearly, and thus we have $\overline{S \cup T} \supseteq \overline{S} \cup \overline{T}$. Why $\subseteq$? This requires primality. Let $L \in \overline{S \cup T}$. Then $L \supseteq \bigcap\{J \in S \cup T\} = \bigcap\{J \in S\} \cap \bigcap\{J \in T\} = \ker(S) \cap \ker(T) \supseteq \ker(S)\ker(T)$ (multiplication as ideals). But $L$ is prime, so $L \subseteq \ker(S)$ or $\ker(T)$, and so $L \in \overline{S} \cup \overline{T}$.

So, on $\mathrm{Prim}(A)$ (for $A$ a $C^*$-algebra), put on the hull-kernel topology. $\mathrm{Prim}(A)$ is not in general Hausdorff. But for $C^*$-algebras, it is locally compact, in the sense that every point has a closed neighborhood such that any open cover of the nbhd has a finite subcover. Now, Baire category theorem works in separable land, and also in this case. $\mathrm{Prim}(A)$ is $T_0$, i.e. if $J \in \overline{\{K\}}$ and $K \in \overline{\{J\}}$, then $K = J$.

**E.g.** Let $A = \mathcal{B}_0(\mathcal{H})^\sim$, i.e. $\mathcal{B}_0(\mathcal{H})$ adjoin an identity element. Then we have (only) two closed ideal 0 and $\mathcal{B}_0(\mathcal{H})$, and $\overline{\{0\}} = \{0, \mathcal{B}_0(\mathcal{H})\}$, and $\{\mathcal{B}_0(\mathcal{H})\}$ is closed.

# 18: March 5, 2008

## 18.1 Examples

We begin with some rather abstract examples, but we won't stay for very long. It's good to have a framework for examples: generators and relations. We can have a (possibly very infinite) collection $\{a_1, a_2, \dots\}$ of *generators*, and we should also have $a_1^*, a_2^*, \dots$. The *relations* are non-commutative polynomials in the generators. Then form the *free algebra* $\mathcal{F}$ on the generators, which is a $*$-algebra: $* : a_i \mapsto a_i^*$ and $*$ reverses the order of words and is anti-linear over $\mathbb{C}$. Let $I$ be the $*$-ideal generated by the relations. Then form $A = \mathcal{F}/I$, which is a $*$-algebra: it is *the universal $*$-algebra for the given generators and relations.*

We can look for $*$-representations of $A$, i.e. $*$-homomorphisms of $A$ into $\mathcal{B}(\mathcal{H})$ for various Hilbert spaces $\mathcal{H}$. For $a \in A$, set

$$\|a\|_{C^*} \stackrel{\text{def}}{=} \sup\{\|\pi(a)\| : \ \pi \text{ is a } *\text{-rep of } A\}$$

This might be $+\infty$. **E.g.** one generator (and its adjoint) and no relations. We might also have $\|a\|_{C^*} = 0$ forall $a$. **E.g.** one generator, with relation $a^*a = 0$.

So the issues are:

1. Do the relations force $\|a\|_{C^*} < \infty$ on the generators (if it's finite on the generators, then it's finite on any polynomial).

**E.g.** $u^*u = 1 = uu^*$ (we have a generators called "1", satisfying all the relations 1 should have). Then $u$ is unitary in any representations, so $\|u\| = 1$.

2. Are there non-zero representations?

   It may happen that $\|a\|_{C^*} = 0$ for certain $a \in A$, and such $a$ form an ideal, by which we can factor out.

If 1. holds, then the quotient in 2. will have a norm satisfying the $C^*$ relation, and factoring out gives a $C^*$-norm, so we can complete. This gives *the $C^*$-algebra for the generators and relations.*

3. There may be a natural class of $*$-representations which give a $C^*$-norm $\|\cdot\|'_{C^*}$, but this $\|\cdot\|'_{C^*}$ does not give the full $\|\cdot\|_{C^*}$. It might be smaller.

**Question from the audience:** Like the atomic norm, taking just irreducible representations? **Answer:** No, that will just give us back the same thing. Indeed, doing this construction to a $C^*$-algebra will leave it intact.

**E.g.** If we just have the one relation $S^*S = 1$ and not on the other side, then we get the $C^*$-algebra for the unilateral shift on $\ell^2(\mathbb{N})$.

**E.g.** Let $G$ be a discrete group, and take the elements of $G$ as generators, with relations as in $G$. Also demand that $x^* = x^{-1}$. Then the representations of this set of generators and relations is the same as unitary representations — well, we never demand that an algebra's identity element go to $\mathbb{1} \in \mathcal{B}(\mathcal{H})$, but it will go to an idempotent, i.e. a projection operator, so we can cut down — on subspaces. All words in $A$ in the the generators are just given by elements of $G$. So $A$ (purely at the algebraic level; we haven't completed) consists of finite linear combinations of elements of $G$. I.e. given by functions $f \in C_c(G)$ (continuous of compact support). ****I would call this $\mathbb{C}[G]$ instead. This construction is covariant in $G$, but $C_c(-)$ is by rights contravariant.**** I.e. an elements of $A$ is $\sum_{x\in G} f(x)x$. **Question from the audience:** Compact support for the discrete group is just ... **Answer:** Finite support. **Question from the audience:** So this is exactly the group ring. **Answer:** Yes.

**E.g.** $G = SL(3, \mathbb{Z})$. Where do we find irreducible unitary representations of this?

In fact, for this setting, there always exist two unitary representations:

(a) The trivial representation, 1-dimensional on $\mathcal{H} = \mathbb{C}$.

(b) The left-regular representation of $G$ on $\ell^2(G)$:

$$(L_x\phi)(y) = \phi(x^{-1}y)$$

We need the inverse to preserve $L_xL_z = L_{xz}$. This really is a unitary operator, satisfying all the necessary relations.

Some group-ring calculations:

$$\left(\sum f(x)x\right)\left(\sum f(y)y\right) = \sum_{x,y} f(x)g(y)xy = \sum_z \left(\sum_{xy=z} f(x)g(y)\right) z = \sum_y \underbrace{\left(\sum_x f(x)g(x^{-1}y)\right)}_{\in C_c(G)} y$$

I.e. we define *convolution* on $C_c(G)$ by

$$(f \star g)(y) \stackrel{\text{def}}{=} \sum_x f(x)g(x^{-1}y)$$

Then $\left(\sum f(x)x\right)\left(\sum f(y)y\right) = \sum (f \star g)(z)z$.

What about the $*$?

$$\left(\sum f(x)x\right)^* = \sum \overline{f(x)}x^* = \sum \bar{f}(x)x^{-1} = \sum \bar{f}(x^{-1})x$$

So on $C_c(G)$ we set

$$f^*(x) \stackrel{\text{def}}{=} \overline{f(x^{-1})}$$

Anyway, for a representation $(\pi, \mathcal{H})$ of $G$, we have $\pi\left(\sum f(x)x\right) \stackrel{\text{def}}{=} \sum f(x)\pi(x)$. So, for $f \in C_c(G)$, set $\pi_f \stackrel{\text{def}}{=} \sum f(x)\pi(x)$. Then $f \to \pi_f$ is a $*$-representation of $C_c(G)$. Conversely, a $*$-representation of $C_c(G)$ must restrict to a unitary representation of $G$, since we can view $G \hookrightarrow C_c(G)$, by $x \mapsto \delta_x$. ****Gah! If $G$ is not discrete, then the $\delta$ functions are not in $C_c(G)$, although they are in $\mathbb{C}[G]$ the group ring.****

Let $L$ be the left-regular representation of $G$ on $\ell^2(G)$, and look at $\delta_e \in \ell^2(G)$ be the vector at the identity. Then

$$L_f\delta_e = \left(\sum f(x)L_x\right)\delta_e = \sum f(x)\delta_x \in \ell^2(G)$$

So if $L_f = 0$, then $f = 0$. So $L$ is a faithful $*$-representation of $C_c(G)$. So left $\|f\|^r_{C^*} \stackrel{\text{def}}{=} \|L_f\|$; this is a legitimate $C^*$-norm on $C_c(G)$. This is an example of 3. above. ($r$ for "reduced")

**Theorem:** $\|\cdot\|^r_{C^*(G)} = \|\cdot\|_{C^*(G)}$ if and only if $G$ is *amenable*.

There are twenty different equivalent definitions of "amenable". Where does the name come from? $G$ is *amenable* if on $\ell^\infty(G)$ there is a *state* ("mean") $\mu$ which is invariant under left translation, e.g. $\mu(L_x\phi) = \mu(\phi)$ forall $\phi \in \ell^\infty(G)$. All abelian groups are amenable. **Exercise:** Why is $\mathbb{Z}$ amenable?

**Question from the audience:** You get the left-invariant representation by looking at .... Where is the other one? What is the norm on $C^*(G)$? **Answer:** We defined $\|L_f\| \stackrel{\text{def}}{=} \|L_f\|_{\mathcal{B}(\ell^2(G))}$.

**Question from the audience:** What is the topology on $G$? **Answer:** Discrete. We will eventually imitate this on locally compact groups.

**Question from the audience:** How is this $\mu$ related to the Haar measure? **Answer:** Every finite group is amenable. Just use the Haar measure.

## 19: March 7, 2008

We left off on this business of amenable groups. Another version:

**Definition/Theorem:** $G$ is amenable if it satisfies Følner's condition:

$$\forall \epsilon > 0, \forall \text{ finite } K \subset G, \exists \text{ finite } U \in G \text{ s.t. } \forall x \in K, \frac{|xU \bigtriangleup U|}{|U|} < \epsilon$$

Where $\bigtriangleup$ is the symmetric difference "xor."

Finite groups and solvable groups are amenable, but $F_n$ the free group and $SL(n, \mathbb{Z})$ for $n \geq 2$ are not.

### 19.1 Tensor products

Let $A$ and $B$ be $C^*$-algebras with 1. We want $A \otimes B$, which should include elements like $a \otimes b$, and $A \hookrightarrow A \otimes B$ via $a \mapsto a \otimes 1_B$. The multiplication has $A$ and $B$ commuting: $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$.

We now consider the $*$-algebra with generators $A \cup B$ and relations those of $A$ and those of $B$ and that $ab = ba$ if $a \in A$ and $b \in B$ (and that $1_A = 1_B$). Then we exactly get $A \overset{\text{alg}}{\otimes} B$. This is a $*$-algebra: $(a \otimes b)^\star = a^* \otimes b^*$.

We haven't yet introduced a norm. Does $A \overset{\text{alg}}{\otimes} B$ have any $*$-reps? There is a natural class: Let $(\pi, \mathcal{H})$ be a $*$-rep of $A$ and $(\rho, \mathcal{K})$ a $*$-rep of $B$. We form $\mathcal{H} \overset{\text{alg}}{\otimes} \mathcal{K}$ the algebraic tensor product of vector spaces, with inner product $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle \overset{\text{def}}{=} \langle \xi_1, \xi_2 \rangle_{\mathcal{H}} \langle \eta_1, \eta_2 \rangle_{\mathcal{K}}$, extended to $\mathcal{H} \overset{\text{alg}}{\otimes} \mathcal{K}$ by (conjugate) linearity. Check that this result is positive definite (not hard by expressing everything in terms of an orthonormal basis). Then complete to get the Hilbert-space tensor product $\mathcal{H} \otimes \mathcal{K}$.

For $S \in \mathcal{B}(\mathcal{H})$, we define $(S \otimes \mathbb{1}_{\mathcal{K}})(\xi \otimes \eta) = (S\xi) \otimes \eta$, and extend by linearity to $\mathcal{H} \overset{\text{alg}}{\otimes} \mathcal{K}$, where it is a bounded operator, so extends by continuity to $\mathcal{H} \otimes \mathcal{K}$. Then we can check that $\|S \otimes \mathbb{1}_{\mathcal{K}}\| = \|S\|$ and $(S \otimes \mathbb{1})^* = S^* \otimes \mathbb{1}$. All this also works for $\mathbb{1}_{\mathcal{H}} \otimes T$. On the algebraic tensor product, these elements commute, so $S \otimes T$ is well-defined,

In any case, we can define $(\pi \otimes \rho)(a \otimes b) = \pi(a) \otimes \rho(b)$ on $\mathcal{H} \otimes \mathcal{K}$, which extends to $A \overset{\text{alg}}{\otimes} B$. If $\pi$ is faithful then $\pi \otimes \rho$ is faithful.

**Question from the audience:** On $\mathcal{H} \otimes \mathcal{K}$, are all the bounded operators of this form? **Answer:** Oh, certainly not. Take finite linear combinations of elementary tensors, and close up in the weak-$*$ topology. By double-commutant theorem, these are dense.

For $t \in A \overset{\text{alg}}{\otimes} B$, we set

$$\|t\|_{\min} \overset{\text{def}}{=} \sup\{\|(\pi \otimes \rho)t\| : (\pi, \mathcal{H}), (\rho, \mathcal{K}) \text{ are representations of } A, B\}$$

This is a $C^*$-norm. It's a nice norm, but it's not the universal norm. We define the closure of $A \overset{\text{alg}}{\otimes} B$ with this norm to be $A \overset{\min}{\otimes} B$.

$\|t\|_{\max} \overset{\text{def}}{=} \sup\{\|(\pi \otimes \rho)t\| : \pi, \rho$ are reps of $A, B$ on same $\mathcal{H}$ s.t. $\pi(a), \rho(b)$ commute $\forall a \in A, b \in B\}$

Then $\|t\|_{\min} \leq \|t\|_{\max}$, since we can always take $\mathcal{H}$ in the second definition to be $\mathcal{H} \otimes \mathcal{K}$ in the first, and we can complete with the latter to define $A \overset{\max}{\otimes} B$.

Are these the same? In 1959, Takesaki ****sp?**** said no: Let $G$ be a discrete group, and let $\pi$ be the left regular representation of $C_r^*(G)$ on $\ell^2(G)$. Let $\rho$ be the right regular representation. Left and right representations commute, so $\pi \otimes \rho$ gives a representation of $C_r^*(G) \overset{\text{alg}}{\otimes} C_r^*(G)$ on $\ell^2(G)$. But this does not split as a tensor product of representations, and e.g. for $G = F_n$, $n \geq 2$, the free group on $n$ generators, we have $\|\cdot\|_{\max} \gneq \|\cdot\|_{\min}$.

A diagram, where $\dashrightarrow$ is $\pi \otimes \rho$ in the definition of $\|\cdot\|_{\max}$:

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr, "\pi"] & & \\
& A \overset{\text{alg}}{\otimes} B \arrow[r, dashed] & \mathcal{B}(\mathcal{H}) \\
B \arrow[ur] \arrow[urr, "\rho"'] & &
\end{tikzcd}$$

Everyone sort of assumed that tensor products were easy, until this example came along. Then min and max products are minimal and maximal in the appropriate sense, but there are many intermediate ones.

**Definition:** A $C^*$-algebra $A$ is *nuclear* if for any $C^*$-algebra $B$ we have $A \overset{\min}{\otimes} B = A \overset{\max}{\otimes} B$. (I.e. the two norms are the same.)

**E.g.** commutative, $\mathcal{B}_0(\mathcal{H})$, any GCR. There are others too.

Given a short-exact sequence
$$0 \to I \to B \to B/I \to 0,$$
we can show that
$$0 \to A \overset{\max}{\otimes} I \to A \overset{\max}{\otimes} B \to A \overset{\max}{\otimes} (B/I) \to 0$$
is exact. So we say that $A$ is *exact* if for any exact $0 \to I \to B \to B/I \to 0$, we have

$$0 \to A \overset{\min}{\otimes} I \to A \overset{\min}{\otimes} B \to A \overset{\min}{\otimes} (B/I) \to 0$$

exact.

Then nuclear implies exact.

If $G$ is discrete, then $G$ is amenable iff $C^*(G)$ is nuclear (this fails for some non-discrete groups). Open question: does there exist $G$ discrete with $C^*(G)$ not exact?

These matter for various differential-geometry questions.

Gromov: "Any statement you can make about all discrete groups is either trivial or false." This question is certainly not trivial; Gromov has some ideas of where to look for a counterexample.

## 20: Problem Set 2: Due March 14, 2008

****The problem set was given out typed. I've retyped it, partly so I could submit my answers set between the questions. I have corrected some typos, and no doubt introduced even more. In doing so, I have changed the formatting slightly.****

**A. Fields of $C^*$-algebras.** Anytime the *center* of a $C^*$-algebra (i.e. the set of its elements which commute with all elements of the algebra) is more than one-dimensional and acts non-degenerately on the algebra, the $C^*$-algebra can be decomposed as a field of $C^*$-algebras over the maximal ideal space of the center (or of any non-degenerate $C^*$-subalgebra of the center). For simplicity we deal here with unital algebras, but all of this works without difficulty in general. So let $A$ be a $C^*$-algebra with 1, and let $C$ be a $C^*$-subalgebra of the center of $A$ with $1 \in C$. Let $C = C(X)$, and for $x \in X$ let $J_x$ be the ideal of functions vanishing at $x$. Let $I_x = AJ_x$ (closure of linear span), an ideal in $A$. Let $A_x = A/I_x$ ("localization"), so $\{A_x\}_{x \in X}$ is a field of $C^*$-algebras over $X$. For $a \in A$ let $a_x$ be its image in $A_x$.

1. Prove that for any $a \in A$ the function $x \mapsto \|a_x\|_{A_x}$ is upper-semi-continuous. (So $\{A_x\}$ is said to be an upper-semi-continuous field.)

2. If $x \mapsto \|a_x\|_{A_x}$ is continuous for all $a \in A$, then the field is said to be continuous. For this part assume that $A$ is commutative. Note that then one gets a continuous surjection from $\hat{A}$ onto $\hat{C}$. Find examples of $A$s and $C$s for which $x \mapsto \|a_x\|$ is not continuous. In fact, find an attractive characterization of exactly when the field is continuous, in terms of the surjection of $\hat{A}$ onto $\hat{C}$ and concepts you have probably met in the past. (It can be shown that an analogous characterization works in the non-commutative case, using the primitive ideal space of $A$.)

3. Let

$$A_1 \overset{\text{def}}{=} \left\{ f : [0,1] \to M_2 \text{ continuous, with } f(1) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right\}$$

$$A_2 \overset{\text{def}}{=} \left\{ f : [0,1] \to M_2 \text{ continuous, with } f(1) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right\}$$

and let $C_i \overset{\text{def}}{=} Z(A_i)$ be the center of $A_i$. Are the corresponding fields continuous? Are all the fiber algebras $A_x$ isomorphic? Show that $A_1$ and $A_2$ are very simple prototypes of behavior that occurs often "in nature", but with higher-dimensional algebras, and more complicated boundary behavior.

4. Determine the primitive ideal space of each of these two algebras, with its topology.

**B. An important extension theorem.** (This will be used in the lectures.) Prove that if $I$ is a $*$-ideal of a $*$-normed algebra $A$, and if $I$ has an approximate identity of norm one for itself, then every non-degenerate $*$-representation of $I$ extends uniquely to a non-degenerate representation of $A$.

**C. The non-commutative Stone-Čech compactification.** (At a few points in the course I may use the results of this problem.) Motivation: If the locally compact space $X$ is an open subset of the compact space $Y$, then $C_\infty(X)$ "is" an ideal in the $C^*$-algebra $C_b(Y)$ of bounded continuous functions on $Y$. Then $X$ is dense in $Y$ exactly if $C_\infty(X)$ is an essential ideal in $C(Y)$, where by definition, an ideal $I$ in an algebra $B$ is *essential* if there is no non-zero ideal $J$ in $B$ with $IJ = 0$ or $JI = 0$. Thus the Stone-Čech compactification of an algebra $A$ without 1 should be a "maximal" algebra with 1 in which $A$ sits as an essential ideal. If $B$ is any algebra in which some (probably non-unital) algebra $A$ sits as an ideal, then each $b \in B$ defines a pair $(L_b, R_b)$ of operators on $A$ defined by $L_b a = ba$, $R_b a = ab$. These operators satisfy, for $a, c \in A$, $L_b(ac) = (L_b(a))c$, $R_b(ca) = c(R_b(a))$, and $a(L_b(c)) = (R_b(a))c$.

**Definition:** By a *double centralizer* (or *multiplier*) on an algebra $A$ we mean a pair $(S, T)$ of operators on $A$ satisfying the above three conditions. Let $M(A)$ denote the set of double centralizers of $A$.

1. Using the example of $A$ as ideal in $B$ as motivation, define operations on $M(A)$ making it into an algebra, with a homomorphism of $A$ onto an ideal of $M(A)$.

2. Show that if $A$ is a Banach algebra with approximate identity of norm one, and if we require $S$ and $T$ to be continuous (which actually is automatic), then $M(A)$ can be made into a Banach algebra in which $A$ sits isometrically as an essential ideal. (This is quite useful for various Banach algebras which are not $C^*$-algebras. For example, if $A = L^1(G)$ for a locally compact group $G$, then it can be shown that $M(A)$ is the convolution measure algebra $M(G)$ of $G$.) Show that if $A$ is a $*$-Banach algebra, then its involution extends uniquely to make $M(A)$ a $*$-algebra. Note then that the theorem of problem B. above says that every nondegenerate $*$-representation of $A$ extends to $M(A)$.

3. Show that if $A$ is a $C^*$-algebra, then so is $M(A)$.

4. Let $A$ be a $C^*$-algebra, and let $X = A_A$ as a right $A$-module, with $A$-valued inner product as defined in class. Let $B_A(X)$ be the algebra of all continuous (which actually is automatic) $A$-module endomorphisms of $X$ that have a continuous enomorphism as adjoint for the $A$-valued inner product (which is not automatic). Show that in a very natural way $M(A) = B_A(X)$.

5. For $A$ a $C^*$-algebra, show that if $B$ is any $C^*$-algebra in which $A$ sits as an essential ideal, then $B$ can be identified as a subalgebra of $M(A)$, so $M(A)$ is maximal in this

sense, and thus ban be considered to be the Stone-Čech compactification of $A$.

6. Determine $M(A)$ when $A = C_\infty(X)$, and when $A = \mathcal{B}_0(\mathcal{H})$, the algebra of compact operators on a Hilbert space $\mathcal{H}$.

**D. Morphisms.** If $X$ and $Y$ are locally compact spaces and $\phi$ is a continuous map from $X$ to $Y$, then $\phi$ determines a homomorphism from $C_\infty(Y)$ to $C_b(X)$, the algebra of bounded continuous functions.

1. Give a characterization of those homomorphisms from $C_\infty(Y)$ to $C_b(X)$ which arise in this way from maps from $X$ to $Y$. Your characterization should be phrased so that it makes sense for non-commutative $C^*$-algebras. (Hint: recall the definition of a representation being non-degenerate.) Such homomorphisms are then called "morphisms". That is, define what is meant by a morphism from a (non-commutative) $C^*$-algebra $A$ "to" a $C^*$-algebra $B$.
2. For the non-commutative case explain how to compose morphisms.

## 21: March 10, 2008

Last time we defined the tensor product of $C^*$ algebras. We also have a free product:

**Definition:** Given $C^*$-algebras $A$ and $B$, we define $A * B$ to be the free algebra with all relations in $A$ and all those in $B$, and that $\mathbb{1}_A = \mathbb{1}_B$, but we do not require that the algebras commute.

Then a representation is just a pair of non-commuting representations on the same Hilbert space. There is also a reduced product $A *_r B$, which we will not go into.

**E.g.** $C(S^1) * C(S^1) = C^*(F_2)$, because $C(S^1)$ is the $C^*$ algebra generated freely by one unitary operator.

### 21.1 $C^*$-dynamical systems

Let $A$ be a $C^*$ algebra, $G$ a discrete group, and $\alpha : G \to \text{Aut}(A)$. **E.g.** Let $M$ be a locally compact space, $\alpha : G \to \text{Homeo}(M)$. Set $A = C_\infty(M)$; then $(\alpha_x(f))(m) \stackrel{\text{def}}{=} f(\alpha_x^{-1}(m))$ where $\alpha : x \in G \mapsto \alpha_x$.

The first discussion of what we are about to say came from quantum physics, where the *observables* of a system are self-adjoint operators (possibly unbounded, but we will duck that question, as well as the philosophy of physics), i.e. they are in some $C^*$-algebra $A$. We have already defined "states" for an algebra, and we will continue that notion here. Symmetries of the system form a group $G$ ****usually a Lie group****.

The physicists want everything acting on a Hilbert space, which in fact is a useful way to understand groups acting on algebras of operators. So we will represent $A$ on a Hilbert space $\mathcal{H}$, via a $*$-rep $\pi$, and let's ask for $U$ to be a unitary representation of $G$ on $\mathcal{H}$. What about the action $\alpha$? From the physicists' point of view, $\alpha$ should be unitarily represented.

Setting $\beta_x(T) = U_x(T)U_{x^{-1}}$ gives an action $G \to \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$, as inner representations. So we demand what the physicists call the *covariance condition*:

$$\pi(\alpha_x(a)) = U_x \pi(a) U_x^{-1}$$

**Definition:** We say that $(\pi, U)$ is a *covariant representation* of $(A, G, \alpha)$ if this condition holds.

We can use the generators of $G$ and $A$ and their relations, along with the covariance relation, which can be rewritten as $xa = \alpha_x(a)x$, and the requirement that $x^* = x^{-1}$. But this says that any word in the generators can be rearranged into normal form with all the $x$s on the right and all the $a$s on the left (just about everyone seems to use this convention); but then we can multiply adjacent $x$s and adjacent $a$s. So the $*$-algebra is just finite linear combinations of $ax$, i.e. sums of the form $\sum f(x)x$ where $f(x) \in A$.

So $f$ contains the data of the element, and so we define operations on $C_c(G, A)$ (= functions of finite support with values in $A$):

$$\begin{aligned}
\left(\sum f(x)\, x\right) \left(\sum g(y)\, y\right) &= \sum_{x,y} f(x)\, x\, g(y)\, y \\
&= \sum f(x)\, \alpha_x(g(y))\, xy \\
&= \sum_{x,y} f(x)\, \alpha_x(g(x^{-1}y))\, y \\
&= \sum_y \left( \sum_x f(x)\, \alpha_x(x^{-1}y) \right)
\end{aligned}$$

So we define the *twisted convolution* ****the standard notation, using $*$ for both the convolution and the adjoint, is unfortunate; I will use $\star$ for convolution****:

$$(f \star g)(y) = \sum f(x)\, \alpha_x(g(x^{-1}y))$$

We also have a $*$ operation:

$$\begin{aligned}
\left(\sum f(x)\, x\right)^* &= \sum x^*\, f(x)^* \\
&= \sum x^{-1}\, f(x)^* \\
&= \sum \alpha_x^{-1}(f(x)^*)\, x^{-1} \\
&= \sum \alpha_x(f(x^{-1})^*)\, x
\end{aligned}$$

So, every covariant representation $(\pi, U)$ of $(A, G, \alpha)$ will give a representation of $(C_c(G, A), \star, ^*)$. For $f \in C_c(G, A)$, we set $\sigma_f \stackrel{\text{def}}{=} \sum \pi(f(x))U_x$; then $\sigma$ is a $*$-rep of $(C_c(G, A), \star, ^*)$.

Then we can estimate norms:

$$\|\sigma_f\| \leq \sum \|f(x)\|_A \stackrel{\text{def}}{=} \|f\|_1$$

where $\|\cdot\|_1$ is the "$\ell^1$" norm in $A$.

In general, we define $\|f\|_{C^*(G,A,\alpha)}$ to be the supremum over all such representations, but it's not clear that there are any.

We can make the following comments. In a suitable sense, $A \hookrightarrow C_c(G, A, \alpha)$ by $a \mapsto a\delta_{1_G}$. If $A$ has an identity element, then $G \hookrightarrow C_c(G, A, \alpha)$ by $x \mapsto 1_A\delta_x$. If $A$ does not have a unit, then $G \to M(C_c(G, A, \alpha))$ where this is the algebraic multiplier algebra, in the sense as on the problem set. All of this works for $*$-normed algebras

Why are there plenty of covariant representations? We need representations on $A$, which for generic $*$-normed algebras might be few and far between. But for each rep $\rho$ of $A$ on $\mathcal{K}$, form the *induced covariant representation* of $(G, A, \alpha)$. (This is induced from $\{e\} \subseteq G$; we can induce from any subgroup.) In particular, we take $\mathcal{H} = \ell^2(G, \mathcal{K}) = \ell^2(G) \otimes \mathcal{K}$. Then the actions are by

$$\begin{aligned}
(U_x\xi)(y) &\stackrel{\text{def}}{=} \xi(x^{-1}y) \\
(\pi(a)\xi)(y) &\stackrel{\text{def}}{=} \rho(\alpha_y^{-1}(a))\xi(y)
\end{aligned}$$

We check the covariance conditions, and sure enough it passes.

Then we define the *reduced norm*:

$$\|f\|_{C_r^*(G,A,\alpha)} = \sup\{\|\pi(f)\| \text{ for all induced covariant reps}\}$$

If we start with a faithful representation of $A$, then our induced representation is faithful on the functions of compact support, so this is a norm. The full norm:

$$\|f\|_{C_r^*(G,A,\alpha)} = \sup\{\|\pi(f)\| \text{ for all covariant reps}\}$$

## 22: March 12, 2008

****I was out sick.****

## 23: March 14, 2008

We have a problem set due today. Many have asked for more time; that is fine. We'd prefer a more complete paper on Monday over a less complete paper today.

## 23.1 Twisted convolution, approximate identities, etc.

We were sketching what happens when $G$ is locally compact but not discrete. We looked at $C_c(G, A)$ the continuous functions (compact support) with values in $A$. We have $(A, G, \alpha)$ with $\alpha$ strongly continuous, and we look for covariant representations. For a covariant representation $\{\pi, U, \mathcal{H}\}$ and $f \in C_c(G, A)$, we set

$$\sigma_f \xi = \int \pi(f(x))\, U_x \xi \, dx$$

where $dx$ is the left Haar measure. Then we define *twisted convolution*:

$$(f \star_\alpha g)(x) \overset{\text{def}}{=} \int f(y)\, \alpha_y\big(g(y^{-1}x)\big)\, dx$$

Then $\sigma_f \sigma_g = \sigma_{f \star_\alpha g}$ and $\|\sigma_f\| \le \|f\|_1 \overset{\text{def}}{=} \int \|f(x)\|_A \, dx$.

Now we look at $G$ and $C_c(G)$. In the discrete case, if $A$ has an identity (and we're using $A = \mathbb{C}$), then $C_c(G)$ has an identity element, given by the $\delta$ function at the identity. But in the non-discrete ****indiscrete?**** case, any neighborhood has infinitely many points, so the Haar measure cannot give any point positive measure. In particular, we do not have an identity in $C_c(G), \star$. All this extends to $L^1(G, A)$ by uniform continuity.

We do have an approximate identity: Let $\mathcal{N}$ be a neighborhood base of $1_G$. For $U \in \mathcal{N}$, choose (Urysson) $f_U \in C_c(G)$ with support in $U$, $f_U \ge 0$, and $\|f_U\|_1 = \int f_U = 1$. By strong continuity of $\alpha$, $f_U \star_\alpha g$ is very close to $g$. Then this is an approximate identity of norm 1 for $L^1(G)$.

In the more general case, if $e_\lambda$ is an approximate identity of norm 1 for $A$, then $\{e_\lambda f_u\}_{\lambda, U}$ is an approximate identity of norm 1 for $L^1(G, A)$.

We've been ducking an issue here.

$$\begin{aligned}
\sigma_f^* \xi &= \int (\pi(f(x)) U_x)^* \, \xi \, dx \\
&= \int U_x^* \pi(f(x)^*) \xi \, dx \\
&= \int U_{x^{-1}} \pi(f(x)^*) \xi \, dx \\
&= \int \alpha_{x^{-1}}(\pi(f(x))^*)\, U_{x^{-1}} \xi \, dx \\
&= \int \alpha_x\big(\pi(f(x^{-1})^*)\big)\, U_x \xi \, d(x^{-1})
\end{aligned}$$

But this isn't quite right. $f \mapsto \int f(x^{-1})\, dx$ is right-translation-invariant, not left-translation-invariant. The problem is that the left Haar measure need not be right-invariant.

**Definition:** $G$ is *unimodular* if left Haar = right Haar.

**E.g.** Abelian groups, compact groups (not immediately obvious), discrete groups, semi-simple Lie groups, nilpotent Lie groups.

But there are many solvable Lie groups that are not unimodular. **E.g.** "$ax + b$" group of affine transformations of $\mathbb{R}$, for $a \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}$. (For more details, take Math 260.) This is the simplest nonabelian solvable Lie group, and it is not unimodular. Exercise: explicitly compute the left and right Haar measures; this gets into "What is Haar on $\mathbb{R}_{>0}$ with respect to $\times$?" — this expression, with respect to Lebesgue measure, pops up all over.

For a non-unimodular group $G$, we have $d(x^{-1}) = \Delta(x)dx$, where $\Delta(x)$ is the "modular function" of $G$. It's nice: it sends $\Delta : G \to \mathbb{R}_{>0}$ under a continuous group homomorphism. It's not hard to show this, but we will not. One has to make a convention, which is not always agreed upon; some people would use $\Delta(x^{-1})$. Well, if $G$ happens to be compact, then there are very few continuous homomorphisms into the positive reals, because there are very few compact subgroups of $\mathbb{R}_{>0}$. Hence compact groups are unimodular.

So, in the non-unimodular case, we must define the involution as:

$$f^*(x) = \alpha_x\big(f(x^{-1})^*\big)\,\Delta(x^{-1})$$

At various cases, this complicates the bookkeeping, and even worse, there are some theorems that work for unimodular and do not work for non-unimodular groups (without becoming substantially more complicated). Whenever someone thinks they have a theorem for locally compact groups, they prove it for unimodular groups and then have to go back and check with the modular functions.

**Question from the audience:** What is a solvable Lie group? **Answer:** Up to discrete subgroups of the center, they are of the form:

$$\begin{pmatrix} * & & & * \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}$$

Nilpotent has 1s on the diagonal. You can always embed a nonunimodular group into a unimodular one by extending by a copy of $\mathbb{R}$: you let the real line act as modular automorphisms, and get a "Type II" algebra (meaning it has traces).

Later on, one very much wants to look at homogeneous spaces $G/H$, which is an extremely rich collection of manifolds. Since $G$ acts on $G/H$, we can ask if there is a measure on $G/H$ that is preserved by the $G$ action. This wraps up the modular functions on $G$ and on $H$; ultimately, the answer is nice, if a bit complicated.

So anyway, we have a $*$ algebra with approximate identity of norm 1. Are there covariant representations of $(G, A, \alpha)$. The operations are all arranged to that covariant representations give us $*$-representations of $C_c(G, A)$. We have the "induced representations" from representations of $A$. We did that for discrete groups; just replace sums by integrals with respect to Haar measure. This gives a nice class of representations, which are faithful on the algebra. We can define the *reduced* $C^*$

algebra $C_r^*(A, G, \alpha) \overset{\text{def}}{=} A \times_\alpha^r G$, where the norm comes from just the induced representations. And we have the full algebra $C^*(A, G, \alpha) = A \times_\alpha G$, which can be different. Even if $A = \mathbb{C}$, we can have $C_r^*(G) \neq C^*(G)$; $G$ is ammenable iff these are equal. **E.g.** $SL(n, \mathbb{R})$ is not ammenable. We always have a quotient map $C^*(G) \to C_r^*(G)$, so representations of $C_r^*$ give representations of $C^*$. We call the ones that come this way *tempered*, but this is still a very active field of investigation. It even got into the newspapers: a huge calculation that made progress into finding the representations of $E_8$. We have essentially a complete list of the semisimple Lie algebras, or at least the real forms of them, but sorting out the representations is hard: we get into representations that are not on Hilbert spaces, or that are not unitary. So be warned: sometimes the word "tempered" is used for non-unitarizable representations.

**Theorem:** There exists a bijection between covariant representations of $(A, G, \alpha)$ (nondegenerate as representations of $A$) and non-degenerate representations of $C^*(A, G, \alpha)$.

**Proof:**

$\sigma$ is nondegenerate iff $\pi$ is.

We have the mapping on one direction.

$A$ does not need to have an identity element, but think about the multiplier algebra $M(C^*(A, G, \alpha))$. Then $G$ and $A$ both sit inside: $G, A \hookrightarrow M$. So $C^*(A, G, \alpha)$, which sits inside as an essential ideal (from the problem set), so any representation of $C^*$ extends to a representation uniquely of $M$ (by problem set), and compose with $G, A \hookrightarrow M$, giving a strongly continuous and non-degenerate covariant pair. If we take its integral form, that's actually equal to the original representation. So we really get a bijection. $\square$

## 24: March 17, 2008

****I was a little late.****

**Theorem:** Let $0 \to I \xrightarrow{i} A \xrightarrow{p} A/I \to 0$ be an exact sequence of $C^*$-algebras. Let $\alpha$ be an action of $G$ on $A$, which caries $I$ into itself; i.e. $i$ is *equivariant*, and also $\alpha$ drops to action on $A/I$. Then

$$0 \to I \times_\alpha G \xrightarrow{i_*} A \times_\alpha G \xrightarrow{p_*} (A/I) \times_\alpha G \to 0$$

is exact.

(This can fail for $\times_\alpha^r$.) **Question from the audience:** Can you get half-exactness? **Answer:** Yes, somewhat.

**Proof:**

$p_* : C_c(A, G, \alpha) \to C_c(A/I, G, \alpha)$ has dense range. When $G$ is not discrete, this is not an immediate fact, but it is true. We approximate functions $f : G \to A/I$ by $f \sim \sum h_j a_j$, which we can do for any continuous function of compact support into a Banach space, for the $L^1$

norm. Thus $p_* : A \times_\alpha G \to (A/I) \times_\alpha G$, which has dense range, and a homomorphism of $C^*$ algebras, but those have closed image, so this must be onto by denseness.

If $f \in C_c(I, G)$, then $p_*(i_*(f))$ is clearly 0, just by following image values. Extending by continuity we still have $p_* \circ i_* = 0 : I \times_\alpha G \to (A/I) \times_\alpha G$.

So, why is $i_*$ injective? And, we've shown that the image of $i_*$ is contained in the kernel of $p_*$; why are these equal?

For exactness at $I \times_\alpha G$, let $(\sigma, \mathcal{H})$ be a (non-degenerate) representation of $I \times_\alpha G$. From what we sketched last time, this must come from a covariant representation: let $(\pi, U, \mathcal{H})$ be the corresponding covariant representation of $(I, G, \alpha)$. Then $\pi$ is non-degenerate. Using the extension theorem from the problem set, let $\tilde{\pi}$ be the unique extension of $\pi$ to a representation of $A$. Claim: $(\tilde{\pi}, U, \mathcal{H})$ is a covariant representation of $(A, G, \alpha)$. Because:

$$\begin{aligned} U_x(\tilde{\pi}(a))(\pi(d)\xi) &= U_x(\pi(ad)\xi) \\ &= \pi(\alpha_x(ad))U_x\xi \\ &= \pi(\alpha_x(a)\alpha_x(d))U_x\xi \\ &= \tilde{\pi}(\alpha_x(a))\pi(\alpha_x(d))U_x\xi \\ &= \tilde{\pi}(\alpha_x(a))U_x(\pi(d)\xi) \end{aligned}$$

where $d \in I$, and the linear span of these things is dense by the nondegeneracy.

**Question from the audience:** Why is $\pi$ nondegenerate? **Answer:** That was something quick from last time. In the correspondence, $\sigma(f)\xi \overset{\text{def}}{=} \int \pi(f(x))U_x\xi$, and if this is nondegenerate, then the places where it was zero would be invariant, so we'll have nondegenerateness of $\sigma$ exactly when we have it for $\pi$.

So let $\tilde{\sigma}$ be the integrated form of $(\tilde{\pi}, U, \mathcal{H})$ a rep of $A \times_\alpha G$. Then $\tilde{\sigma}|_{I \times_\alpha G} = \tilde{\sigma} \circ i_* = \sigma$. If $\sigma$ is faithful on $I \times_\alpha G$, then $i_*$ has kernel 0.

Now we want exactness at $A \times_\alpha G$, i.e. that the kernel of $p_*$ is the range of $i_*$. Since we know that $i_*$ is injective, we should think of the range as $I \times_\alpha G \subseteq A \times_\alpha G$ as an ideal (we didn't do this part, but it's not hard that at the level of functions, this is an ideal). We saw at the outset that $\supseteq$ is easy. Here we will need the full force of $C^*$ algebras. We look at $(A \times_\alpha G)/(I \times_\alpha G)$, which is a $C^*$-algebra, so it has a faithful representation $\sigma$ on $\mathcal{H}$. (I.e. the kernel of $\sigma$ is exactly $I \times_\alpha G$). Then $\sigma$ is the integrated form of some $(\pi, U, \mathcal{H})$ where $\pi$ is a rep of $A$. If $d \in I$, then for any $h \in C_c(G, \mathbb{C})$, we have $dh \in C_c(G, I)$. So $0 = \sigma(hd)\xi = \int h(x)\pi(d)U_x\xi dx$ for all $h$ and $\xi$. So $\pi(d) = 0$. So $\pi(I) = 0$. Thus we can look at $(\pi, U, \mathcal{H})$ as a covariant rep of $(A/I, U, \mathcal{H})$, with integrated form $\tilde{\sigma}$, a representation of $(A/I) \times_\alpha G$.

Then

$$\begin{tikzcd} A \times_\alpha G \arrow[rr, "p_*"] \arrow[dr, "\sigma"'] & & (A/I) \times_\alpha G \arrow[dl, "\tilde{\sigma}"] \\ & \mathcal{B}(\mathcal{H}) & \end{tikzcd}$$

□

## 25: March 19, 2008

We review classical dynamical systems: a group $G$ acts as diffeomorphisms on a locally compact space $M$, thought of as the phase space of the system. Then we get an action on $A = C_\infty(M)$, since $C_\infty(-)$ is contravariant: if $\alpha : G \to \text{Homeo}(M)$, then $G$ acts on $A$ via $\alpha_x(f)(m) = f(\alpha_{x^{-1}}(m))$. So we can form $A \times_\alpha G$. If the action on $M$ is sufficiently continuous, then $\alpha$ is strongly continuous on $A$.

**Theorem:** For $(M, G, \alpha)$, with $M$ *second countable* (i.e. a countable base for its topology): let $(\sigma, \mathcal{H})$ be an irreducible representation of $A \times_\alpha G$, where $A = C_\infty(M)$; let $\sigma$ be the integrated form of $(\pi, U, \mathcal{H})$. Let $I = \ker(\pi)$. (**Question from the audience:** Is $\pi$ irreducible? **Answer:** Absolutely not.) Then $I$ is a closed ideal of $A$; let $Z_I = \text{hull}(I)$ (i.e. maximal ideals that contain $I$ — maximal ideals of $A$ correspond to points in $M$), so $I = \{f \in A : f|_{Z_I} = 0\}$.

Then $Z_I$ is the closure of an orbit in $M$, i.e. $\exists\, m_0 \in M$ s.t. $\overline{\{\alpha_x(m) : x \in G\}} = Z_I$. (There's no reason the orbit ought to be closed, e.g. an action of $\mathbb{Z}$ on a compact space.)

**Proof:**

Note: $\alpha_x(I) \subseteq I$ for all $X$. ($d \in I$, then $\pi(\alpha_x(d)) = U_x \pi(d) U_{x^{-1}} = 0$; this uses only the covariance relation.) We say that $I$ is "$\alpha$-invariant". We say $\subseteq$, but it's true for $x^{-1}$, so we get equality. This implies that $\alpha_x(Z_I) = Z_I$.

Choose a countable base for the topology of $M$; let $\{B_n\}$ be (an enumeration of) those elements of the base that meet $Z_I$. (Thus $\{B_n \cap Z_I\}$ is a base for the relative topology of $Z_I$.) For each $n$, let $O_n = \bigcup_{x \in G} \alpha_x(B_n) = \alpha_G(B_n)$. Since each $B_n$ is open and $\alpha$ is homeo, this is open; it's also clear that $O_n$ is $\alpha$-invariant, in that it's carried into itself by the $G$-action. Let $J_n = C_\infty(O_n)$. We view these as continuous functions on $M$ that vanish outside $O_n$; $J_n$ is exactly those functions that vanish on the closed set $M \smallsetminus O_n$. So $J_n$ is an ideal of $A$.

Furthermore, because $B_n \cap Z_I \neq \emptyset$, we can find $f \in C_\infty(B_n)$ so that $f|_{Z_I} \neq 0$. Thus $J_n \not\subseteq I$. So $J_n \times_\alpha G$ is an ideal in $A \times_\alpha G$, and it is not a subideal of $I \times_\alpha G = \ker(\sigma)$. Since $\sigma$ is irreducible, $\sigma|_{J_n \times_\alpha G}$ is non-degenerate. Thus $\pi|_{J_n}$ is non-degenerate.

Choose $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Define $\mu \in S(A)$ to be the vector state: $\mu(f) = \langle \pi(f)\xi, \xi \rangle$. I.e. $\mu$ is a probability Radon measure on $M$. Since $\pi|_{J_n}$ is non-degenerate, choose $\{e_\lambda\}$ a positive approximate identity of norm 1. Then $\mu(e_\lambda) = \langle \pi(e_\lambda)\xi, \xi \rangle \xrightarrow{\lambda} \langle \xi, \xi \rangle = 1$. So $\left\| \mu|_{J_n} \right\| = 1$. Let $\mu$ also be the corresponding Borel measure. ****huh?**** I.e. we view $\mu$ as giving sizes of sets: $\mu(O_n) = 1$. Then $\mu(M \smallsetminus O_n) = 0$. So $\mu\left(\bigcup_n (M \smallsetminus O_n)\right) = 0$ — this is where we use the separability hypothesis —, so $\mu(\bigcap_n O_n) = 1$, so $\bigcap O_n \neq \emptyset$. Pick any $m_0 \in \bigcap O_n$.

If $f \in I$, then $\pi(f) = 0$, so $\mu(f) = 0$. Thus, $\mu(M \smallsetminus Z_I) = 0$, so $\mu(Z_I) = 1$, and we should have intersected all our $O_n$ in the previous paragraph with $Z_I$. So we have $m_0 \in \bigcap (O_n \cap Z_I)$.

So $\alpha_G(m_0) \subseteq \bigcap(O_n \cap Z_I) = \bigcap(\alpha_G(B_n \cap Z_I))$. So for each $n$, $\alpha_G(m_0) \cap B_n \neq \emptyset$. So $\alpha_G(m_0)$ meets each elements of a base for the topology of $Z_I$, and so is dense in $Z_I$. $\square$

(For the record, this argument works for "factor representations" of von-Neuman algebras.)

## 26: March 21, 2008

****I was out sick.****

## 27: March 31, 2008

We consider $C_\infty(G) \times_\alpha G \cong \mathcal{B}_0(L^2(G)) = C_\infty(G) \times_\alpha^r G$, so action of $G$ on $C_\infty(G)$ is amenable.

**E.g.** $G = \mathbb{R}$, $X = \mathbb{R} \cup \{+\infty\}$, let $\alpha$ be translation, fixing $+\infty$. Then $\mathbb{R}$ sits in $X$ as an open $\alpha$-invariant set, so $C_\infty(\mathbb{R})$ is an $\alpha$-invariant ideal in $C_\infty(X)$. From the theorem, exact sequences:

$$0 \longrightarrow C_\infty(\mathbb{R}) \longrightarrow C_\infty(X) \longrightarrow C_\infty(\{-\infty\}) = \mathbb{C} \longrightarrow 0$$

$$0 \longrightarrow C_\infty(\mathbb{R}) \times_\alpha \mathbb{R} \longrightarrow C_\infty(X) \times_\alpha \mathbb{R} \longrightarrow \mathbb{C} \times_\alpha \mathbb{R} = C^*(\mathbb{R}) \longrightarrow 0$$

But $C^*(\mathbb{R}) = C_\infty(\mathbb{R})$ by Fourier. So we can read off the irreps. Let $\sigma$ be an irreducibble representation of $C_\infty(X) \times_\alpha \mathbb{R}$.

**Case 1:** $\sigma|_{C_\infty(\mathbb{R}) \times_\alpha \mathbb{R}} \neq 0$ rep. Then $\sigma$ is iso to the irrep on $L^2(\mathbb{R})$; indeed, $\sigma(C_\infty(\mathbb{R}) \times_\alpha \mathbb{R}) = \mathcal{B}_0(L^2(G))$. This has a unique extension to $C_\infty(X) \times_\alpha \mathbb{R}$ as a covariant rep $(\pi, U, L^2(\mathbb{R})$, where $\pi$ is multiplication by functions in $C_\infty(X)$. If $f \in C_\infty(X)$, $f(+\infty) \neq 0$, and if $\phi \in C_c(G)$, then $f \times \phi \in C_c(G, C_\infty(X))$. Well, $\sigma(f \times \phi)$ is not compact. So $C_\infty(X) \times_\alpha \mathbb{R}$ s not CCR.

**Case 2:** $\sigma|_{C_\infty(\mathbb{R}) \times_\alpha \mathbb{R}} = 0$. So $\sigma$ drops to an irreducible representation of $C^*(\mathbb{R}) \cong C_\infty(\mathbb{R})$. So $\sigma$ is given by evaluation at some point in $\mathbb{R}$. These are one-dimensional, so certainly give compact operators.

This then is the full list of irreps. Thus $C_\infty(X) \times_\alpha \mathbb{R}$ is GCR. (Every irrep includes compact operators.) The primitive ideals? Prim $= \{0, \mathbb{R}\}$ = the zero ideal together with one point for each point on the real line. Closure of 0-ideal is all of everything, and each ideal in $\mathbb{R}$ is closed.

Let $N, Q$ be locally compact groups. Let $\alpha : Q \to \mathrm{Aut}(N)$ with $(n, q) \mapsto \alpha_q(n)$ is continuous ("joint continuity", i.e. continuity in each variable). We can form the *semi-direct product*: let $G = N \times Q$ as a set, and indeed as a space with the product topology. The multiplication is

$$(n, q)(n', q') \overset{\text{def}}{=} (n\alpha_q(n'), qq')$$

Then $N = N \times \{1_Q\}$ is a closed normal subgroup, and the following sequence is exact:

$$0 \to N \to N \times_\alpha Q \to Q \to 0$$

But this is not, of course, the most general sequence. If $0 \to N \to G \to G/N \to 0$, there's no reason for this to split. But for semi-direct, it does: $Q = \{1_N\} \times Q$ is a closed subgroup of $N \times_\alpha Q$. So we have a split extension:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & N \times_\alpha Q \arrow[r] & Q \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd}$$

Let $(U, \mathcal{H})$ be a unitary rep of $N \times_\alpha Q$; then $(U|_N, \mathcal{H})$ is a unitary rep on $N$, and ditto $(U|_Q, \mathcal{H})$ on $Q$. Then $(U|_N, \mathcal{H})$ gives a rep of $C^*(N)$; the action of $\alpha$ gives an action of $Q$ on $C_c(N)$ with convolution. So $\alpha$ gives an action of $Q$ on $C^*(N)$ by functoriality (you have to check that this action is strongly continuous), "by transport of structure". Let $\pi$ be the integrated form of $U|_N$, i.e. $\pi$ is a rep of $C^*(N)$. You find (but have to check): $(\pi, U|_Q, \mathcal{H})$ is a covariant representation of $(C^*(N), Q, \alpha)$. The details are straightforward, but we will not take the time to do them on the board. In all, we have a representation of $C^*(N) \times_\alpha Q$.

Can we go backwards? Of course. If we have a rep of $C^*(N) \times_\alpha Q$, we get a $\pi$, and work our way up to a $U$.

**Prop:** $C^*(N \times_\alpha Q) = C^*(N) \times_\alpha Q$. ****naturally isomorphic****

**E.g.** When $N$ is abelian, so $C^*(N) = C_\infty(X)$. How far did y'all go in Math 206? This is really Math 260: Abstract Harmonic Analysis. It turns out that $X = \hat{N} = \mathrm{Hom}(N \to T)$, where $T$ is the unit circle in $\mathbb{C}$. **Question from the audience:** Could you say something about the natural topology on $\hat{N}$? **Answer:** Yes. There are two views that must be reconciled. The more accessible one: $\phi \in \hat{N}$ lifts to a homomorphism $\phi^* : C^*(N) \to \mathbb{C}$, and they all come about this way. So $\hat{N} = (\widehat{C^*(N)})$. An equivalent description of topology requires knowledge of characters. $\hat{N}$ has multiplication pointwise as functions.

**Theorem:** Pontyang ****sp?**** duality

$\hat{\hat{N}} = N$ as a natural isomorphism.

The injection $N \hookrightarrow \hat{\hat{N}}$ is obvious, but that it is a bijection is exciting. **E.g.** $\widehat{\mathbb{Z}^n} = T^n$, and $\widehat{T^n} = \mathbb{Z}^n$. $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$, but not naturally.

**E.g.** A famous example: $L =$ Lorentz group.

$$L = \{T \in \mathrm{End}(\mathbb{R}^4) \text{ s.t. } B(Tv, Tw) = B(v, w)\}$$

where $B$ is symmetric bilinear, but you make a choice of sign convention: $B((r_0, r_1, r_2, r_3), (s_0, s_1, s_2, s_3)) = -r_0 s_0 + r_1 s_1 + r_2 s_2 + r_3 s_3$. ****This, I believe, is the East Coast convention.**** $L$ acts in a natural way on $\mathbb{R}^4$, so we can form $\mathbb{R}^4 \times_\alpha L$. We need this group if we are physicists in the 30s: We need an irreducible representation (so that we have a single particle) that is translation ($\mathbb{R}^4$) and special-relativistic ($L$) invariant. Paper by Wigner in 1939. He didn't find all of them, because there are non-physically interesting reps that kill $\mathbb{R}^4$, and it's rather difficult to find all reps of $L$. It turns out that you can find the representations that don't kill $\mathbb{R}^4$; we will sketch this next time.

## 28: April 2, 2008

Any questions? We hand back some problem sets from before.

### 28.1 A landscape sketch

Last time we defined the Poincaré group $\mathbb{R}^4 \times_\alpha L$, where $L$ is the Lorentz group ****or rather the connected component****. If you want a quantum mechanics, you would like a Hilbert space. An elementary particle, for quite some time, was a representation of this group — so that physics would be symmetric — and irreducible — so that the particle was really just a unique particle. In 1939, Wigner listed the representations (that are non-trivial as representations of $\mathbb{R}^4$), which consists of understanding the $C^*$-algebra $C^\infty(\widehat{\mathbb{R}}) \times_\alpha L$. In fact, we want to use the simply connected cover of $L$, which is $SL(2, \mathbb{C})$.

We can understand the orbits of $L$ ****picture of concentric hyperbolas, with the degenerate one labeled the "light cone"****. But the action of $L$ is not free on these orbits; rather, if $G$ acts on $M$ via $\alpha$, we can construct the *stability subgroup* $G_m = \{x \in G : \alpha_x(m) = m\}$; then the orbit of $m \in M$ bijects to $G/G_m$ by $\alpha_x(m) \leftrightarrow x$. The stability subgroup of $L$ is $SO(3) \subseteq L$; this lifts to $SU(2) \subseteq SL(2, \mathbb{C})$. So the orbits are $SL(2, \mathbb{C})/SU(2)$.

Given $G$ acting via $\alpha$ on $M$, and $m \in M$, we understand what happens when $G_m = \{1_G\}$, and if $G_m = G$, everything is trivial. We want to understand the intermediate case. But the orbit $G/G_m = \mathrm{Orbit}_\alpha(m) \subseteq M$ can be very bad. **E.g.** $M = T = S^1$ and $G = \mathbb{Z}$ acting by rotation by an irrational multiple of $\pi$. The action is free, and the orbit is dense in $T$ — countable dense subsets are very bad from the point of view of the things we've been doing. We'll look at that later.

The good situation: we want the orbit, with the relative topology, to be locally compact. (Contemplate pulling back the topology of the circle to the integers.) For the experts:

**Theorem:** Given $M$ locally compact and $S \subseteq M$, then $S$ is locally compact for the relative topology if and only if $S$ is open in its closure.

If $G$ is second countable ****and we are in this good case****, we can use Baire Category to show that $G/G_m \to$ Orbit is a homeomorphism. **E.g.** $\mathbb{R}_\delta$ is $\mathbb{R}$ with the discrete topology, and is not second-countable; it acts on $\mathbb{R}$ in the natural way, but we don't have a homeomorphism.

We can look at the ideal of functions on $M$ that vanish on the closure of the orbit, and if the open orbit is dense in its closure, we can generalize the picture we had last time of $\mathbb{R} \cup \{+\infty\}$. We reduce to considering $G$ action on $G/G_m$.

In general, for $H$ a closer subgroup of $G$, $G$ acts on $G/H$ (call action $\alpha$) and we consider the covariant representations. From our point of view, we're curious to understand $C(G/H) \times_\alpha G$.

**Question from the audience:** So $G$ acts on $M$, which we decompose into orbits. So we're just looking at one orbit at a time? **Answer:** We showed that irreducible representations live on orbit-closures. As long as your orbit is open, and so locally compact, the irreducible representations all come from this mechanism. To be specific: an irreducible representation comes from an orbit closure, and if there is a dense open set, that gives us an essential ideal. And if you have the right orbit closure for the representation, this essential ideal cannot act as zero, so the representation comes from the cross-product with that ideal.

Mackey worked out the general theory measure-theoretically. It involves induced representations — Frobenius understood these for finite groups in the late 1800s, but Mackey had to work it out for locally-compact groups. Assume for simplicity that there is a $G$-invariant measure on $G/H$. (Otherwise, you have to introduce modular functions, and the bookkeeping is too complicated for this exposition.) Let $(\rho, \mathcal{K})$ be a representation of $H$. (We think of $H = G_m$; the phycisists call these "little groups".) Set

$$\mathcal{H} \overset{\text{def}}{=} \{\xi : G \to \mathcal{K} \text{ measureable s.t. } \xi(xs) = \rho_s^{-1}(\xi(x)) \text{ for } s \in H, x \in G\}$$

We have $\langle \xi(xs), \eta(xs) \rangle_{\mathcal{K}} = \langle \xi(x), \eta(x) \rangle_{\mathcal{K}}$ as functions on $G/H$, so we define the inner product on $\mathcal{H}$:

$$\|\xi\|^2 \overset{\text{def}}{=} \int_{G/H} \langle \xi(x), \xi(x) \rangle_{\mathcal{K}} \, dx$$

where $dx$ is the invariant measure on $G/H$.

Let $\pi$ be the action of $C_\infty(G/H)$ on $\mathcal{H}$ by pointwise multiplication. Let $U$ be the action of $G$ by translation; then $(\pi, U, \mathcal{H})$ is a covariant representation. Thus, as before, we get a representation of $C_\infty(G/H) \times_\alpha G$. If $(\rho, \mathcal{K})$ is irreducible, then so is $(\pi, U, \mathcal{H})$. **Question from the audience:** Sorry, what is $\alpha$? **Answer:** The action by translation. **$\alpha = U$?**

**Theorem:** (Mackey, phrased in terms of $L^\infty$, not $C^*$-algebras)

Every irreducibly representation arises in this way:

$$\{\text{Irreducible representations of } C_\infty(G/H) \times G\} \leftrightarrow \{\text{irreps of } H\}$$

**E.g.** Every irrep of the Poincaré group has an inherent *spin*, which is the representation of $SU(2)$.

We can reformulate this, which is a nice way to do it because M.R. did it.

**Theorem:** (M.R.)

$C_\infty(G/H) \times_\alpha G$ and $C^*(H)$ are strongly morita equivalent.

To say this, we need bimodules. We think of $C_c(H)$ with measure $dx_H$, which acts on the right by convolution $\star$ on $C_c(G)$. But $C_\infty(G/H) \times_\alpha G$ acts on the left by point-wise multiplication, so at least at the level of functions, we have $C_c(G)$ as a bimodule. But on the level of inner products? If we have $f, g \in C_c(G)$, we can define $\langle f, g\rangle_{C^*(H)}$? For better or for worse, it pays off to work with continuous rather than measurable functions, because if $H$ is a null-set, restricting a measurable function doesn't make sense. In the unimodular case, we can define

$$\langle f, g\rangle_{C^*(H)} \stackrel{\text{def}}{=} f^* \star g|_H$$

In fact, everything fits together nicely, although there are many things to check, and in the non-unimodular case you have to sprinkle in modular functions.

So, basically, and you have to define things right: if you have a representation of $B = C^*(H)$, you can tensor it with this bimodule to get a representation of $A = C_\infty(G/H) \times_\alpha G$. What you need to check is that $\langle f, g\rangle_A h = f\langle g, h\rangle_B$.

And the long and the short of it is that the representation theories are the same.

**Question from the audience:** So if you embed $H$ as a subgroup of different groups $G$, you can look at their relationships? **Answer:** Yes, there are many interesting games you can play. For instance, you can sort out that $C_\infty(G/H) \times_\alpha K$ is Morita equivalent to $C_\infty(G/K) \times_\alpha H$ if $H, K \subseteq G$.

We've been looking at an action $G, \alpha$ on a space $M$. This leads to a simple class of groupoids, although we won't tell you what a groupoid is. Much of the above story generalizes to groupoids. (Groupoids come from gluing together group-type things and space-type things, and arise in many interesting places. Locally compact ones have a $C^*$-algebra, and by the now there is a very substatially developed theory, which imitates the theory of groups acting on spaces.)

## 28.2 Heisenberg commutation relations

Planck wrote his paper in 1909; in 1926, Heisenberg basically suggested that, for $\mathbb{R}^n$ (really $\mathbb{R}^{3n}$) one do the following.

We have unbounded self-adjoint (in a sense not made precise by Heisenberg) "position" operators $Q_1, \ldots, Q_n$ and "momentum" operators $P_1, \ldots, P_n$ all on a Hilbert space $\mathcal{H}$. Since these operators are unbounded and hence only defined on dense domains, we will need to make this precise or avoid the problems entirely. Even saying these operators commute is hairy. But, naively, we want that $P_j, Q_k$ commute if $j \neq k$. For the same index:

$$[P_j, Q_j] = i\hbar \mathbb{1}_{\mathcal{H}}$$

where $i = \sqrt{-1}$, and $\hbar$ is an experimental fudge factor and $\mathbb{1}_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. Shortly thereafter, Weyl suggested how to bypass some of the difficulties here.

****I prefer to use slightly more tensorial notation. We have what a physicists would call a "vector" of operators $Q^i$ and a "covector" $P_j$. Then the canonical commutation relations are**

$$[P_j, Q^k] = i\hbar\delta_j^k \mathbb{1}_{\mathcal{H}}$$

**where $\delta_j^k$ is Kronecker.****

## 29: April 4, 2008

****I was out sick.****

## 30: Problem Set 3: Due April 11, 2008

****The problem set was given out typed. I've retyped it, partly so I could submit my answers set between the questions. I have corrected some typos, and no doubt introduced even more. In doing so, I have changed the formatting slightly.****

1. (a) Show that the following $C^*$-algebras are isomorphic:

    i. The universal unital $C^*$-algebra generated by two (self-adjoint) projections

    ii. The universal $C^*$-algebra generated by two self-adjoint unitary elements

    iii. The group algebra $C^*(G)$ for $G = \mathbb{Z}_2 * \mathbb{Z}_2$, the free product of two copies of the 2-element group.

    iv. The crossed-product algebra $A \times_\alpha G$ where $A = C(T)$ for $T$ the unit circle in the complex plane, $G = \mathbb{Z}_2$, and $\alpha$ is the action of taking complex conjugation. (So $T/\alpha$ exhibits the unit interval as an "orbifold", i.e. the orbit-space for the action of a finite group on a manifold, and $A \times_\alpha G$ remembers where the orbifold comes from.) Hint: In $\mathbb{Z}_2 * \mathbb{Z}_2$ find a copy of $\mathbb{Z}$.

   (b) Determine the primitive ideal space of the above algebra, with its topology.

   (c) Use the center of the algebra above to express the algebra as a continuous field of $C^*$-algebras.

   (d) Use part (c) to prove that if $p$ and $q$ are two projections in a unital $C^*$-algebra such that $\|p - q\| < 1$, then they are unitarily equivalent, that is, there is a unitary element $u$ in the algebra (in fact, in the subalgebra generated by $p$ and $q$) such that $upu^* = q$.

   (e) Use part (d) to show that in a unital separable $C^*$-algebra the set of unitary equivalence classes of projections is countable.

2. For any $n \times n$ real matrix $T$ define an action $\alpha$ of $\mathbb{R}$ on the group $\mathbb{R}^n$ by $\alpha_t = \exp(tT)$ acting in the evident way. Let $G = \mathbb{R}^n \times_\alpha \mathbb{R}$. Then $G$ is a solvable Lie group. For the case of $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ determine the equivalence classes of irreducible unitary representations of $G$, i.e. the irreducible representations of $C^*(G)$. Determine the topology on $\mathrm{Prim}(C^*(G))$. Discuss whether $C^*(G)$ is CCR or GCR, and why.

# 31: April 7, 2008

## 31.1 Some group cohomology

We had looked at the Heisenberg commutation relations in the form that Herman Weyl gave:

$U$ a rep of $G$ (Abelian) on $\mathcal{H}$, $V$ a rep of the dual group $\hat{G}$ on $\mathcal{H}$; we declare

$$V_x U_s = \langle s, x\rangle U_s V_x$$

We saw, and basically gave the proof, that when $G$ is $\mathbb{R}^n$, and more generally when $\hat{\hat{G}} = G$, there's one irreducible representation ("Schrodinger representation") on $L^2(G)$, and every representation comes from one of these.

Looking at this in a slightly different way, define unitary $W$ on $G \times \hat{G}$ by $W_{(x,s)} \stackrel{\text{def}}{=} V_x U_s$. Then

$$W_{(x,s)} W_{(y,t)} = \langle s, y\rangle W_{(x,s)+(y,t)}$$

and $\langle s, y\rangle \in T \stackrel{\text{def}}{=} \{e^{i\theta} \in \mathbb{C} : \theta \in \mathbb{R}\}$.

For any group $G$ (e.g. $G \times \hat{G}$), we can consider $W : G \to \mathcal{U}(\mathcal{H})$ the unitary operators on $\mathcal{H}$ such that $W_x W_y = c(x, y) W_{xy}$ for $c(x, y) \in T$. The associativity in $G$ implies that $c$ is a $T$-valued 2-cocycle, meaning

$$c(x, yz)\, c(y, z) = c(xy, z)\, c(x, y)$$

It's natural to assume $W_e = \mathbb{1}_{\mathcal{H}}$: $c(x, e) = 1 = c(e, x)$.

There's a homology theory of groups ("group cohomology"). We're looking at $[c] \in H^2(G, T)$, which we won't really define. For a function of one variable $h : G \to T$, we define the boundary of $h$ by

$$\partial h(x, y) \stackrel{\text{def}}{=} h(x)\, h(y)\, \overline{h(xy)}$$

**Definition:** $W$ is a *projective* representation of $G$ on $\mathcal{H}$ with cocycle $c$

Since $T$ is abelian, $H^2(G, T)$ is a group. If $G$ is topological, we do not demand that $c$ be continuous. This machinery works best when $G$ is second-countable locally-compact, and then we want $c$ to be measurable. Such $c$ correspond to extensions:

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] & E_c \arrow[r] & G \arrow[r] \arrow[l, bend left, "c"] & 0 \end{tikzcd}$$

**E.g.** $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \rightleftarrows T \longrightarrow 0$

This is important for physics. In QM, $\mathcal{B}_0(\mathcal{H})$, with pure state the vector states. Because the only irrep is the one on the Hilbert space. But for any algebra, the pure states via GNS give an irreducible rep for which the state is a vector state. An operator gives a projection:

$$T \mapsto \frac{\langle T\xi, \xi \rangle}{\langle \xi, \xi \rangle}$$

These give one-dim subspaces of $\mathcal{H}$, and $\mathcal{PH}$, the projective space, is the space of states. Automorphisms of QM are automorphisms of $\mathcal{PH}$.

**Theorem:** (Wigner, 1930s)

> This are given by unitary or anti-unitary (conjugate-linear but length-preserving) operators (unique up to multiplication by an element of $T$.

If we have a one-parameter family of automorphisms of $\mathcal{PH}$, then for each auto, $U^2$ is linear. So anti-unitary operators only come up in discrete situations, usually as autos of order 2. For example:

- Charge conjugation $C$
- Parity $P$ — weak force does not respect parity
- Time reversal $T$

Then $C^2 = P^2 = T^2 = 1$, and $CPT$ often comes up.

In any case, we've found irreducible projective representations with non-trivial cocycle: $H^2(\mathbb{R}^2, T) \neq \{0\}$.

**Theorem:** For $G$ a semi-simple connected simply-connected Lie group, then $H^2(G,T) = \{0\}$.

For example, $SO(3) \to \mathrm{Aut}(\mathcal{PH})$. The double cover $SU(2) \xrightarrow{2} SO(3)$ is simply connected and semi-simple. So any projective representation of $SO(3)$ gives an ordinary representation of $SU(2)$. Similarly, the (connected component of the) Lorentz group $\mathcal{L}$ is covered by simply-connected $SL(2,\mathbb{C})$, so has the same story. And it's much easier to work with ordinary representations than with projective representations. (The story does not work with $\mathbb{R}^{2n}$, which has an infinite irreducible projective representation, even though any ordinary irrep is one-dimensional.)

**Question from the audience:** How do we get a cocycle? **Answer:** We have $\alpha : G \to \mathrm{Aut}(\mathcal{PH})$. For each $x$, chose $U_x$ implementing $\alpha(x)$. This is only defined up to scalar multiple. $U_x U_y = c(x,y)U_{xy}$. Associativity in Aut implies the cocycle condition ****and the unknown scalars are the boundaries****. When $G$ is topological, you cannot make this choice continuous, but you'd like to make it at least measurable. If $\mathcal{H}$ is separable, Aut can be given topology of a complete metric space (not locally compact), and from that there are theorems that can go and chose $c$ to be measureable.

Incidentally, the complete metric space for Aut makes it into a *Polish space*; these do not have Haar measure, but the homology was worked out nicely by Prof Moore in our department.

We should mention another aspect of this story. Given $G$ and a cocycle $c : G \to T$, we can define the convolution

$$(f \star_c g)(x) \stackrel{\text{def}}{=} \int f(y)\, g(y^{-1}x)\, c(y, y^{-1}x)\, dy$$

This is associative iff $c$ is a 2-cycle almost everywhere. So we get a $C^*(G, c)$, and if $c'$ and $c$ are homologous, then the corresponding algebras are isomorphic (indeed, the boundary tells how to build the isomorphism). Look at $c(s,t) = e^{2\pi i s t}$ on $\mathcal{R}^2$; then $C^*(\mathcal{R}^2, c) \cong \mathcal{B}_0(L^2(\mathcal{R}))$. We can, of course, stick in a constant, and promote the product to a dot-product: $c(s,t) = e^{2\pi\hbar i\langle s,t\rangle}$. This is one view on what we've been doing. Even more generally, we can build $C^*(G, A, \alpha, c)$ where $c$ is an $A$-valued cocycle and $\alpha$ a representation. There is a very nice treatment in this language of the Quantum Hall Effect.

For the last five minutes, some special example. Let $G = \mathbb{Z}^d$ (we use $m$, $n$ for elements of $G$, not the dimension). Let $\theta \in M_d(\mathbb{R})$ be a $d \times d$ matrix. Define

$$c_\theta(m,n) \stackrel{\text{def}}{=} e^{2\pi i (m\cdot\theta n)}$$

This is a bicharacter, i.e. for $n$ fixed, $m \mapsto c_\theta(m,n)$ is a character (element of $\widehat{\mathbb{Z}^d}$). An easy check: a bicharacter is a 2-cocycle. We will not prove:

**Theorem:** Every 2-cocycle on $\mathbb{Z}^d$ with values in $T$ is homologous to a bicharacter.

Now we will study $C^*(\mathbb{Z}^d, c_\theta)$. For $\theta = 0$ (or all integers), this is just $C^*(\mathbb{Z}^d) = C(T^d)$ continuous functions on the $d$-dim torus. In general, $C^*(\mathbb{Z}^d, c_\theta)$ are called *non-commutative tori* (or "quantum tori"). These are the easiest examples of non-commutative differentiable manifolds.

## 32: April 4, 2008

We begin with some homological algebra. We have $C_2(G, A) = \{c : G \times G \to A\}$ and we define $\partial c(x,y,z) = c(xy,z)c(x,y) - c(y,z)c(x,yz)$. For $c \in C_1 = \{c : G \to A\}$, we define $\partial c(x,y) = c(x)c(y)c(xy)$. Then the second homology $H_2 = Z_2/B_2 = \ker\partial/\operatorname{im}\partial$ classify extensions $0 \to A \to E_c \to G \to 0$, at least at the algebraic level. Since $B_n$ may not be closed in $Z_n$, the quotient can get a non-Hausdorff topology; this adds difficulty to the theory.

### 32.1 A specific class of examples

We have matrices $\theta \in M_d(\mathbb{R})$ and $\mathbb{Z}^2$, and we define a cocycle $c_\theta(m,n) = e^{2\pi i m\cdot\theta n}$. ****We should call $\theta$ a $0,2$-tensor.**** We look at projective unitary reps of $\mathbb{Z}^d$ for bicharacter $c_\theta$:

$$U_m U_n = c_\theta(m,n) U_{m+n} \tag{4}$$

We look at the universal $C^*$-algebra generated by the unitary symbols $U_m$ ($U_0 = \mathbb{1}$) and the the relation (4).

We can construct this. $\mathbb{Z}^d \hookrightarrow C_c(\mathbb{Z}^d)$ with the twisted convolution

$$(f \star_\theta g)(m) \stackrel{\text{def}}{=} \sum_n f(n)g(m-n)c_\theta(n, m-n)$$

We need a $*$-operation. The relation (4) gives $U_m U_{-m} = c_\theta(m, -m) = \overline{c_\theta(m, m)}$, so $(U_m)^* = (U_m)^{-1} = c_\theta(m, m)U_{-m}$. Saying this again, for $C_c$:

$$f^*(m) = c_\theta(m, m)f(-m)$$

Then on this algebra the universal $C^*$-norm $\|f\|_{C^*}$ is well-defined. Indeed, looking at (the integrated form of) a representation $f \mapsto \sum f(m)U_m$, we see that $\|U_f\| \leq \|f\|_{\ell^1}$.

**Question from the audience:** Do you have a specific representation in mind? **Answer:** No. This is for any rep. **Question from the audience:** And the cocycle condition is equivalent to associativity? **Answer:** Yes. This is a bicharacter, so certainly a cocycle.

So, we can complete $C_c(\mathbb{Z}^d)$ — by the way, the "$c$" here means "of finite support", it doesn't have anything to do with the cocycle — to get $C^*$-algebra $A_\theta$, and this is the universal $C^*$-algebra as above. Are there any projective representations? Look at the left-regular representation, and see what you can do. Let $C_c(G)$ act on $\ell^2(\mathbb{Z}^d)$, a fine Hilbert space by left convolution:

$$(f, \xi) \mapsto f \star_\theta \xi$$

for $\xi \in \ell^2(\mathbb{Z}^d)$

**Question from the audience:** I'm still confused. Why didn't we just define $A_c$ as the universal algebra? **Answer:** We did. This is a description, using (4). If you take any juxtaposition of these symbols, we just get a scalar times another symbol. So any combination is a linear combination.

In any case, the norm you would get on this space is the "reduced" norm. We will see later that the full norm is the reduced norm.

We'd like to understand better the structure of the algebra $A_\theta$. Certainly this will depend on $\theta$ — when $\theta$ is 0 ****mod $\mathbb{Z}$**** we get the commutative algebra with the usual convolution; when $\theta$ is not zero, we do not expect a commutative answer.

The dual group $\widehat{\mathbb{Z}^d} \cong T^d = \mathbb{R}^d/\mathbb{Z}^d$, where we identify the character $e_t(n) = e(n \cdot t) \stackrel{\text{def}}{=} \langle n, t \rangle$ ****bah, dot products****, and have adopted the notation $e(\tau) \stackrel{\text{def}}{=} e^{2\pi i \tau}$. There is an action $\alpha$ of $T^d$ on $A_\theta$ via

$$(\alpha_t(f))(m) = \langle m, t \rangle f(m)$$

This action is independent of $\theta$. We can check

$$\begin{aligned}
(\alpha_t(f \star_\theta g))\,(m) &= \langle m,t\rangle \sum f(n)g(m-n)c_\theta(n,m-n) \\
&= \sum \langle n,t\rangle f(n)\,\langle m-n,t\rangle g(m-n)\,c_\theta(n,m-n) \\
&= (\alpha_t(f) \star_\theta \alpha_t(g))\,(m)
\end{aligned}$$

In general, for any $G$ (discrete, or even non-discrete) abelian, and cocycle $c$, then $\hat{G}$ acts on $C^*(G,c)$ exactly by the analog of this pairing. This is because if $G$ abelian acts on a $C^*$-algebra $A$ by an action $\beta$, then $\hat{G}$ acts on the cross-product algebra $A \times_\beta G$. The formula is the same — we get the "dual action".

**Question from the audience:** What can we say about $A \times_\beta G \times_\alpha \hat{G}$? **Answer:** Quite a lot. More generally, we can consider $G \rightsquigarrow \mathbb{C}[G]$ a Hopf algebra, and $\hat{G}$ the dual Hopff algebra. There's a lot to be said about $C^*$-Hopff algebras. Compact case is more-or-less understood, but locally compact quantum groups are hard even to define. E.g. no one knows how to prove the existence of a Haar measure in the non-compact case.

In any case, the action $(\alpha_t(f))\,(m) = \langle m,t\rangle f(m)$ is strongly continuous: $t \mapsto \alpha_t(f)$ is continuous for the norm $\|\cdot\|_{A_\theta}$. It's worth generalizing. Let $G$ be a compact Abelian group with an action $\alpha$ on some $C^*$-algebra $A$. We can try to do "Fourier analysis" We have $\hat{G}$ discrete. For $a \in A$ and $m \in \hat{G}$, the $m$th Fourier coef of $a$ is

$$a_m \overset{\text{def}}{=} \int_G \alpha_t(a)\,\overline{\langle t,m\rangle}\,dt \in A$$

## 33: April 11, 2008

****I arrived late.**** When $G$ is compact, $\hat{G}$ is discrete, (if $G$ is not abelian, $\hat{G}$ are the equivalence classes of irreps of $G$) and given the right set-up ****an action of $G$ on Banach space $B$?****, we can average and define an "isotypic subspace": $B_m = \{\xi : \alpha_x(\xi) = \langle x,m\rangle\xi\}$.

Another view: Let $e_m(x) = \overline{\langle x,m\rangle}$. Then $e_m \in L^1(G)$, and $(e_m \star e_n)(x) = \int_G e_m(y)\,e_n(x-y)\,dy = \int_G \overline{\langle y,m\rangle\langle x-y,n\rangle}dy = \overline{\langle x,n\rangle} \int \overline{\langle y,m-n\rangle}dy$. Based on experience with these things over, e.g. the torus, we can show that this integral is $\int \overline{\langle y,m-n\rangle}dy = \delta_{m,n}$. So $e_m$ is an idempotent in $L^1(G)$ and $e_m \star e_n = 0$ if $m \neq n$. So set $\xi_m = \alpha_{e_m}(\xi)$; then $\alpha_{e_m}$ is a projection of $B$ onto $B_m$.

**Prop:** If $\xi_m = 0$ for all $m$, then $\xi = 0$.

**Proof:**

> $\alpha_{e_m}(\xi) = 0$. The finite linear combinations of the $e_m$s form a subalgebra under pointwise multiplication — $e_m e_n = e_{m+n}$ — and under complex conjugation. There are lots of characters of a compact group: we even have the machinery to show this ****and sketched the proof verbally, but I didn't catch it****. This algebra separates points, hence is dense in

$C(G)$ with $\infty$-norm, so dense in $L^1(G)$. Thus $\alpha_f(\xi) = 0$ for any $f \in L^1(G)$. But let $f_\lambda$ be an approximate identity for $L^1(G)$, so $0 = \alpha_{f_\lambda}(\xi) \to \xi$. $\square$

**Question from the audience:** So this is saying that if all the Fourier coefficients of a function are zero, then it's zero? **Answer:** Precisely. And more generally.

**Cor:** If $a \in A_\theta = C^*(\mathbb{Z}^d, c_\theta)$, and if $a_n = 0$ for all $n$, then $a = 0$.

If $G$ compact Abelian, $\alpha$ and action on $C^*$-algebra $A$, we can define $A_m$ for each $m \in \hat{G}$. We pick $a \in A_m$ and $b \in A_n$; then $\alpha_x(ab) = \alpha_x(a)\alpha_x(b) = \langle x, m\rangle a \, \langle x, n\rangle b = \langle x, m+n\rangle ab$. So $ab \in A_{m+n}$. And $a^* \in A_{-m}$. So then $\bigoplus A_m$ is a dense subalgebra fibered over $\hat{G}$. ****I would call this "graded".**** We can even do this in the nonabelian case. These are often called "Fell bundles".

So, back in our torus case, let $U_m$ be a unitary generator for $A_\theta$ (corresponds to $\delta_m$).

$$\begin{aligned}
(U_m)_n &= \int \overline{\langle x, m\rangle} \, \alpha_x(U_m) \, dx \\
&= \int \overline{\langle x, b\rangle} \langle x, m\rangle U_m \, dx \\
&= \int \langle x, m-n\rangle dx \, U_m \\
&= \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}
\end{aligned}$$

So $(A_\theta)_m = \operatorname{span}(U_m)$. We can try to ask at a convergence level whether $a \sim \sum a_m U_m$. This doesn't have a good answer, even in the continuous case: Which collections of Fourier coefficients come from continuous functions?

For $G$ acting on a $C^*$-algebra $A$, the fiber over 0 is a $C^*$-subalgebra. $A_0 = \{a : \alpha_x(a) = a \, \forall x\} \stackrel{\text{def}}{=} A^G$. If $P = \alpha_{e_0}$, then $P(a) = \int_G \alpha_x(a) \, dx$.

**Prop:** $P$ is a *conditional expectation* from $A$ onto $A^G$:

(a) If $a > 0$, then $P(a) > 0$.

(b) If $a \in A$ and $b \in A^G$, then $P(ba) = bP(a)$ and $P(ab) = P(a)b$.

(c) $P(\alpha_x(a)) = P(a) \, \forall x \in G$.

For $A_\theta$, $P(a) = \int_{T^d} \alpha_x(a) \, dx = a_0 U_0 = a_0 \mathbb{1}$. So we can view $P$ as a linear functional $\tau : a \mapsto a_0 \in \mathbb{C}$. It's positive, from what we've seen, and $P(\mathbb{1}) = P(U_0) = 1$, so it's a state, but also *tracial*: $\tau ab = \tau(ba)$. It's enough to check this on generators:

$$\tau(U_m U_n) = \int \langle m, x\rangle U_m \langle n, x\rangle U_n \, dx = \begin{cases} 0, & m \neq -n \\ U_m U_{-m} = c_\theta(m, -m), & m = -n \end{cases}$$

**Cor:** $A_\theta$ contains no proper $\alpha$-invariant ideal.

**Proof:**

If $I$ is an ideal, $a \in I$, $a \neq 0$, then $a^*a \in I$ and $a^*a \neq 0$. So $P(a^*a) = \int \alpha_x(a^*a)\,dx > 0$, but it is in $\mathbb{C}\mathbb{1}$, so $1 \in I$, so $I \in A_\theta$. $\square$

**Cor:** The rep of $A_\theta$ on $L^2(\mathbb{Z}^d)$ is faithful.

Next time.

**Cor:** $\tau$ is the only $\alpha$-invariant tracial state.

**Proof:**

If $\tau_0$ is another one, then $\tau_0(a) = \tau_0(\alpha_x(a)) = \int_G \tau_0(\alpha_x(a))dx = \tau_0\left(\int \alpha_x(a)dx\right) = \tau_0(a_0) = \tau_0(\tau(a)1) = \tau(a)$. $\square$

## 34: April 14, 2008

Recall we have $A_\theta$ and the "dual" action $\alpha$ of $T^d$. $A_\theta$ has no proper $\alpha$-invariant ideals.

**Theorem:** The rep $\pi$ of $A_\theta$ on $\ell^2(\mathbb{Z}^d)$ (the GNS rep of the (unique) tracial state)

$$\pi(f)\xi \stackrel{\text{def}}{=} f \star_{c_\theta} \xi$$

is faithful, i.e. kernel $= 0$.

**Proof:**

Slogan: "$\alpha$ is unitarily implemented on $\ell^2(\mathbb{Z}^d)$." I.e. there is a unitary representation $W$ of $T^d$ on $\ell^2(\mathbb{Z}^d)$ such that, for $x \in T^d$:

$$\pi(\alpha_x(a)) = W_x \pi(a) W_x^* \tag{5}$$

What this is saying is that $(\pi, W, \ell^2(\mathbb{Z}^d)$ is a covariant rep for $(A_\theta, T^d, \alpha)$.

If so (we haven't justified the above yet), then if $a \in \ker \pi$, then $\alpha_x(a) \in \ker$ for all $x \in T^d$, so kernel is $\alpha$-invariant. But the kernel is not the whole algebra — there are non-zero operators — then by last time, $\ker = 0$.

Ok, so for unitary equivalence, set:

$$(W_x\xi)(m) \stackrel{\text{def}}{=} \langle m, x\rangle \xi(m)$$

This is almost the same formula as for $\alpha$: $(\alpha_x(f))(m) \stackrel{\text{def}}{=} \langle m, x\rangle f(m)$. From these, it's an easy exercise to sort out the slogan (5). $\square$

Well, so, from before, $U_m U_n = c_\theta(m,n)U_{m+n}$, and $U_m^* = c_\theta(m,n)U_{-m}$. These are unitary generators of the algebra; we can multiply each by a complex number of modulus 1, and we'll still have unitary generators. So, set

$$V_m \stackrel{\text{def}}{=} c_{\theta/2}(m,m)U_m = e^{2\pi i m\cdot \frac{\theta}{2} n}U_m$$

**Claim:**

- $V_m^* = \left(c_{\theta/2}(m,m)U_m\right)^* = \overline{c_{\theta/2}}(m,m)\, c_\theta(m,m)\, U_{-m} = c_{\theta/2}(-m,-m)\, U_{-m} = V_{-m}$
- $V_m V_n = c_\zeta(m,n)V_{m+n}$, where $\zeta = \left(\theta - \theta^t\right)/2$ is the skew-symmetric. What's going on is that the cocycle $c_\theta$ is homologous to $c_\zeta$.

We're using the fact that in $T^d$ everything has a square root. This is not always the case for dual groups, e.g. of finite groups. We won't check the second fact in the claim on the board. In any case, when convenient, we can always insist that $\theta$ be skew-symmetric. (Recall that $\theta$ is a real $d\times d$ matrix, and we can take it up to mod $\mathbb{Z}$.)

For a given $n$, consider the conjugation of $A_\theta$ by $U_n$.

$$\begin{aligned} U_n U_m U_n^* &= c_\theta(n,m)U_{m+n}c_\theta(n,n)U_{-n} \\ &= c_\theta(n,m)c_\theta(n,n)c_\theta(m+n,-n)U_m \\ &= e^{2\pi i(n\cdot\theta m - m\cdot\theta n)}U_m \\ &= \overline{c_{\theta-\theta^t}}(m,n)U_m \\ &\stackrel{\text{def}}{=} \rho_\theta(m,n)U_m \end{aligned}$$

Writing "$\sim$" for "homologous", we see that $\overline{\rho_\theta} = \left(c_{(\theta-\theta^t)/2}\right)^2 \sim c_\theta^2$. Then $\rho_\theta(\cdot,n) \in \widehat{\mathbb{Z}^d} \cong T^d$ is a character. And indeed

$$U_n U_m U_n^* = \alpha_{\rho_\theta(\cdot,n)}(U_m)$$

(We can turn things around and get rid of the complex conjugate sign.)

So, view $\rho_\theta$ as a map $\mathbb{Z}^d \to T^d$ by $n \mapsto \rho_\theta(\cdot,n)$. There's no reason whatsoever why the image should be a closed subset. Let $H_\theta = \overline{\{\rho_\theta(\cdot,n) : n\in\mathbb{Z}^d\}}$ be the closure of the image in $T^d$. So $H_\theta$ is a closed subgroup of $T^d$. Then there's a little taking duals: any closed subgroup has a connected component of the identity, and any connected closed subgroup is another torus stuck in skew-wise. So

$$H_\theta \cong T^e \times F$$

where $e \le d$ and $F$ is finite abelian. We like this version, because we have a compact group and we'd like to average over it, and we know how to do so on each piece.

Let $J$ be any closed 2-sided ideal in $A_\theta$. Then $\alpha_{\rho_\theta(\cdot,n)}(J) = U_n J U_n^* = J$. But $\alpha$ is continuous, and the $\rho_\theta(\cdot,n)$ are dense in $H_\theta$, so $\alpha_x(J) = J$ for all $x\in H_\theta$. On $a\in A_\theta$, define the average

$$Q(a) = \int_{H_\theta} \alpha_x(a)\, dx$$

This is a conditional expectation, and $Q \geq 0$ and $Q$ is faithful. Then $Q(J) \subseteq J$.

If $H_\theta = T^d$ (big if), then $Q(A_\theta) = P(A_\theta) = \mathbb{C}1$ from last time. So if $J$ is not the zero ideal, still in the $_\theta = T^d$ case, then $1 \in J$. In sum:

**Theorem:** If $\{(\theta - \theta^t)(n) : n \in \mathbb{Z}^d\}$ is dense in $\mathbb{R}^d/\mathbb{Z}^d$, then $A_\theta$ has no proper ideals, i.e. is a simple $C^*$ algebra. It's certainly unital and $\infty$-dimensional, but definitely not GCR. Nevertheless, we can write down many irreducible representations.

# 35: April 16, 2008

## 35.1 Irreducible representations of algebras $A_\theta$

Let $\mathcal{H} = L^2(\mathbb{R})$, and pick $\theta \in \mathbb{R} \smallsetminus \{0\}$. Let $U$ be the operator that translates by $\theta$:

$$(U\xi)(t) \stackrel{\text{def}}{=} \xi(t - \theta)$$

Let $V$ be the operator that multiplies by a phase:

$$(V\xi)(t) \stackrel{\text{def}}{=} e^{2\pi i t}\xi(t)$$

Then the $C^*$ algebra generated by $V$ is $C(T)$, where $T$ is the circle $T = \mathbb{R}/\mathbb{Z}$.

**Question from the audience:** Why? The closure is dense in the sup norm, not the operator norm. **Answer:** The sup norm is the operator norm for any of these pointwise multiplication, as long as your measure has full support.

Then, if $f \in C(T)$, we have $V_f = f \times (-)$. And $UV_f = V_{\alpha(f)}U$, where $(\alpha(f))(t) \stackrel{\text{def}}{=} f(t - \theta)$. In particular, taking $V_f = V$ itself, i.e. $f = e^{2\pi i t}$, then we conclude the commutation relation:

$$UV = e^{-2\pi i\theta} VU$$

So we let $W(p, q) \stackrel{\text{def}}{=} U^p V^q$, and $W(p, q)W(p', q') = U^p V^q U^{p'} V^{q'} = e^{2\pi i q p' \theta} W(p + p', q + q')$. This generates the $C^*$ algebra:

$$C^*(\mathbb{Z}^2, \begin{pmatrix} 0 & 0 \\ \theta & 0 \end{pmatrix})$$

or perhaps the transpose of that matrix. But iterating the action $\alpha$, it's clear that this algebra is a crossed product:

$$C(T) \times_\alpha \mathbb{Z}$$

Given a discrete group $G$, and $\alpha$ and action of $G$ on $M$ compact, we get an action $\alpha$ on $C(M)$. Hence, we can form $C(M) \times_\alpha G$. How can we construct *some* irreps of this algebra?

Well, pick some point $m_0 \in M$, and consider its orbit $\mathcal{O}_m$. We have a bijection $G/G_{m_0} \to \mathcal{O}_m$, where $G_{m_0}$ is the stabilizer subgroup. Of course, the orbit might be infinite, so will have limit points. We form $\ell^2(G/G_{m_0})$, which we view as $\ell^2(\mathcal{O}_m)$, with the counting measure — this gives the measure of a compact space to be $\infty$. In any case, we can pull back continuous functions to bounded functions, and hence to bounded operators (multiplication):

$$\begin{tikzcd} C(M) \arrow[r] \arrow[rr, bend right, "\pi: f \mapsto f \times (-)"'] & C_b(\mathcal{O}_m) \arrow[r] & \mathcal{B}(\ell^2(G/G_{m_0})) \end{tikzcd}$$

Then the $\pi(f)$s separate points of $G/G_{m_0}$, and so we get a covariant rep of $C(M), G$. Exercise: this representation is irreducible.

**E.g.** $\theta$ is irrational. Then there are uncountably many different orbits, and each will give a different irrep of $C(T) \times_\alpha \mathbb{Z}$ above. Similarly, for $L^2(T, \text{Lebesgue})$; this is a different Hilbert space, but we can play the same game, so we get more irreps inequivalent to any of these. Classification theorem: you will never explicitly construct all irreducible representations.

****comment on von Neuman algebras, that I missed****

These algebras — $C(T) \times_\alpha \mathbb{Z}$ — are called *rotation algebras*. Even the rational rotation algebras are interesting, although not as much as the irrational ones. More generally, we can look at $C(T^m) \times_\theta \mathbb{Z}^n$ where these rotate at different speeds; this is a special case, because on each of $C(T^m)$ and $\mathbb{Z}^n$ have commuting generators.

**Question from the audience:** Does any measure on the circle give an irrep? **Answer:** No, I need it to be invariant under the rotations.

We saw that $U_m U_n U_m^* = \rho_\theta(m,n) U_n$, with perhaps a different convention last time, where $\rho_\theta$ is a bicharacter, and $\rho_\theta(m,n) = \alpha_{\rho_\theta(m,n)}$. Let $Z_{\rho_\theta} \stackrel{\text{def}}{=} \{m : \rho_\theta(m,n) = 1 \, \forall n\}$; then $U_m \in Z(A_\theta)$ the center iff $m \in Z_{\rho_\theta}$. ****Lecture uses the same symbol for the integers $\mathbb{Z}$ and the variable $Z$; either is reasonable in this context.**** $U_n$ is central iff $\alpha_t(U_n) = U_n$ for any $t \in H_{\rho_\theta}$. Recall $\alpha_t(U_n) = \langle n, t\rangle U_n$. We defined

$$Q(a) = \int_{H_\rho} \alpha_t(a)\, dt$$

and so

$$Q(U_n) = \begin{cases} U_n, & n \in Z_{\rho_\theta} \\ 0, & n \notin Z_{\rho_\theta} \end{cases}$$

This requires a little bit of Fourier analysis. ****Recall that $H$ is the closure of the image of $Z^d$ in $T^d$ under the pairing $\rho_\theta$.****

In any case, $Z_{\rho_\theta}$ is a subgroup of $\mathbb{Z}^d$, and $\text{Range}(Q) \subseteq C^*(Z_{\rho_\theta}) \subseteq Z(A_\theta)$ the center, and if $a \in Z(A_\theta)$, then $Q(a) = a$. Hence $C^*(Z_{\rho_\theta})$ is exactly the center of $A_\theta$.

Ok, so we now can decompose the algebra $A_\theta$ as a field of algebras over the center, and it turns out that each of the fibers is one of these simple algebras.

### 35.2 Differentiation

Smooth structures, in our experience, come from differentiation. We have $A_\theta$ and an action $\alpha$ of $T^d$. We have a surjection $\mathbb{R}^d \to T^d$. Let's generalize a little.

Let $B$ be a Banach space, and $\alpha$ a strongly continuous action of $\mathbb{R}$ on $B$. We don't really need this, but for simplicity, let's think of this action as by isometries. Let $b \in B$, and look at $t \mapsto \alpha_t(b)$, which is norm-continuous on $\mathbb{R}$ with values in $B$. Is this function once-differentiable (at 0 is enough)? I.e., we want to know if

$$\lim_{t \to 0} \frac{\alpha_t(b) - b}{t}$$

exists for the norm $\|\cdot\|_B$ on $B$. In other words, does this limit equal some $c \in B$? Certainly, we'll want $B$ to be complete. If the limit exists, we'll say that $b$ is *differentiable*, and we'll write the limit at $D(b)$.

If $D(b)$ exists, we can ask whether $D(b)$ is differentiable. I.e. $D(D(b))$. And so on: does $D^n(b)$ exist?

****Picture this as $B = C(\mathbb{R})$ and $\alpha$ is by translation.****

Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ (which we think of as $\mathbb{R}^d$, but we don't want to be prejudicial about the basis). Let $\alpha$ be an action of $V$ on $B$. For $v \in V$, we can ask for the *directional derivative* in the direction of $v$:

$$D_v(b) \stackrel{\text{def}}{=} \lim \frac{\alpha_{tv}(b) - b}{t}$$

if the RHS exists. Given $v_1, \ldots, v_n$, we can talk about $D_{v_n} \ldots D_{v_1} b$.

We won't need this generality, but it really does work: Let $G$ be a connected Lie group, and take $G \subseteq GL(n, \mathbb{R})$ closed connected (we can do this up to a discrete subgroup). Then the Lie algebra $\mathfrak{g}$ of $G$ is

$$\mathfrak{g} \stackrel{\text{def}}{=} \{X \in gl(n, \mathbb{R}) : \exp(X) \in G\}$$

There are substantial theorems about this. Then $t \mapsto \exp(tX)$ gives a 1-parameter subgroup of $G$ for each $X$.

Let $\alpha$ be an action of $G$ on $B$. Restrict to $t \mapsto \exp(tX)$. We can define, if it exists:

$$D_X(b) \stackrel{\text{def}}{=} \lim \frac{\alpha_{\exp(tX)}(b) - b}{t}$$

If $D_X(b)$ exists, we can ask about its differentiability, and so on, and let

$$B^\infty = \{b \in B : D_{X_n} \ldots D_{X_1} b \text{ exists for all } n \text{ and all } X_1, \ldots, X_n\}$$

**Theorem:** (Gärding ****sp?****)

$B^\infty$ is dense in $B$.

## 36: April 18, 2008

**Question from the audience:** When we defined the lie group of a Lie Algebra, we said $X \in \mathfrak{g}$ iff $\exp X \in G$. Whenever I've seen this defined, the latter part was $exp(tX) \in G$. Are they equivalent?
**Answer:** No, you want $tX$. E.g. there are matrices so that $\exp X = 1$, but $\exp(X/2) \notin G$.

### 36.1 Smooth structures

Let $G$ be a connected closed subgroup of $GL(n, \mathbb{R})$. In particular, we elide a course on Lie group theory, but $G$ is a submanifold of $GL$. Let $\mathrm{Lie}(G) = \mathfrak{g} = \{X \in M_n(\mathbb{R}) : \exp(tX) \in G \, \forall t \in \mathbb{R}\}$. Then if $X, Y \in \mathfrak{g}$, then $[X, Y] = XY - YX \in \mathfrak{g}$.

The situation we were in: $\alpha$ is a strongly-continuous action of $G$ on a Banach space $B$ by bounded linear maps: $\alpha : G \to \mathcal{B}(B)$. (The same theory with semigroups comes up as well; when $G$ is a group, clearly we map into the invertible maps.) We defined the derivative $D_X b$, if it exists. From there we could define multiple derivatives, and hence the class of $C^\infty$ elements $B^\infty \subseteq B$. This is obviously a linear subspace of $B$.

**Theorem:** (Gärding)

$B^\infty$ is a dense linear subspace in $B$.

**Proof:**

The buzzwords are "smoothly" and "molifies".

(**E.g.** Let $\alpha$ be an action of $\mathbb{R}$ on $M$ a manifold, which could be chaotic evolution. (We won't define this, but see the paper: Lorenz just died, and he brought to life chaotic theory.) We get an action on $C_\infty(M) = B$, and there will be functions that are differentiable in this sense.)

Let $b \in B$ be given, and let $f \in C_c^\infty(G)$. Then we claim $\alpha_f(b) \in B^\infty$. But let $f$ run over an approximate identity; then these will converge to $b$. Recall:

$$\alpha_f(b) \stackrel{\text{def}}{=} \int_G f(x) \, \alpha_x(b) \, dx$$

where $dx$ is Haar measure; $f$ has compact support, so this is a continuous $B$-values function.

So, why is $\alpha_f(b) \in B^\infty$? Go back to the definition: given $X \in \mathfrak{g}$, we look at

$$\frac{1}{t} \left( \alpha_{\exp(tX)}(\alpha_f(b)) - \alpha_f(b) \right)$$

Does this have a limit? For fixed $t$, we can pull into the integral sign, and commute past the number $f(x)$:

$$\begin{aligned}
\frac{1}{t}\left(\alpha_{\exp(tX)}(\alpha_f(b)) - \alpha_f(b)\right) &= \frac{1}{t}\left(\int_G f(x)\,\alpha_{\exp(tX)}\alpha_x(b)\,dx - \text{something}\right)\\
&= \frac{1}{t}\left(\int_G f(x)\,\alpha_{\exp(tX)x}(b)\,dx - \text{something}\right)\\
&= \frac{1}{t}\left(\int_G f(\exp(-tX)x)\,\alpha_x(b)\,dx - \text{something}\right)\\
&= \int_G \frac{f(\exp(-tX)x) - f(x)}{t}\,\alpha_x(b)\,dx
\end{aligned}$$

And the inside fraction is just a derivative of a scalar-function in the Lie group: it's just $D_X(f)(x) \stackrel{\text{def}}{=} \lim \frac{f(\exp(-tX)x)-f(x)}{t}$. We have Taylor series: $f(\exp(-tX)x) = f(x)+(D_X f)(x)+\frac{1}{2}(D_X^2 f)(x) + O(t^3)$. Thus the difference is

$$\frac{f(\exp(-tX)x) - f(x)}{t} - (D_X f)(x) = (\text{continuous})(t) \underset{t\to 0}{\longrightarrow} 0$$

uniformly in $x$ (we have compact support). Thus we can integrate, so

$$\int_G \frac{f(\exp(-tX)x) - f(x)}{t}\,\alpha_x(b)\,dx \underset{t\to 0}{\longrightarrow} \int_G (D_X f)(x)\,\alpha_x(b)\,dx = \alpha_{D_X f}(b)$$

But $D_X f \in C_c^\infty(B)$, so $D_Y(\alpha_{D_X f}(b)) = \alpha_{D_Y D_X f}(b)$. Iterate, and $\alpha_f(b) \in B^\infty$. $\square$

**Definition:** The *Gärding domain* is the linear span of $\{\alpha_f(b) : f \in C_c^\infty(B), b \in B\}$.

This is certainly dense in $B$, and contained in $B^\infty$. Did everybody catch why? **Question from the audience:** Don't you need that your representation is nondegenerate? **Answer:** The representation is coming from an action. And such things are always nondegenerate, because we have approximate identities. For instance, let $f$ approximate a delta function at the identity in $G$. Then

$$\begin{aligned}
\alpha_f(b) - b &= \int (f(x)\alpha_x(b) - b)dx\\
\|\alpha_f(b) - b\| &= \|\int (f(x)\alpha_x(b) - b)dx\|\\
&\leq \sup\{\|\alpha_x(b) - b\| : x \in \text{support}(f)\}
\end{aligned}$$

**Question from the audience:** These are unbounded operators. If we're in a $C^*$ algebra, and give it the standard Hilbert structure, we can ask if these are adjointable? **Answer:** I don't know if that's been looked at. We can ask if the Gärding domain is equal to $B^\infty$. Dixmier-Melhann ****??**** looked at a related question. They asked something like whether each element of $C_c^\infty(G)$ is a convolution of things in there. They found groups for which that's false, although any element is a finite sum of convolutions. There are places where knowing things like that it useful. But we will be working where the Lie algebra is abelian.

**Theorem:** On $B^\infty$, $[D_X, D_Y] = D_X D_Y - D_Y D_X = D_{[X,Y]}$.

This is a basic and important fact, and takes some analysis.

Suppose that $A$ is a Banach algebra, and $\alpha$ is a (strongly continuous) action of $G$ on $A$ by Banach-alg automorphisms. It's easy to prove (a la freshman calculus) that

$$D_X(ab) = (D_X(a))b + a(D_X(b))$$

for $a, b \in A^\infty$. **Cor:** $A^\infty$ is a subalgebra of $A$.

Also, if $A$ is a $*$-algebra and $\alpha$ is by $*$-automorphisms, then $A^\infty$ is a $*$-subalgebra.

**Question from the audience:** It seems like our notion of smoothness depends on the group. Should we look for a maximal action of some sort to make sure we have the right functions? **Answer:** That seems like a good idea, but nobody knows how to do that. If you look at examples (interesting ones, nothing pathological), you find that they may have very few actions by Lie groups. Then there doesn't appear to be much differentiable structure. But the leap that Alain Connes has taken is to say that $A$ is a $C^*$-algebra, and view $A \subseteq \mathcal{B}(\mathcal{H})$. Let $D$ be an unbounded self-adjoint operator on $\mathcal{H}$. Let $U_t = e^{2\pi i t D}$, and let $\alpha_t(T) = U_t T U_t^*$ for $T \in \mathcal{B}(\mathcal{H})$. We can talk about smooth vectors $\mathcal{B}(\mathcal{H})^\infty$. There's no reason the action should carry the algebra into itself, but it may happen that $A \cap \mathcal{B}^\infty(\mathcal{H})$ is dense in $A$. $a \in A \cap \mathcal{B}^\infty(\mathcal{H})$ iff, more or less (only densely define), $[D, a]$ is a bounded operator (on a dense domain, so extends — well, this is once differentiability, so need to repeat. This picks out a smooth subalgebra of $A$. Then an operator is being used in this way, Connes calls this a *Dirac operator*. Because it matches the notion on a Riemannian manifold, and indeed we can recover the metric from the Dirac operator. So Connes says that this is the way to do non-commutative Riemannian geometry. **Question from the audience:** What is $\mathcal{H}$ in this manifold case? $L^2(M)$? **Answer:** No, it's $L^2$ with values in the spinor bundle.

### 36.2 Returning to our main example

Ok, let's return to the case at hand. We have $A_\theta, \alpha, T^d$. $\alpha$ is an action of $T^d$ on a Banach space $B$. For $n \in \widehat{T^d} = \mathbb{Z}^d$, we have $B_n$ defined by

$$B_n = \{b \in B : \alpha_x(b) = \langle x, n\rangle b\}$$

where, of course, $\langle x, n\rangle = e^{2\pi i x\cdot n}$. For fixed $b \in B_n$, its span is an invariant one-dimensional subspace in $B_n$.

On $\mathbb{R}^d$, we vie the Lie algebra and Lie group as

$$\begin{pmatrix} 0 & 0 & \vec{v} \\ 0 & \ddots & \\ 0 & & 0 \end{pmatrix} \overset{\exp}{\mapsto} \begin{pmatrix} 1 & 0 & \vec{v} \\ 0 & \ddots & \\ 0 & & 1 \end{pmatrix}$$

So really $\exp(X) = X$. Then for $b \in B_n$,

$$D_X(b) = \lim \frac{\alpha_{tX}(b) - B}{t} = \lim \frac{e^{2\pi i t X \cdot n} - 1}{t} b = (2\pi i X \cdot n)b$$

**Question from the audience:** Fourier transform takes differentiation to multiplication? **Answer:** Precisely.

## 37: April 21, 2008

We were in the situation of having $T^d$ and $\alpha$ an action on $B$ a Banach space. We had seen from general considerations of Lie groups that you can form the space $B^\infty$ of smooth vectors, dense in $B$.

On the other hand, for $n \in \mathbb{Z}^d = \widehat{T^d}$, we had $B_n = \{b : \alpha_t(b) = \langle n, t\rangle b\}$. (Recall: $\langle n, t\rangle = e^{2\pi i t \cdot n}$.) Then if $X \in \mathbb{R}^d = \mathrm{Lie}(T^d)$, we defined

$$D_X b \stackrel{\text{def}}{=} \lim_{r \to 0} \frac{\alpha_{rX}(b) - b}{r}$$

and on $b \in B_n$ we have $D_X b = (2\pi i X \cdot n)b$.

Remember, on $A_\theta$ the noncom torus, we had a dual action

$$\alpha_t(U_n) \stackrel{\text{def}}{=} \langle n, t\rangle U_n$$

and so $(A_\theta)_n = \mathbb{C}U_n$ is just a one-dimensional span.

**Question from the audience:** What is $\alpha_{rX}$? **Answer:** $\alpha_{\exp rX}$. But we're writing $T^d$ additively.

So $D_{X_1} \ldots D_{X_k} b = (2\pi i)^k \left(\prod_{j=1}^k n \cdot X_j\right) b$.

For $b \in B^\infty$, we expect to write incomponents: $b \sim \{b_n\}$. Then we expect $\left(D_{X_1} \ldots D_{X_k} b\right)_n \sim D_{X_1} \ldots D_{X_k} b_n = (2\pi i)^k \left(\prod_{j=1}^k n \cdot X_j\right) b_n$. But the right and left should be in $b$; taking norms:

$$(2\pi)^k \prod |n \cdot X_j| \, \|b_n\| \leq \|D_{X_1} \ldots D_{X_k} b\|$$

and the RHS is indep of $n$. The norm on the RHS is some constant, adjusting it we can say that

$$\|b_n\| \leq \frac{c}{(1 + \prod |n \cdot X|^2)^n}$$

so the coefficients $b_n$ must die faster than any polynomial.

We define the Schwartz space: $\mathcal{S}(\mathbb{Z}^d) = \{f : \mathbb{Z}^d \to \mathbb{C}$ such that $n \mapsto |f(n)p(n)$ is a bounded function for all polys $p\}$.

**Theorem:** $B^\infty$ is the $B$-valued Schwartz space: it consists of all functions $c : \mathbb{Z}^d \to B$ such that $c(n) \in B_n$ and $\{n \mapsto \|c_n\|\} \in \mathcal{S}(\mathbb{Z}^d)$.

Then in particular $(A_\theta)^\infty$ "is" $\mathcal{S}(\mathbb{Z}^d)$ by $f \mapsto \sum f(n)U_n$. In any case, for $p$ big enough, it's clear that $\mathcal{S}(\mathbb{Z}^d) \subseteq \ell^1(\mathbb{Z}^d)$, and this sum converges.

This relates to a well-known fact: for $g \in C(T^d)$, $g \in C^\infty(T^d)$ iff the Fourier transform $\hat{g} \in \mathcal{S}(\mathbb{Z}^d)$.

**Proof of Theorem:**

If we have $c \in \mathrm{RHS}$, we certainly have $\{n \mapsto \|c(n)\|\} \in \ell^1(\mathbb{Z}^d)$, so set $b = \sum c(n)$ converges just fine, and $b_n = c(n)$. Using the differential quotient, you find that $(D_X b)_n = (2\pi i)n \cdot X\, c(n)$, and the sum of these things since we're in the Schwartz space converges. So we have $\supseteq$. (We did not complete the proof, but it goes along pretty straightforwardly.)

In the opposite direction, let $b \in B^\infty$. ****We spend some time on a calculation we had done before, going the wrong direction.**** We're trying to show that $\{n \mapsto \|b_n\|\} \in \mathcal{S}(\mathbb{Z}^d)$. Then $D_X b$ exists, and we need to show that $(D_X b)_n = 2\pi i\, n \cdot X\, b_n$. Write $f(t) = e^{2\pi i\, n \cdot t}$. The limit $D_X b = \lim(\alpha_{rX}(b) - b)/r$ is a uniform limit, so:

$$\begin{aligned}
(D_X b)_n &= \int f(t)\, \alpha_t(D_X b)\, dt \\
&= \lim \int f(t)\, \alpha_t\left(\frac{\alpha_{rX}(b) - b}{r}\right) dt \\
&= \lim \frac{1}{r}\left(\int f(t)\, \alpha_t \alpha_{rX}(b)\, dt - \int f(t)\, \alpha_t(b)\, dt\right) \\
&= \lim \frac{1}{r}\left(\int f(t - rX)\, \alpha_t(b)\, dt - \int f(t)\, \alpha_t(b)\, dt\right) \\
&= \lim \int \frac{f(t - rX) - f(t)}{r} \alpha_t(b)\, dt \\
&= 2\pi i\, n \cdot X\, b_n, \text{ since } f = e^{2\pi i n \cdot t}
\end{aligned}$$

Thus $(D_{X_1} \dots D_{X_n} b)_n = p(n)\, b_n$, and so take norms and observe that we're in $\mathcal{S}(\mathbb{Z}^d)$. $\square$

Let $A$ be a $C^*$ algebra, and let $\alpha$ be an action of $T^d$ on $A$. (**E.g.** let $\alpha$ be an action of $T^d$ on a locally compact space $X$, e.g. a manifold, so can take $A = C_\infty(X)$. Even this commutative case is interesting.) So we have all these spaces $A_n$, and let $a_m \in A_m$, $a_n \in A_n$, then $\alpha_t(a_m a_n) = \alpha_t(a_m)\alpha_t(a_n) = \langle m, t\rangle a_m \langle n, t\rangle a_n = \langle m + n, t\rangle a_{m+n}$, and the multiplication is graded: $A_m A_n \subseteq A_{m+n}$. Similarly, $A_m^* = A_{-m}$.

Let $\theta$ be given, and build the cocycle $c_\theta(m, n)$. Let $(\pi, U, \mathcal{H})$ be a faithful covariant representation of $(A, T^d, \alpha)$. $T^d$ acts on $\mathcal{H}$, so we can factor

$$\mathcal{H} = \bigoplus_{n \in \mathbb{Z}^d} \mathcal{H}_n$$

You can check: if $a_m \in A_m$ and $\xi_n \in \mathcal{H}_n$, then $\pi(a_m)\xi_n \in \mathcal{H}_{m+n}$.

Now we will do something weird. Continuing to use these labels to tell you where things come from, define:

$$\pi^\theta(a_m)\xi_n \stackrel{\text{def}}{=} \pi(a_m)\xi_n c_\theta(m,n) \in \mathcal{H}_{m+n}$$

. Next time, we will explore this. We will find that all of this is well-defined on $A^\infty$, and we're twisting this algebra by a cocycle.

# 38: April 23, 2008

I keep forgetting, you need to turn in a third problem set, and I don't want it at the end of the semester. Is it unreasonable to ask for it on Monday? If you won't be turning it in on Monday, please talk to me.

## 38.1 We were doing somewhat strange things

We have $T^d$ and $\alpha$ an action on $C^*$-algebra $A$. This gives us a smooth algebra $A^\infty$, which sort of looks like the Schwartz space, except for functions values in $A_n$. Then we get subspaces $A_n \subseteq A^\infty$ for $n \in \mathbb{Z}^d$. We had $(\pi, U, \mathcal{H})$ a covariant representation of $(A, T^d, \alpha)$, and we assume $\pi$ is faithful. Then we have Hilbert spaces $\mathcal{H}_n$, and if $m \neq n$, then $\mathcal{H}_n \perp \mathcal{H}_m$. Exactly the same proof that eigenspaces of self-adjoint operators are orthogonal.

**Question from the audience:** The direct sum of $A_n$s is dense in $A^\infty$? Is it all of it? **Answer:** $A^\infty$ is the set of sequences $\{a_n\}$ where each $a_n \in A_n$, and where the map $\{n \mapsto \|a_n\|\} \in \mathcal{S}(\mathbb{Z}^d) \subseteq \ell^1(\mathbb{Z}^d)$. For any $a$ we can get a sequence of $a_n$, but it's almost impossible to say what sequences come from general functions. So the direct sum is dense, but not complete in the Frechet topology; the direct sum is all the finite ones. **Question from the audience:** $A^\infty$ is the sums of these sequences? **Answer:** Yes. Or define the space of these sequences as a graded algebra, and there's a bijection between $A^\infty$ and these sequences, by summing in one direction, and in the other direction by taking Fourier modes.

In the non-commutative case, it's more complicated, but you still get a decomposition of the $C^*$ algebra of functions via the group action. **Question from the audience:** With a non-compact Lie group? **Answer:** For each irreducible representation, you can define a space $A_n$. Where you get into trouble: if we multiply two characters, you get a character, but in higher dimension the tensor product of two irreducible representations may not be irreducible, and the bookkeeping gets harder.

So, given $\theta \in M_d(\mathbb{R})$ and cocycle $c_\theta$. For any $\xi_n \in \mathcal{H}_n$, and $a_m \in A_m$, define

$$\pi^\theta(a_m)\xi_n \stackrel{\text{def}}{=} \pi(a_m)\xi_n c_\theta(m,n)$$

For any $\xi \in \mathcal{H}$, we have $\xi = \sum \xi_n$.

$$\begin{aligned}
\left\| \pi^\theta(a_m) \sum_{\|n\| \le M} \xi_m \right\|^2 &= \left\| \sum \pi(a_m)\xi_n c_\theta(m,n) \right\|^2 \\
&= \sum \|\pi(a_m)\xi_n c_\theta(m,n)\|^2 \text{ by orthogonality} \\
&\le \sum \|\pi(a_m)\| \, \|\xi_n\| \\
&= \|\pi(a_m)\| \, \|\xi\| \text{ where } \xi = \sum \xi_n
\end{aligned}$$

So $\sum_{n\in\mathbb{Z}^d} \pi^\theta(a_m)\xi_n$ converges, and we call the limit $\pi^\theta(a_m)\xi$.

$$\left\| \pi^\theta(a_m)\xi \right\| \le \|\pi(a_m)\| \|\xi\| \le \|a_m\| \|\xi\|$$

Thus for $a \in A^\infty$, set

$$\pi^\theta(a)\xi \stackrel{\text{def}}{=} \sum \pi^\theta(a_m)\xi$$

and $\pi^\theta(a_m)$ is $\ell^1$. Then

$$\begin{aligned}
\pi^\theta(a_m)\pi^\theta(a_n)\xi_p &= \pi^\theta(a_m) \left( \pi(b_n)\xi_p c_\theta(n,p) \right) \\
&= \pi(a_m)\pi(b_n)\xi_p c_\theta(n,p) c_\theta(m, n+p) \\
&= \left( \pi(a_m)\pi(b_n) c_\theta(m,n) \right) \xi_p c_\theta(m+n, p)
\end{aligned}$$

So, on $A^\infty$, we define a product

$$a \star_\theta b = \sum a_m \, b_n \, c_\theta(m,n)$$

and since these sequences in norm are $\ell^1$, we see that this series converges without any difficulty. Then

$$\pi^\theta(a)\pi^\theta(b) = \pi^\theta(a \star_\theta b)$$

Moreover, the $*$:

$$\left( \pi^\theta(a_n) \right)^* = a_n^* c_\theta(n,n) \stackrel{\text{def}}{=} a_n^{*_\theta}$$

In any case, this gives a $*$-algebra structure on $A^\theta$ and a $*$-rep on $\mathcal{H}$. We want $\pi$ faithful. Then we get a $C^*$-norm on $A^\infty$. Complete this to get a $C^*$-algebra $A^\theta$. This is not the same as $A_\theta$ from earlier.

**Question from the audience:** Why do you do this just for $A^\infty$ and not all of $A$? **Answer:** The twisted $C^*$ norm is not continuous for all of $A$. **E.g.** $A = C(T^d)$, then $A^\theta = A_\theta$, but the norm on $A_\theta$ is not equivalent to the sup norm on $A$, just on $A^\infty$. Well, we could work on $\ell^1$.

A bit of context: where does this come from? $\theta$ defines a "Poisson bracket" on $A^\infty$ in the obvious sense: choose an orthonormal basis for $\mathbb{R}^d = \mathrm{Lie}(T^d)$, which might as well be the "standard" basis $\{E_j\}$. Then the Poisson bracket of $a$ and $b$ is

$$\{a,b\}_{\theta,\alpha} \stackrel{\text{def}}{=} \sum \theta_{jk} D_{E_j}(a) D_{E_k}(b)$$

This is really best after we change so that $\theta^t = -\theta$ is skew-symmetric. If $M$ is a manifold, and $T^d$ acts smoothly on $M$, then $T^d$ acts on $C_\infty(M)$, and it's very natural to define a Poisson bracket $\{f, g\} = \sum \theta_{jk} D_{E_j}(f) D_{E_k}(g)$. The first person to do our more general case carefully was a student of Alan Weinstein's, by the name of ****missed****.

When $A = C_\infty(M)$, we have $A^\theta$, which we should view as a "quantization" of $C_\infty(M)$ in the direction of the Poisson bracket. To make this precise, we need to get Plank's constant $\hbar \in \mathbb{R}$ in here. Then $\hbar\theta$ is again skew-symmetric, so we can define $A^{\hbar\theta}$. I.e. $\hbar \mapsto A^{\hbar\theta}$ is a one-parameter family of algebras, and when $\hbar = 0$ we get the original algebra. Then for $a, b \in A^\infty$,

$$\left\| \frac{a \star_{\hbar\theta} b - b \star_{\hbar\theta} a}{\hbar} - 2i\{a, b\}_\theta \right\| \xrightarrow{\hbar \to 0} 0 \tag{6}$$

Says that the "semi-classical limit" of the $A^{\hbar\theta}$ is $A$ equipped with the Poisson bracket $\{\}_\theta$.

So if you take the opinion that the world is quantum, then in the classical limit, the remnant of the quantum world is the Poisson bracket in the ordinary world.

Another way of putting equation (6):

$$a \star_{\hbar\theta} b = ab + i\hbar\{a, b\} + O(\hbar^2)$$

The limit (6) is often called the "correspondence principle".

For $\mathbb{R}^n$ acting on $A$, we again have $A^\infty$, and for $\theta$ we can define $a \star_\theta b$. This is technically more difficult, because we don't have subspace $A_n$.

By the same formula as before, $T^d$ acts on $A^\theta$ by multiplying by the corresponding character independent of $\theta$. We see that $(A^\theta)^\infty = A^\infty$. And so given $\theta_1$, $\theta_2$, via the $\mathbb{R}^d$ action,

$$\left(A^{\theta_1}\right)^{\theta_2} = A^{\theta_1 + \theta_2}$$

and in particular we can twist by $\theta$ and then twist back by $-\theta$.

These are "uniform deformation formulas", also called "deformation quantization". There are other kinds of quantization, e.g. by approximating an algebra by an algebra of matrices. Want: for any Lie group $G$ with "compatible" Poisson bracket, i.e. for any "Poisson Lie group", and any action $\alpha$ of $G$ on a $C^*$-algebra $A$, we would want a construction to deform $A^\infty$ in the direction of the Poisson bracket.

Most quantum groups people construct are made by doing this at the purely algebraic level, where a Lia algebra acts on an algebra. At this algebraic level, that's tough. It's even tougher in our analytical context. Some interesting papers exist, but it's presently under research.

# 39: April 25, 2008

## 39.1 Further comments on deformation quantization

$SO(n+1)$ acts on $S^n$ in a natural way, and so long as $n \geq 3$, we can find inside $SO(n+1)$ two or more copies of the rotation group:

$$\begin{pmatrix} \cos 2\pi r & \sin 2\pi r & & & \\ -\sin 2\pi r & \cos 2\pi r & & & \\ & & \cos 2\pi s & \sin 2\pi s & \\ & & -\sin 2\pi s & \cos 2\pi s & \\ & & & & \ddots \end{pmatrix}$$

So we have roughly a $(n+1)/2$-dimensional torus acting on $S^n$, so pick a $\theta$ and build $(S^n)^\theta$. M.R. had described our deformation quantization in general; Alain Connes became interested in examples, and in particular the quantum spheres $(S^n)^\theta$.

Moreover, M.R. gave the prescription for building quantum groups like $(SO(n+1))^\theta$, a quantum group, but others did the examples, and $(SO(n+1))^\theta$ acts on $(S^n)^\theta$. This is a relatively tame situation.

**Question from the audience:** So when you deform a group to get a quantum group, you have different multiplication by same comultiplication? **Answer:** Well, there are different versions. If we have a compact group, we use

$$\begin{array}{rcl} C(G) & \overset{\Delta}{\to} & C(G) \underset{C^*}{\otimes} C(G) = C(G \times G) \\ f & \mapsto & (\Delta f)(x,y) \overset{\text{def}}{=} f(xy) \end{array}$$

Then the comultiplication encodes the group structure, and a quantum group is some algebra with a coassociative comultiplication.

## 39.2 Differential forms

Let $G$ be a Lie group, $\alpha$ and action on $A$. Then we have $A^\infty$, $\alpha_X = D_X$ for $X \in \mathfrak{g} = \mathrm{Lie}(G)$. Given $a \in A^\infty$, define $da : \mathfrak{g} \to A^\infty$, i.e. $da \in \mathfrak{g}' \otimes A^\infty$ (where $\mathfrak{g}'$ is the dual algebra of $\mathfrak{g}$ ****why not use $\hat{\mathfrak{g}}$?**** by

$$(da)X \overset{\text{def}}{=} \alpha_X(a)$$

Then

$$d(ab)\,X = \alpha_X(ab) = \alpha_X(a)b + a\alpha_X(b) = ((da)b + a(db))\,(X)$$

**Definition:** For any algebra $A$, a *first-order differential calculus* over $A$ is a pair $(\Omega^1, d)$ where $\Omega^1$ is an $A$-bimodule and $d$ is a map $d : A \to \Omega^1$ satisfying the Leibniz rule $d(ab) = a\,db + da\,b$.

Often, we require additionally that $\Omega^1$ be generated as a bimodule by $d(A)$. In this case, if $1 \in A$, the Leibniz rule provides that $\Omega^1$ is generated by $d(A)$ as a left (or as a right) module.

Every $A$ (with 1) has a universal first-order calculus: we let $\Omega^1$ is (a subspace of) the algebraic tensor product $A \otimes A$, where $da \stackrel{\text{def}}{=} 1 \otimes a - a \otimes 1$. Then in particular $a\,db = a \otimes b - ab \otimes 1$. We have the algebra multiplication a map $m : A \otimes A \to A$, and then $m(a\,db) = 0$. You can check: $\operatorname{span}\{a\,db\} = \ker(m)$. Sometimes people define $\Omega^1$ as this kernel, but this, at a philosophical level, is from convenience rather than general principles.

**Question from the audience:** In what notion is this universal? **Answer:** Any other first-order differential calculus is a quotient of this one.

In any case, above (Lie group) is an example. In non-commutative geometry, the notion of "tangent space" becomes less useful. Any algebra $A$ might have lots of derivations, but the space of derivations is not really a module over $A$. By non-commutativity, if $D$ is a derivation, then $aD$ probably is not.

But here we have cotangent spaces: differential forms. Indeed, we have a proliferation of them. Without getting too deep, we certainly have higher-order differential calculi:

$$A \longrightarrow \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \cdots$$

where we demand that $d^2 = 0$. Once we have this type of structure, we can define a cohomology for our differential calculus: $Z^n \stackrel{\text{def}}{=} \ker(\Omega^n \xrightarrow{d} \Omega^{n+1})$ and $B^n \stackrel{\text{def}}{=} \ker(\Omega^{n-1} \xrightarrow{d} \Omega^n)$, and $H^n \stackrel{\text{def}}{=} Z^n/B^n$.

For the universal calculus, we can take $\Omega^2$ as the span of symbols of the form $a_0\,da_1\,da_2$, manipulated in the obvious way (where we be careful about keeping the order in tact, as we are noncommutative). For $G$ and an action $\alpha$ on $A$, we can use $\Omega^n = (\bigwedge^n \mathfrak{g}') \otimes A^\infty$. **Question from the audience:** Normally we let the wedge product be anti-commutative. In the non-commutative case, shouldn't this be worse? **Answer:** Well, we want $n$-linear alternating $A$-valued forms on $\mathfrak{g}$. The Leibniz rule is complicated:

$$d(\omega_p \omega) = (d\omega_p)\omega + (-1)^p \omega_p (d\omega)$$

where $\omega$ can be any form, and $\omega_p$ is homogeneous of degree $p$.

Well, this is all somewhat weird. I'm sure you've heard that even in ordinary differential geometry, as soon as you get to dimension 7, the 7-sphere and higher have exotic differential structures. This happens in non-com-land, e.g. for non-commutative tori, when $n \geq 4$: On $T^4$, we can have $\theta_1$ and $\theta_2$ where $A_{\theta_1} \cong A_{\theta_2}$ but $A^\infty_{\theta_1} \not\cong A^\infty_{\theta_2}$. Finding the right invariants to show all this is hard, and gets into K-Theory. It is in the direction we want to go in.

## 39.3 Vector Bundles

We won't assume that you know too much about vector bundles in detail, but the general picture is that you have some space $M$ and a bundle of vector spaces over $M$ ****a standard picture****

$$\begin{tikzcd} E \arrow[d] \\ M \end{tikzcd}$$

so that locally $E \cong \mathcal{O} \times \mathbb{R}^n$ or $\mathbb{C}^n$, i.e. local triviality. We can think about smooth cross-sections. We write $\Gamma(E)$ for the space of continuous cross-sections, and by local triviality we can take bump functions, giving lots of continuous cross-sections (certainly we also have the 0-section).

Take $M$ compact for simplicity. Then we have $C(M)$, and given $f \in C(M)$ and $\xi \in \Gamma(E)$, we can define $f\xi$ in the obvious way. By looking at an open neighborhood of each point, it's clear that this again is a continuous cross-section. Written briefly: $\Gamma(E)$ is a module over $C(M)$.

In fact, these are somewhat special modules. We know from working with vector spaces that it's useful to have inner products. Since we're assuming compactness, we can cover $M$ with a finite number of open sets $\mathcal{O}_1, \ldots, \mathcal{O}_k$ over which $E$ is trivial. Then we can find a continuous partition of unity $\{\phi_j\}$ subordinate to $\{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$: i.e. the support of $\phi_j$ is contained in $\mathcal{O}_j$ for each $j$, and $\sum \phi_j = 1$ and $0 \le \phi_k \le 1$.

Then for $\xi, \eta \in \Gamma(E)$, we look at $E$ over $\mathcal{O}_j$, over which it looks like $\mathcal{O}_j \times \mathbb{R}^n$ (or perhaps $\mathbb{C}^n$), then we can view $\xi, \eta|_{\mathcal{O}_j}$ as living in $\mathcal{O}_j \times \mathbb{R}^n$, and then we can form the standard inner product in terms of our choice of trivialization and get a function $\langle \xi, \eta \rangle_{\mathbb{R}^n}$. Multiplying by $\phi_j$ gives us a function that's 0 near the boundary, and so extends to the whole space. Then we can get a global inner product:

$$\langle \xi, \eta \rangle_{C(M)} \stackrel{\text{def}}{=} \sum_j \phi_j \langle \xi|_j, \eta|_j \rangle_{\mathbb{R}^n} \tag{7}$$

This is a continuous function, i.e. it is an element of $C(M)$. This is a good example of an "$A$-valued inner product" (for $A = C(M)$) on $\Gamma(E)$. In the real case, these are called "Riemannian metrics" on the bundle, and in the complex case called "Hermetian metrics". A good neutral term is *bundle metric*.

We should set this machinery up to avoid the following stupid possibility: $A = C([0,1])$ and $E = [0,1] \times \mathbb{R}^n$, and we could set $\langle \xi, \eta \rangle_A(t) = t\langle \xi(t), \eta(t) \rangle$. This is an inner product, and has the property that if $\langle \xi, \xi \rangle = 0$ then $\xi = 0$. So this $A$-valued inner product satisfies all the right conditions for an inner product, but it seems wrong to have the zero inner product even at a point. What's wrong is that it's not self-dual. Our earlier inner product (7) is *self-dual* in the sense that if $F \in \mathrm{Hom}_A(\Gamma(E), A)$ then there is (unique) $\eta \in \Gamma(E)$ such that $F(\xi) = \langle \xi, \eta \rangle_A$ for every $\xi$. (We are in the commutative case, so the order we write in doesn't really matter.) These modules are called *projective*.

# 40: April 28, 2008

## 40.1 More on vector bundles

We had been talking about vector bundles, in preparation of the non-commutative case. It will be more convenient to use right modules.

**Definition:** Let $A$ be a unital $C^*$-algebra (or a "nice" $*$-subalgebra thereof), and $\Xi$ a right $A$-module. An $A$-valued *inner product* on $\Xi$ is a map

$$\langle \cdot, \cdot \rangle_A : \Xi \times \Xi \to A$$

satisfying "$A$-sesquilinearity" and "positivity":

(a) Bi-additivity ****bilinearity over $\mathbb{Z}$****

(b) $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$

(c) $\langle \xi, \eta \rangle_A^* = \langle \eta, \xi \rangle_A$ (Hence $\langle \xi a, \eta \rangle_A = a^* \langle \xi, \eta \rangle_A$)

(d) $\langle \xi, \xi \rangle_A \geq 0$ (the notion of positivity requires something about $C^*$-algebras)

(e) Sometimes: $\langle \xi, \xi \rangle_A = 0$ implies $\xi = 0$

**E.g.** $E$ a vector bundle over $M$ compact, and $\Xi = \Gamma(E)$, $A = C(M)$. Then take the inner-product that's $\mathbb{C}$-linear in the second variable.

We say that $\Xi$ is a "Hilbert $C^*$-module over $A$". **Question from the audience:** In order to use the name "Hilbert", shouldn't there be some sort of completeness? **Answer:** Yes. So perhaps above is a "pre-Hilbert module". We can set a norm

$$\|\xi\|_\Xi \stackrel{\text{def}}{=} \|\langle \xi, \xi \rangle_A\|_A^{1/2}$$

We can show this is a norm, and for Hilbert we need some sort of Cauchy-Schwarts condition. Our above example will be complete once you do all that.

Well, if you have a Hilbert space, it's common to discuss rank-one operators. Here we can do the analogous thing: Given $\xi, \eta \in \Xi$, we set $\langle \xi, \eta \rangle_0$ ($= \langle \xi, \eta \rangle_E$, for "endomorphism", but a different "$E$" than in the above example) to be the element of $\mathrm{End}_A(\Xi)$ defined by

$$\langle \xi, \eta \rangle_0 \zeta = \xi \langle \eta, \zeta \rangle_A$$

We write $A$-things on the right so that we can put endomorphisms on the left; then there is no crossing. The formalism works just like with rank-one operators.

We write $\mathcal{B}(\Xi)$ for the bounded operators for the above norm, except that sometimes operators don't have adjoints, and this is sad. Hence, we use:

- The *adjoint* (with respect to the norm $\langle , \rangle_A$) of an operator $T \in \mathrm{End}_A(\Xi)$ is an operator $S \in \mathrm{End}_A(\Xi)$ such that
$$\langle T\xi, \eta \rangle_A = \langle \xi, S\eta \rangle_A$$
for every $\xi, \eta \in \Xi$. If $\langle , \rangle$ is definite, then $S$ is unique, and we write $S = T^*$.

- Then
$$\mathcal{B}(\Xi) \stackrel{\text{def}}{=} \{T \in \mathrm{End}_A(\Xi) : \|T\|, \|T^*\| < \infty\}$$

In any case, we see that $\langle \xi, \eta \rangle_0^* = \langle \eta, \xi \rangle_0$. If $T \in \mathcal{B}(\Xi)$, then $T\langle \xi, \eta \rangle_0 = \langle T\xi, \eta \rangle_0$. So "$\langle \cdot, \cdot \rangle_0$ is a $\mathcal{B}(\Xi)$-valued inner-product, where we consider $\Xi$ as a left-module over $\mathcal{B}(\Xi)$."

**E.g.** In the above vector-bundle example, this works, and is appropriately continuous (the notion of continuity can be derived from a suitable open cover).

**Question from the audience:** Most of these notions are in your paper? **Answer:** Various papers, yes.**Question from the audience:** Do the rank-one operators form an ideal? **Answer:** No, you have to take sums. The rank-one operators span an ideal; denote its closure $\mathcal{K}(\Xi)$ for "compact": these are not compact in the usual range sense, but it's an extremely useful ideal. If $\mathrm{span}\langle \xi, \eta \rangle_A$ is dense in $A$ (we never said how big a module we had; this means it's not tiny), then
$${}_{\mathcal{K}(A)}\Xi_A$$
is a Morita equivalence. ****Perhaps the left subscript should be $\mathcal{K}(\Xi)$? The board says $\mathcal{K}(A)$, which is a natural notion, as in the subsequent question.****

**Question from the audience:** Is this a simple ideal, topologically? **Answer:** No. For instance, take $A$, with the obvious right-action and inner product. If $A$ is unital, then $\mathcal{K}(A) = A$, so you can't say much.

In our vector-bundle $E \xrightarrow{\pi} M$ example, we pick a cover $\mathcal{O}_j$ with partition-of-unity $\phi_j$ and trivialization $\pi^{-1}(\mathcal{O}_j) \cong \mathcal{O}_j \times \mathbb{C}^n$. Then pick unit vectors $e_k$ of $\mathbb{C}^n$, and set $\zeta_k^j = \phi_j(x)\, e_k$. Then
$$T_j \stackrel{\text{def}}{=} \sum_k \left\langle \zeta_k^j(x), \zeta_k^j(x) \right\rangle_0 \geq c(x)\mathbb{1}$$
for $c(x) \neq 0$ if $\phi_j(X) \neq 0$. Then $T \stackrel{\text{def}}{=} \sum T_j$ has
$$T(x) = \sum T_j(x) \geq c(x)\mathbb{1} \geq \epsilon\mathbb{1}$$
where $c(x)$ is some always-positive function, and $M$ is compact, hence $c(x) \geq \epsilon > 0$.

Now set $S(x) = T(x)^{-1/2}$, and $\eta_k^j = S\zeta_k^j$. Then
$$\sum_{j,k} \left\langle \eta_k^j, \eta_k^j \right\rangle_0 = \sum_{j,k} \left\langle S\zeta_k^j, S\zeta_k^j \right\rangle_0 = S \sum_{j,k} \langle \zeta_k^j, \zeta_k^j \rangle_0 S = STS = \mathbb{1}$$
so $\mathcal{K}(E)$ includes the identity operator.

**Definition:** Given unital $C^*$-algebra $A$ and a right-module $\Xi$ with $\langle , \rangle_A$. By a "standard module frame" for $\Xi$ we mean a finite set $\{\eta_j\}$ of elements of $\Xi$ such that $\mathbb{1}_\Xi = \sum_j \langle \eta_j, \eta_j \rangle_0$.

This is not entirely standard language, but is catching on. Some people think about infinite sums, with all their subsequent convergence questions. We've seen that any vector bundle over a compact space can receive an inner product with a standard module frame. In general, a frame has many more vectors than the dimension; nevertheless, frames are like bases, and are increasingly used in simple old Hilbert land.

**Equivalent Definition:** For any $\xi \in \Xi$,

$$\xi = \mathbb{1}_\Xi \xi = \sum \langle \eta_j, \eta_j \rangle_0 \xi = \sum \eta_j \langle \eta_j, \xi \rangle_A$$

which looks just like the reconstruction formula for a basis in finite-dimensional-land. This stuff is useful for, e.g., error-correction and signal processing.

**Definition:** Let $R$ be a unital ring (possibly non-commutative). We will always deal with finitely-generated modules. A *free module* (right or left) over $R$ is a (right- or left-) module isomorphic to $R^n$ (as a right- or left-) module, for some $n$.

**Question from the audience:** Does finitely-generated assure a unique $n$? **Answer:** Absolutely not. E.g. $R = \mathcal{B}(\mathcal{H})$.

**Definition:** A *projective module* is a direct summand of a free module.

Next time:

**Theorem:** Let $A$ be a unital $C^*$-algebra (or nice subalgebra), and $\Xi$, $\langle , \rangle$ a (Hilbert, but we don't so much need this, by finite-generated-ness) $C^*$-module over $A$. If $\Xi$ has a standard module frame, then $\Xi$ is a projective $A$-module and is self-dual for $\langle , \rangle_A$.

**Corollary:** (Swan's theorem)

For $M$ a compact space and $E$ a vector bundle, $\Gamma(E)$ is a projective $C(M)$-module (and conversely).

## 41: April 30, 2008

Last time we defined a *standard module frame*:

**Theorem:** $A$ is a unital $C^*$-algebra (or smooth subalgebra), and $\Xi$ a right $A$-module equipped with $\langle , \rangle_A$ an $A$-valued inner product. If $\Xi$ has a (finite) standard module frame $\{\eta_j\}_{j=1}^n$ — i.e. the sum of the corresponding rank-one operators $\sum \langle \eta_j, \eta_j \rangle_0 = \mathbb{1}_\Xi$, or equivalently there's a reconstruction formula $\xi = \sum \eta_j \langle \eta_j, \xi \rangle_A$ — then $\Xi$ is projective, and in fact "isometric" to a direct summand of $A^n$ (viewed as a right module), and $\Xi$ is self-adjoint for $\langle , \rangle_A$.

**Proof:**

(Because of our conventions with left and right, everything is simple; using other conventions makes for painful bookkeeping.) Define $\Phi : \Xi \to A^n$ by $\xi \mapsto (\langle \eta_j, \xi \rangle)_{j=1}^n$. It's clear that $\Phi$ is an $A$-module homomorphism. Furthermore, by the reconstruction formula, this is injective. So $\Xi$ is (equivalent to) a submodule of $A^n$, and to show it's projective, we need to show it's a summand. This consists of displaying a projection $A^n \twoheadrightarrow \Xi$. Let $P \in M_n(A)$ the matrix algebra, acting on $A^n$ from the left. Let $P$ be given by $P_{jk} = \langle \eta_j, \eta_k \rangle_A$. Then

$$\begin{aligned}
(P^2)_{ik} &= \sum_j P_{ij} P_{jk} \\
&= \sum_j \langle \eta_i, \eta_j \rangle_A \langle \eta_j, \eta_k \rangle_A \\
&= \left\langle \eta_i, \sum_j \eta_j \langle \eta_j, \eta_k \rangle_A \right\rangle_A \\
&= \langle \eta_i, \eta_k \rangle = P_{ik}
\end{aligned}$$

Moreover, $(P^*)_{ij} = (P_{ji})^* = \langle \eta_j, \eta_i \rangle^* = \langle \eta_i, \eta_j \rangle = P_{ij}$. So we have a self-adjoint projection. But

$$\begin{aligned}
(P(\Phi\xi))_j &= \sum_k P_{jk} (\Phi\xi)_k \\
&= \sum \langle \eta_j, \eta_k \rangle_A \langle \eta_k, \xi \rangle_A \\
&= \left\langle \eta_j, \sum_k \eta_k \langle \eta_k, \xi \rangle \right\rangle \\
&= \langle \eta_j, \xi \rangle = (\Phi\xi)_j
\end{aligned}$$

So $P\Phi = \Phi$, so $\Phi(\Xi)$ is contained in the range of $P$. On the other hand, if $v \in A^n$ and if $v \in \text{range}(P)$, so $Pv = v$, then

$$\begin{aligned}
v_j &= (Pv)_j \\
&= \sum_k \langle \eta_j, \eta_k \rangle v_k \\
&= \left\langle \eta_j, \underbrace{\sum_k \eta_k v_k}_{\xi} \right\rangle \\
&= (\Phi\xi)_j
\end{aligned}$$

So $P$ is the self-adjoint projection onto $\Phi(\Xi)$. And

$$A^n = \underbrace{P(A^n)}_{\cong \Xi} \oplus (1 - P)(A^n)$$

For isometricity, we use the standard inner-product $\langle a, b\rangle_A = \sum_j a_j^* b_j$ on $A^n$. Then

$$\begin{aligned}
\langle \Phi\xi, \Phi, \zeta\rangle_A &= \sum (\Phi\Xi)_j^* (\Phi\zeta)_j^* \\
&= \sum \langle \xi, \eta_j\rangle \langle \eta_j, \zeta\rangle \\
&= \left\langle \xi, \sum \eta_j \langle \eta_j, \zeta\rangle \right\rangle \\
&= \langle \xi, \zeta\rangle
\end{aligned}$$

And last to prove is self-adjointness: Let $\phi \in \mathrm{Hom}_A(\Xi, A_A)$. Then for $\xi \in \Xi$, we have

$$\begin{aligned}
\phi(\xi) &= \phi\left(\sum \eta_j \langle \eta_j, \xi\rangle\right) \\
&= \sum \phi(\eta_j)\langle \eta_j, \xi\rangle \\
&= \left\langle \underbrace{\sum \eta_j \phi(\eta_j)^*}_{\in \Xi}, \xi \right\rangle \quad \square
\end{aligned}$$

**Question from the audience:** This $P(A)$ is closed, because it's continuous. So any pre-Hilbert module is a Hilbert module? **Answer:** Certainly if $A$ is $C^*$, yes. But all of this works for any $*$-subalgebra of a $C^*$-algebra.

The smooth algebra are spectrally invariant: if $A^\infty \subseteq A$ a $C^*$-algebra, and if $a \in A^\infty$ and $a$ is invertible in $A$, then $a$ is invertible in $A^\infty$. This implies that the spectrum of $a$ in $A$ agrees with that in $A^\infty$. Hence, we have a good notion of positivity. **E.g.** $C(T) \supseteq C^\infty(T) \supseteq$ trigonometric polynomials; the MHS is spectrally invariant in the LHS, but the RHS is not, even though it's dense.

Let $Q$ be any self-adjoint projection in $A^n$, and let $\Xi = Q(A^n)$. Let $\{e_j\}$ be the standard basis for $A^n$, and $\eta_j = Qe_j$. Then $\{\eta_j\}$ is a standard module frame for $\Xi$. Some cultural remarks: Let $\mathcal{H}$ be an $\infty$-dim Hilbert space and $Q \in \mathcal{B}(\mathcal{H})$ be a self-adjoint projection, and let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis for $\mathcal{H}$. Set $\eta_j = Qe_j$; then $\{\eta_j\}$ is a "normalized frame" for $Q\mathcal{H}$, in the sense that we have a (convergent) reconstruction formula:

$$\xi = \sum^\infty \eta_j \langle \eta_j, \xi\rangle_{\mathcal{H}}$$

Conversely, given a Hilbert space and a normalized frame, then there exists a bigger Hilbert space so that the frame is the projection of an orthonormal basis.

Let $A = C(T)$ be the continuous functions on the circle $T = \mathbb{R}/\mathbb{Z}$; so $A$ is the 1-periodic functions on $\mathbb{R}$. Then simplest non-trivial vector bundle is the Möbius strip:

$$\{\xi \in C_{\mathbb{R}}(\mathbb{R}) \text{ s.t. } \xi(t-1) = -\xi(t)\}$$

This is not a free module. Define the inner product in the obvious way: $\langle \xi, \eta \rangle_A(t) = \xi(t)\eta(t)$. Then find a standard module frame. More generally, we can write

$$\Xi_p^- = \{\xi \in C_{\mathbb{R}}(\mathbb{R}) \text{ s.t. } \xi(t-p) = -\xi(t)\}$$

$$\Xi_p^+ = \{\xi \in C_{\mathbb{R}}(\mathbb{R}) \text{ s.t. } \xi(t-p) = +\xi(t)\}$$

Don't turn these in — there are no more problem sets — but try them anyway.

**Question from the audience:** Do you want a bar somewhere? **Answer:** No, over real numbers. Over the complex numbers, the Möbius bundle is trivial. Do that example too.

Next time we will have more examples. We are heading towards the non-commutative torus.

# 42: May 5, 2008

## 42.1 Vector bundles and projective modules

**Theorem:** (Swan, 1962)

> Let $E$ be a ($\mathbb{R}$ or $\mathbb{C}$) vector bundle over $X$ compact. Then $\Gamma(E)$ is a projective $A$ module for $A = C(X)$. Conversely, suppose $\Xi$ is a projective $A$ module. Then by definition there exists $\Xi_1$ such that $\Xi \oplus \Xi_1 \cong (A^n)_A$. Let $Q$ be the projection of $A^n$ onto $\Xi$ along $\Xi_1$. I.e., $Q \in \mathrm{End}_A(A^n) = M_n(A)$ and $Q^2 = Q$ (in this generality we don't have self-adjointness; in this case $Q$ is called "idempotent"); view $Q$ as a matrix of functions: $M_n(C(X \to k)) = C(X \to M_n(k))$ for $k =$ ground field $\mathbb{R}$ or $\mathbb{C}$. Define $E$ a vector bundle by: the fiber $E_x$ above $x$ is the range of $Q(x)$ in $k^n$. This function is clearly continuous in $x$, so essentially it is a bundle. We check local triviality: if $b_1, \ldots, b_k$ is a basis for range$(Q(x_0))$, then view $b_i \in k^n$, and then write down $Q(x)b_j$ and $(1 - Q(x))b_j$. We take the determinant of these $n$ vectors; they're a basis at $x_0$, so this determinant is non-zero, and determinant is continuous in the coefficients, so it's a basis in a small neighborhood. □

**Question from the audience:** If $X$ is not connected, there might be dimension change from component to component? **Answer:** Absolutely. Our definition of "vector bundle" allows for this

When $X$ is not compact, the story is more complicated. But usually when we have non-compact spaces, we control the behavior at infinity by having in mind a particular compactification, and that throws us back into this story. For instance, we might use the one-point compactification; this makes our bundle trivial at infinity, i.e. there's a large enough compact set in $X$ so that on the complement, the bundle is trivial.

For any ring $R$ with 1, we can consider the finitely generated projective modules (if you're very careful, that's not a set, but you know how to deal with this), and we consider them up to isomorphism class: $\mathcal{S}(R)$ is the set of isomorphism classes. Given projective modules $\Xi_1$, $\Xi_2$, it's obvious that $\Xi_1 \oplus \Xi_2$. (Everywhere finitely generated, but I don't want to go into that. **Question from the audience:** Meaning $n$ is finite? **Answer:** Well, a little more complicated. **Question from the audience:** Every kind of projective module we've defined is f.g. **Answer:** Yes) This sum interpreted as bundles is fiber-wise, called the "Whitney sum". This defines an addition on $\mathcal{S}(R)$, which is certainly commutative, and the 0 module is an identity element. So $\mathcal{S}(R)$ is a commutative semigroup with 0. This is an invariant of $R$. I.e. this is all functorial, but I'm glossing over that.

$\mathcal{S}(R)$ is interesting to calculate. As a teaser, let $\theta \in M_d(\mathbb{R})$, and build $A_\theta$. If $\theta$ has at least one irrational entry, then we can describe $\mathcal{S}(A_\theta)$ in pretty explicit terms; indeed, up to isomorphism we can construct all the projective modules. The description depends on $\theta$ and is a little bit complicated. On the other hand, for $\theta = 0$, we have $C(T^d)$, and for $d \gtrsim 10$, $\mathcal{S}(C(T^d))$ is basically unknown: it corresponds to homotopy classes of something, but it's way too complicated. Similarly for the $d$-sphere above a certain dimension. In a surprising number of cases, the quantum world ends up being nicer like this than the classical world: the classical world ends up being "degenerate".

Let's indicate some of the obstructions. Last time, we gave some projective modules over the circle. Let's look at $S^2 \subseteq \mathbb{R}^3$ the unit two-sphere. Then we have the tangent bundle and cross-sections $\Gamma(TS^2) = \{\xi : S^2 \to \mathbb{R}^3 \text{ s.t. } \xi(x) \cdot x = 0 \, \forall x \in S\}$. We know the hairy ball theorem: this is not the trivial bundle, i.e. it's not $A^2_A$, where $A = C(S^2 \to \mathbb{R})$. We can also define the normal bundle $\Gamma(NS^2) = \{\xi : S^2 \to \mathbb{R}^3 \text{ s.t. } \xi(x) \in \mathbb{R}x\}$. This is the trivial bundle $A_A$. Well, $\Gamma(TS^2) \oplus \Gamma(NS^2) = A^3_A$ is a trivial (i.e. free) bundle. So $\Gamma(TS^2) \oplus A \cong A^2 \oplus A$, but $\Gamma(TS^2) \ncong A^2$, so $\mathcal{S}(C(S^2 \to \mathbb{R}))$ is not cancelative. Even presenting semigroups in which cancelation fails is complicated. We can play the same game over $\mathbb{C}$, but have to get to $d \geq 5$ for cancelation in $\mathcal{S}(C(S^d \to \mathbb{C}))$ to fail. So the moral of the story: calculating $\mathcal{S}(R)$ can be hard.

On the other hand, in a paper some years ago by R., we show that in the noncommutative torus and a non-zero ****or non-rational, I didn't hear**** entry in $\theta$, cancelation holds.

Given a semigroup $S$ commutative with 0, force cancellation. I.e. consider $s \sim t$ if $\exists r$ with $s + r = t + r$. Check: then $S/\sim$ is a commutative unital cancelative semigroup. Call it $cS$, standing for cancelation. ****Board says "$C(S)$", but also "there are too many $C$s around", so I'll use this notation.**** So we set $\mathcal{C}(R) = c\mathcal{S}(R)$. This is also an invariant of $R$, and can be a bit easier to calculate, but still possibly daunting.

Ok, remember how to construct the integers from the positive integers? That procedure works for any semigroup with cancelation. Recall: we look at pairs $(m, n)$ which we think of as $m - n$, and consider $(m, n) \sim (m', n')$ if $m + n' = m' + n$. For a cancelative commutative semigroup $C$, we can embed it in an abelian group $gC$. ****"groupify"**** This procedure again loses information. We define $K_0(R) \stackrel{\text{def}}{=} g\mathcal{C}(R) = gc\mathcal{S}(R)$. This is the 0-group of $K$-theory, and finially gets us to a homology theory. For complicated examples, this can still be difficult to calculate. $\mathcal{C}(R)$ is a "positive cone" inside $K_0(R)$; it may be degenerate (e.g. it can be all of $K_0$). So denote $\mathcal{C}(R) = K^+(R)$, and we often see written the pair $(K_0(R), K^+(R))$, which of course has exactly the data of $\mathcal{C}(R)$.

Everything is functorial: Given rings $R_1$ and $R_2$ and a unital map $\phi : R_1 \to R_2$, we have $\mathcal{S}(\phi) : \Xi_{R_1} \mapsto \Xi_{R_1} \underset{R_1 R_1}{\otimes} (R_2)_{R_2}$, where we view $R_2$ as a left-$R_1$-module using $\phi$. This extends to $K_0$. Given a short exact sequence

$$0 \to J \to R \to R/J \to 0$$

we want a long exact sequnce in $K_*$. We need to define $K_0(J)$. First we form $\tilde{J}$ by adjoining a unit. Then we have a homomorphism $\tilde{J} \to \mathbb{Z}$ (it's really better if everything is with algebras over a field $k$; certainly this works in that case, but probably works if $k = \mathbb{Z}$). Then we have $K_0(\tilde{J}) \to K_0(\mathbb{Z})$, and we define $K_0(J) = \ker(K_0(\tilde{J}) \to K_0(\mathbb{Z}))$. E.g. if $J = C_\infty(X)$, then $\tilde{J} = C(\tilde{X})$, where $\tilde{X}$ is the one-point compactification. Anyway, then we get

$$K_0(J) \to K_0(R) \to K_0(R/J)$$

but to extend that takes more work. This is an interesting direction, but not one we will pursue.

# 43: May 5, 2008

## 43.1 Some $K$ theory

It's been asked that we define $K_1$.

When $\phi : A \to B$, we get a map $K_0(A) \xrightarrow{\phi} K_0(B)$, because if $[\Xi_1] - [\Xi_2] \in \ker(\phi)$, then $[\phi(\Xi_1)] \sim [\phi(\Xi_2)]$ in $K_0(B)$.

On the other hand, given an isomorphism $\phi(\Xi_1) \cong \phi(\Xi_2)$ over $B$, one can ask whether we can lift this to an isomorphism over $A$ between $\Xi_1$ and $\Xi_2$. What this comes down to is whether given an invertible element $S$ of $M_n(B)$, is there an invertible element $T$ of $M_n(A)$ so that $\phi(T) = S$. I.e. "can you lift invertible elements?" We're asking to what extent the map $GL_n(A) \xrightarrow{\phi} B$ is onto. More or less, vaguely, $K_1$ measures the invertible elements that cannot be lifted. This is a very vague statement.

Let's make it more precise. We look for universally liftable elements of $GL_n(A)$ (which was the $B$ up above). We want $\phi : A \to B$ to be onto, and for the moment these are unital algebras without topology. Let's give some examples:

$$\begin{pmatrix} 1 & & 0 & \\ & 1 & & r_{ij} \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

These clearly can all be lifted, since $A \to B$ is onto, and is invertible for any single value $r_{ij}$. Call the (normal) subgroup generated by such things $El_n(A)$: then

$$GL_n(A)/El_n(A) \to GL_{n+1}(A)/El_{n+1}(A) \to \cdots \to \text{limit} = GL_\infty(A)/El_\infty(A)$$

under

$$T \mapsto \begin{pmatrix} T & \\ & 1 \end{pmatrix}$$

and

$$GL_\infty(A) = \begin{pmatrix} \boxed{\text{invertible}} & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

and we can look at the image of $El_n(A)$ in $GL_{2n}(A)$, which sits in $[GL_{2n}(A), GL_{2n}(A)] \subseteq El_{4n}(A)$. Then we define

$$K_1^{\text{alg}}(A) \stackrel{\text{def}}{=} GL_\infty(A)/[GL_\infty(A), GL_\infty(A)]$$

The denominator is the commutator subgroup, so this is abelian.

There is no good algebraic definition of $K_1$ for non-unital algebras. One way to do it is to use an ideal $J \leq A$, and then writing down the sequence

$$K_1^{\text{alg}}(J, A) \to K_1^{\text{alg}}(A) \to K_1^{\text{alg}}(J)$$

but it becomes even harder to get $K_2$, etc., indeed someone won a Fields Medal for such stuff.

For unital Banach algebras, again we look for universally liftable elements of $M_n(A)$. If $T \in M_n(A)$ with an appropriate Banach norm on $M_n(A)$, and if $\|T - \mathbb{1}\| < 1$, then we can use holomorphic functional calculus to define $\log(T) = S$. Then $T = e^S$, and any $e^S$ is liftable, because $S$ is just some matrix and we have a Banach homomorphism that's onto. So everything close to $\mathbb{1}$ is universally liftable; this is an open neighborhood of $\mathbb{1}$ in the group of invertible elements. And the point is that the connected component of $\mathbb{1}$ in $GL_n(A)$ is algebraically generated by any open neighborhood of the identity. Thus everything in $GL_n^0(A)$ is universally liftable. So in this context we define the topological $K_1$ by the sequence of discrete groups:

$$GL_n(A)/GL_n^0(A) \to GL_{n+1}(A)/GL_{n+1}^0(A) \to \cdots \to GL_\infty(A)/GL_\infty^0(A)$$

and, of course, $[GL_\infty, GL_\infty] \subseteq GL_\infty^0$. What happens is that we're deviding out by more: $K_1^{\text{alg}} \twoheadrightarrow K_1^{\text{top}}$. And

$$K_1(\text{non-unital } A) = \ker\Big(K_1(\tilde{A}) \to K_1(\text{field})\Big)$$

Then we have the famous

**Bott periodicity theorem:** If we are over $\mathbb{C}$, then $K_2(A) \cong K_0(A)$.

So we don't have to worry about $K_2$ and higher. The surprise is that the following six-term sequence is exact everywhere:

$$\begin{tikzcd} & J & A & A/J \\ K_1 & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[d] \\ K_0 & \bullet \arrow[u] & \bullet \arrow[l] & \bullet \arrow[l] \end{tikzcd}$$

Over $\mathbb{R}$, the iso is $K_8(A) \cong K_0(A)$, because you get tied up in quaternions and Clifford algebras.

Good questions: if $G$ is discrete and we take $C^*(G)$ or $C^*_r(G)$, what are the $K$-groups of these? By now there is a large literature using non-commutative geometry in intense ways, e.g. Dirac operators, to answer those questions at least for large classes of groups in a way that you could imagine you might be able to actually compute these. Part of the difficulty is figuring out what all the projective modules over these, e.g. $\mathbb{Z}^10$ no one has in an effective way shown how to list all of the projective modules over the commutative 10-torus.

## 43.2 Return to tori and projective modules

For $\widehat{\mathbb{Z}^2} = T^2$, we have a commutative $C^*$-algebra $A = C(T^2)$, where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then we skip the proofs, and have:

$$\Xi(q,a) \stackrel{\text{def}}{=} \{\xi \in C(\mathbb{R}^2 \to \mathbb{C}) : \xi(s+q,t) = \xi(s,t),\, \xi(s,t+1) = e^{2\pi i a s}\xi(s,t)\}$$

**Theorem:** Every projective module over $C(T^2)$ is a free module or isomorphic to a $\Xi(q,a)$. And when $a \geq 0$, the $\Xi(q,a)$ are all isomorphic.

This does not give a particularly good clue how to deal with non-commutative tori. We have $\mathbb{Z}^d$ and a matrix $\theta \in M_d(\mathbb{R})$, and we form $A_\theta$ as before in terms of the bicharacter $c_\theta$.

In any case, $\mathbb{Z}^d$ fits inside $A_\theta$, not comfortably as a subgroup, because of the twisting, but as a subgroup. And precisely this means that a projective module will give a "$c_\theta$-projective representation of $\mathbb{Z}^d$", although we don't have a Hilbert space. (This is a way of thinking of this stuff in hindsight.) We can look for $c_\theta$-projective representations, and there aren't a lot of ways to construct these:

Let $M$ be a locally compact Abelian group, and $\hat{M}$ its dual group. Let $G = M \times \hat{M}$; then on $L^2(M)$ we have

$$(\pi_{(x,s)}\xi)(y) \stackrel{\text{def}}{=} \langle y, s\rangle\, \xi(y-x)$$

the "Schrodinger representation." Then $\pi$ is a projective representation of $G$ on this Hilbert space, with bicharacter $\beta$ (easily enough computed).

Strategy:

- Find embeddings of $\mathbb{Z}^d$ into $M \times \hat{M}$ such that $\beta|_{\mathbb{Z}^d} = c_\theta$.
- If $\mathbb{Z}^d$ is a lattice in $M \times \hat{M}$, restrict attention to $C_c(M)$; then this leads to a projective module.

The difficulty: this gives zillions of projective modules, and it's hard to figure out when two such things are isomorphic.

# 44: May 7, 2008

## 44.1 We go through a careful computation

Recall, we're interested in $\mathbb{Z}^d$ and a cocycle $c_\theta$, and we want projective modules.

The strategy: let $M$ be a locally compact abelian group: $M = \mathbb{R}^m \oplus \mathbb{Z}^n \oplus F$ for $F$ finite abelian. ****If $F$ is torsion but not finite, what do we get?**** Then let $G = M \times \hat{M}$, where $\hat{M}$ is the dual group, $\hat{M} \cong \mathbb{R}^m \oplus T^m \oplus \hat{F}$, where $\hat{F} \cong F$, but not in a canonical way — none of these equalities is canonical. Hence $G \cong \mathbb{R}^{2m} \times \mathbb{Z}^n \times T^n \times F \times \hat{F}$.

Then $G$ has a projective "Schrodinger" unitary rep on $L^2(M)$ with cocycle $\beta$. Then what we will attempt is to find embeddings (as closed subgroup) $\mathbb{Z}^d \hookrightarrow M \times \hat{M}$, such that $\beta|_{\mathbb{Z}^d} = c_\theta$. Then we can hope that $C_c(M) \subseteq L^2(M)$ gives a projective module. The condition will be that if $\mathbb{Z}^d$ is *cocompact* in $G$ — i.e. $G/\mathbb{Z}^d$ is compact — then we do get a projective module. A necessary condition is that $2m + n = d$. This all, at least in the abelian case, defines a *lattice* in $G$.

This generates a whole bunch of projective modules. It's hard to tell, and we will not in this class, whether these modules are isomorophic; to sort them out requires a non-commutative Chern class. You can prove: if $\theta$ has an irrational entry, then every projective module is a direct sum of things like this. But if $\theta$ is entirely rational, then the situation is Morita-equivalent to the commutative ($\theta = 0$) case, and you do not get all of the modules in this way, just a lot of modules.

Ok, so we begin the calculation. Let $\left(\pi_{(x,s)}\xi\right)(y) \stackrel{\text{def}}{=} \langle y, s\rangle\, \xi(y-x)$ for $x, y \in M$, $s \in \hat{M}$. We find the cocycle:

$$\begin{aligned}
\left(\pi_{(x,s)}\pi_{(y,t)}\xi\right)(z) &= \langle z, s\rangle \left(\pi_{(y,t)}\xi\right)(z-x) \\
&= \langle z, s\rangle \langle z-x, t\rangle \xi(z-x-y) \\
\left(\pi_{(x+y,s+t)}\xi\right)(z) &= \langle z, s+t\rangle\, \xi(z-(x+y)) \\
\left(\pi_{(x,s)}\pi_{(y,t)}\xi\right)(z) &= \overline{\langle x, t\rangle} \left(\left(\pi_{(x+y,s+t)}\xi\right)(z)\right)
\end{aligned}$$

Hence we define

$$\beta((x,s),(y,t)) = \overline{\langle x, t\rangle}$$

which is not skew-symmetric. Using $u, v$ for letters in $G = M \times \hat{M}$, we set

$$\pi_u^* \stackrel{\text{def}}{=} \beta(u,u)\pi_{-u}$$

Now, let $D$ (e.g. $D \cong \mathbb{Z}^d$) be a discrete subgroup of $M \times \hat{M} = G$. Much of what we do works with any closed subgroup, which is fine, but everywhere where we'll have a sum, you'll need an integral. We don't need that generality, so we skip it. In any case, let's restrict $\pi$ to $D$, and we're not going to worry about how to match up $\beta$ with $c_\theta$. In any case, restrict $\beta$ to $D$, and then $\pi$ in $D$ is a $\beta$-projective representation of $D$ on $C_c(M) \subseteq L^2(M)$.

Another bookkeeping: we will use right-modules, since we need to consider endomorphisms (for us, acting from the left) in order to show projectivity. So let's make $C_c(M)$ into a right $C_c(D)$-module (a certain amount carries over to $L^2$; of course, $C_c(D)$ for $D$ discrete is just the functions of finite support): for $\xi \in L^2(M)$ and $f \in C_c(D)$, we set

$$\xi \cdot f \stackrel{\text{def}}{=} \sum_{u \in D} (\pi_u^* \xi) \, f(u)$$

the $*$ makes it a right-action. Check that this works out:

$$\begin{aligned}
((\xi \cdot f) \cdot g) &= \sum_u \pi_u^* (\xi \cdot f) \, g(u) \\
&= \sum_u \pi_u^* \left( \sum_v (\pi_v^* \xi) \, f(v) \right) g(u) \\
&= \sum_{u,v} \pi_u^* \pi_v^* \xi \, f(v) \, g(u) \\
&= \sum_{u,v} (\pi_v \pi_u)^* \xi \, f(v) \, g(u) \\
&= \sum_{u,v} (\beta(v,u) \pi_{v+u})^* \, \xi \, f(v) \, g(u) \\
&= \sum \bar{\beta}(v,u) \pi_{v+u}^* \, f(v) \, g(u) \\
&= \sum \bar{\beta}(v, u-v) \pi_u^* \xi \, f(v) \, g(u-v) \\
&= \sum_u (\pi_u^* \xi) \underbrace{\sum_v f(v) \, g(u-v) \, \bar{\beta}(v, u-v)}_{f \star_{\bar{\beta}} g \text{ restricted to } D}
\end{aligned}$$

Ok, so this works for $\xi \in L^2$, but let's move in the $C_c$ direction. So we let $A = (C_c(D), \star_{\bar{\beta}})$, and later complete to a $C^*$ algebra. We leave Hilbert space: let $\Xi = C_c(M)$, later on completed. Let's pick the ordinary inner product $\langle , \rangle_{L^2}$ on $L^2(M)$ to be linear in the first variable. Define a "bundle metric", i.e. an $A$-valued inner product on $\Xi$, by:

$$\langle \xi, \eta \rangle_A (\underset{\in D}{u}) \stackrel{\text{def}}{=} \overline{\langle \xi, \pi_u^* \eta \rangle_{L^2(M)}} = \overline{\langle \pi_u \xi, \eta \rangle_{L^2(M)}} = \int_{y \in M} \overline{(\pi_u \xi)(y)} \, \eta(y) \, dy$$

If $u = (x, s)$, then

$$\int_{y \in M} \overline{(\pi_u \xi)(y)} \, \eta(y) \, dy = \int_M \overline{\langle y, s \rangle \, \xi(y - x)} \, \eta(y) \, dy$$

Now, if $\eta$ and $\xi$ are each of compact support on $M$, then for each $s$ this is certainly of compact support in $x$. This is more interesting; we still have this $s$ to deal with, and there's no reason why this should be of compact support in $s$. Ok, so the solution is that we need to take a bigger space: when $M = \mathbb{R}^m \times \mathbb{Z}^n \times F$, we need the *Schwartz space* $\mathcal{S}(M)$: all derivatives in the $\mathbb{R}^m$ direction should exist, and everything (including all derivatives) in the $\mathbb{R}^m$ and $\mathbb{Z}^n$ directions should vanish at infinity faster than any polynomial.

**Lemma:** If $\xi, \eta \in \mathcal{S}(M)$, then $\langle \xi, \eta \rangle_A \in \mathcal{S}(D)$.

We'll skip this proof.

# 45: May 9, 2008

MR will not be here next time; Prof. A will give the final lecture, on not this material but stuff related to this course.

## 45.1 Continuation of last time

We make a correction from last time — a $*$ was mis-placed in the bookkeeping — and simplify to $M = \mathbb{R}$ (the general case is just like $\mathbb{R} \times \mathbb{Z}/p$, but even there the bookkeeping is hard). So $M \times \hat{M} \cong \mathbb{R}^2$, and $L^2(M) = L^2(\mathbb{R})$, and we have $\theta \in \mathbb{R}$ with $\theta \neq 0$ (we can have $\theta = 1, 2, \ldots$; then we get non-trivial bundles on the commutative torus).

We have $\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ generated by two generators $\pi_{(1,0)}$ and $\pi_{(0,1)}$. We pick $(1,0) \mapsto (\theta, 0) \in \mathbb{R}^2$ and $(0,1) \mapsto (0,1) \in \mathbb{R}^2$. We write $e(s) \stackrel{\text{def}}{=} e^{2\pi i s}$, and take $\xi \in \mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$. Then

$$(\pi_{(m,n)}\xi)(t) = e(nt)\,\xi(t - m - \theta)$$

and hence

$$\beta((m,n),(p,q)) = \bar{e}(mq\theta)$$

For $f \in C_c(\mathbb{Z}^2)$, or more generally in $\mathcal{S}(\mathbb{Z}^2)$, we have

$$\begin{aligned}
(\xi \cdot f)(t) &= \sum_{m,n} (\pi^*_{m,n}\xi)(t)\, f(m,n) \\
&= \sum \beta((m,n),(m,n))\,(\pi_{-m,-n}\xi)(t)\, f(m,n) \\
&= \sum \bar{e}(mn\theta)\,\bar{e}(nt)\,\xi(t + m\theta)\, f(m,n) \\
&= \sum \bar{e}((t + m\theta)n)\,\xi(t + m\theta)\, f(m,n) \\
&= \sum_m \xi(t + m\theta) \sum_n \bar{e}((t + m\theta)n)\, f(m,n)
\end{aligned}$$

This looks like a Fourier mode. For $g \in \mathcal{S}(\mathbb{Z}^2)$, we set

$$\grave{g}(m,t) \stackrel{\text{def}}{=} \sum_n \bar{e}(nt)\, g(m,n)$$

which is periodic in $t$ with period 1. (The grave accent is half a hat, because we're only transforming one variable.) Then

$$(\xi \cdot f)(t) = \sum_m \xi(t + m\theta) \sum_n \bar{e}((t+m\theta)n)\, f(m,n) = \sum_m \xi(t+m\theta)\, \grave{f}(m, t+m\theta)$$

Ok, then we have an action like $C^\infty(T) \times_\alpha \mathbb{Z}$. We have an inner product:

$$\begin{aligned} \grave{\langle \xi, \eta \rangle}_A(m,t) &= \sum \bar{e}(nt) \overline{\langle \xi, \pi_{m,n}\eta \rangle_{L^2(\mathbb{R})}} \\ &= \sum_n \bar{e}(nt) \int_{\mathbb{R}} \bar{\xi}(s)\, e(ns)\, \eta(s - m\theta)\, ds \\ &= \sum_n \int \bar{\xi}(s)\, \eta(s-m\theta)\, \bar{e}((t-s)n)\, ds \\ &= \sum_n \int \bar{\xi}(s+t)\, \eta(s+t-m\theta)\, e(sn)\, ds \end{aligned}$$

We think of $\bar{\xi}(s+t)\,\eta(s+t-m\theta)$ as some function $h(s)$. Then we have

$$\sum_n \int_{\mathbb{R}} h(s)\, e(ns)\, ds = \sum_n \hat{h}(n) = \sum_n h(n)$$

by the Poisson summation formula. So

$$\begin{aligned} \grave{\langle \xi, \eta \rangle}_A(m,t) &= \sum_n \int \bar{\xi}(s+t)\, \eta(s+t-m\theta)\, e(sn)\, ds \\ &= \sum_n \bar{\xi}(n+t)\, \eta(n+t-m\theta) \end{aligned}$$

is obviously periodic in $t$.

**Question from the audience:** The Poisson summation formula just expresses that Fourier transform is an isometry? **Answer:** No, it is more subtle. For instance, $L^2$ functions aren't defined at points, so plugging in $n$ doesn't work; it uses that we are in Schwartz space, and generalizes slightly.

Continuing on:

$$
\begin{aligned}
(f \star_\beta g)^{\grave{}}(m,t) &= \sum_n \bar{e}(nt) \sum_{p,q} f(p,q)\, g(m-p, n-q)\, e(p(n-q)\theta) \\
&= \sum_{n,p,q} f(p,q)\, \bar{e}(qt)\, g(m-p,n-q)\, \bar{e}((n-q)t)\, e(p(n-q)\theta) \\
&= \sum_{n,p,q} f(p,q)\, \bar{e}(qt)\, g(m-p,n-q)\, \bar{e}((n-q)(t-p\theta)) \text{ sum in } n \\
&= \sum_{p,q} f(p,q)\, \bar{e}(qt)\, \grave{g}(m-p, t-p\theta) \text{ sum in } q \\
&= \sum_{p} f(p,t)\, \grave{g}(m-p, t-p\theta)
\end{aligned}
$$

This is exactly the cross-product formula for $C(T) \times_{\alpha^\theta} \mathbb{Z}$, $(\alpha_p^\theta \phi)(t) \overset{\text{def}}{=} \phi(t-p\theta)$:

$$
(\grave{f} \star \grave{g})(m,t) = \sum_p \grave{f}(p,t)\, \grave{g}(m-p, t-p\theta)
$$

Now we take a leap of faith, and ask if we can find $\xi \in \mathcal{S}(\mathbb{R})$ so that $\langle \xi, \xi \rangle_{A_\theta}$ is a projection in $A$. If we have the ordinary torus, then there are no projections. Suppose that $0 < \theta < 1$. Then we take $\xi$ to be a bump on $[0,1]$ that is 0 at 0, 1 at $\theta$, and 0 again at 1 and $2\theta$. Then when translated by $\theta$, $\xi$ doesn't intersect itself. So

$$
\langle \xi, \xi \rangle_A(p,t) = \sum \bar{\xi}(t+n)\, \xi(t+n-p\theta)
$$

has support only at $p = -1, 0, 1$.

Now look for projections in $C(T) \times_{\alpha^\theta} \mathbb{Z}$ of the form $P = \delta_{-1}\phi + \delta_0 \psi + (\delta_{-1}\phi)^*$ with $\psi = \bar{\psi}$, and $\psi, \phi \in C(T)$. Then

$$
\begin{array}{rclclcl}
P^2 & = & \delta_{-1}\phi\delta_{-1}\phi & + & \delta_{-1}\phi\, \delta_0\psi + \delta_0\psi\, \delta_{01}\psi & + & \text{adjoints} \\
& = & \delta_{-2}\, \alpha_1^\theta(\phi)\, \phi & + & \delta_{-1}\left(\phi\psi + \psi\alpha_1^\theta(\phi)\right) + \delta_0\,() & + & \dots \\
& & \text{need} = 0 & & \text{want} = \phi & &
\end{array}
$$

****I got a little lost in this next remark.**** Then we have $A_\theta$ with a tracial state $\tau$ — given $f(p,t)$, we have $\tau(f) = \int_T f(0,t)$, and we can graph $\phi$ and $\psi$, and what we discover is that $\tau(P) = \int \psi(t) = \Theta$. All of these projections correspond to projective modules.

## 46: May 12, 2008

****I was a little late.****

## 46.1 Guest lecture by W. Arveson: Operator Spaces, "Quantized Functional Analysis"

Lecture notes are available at http://math.berkeley.edu/∼arveson/Dvi/opSpace.pdf.

Let $\mathcal{S} \subseteq \mathcal{B}(H)$ be a linear subspace that is $\|\cdot\|$-closed. We have a notion of *completely contractive* maps, which form a category.

We will see some examples, which illustrate the non-commutativity in finite-dimensional setting.

Consider $\mathcal{B}(\mathbb{C}^p)$ for $p = 1, 2, \ldots$. We have two particular operator spaces, the "row"-space $\mathcal{R}$ and the "column" space $\mathcal{C}$:

$$\mathcal{R} = \left\{ \begin{pmatrix} z_1 & z_2 & \dots & z_p \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{pmatrix} z_1 & 0 & \dots & 0 \\ z_2 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ z_p & 0 & \dots & 0 \end{pmatrix} \right\}$$

Given $z = (z_1, \ldots, z_p) \in \mathbb{C}^p$, we have $R_z$ and $R_c$ as above. Then $\|R_z\| = \|R_z R_z^*\|^{1/2} = \|z\| = \|C_z\|$. Let $\phi : \mathcal{R} \to \mathcal{C}$ be this isometry.

Now, what is $M_n(\mathcal{R})$? Well, they are $n \times n$ matrices with entries in $\mathcal{R}$, but equivalently they are

$$M_n(\mathcal{R}) = \left\{ \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} : A_i \in M_n(\mathbb{C}) \right\}$$

and similarly for $M_n(\mathcal{C})$. We can extend $\phi$ to $\phi_n$:

$$\phi_n : \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \mapsto \begin{pmatrix} A_1 & 0 & \dots & 0 \\ A_2 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ A_p & 0 & \dots & 0 \end{pmatrix}$$

Is $\phi_n$ an isometry?

$$\left\| \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \right\| = \sqrt{\|A_1 A_1^* + \cdots + A_p A_p^*\|}$$

and it's important that the $*$s are on the right. On the other hand, in $M_n(\mathcal{C})$, the $*$s are on the left. And if $n$ is large enough, in particular if $n \geq p$, we can make these different. E.g. if $n = p$, and if $A_i$ are rank-one partial isometries with mutually orthogonal matrices $A_i : e_1 \mapsto e_i$, then $\{A_i A_i^*\}$ are mutually orthogonal projections, so their sum has norm 1. On the other hand, $A_i^* A_i$ is the projection onto $e_1$, so the sum has norm $p$.

Incidentally, we can do the same thing with $\phi^{-1}$. In particular, $\|\phi_n\| \geq \sqrt{p}$ and $\|\phi_n^{-1}\| \geq \sqrt{p}$. And so $\phi$ is not a *complete isometry*. Remark: this does not show that $\mathcal{R}$ and $\mathcal{C}$ are not completely isometric. But an easy generalization of the above calculation does show that $\mathcal{R}$ and $\mathcal{C}$ are not completely isometric. So even a finite-dimensional Hilbert space can be realized in many different ways as an operator space: "The same Banach space has many quantizations."

Recall the basic tool of functional analysis: Hahn-Banach. We need such a theorem in this context; this is what makes the theory fly:

**Theorem:** The non-commutative Hahn-Banach theorem

> Let $\mathcal{S} \subset \mathcal{B}(H)$ be an operator system and $\phi : \mathcal{S} \to \mathcal{B}(K)$ be an operator map. Then this map has a completely bounded norm — it might be a complete contraction — as we defined and erased. Then there exists an extension
>
> $$\tilde{\phi} : \mathcal{B}(H) \to \mathcal{B}(K)$$
>
> with $\|\tilde{\phi}\|_{\mathrm{CB}} = \|\phi\|_{\mathrm{CB}}$.

In particular, any completely contractive map can be extended to a completely contractive map of the ambient space:

$$\begin{tikzcd} \mathcal{S}_1 \arrow[r, hook] \arrow[d, "\text{c.c. } \phi"'] & \mathcal{S}_2 \arrow[r, hook] \arrow[dl, dashed, "\tilde{\phi} \text{ c.c.}"] & \mathcal{B}(H) \\ \mathcal{B}(K) & & \end{tikzcd}$$

Let's take a moment to talk about Stinespring's Theorem. We have a GNS construction. On the other hand,

**Theorem:** (Sz.-Nagy)

> Suppose $\phi : C(X) \to \mathcal{B}(H)$ is linear with $f \geq 0 \Rightarrow \phi(f) \geq 0$ (hence $\phi$ is bounded). Then there exists a representation $\pi : C(X) \to \mathcal{B}(K)$ and $V : H \to K$ such that $\phi(f) = V^* \pi(f) V$. I.e.
>
> $$\langle \phi(f)\xi, \eta \rangle_H = \langle \pi(f) V\xi \, V\eta \rangle_K$$

This looks a lot like the GNS construction: if $\phi : A \to \mathbb{C}$ is positive linear, then we get a representation $\pi$ such that $\phi(a) = \langle \pi(a)\xi, \xi \rangle$. Our formula is a lot like that, except involves $V$ because of the non-commutativity.

**Theorem:** (Stinespring, 1955)

> Let $A$ be a unital $C^*$-algebra, and $\phi : A \to \mathcal{B}(H)$ a completely positive map. (A *positive* map is a map that takes self-adjoint positive elements of $A$ to the same; *completely positive* maps are positive on all $M_n(A)$.) Then there exists a rep $\pi : A \to \mathcal{B}(K)$ and $V : H \to K$ such that $\phi(a) = V^*\pi(a)V$ for all $a \in A$.

By the way, the converse is true: if there is such a $\pi$ and a $V$, then $\phi$ is completely positive. Stinespring generalizes both GNS and Sz.-Nagy. Even though he assumes more (complete positivity)? Yes, because he proved in the same paper that a positive linear function is completely positive (i.e. if $H = \mathbb{C}$). And he proves that a positive linear map from a commutative $C^*$-algebra to an operator space is completely positive.

This was the first penetration into the area of non-commutative functional analysis. There was no further work for many years. A. was assigned the paper be thesis advisor, and it was beautiful, but no one really understood it. In late 1970s, this theory started to take hold, and has become popular in recent years.

All of this is discussed in more detail in the notes.

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