Errata and corrections:

0. Many years after I produced this document, Deligne gave it his blessing, so it is no longer an "unauthorized" translation. A copy of the French original, without a paywall, is on Deligne's website at https://publications.ias.edu/sites/default/files/Tensorielles.pdf.

1. On page 3, proposition 1.1: "... that n linearly independent morphisms f_i: X \rightarrow Y define a morphism ..." should say "monomorphism" instead. I thank Sonja Farr for pointing out the error.

2. On page 5, (1.5.2): "... necessary and sufficient that [\lambda] has at least r rows." should be "at most". I thank Sonja Farr for pointing out the error.

3. On page 8, proposition 1.19: "If S_{\lambda}(X) \neq 0, X is therefore of length < n." should be "S_{\lambda}(X) = 0". I thank Sonja Farr for pointing out the error.

4. In the bibliography, "Gelako" should be "Gelaki".

CATÉGORIES TENSORIELLES

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0. Introduction

0.1. Fix an algebraically closed field k of characteristic 0. We term a *k-tensorial category* a system ($\mathcal{A}$, $\otimes$, some auxiliaries) of the following type:

(0.1.0) $\mathcal{A}$ is an essentially small category (equivalent to a small category);

(0.1.1) it is abelian k-linear;

(0.1.2) the tensor product $\otimes$ is a functor from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$ exact and k-linear in each variable;

(0.1.3) it has associativity, commutativity, and unit constrants (the unit is denoted 1);

(0.1.4) $(\mathcal{A}, \otimes)$ is *rigid*: each object X in $\mathcal{A}$ is *dualizable* in the sense that there exists $X^\vee$, the *dual* of X, with $\delta : 1 \to X \otimes X^\vee$ and $\mathrm{ev} : X^\vee \otimes X \to 1$, such that the morphisms composed of δ and ev:

$$X \to X \otimes X^\vee \to X \text{ and } X^\vee to X^\vee \otimes X \otimes X^\vee \to X^\vee$$

are the identity;

(0.1.5) $k \xrightarrow{\sim} \mathrm{End}(1)$.

A k-tensorial category is said to be *finitely $\otimes$-generated* if it admits a *$\otimes$-generator*: an object X such that each object is built by iterated application of the operations of direct sum, tensor product, dual, passage to a sub-object or to a quotient object.

0.2. Example. The category $\mathrm{Rep}(G)$ of linear representations of finite dimenion of a scheme of affine groups G over k is k-tensorial. It is of finite $\otimes$-generation if and only if G is of finite type over k, i.e. a linear algebraic group.

According to Saavedra (1972), the k-tensorial category $\mathrm{Rep}(G)$ determines G, up to an isomorphism unique up to an inner automorphism. The notion of k-tensorial category can thus be regarded as a generalization of that of a scheme of affine groups. Vague question: can we describe the classification of k-tensorial categories in terms of more concrete objects?

This is an unauthorized translation of P. Deligne, Catégories tensorialles, *Moscow Mathematical Journal*, Vo. 2, No. 2, April–June 2002, pp. 227–248, http://www.ams.org/distribution/mmj/vol2-2-2002/deligne.pdf. Numbering has been preserved, including inconsistencies like whether to write "Notation 1.16." or "2.4. Notation." I have, though, corrected a few (not many) typographical errors in mathematics formulae. If you are reading this as a PDF, all references are hyperlinked. I regret all errors I have introduced during the translation.

0.3. Recall that, systemmatically replacing commutative rings by $\mathbb{Z}/2$-graded rings which are commutative in the graded sense ($xy = (-1)^{\deg(x)\deg(y)}yx$ of x ad y homogeneous), we can, paraphrasing a part of algebraic geometry, obtain "super algebraic geometry". If we are working over k, it comes down to replacing the category with tensor product of vector spaces over k by that of super vector spaces: the $\mathbb{Z}/2$-graded vector spaces, the commutativity of the tensor product being given by the Koszul rule.

In super algebraic geometry over k, the group $\mu_2 = \{\pm 1\}$ acts on each object; on a super vector space, -1 acts by the *parity automorphism* $x \mapsto (-1)^{\deg(x)}x$ for x homogeneous.

The super analog of 0.2 is the following. Let G be a super scheme of affine groups over k. Then it is the spectrum of a super commutative Hopf algebra $\mathcal{O}(G)$, the affine algebra of G. Let ϵ be an element of $G(k)$ of order dividing 2 and such that the automorphism $\mathrm{int}(\epsilon)$ of G is the parity automorphism. Let $\mathrm{Rep}(G, \epsilon)$ be the category of super representations of finite dimension (V, ρ) of G, such that $\rho(\epsilon)$ is the partity automorphism of V. This is a k-tensorial category and it is of finite $\otimes$-generation if and only if G is of finite type over k.

0.4. Examples. (i) If $\mathcal{O}(G)$ is purely even, i.e. if G is a scheme of affine groups seen as a super scheme of groups, ϵ is central. The k-tensorial category $\mathrm{Rep}(G, \epsilon)$ is identified with $\mathrm{Rep}(G)$, with a new commutativity constraint: for each representation (V, ρ) of G, the involution $\rho(\epsilon)$ defines a $\mathbb{Z}/2$-grading on V and the commutativity isomorphism for the tensor product is given by the Koszul rule.

For ϵ trivial, we recover the k-tensorial category $\mathrm{Rep}(G)$ from 0.2.

(ii) Let H be a super scheme in affine groups, and act by μ_2 on H by the the parity action. The k-tensorial category $\mathrm{Rep}(\mu \ltimes G, (-1, \epsilon))$ is the category of super representations of H.

Our principal result is an internal characterization of which k-tensorial categories of finite $\otimes$-generation are of the form $\mathrm{Rep}(G, \epsilon)$ — and precisely, they are $\otimes$-equivalent to such a category.

Proposition 0.5. *(i) Let X be an object of a k-tensorial category $\mathcal{A}$. The following conditions are equivalent:*

- (a) *There exists a Schur functor (1.4) that annihilates X.*
- (b) *The tensor powers of X are of finite length and there exists N such that for all $n \geq 0$ we have $\mathrm{length}(X^{\otimes n}) \leq N^n$.*

(ii) The colleciton of objects of $\mathcal{A}$ verifying (a) is stable for direct sums, tensor products, passage to duals, extensions and subquotients.

Theorem 0.6. *For a k-tensorial category of finite $\otimes$-generation to be of the form $\mathrm{Rep}(G, \epsilon)$, it is necessary and sufficient that every object verifies the equivalent conditions from 0.5 (i).*

A painful reduction to the case of finite $\otimes$-generation verifies that 0.6 remains true without the finite generation hypothesis. We have not written.

Corollary 0.7. *Let $\mathcal{A}$ be a k-tensorial category of finite $\otimes$-generation all of whose objects are of finite length. If it has a finite number of isomorphism classes of simple objects, then it is of the form $\mathrm{Rep}(G, \epsilon)$.*

Corollary 0.8. *If furthermore $\mathcal{A}$ is semisimple, then there exists a finite group G and $\epsilon \in G$ central of order dividing 2, such that $\mathcal{A}$ is $\otimes$-equivalent to $\mathrm{Rep}(G, \epsilon)$.*

Etingof and Gelaki (1998) prove an analogous theorem for triangular semisimple Hopf algebras of finite dimension and their categories of modules.

0.9. Here's the plan of the proof, and of the article. In section 1, we prove 0.5 (ii) (1.13, 1.18, and 1.19) and that for each object X of a k-tensorial category, condition (b) of 0.5 (i) implies condition (a) (1.12). We prove also that to give in a category $\mathrm{Rep}(G, \epsilon)$ each object verifies both conditions (a) and (b) (1.21).

A *super fiber functor* of a k-tensorial category $\mathcal{A}$ over a super commutative k-algebra R is a k-linear exact $\otimes$-functor ω from A to the monoidal category of R-super modules. Here and in the rest of the article, a *$\otimes$-functor* is a functor F with an isomorphism $1 \to F(1)$ and a natural isomorphism $F(X) \otimes F(Y) \to F(X \otimes Y)$ compatible with the associativity, commutativity, and unit constraints. A super fiber functor over $R \neq 0$ is automatically faithful: note that if $X \neq 0$, $\mathrm{ev} : X^\vee \otimes X \to 1$ is an epimorphisms and that $\omega(X)$ is hence nontrivial. For $\mathrm{Rep}(G, \epsilon)$, the functor "underling super vector space" is a super fiber functor over k. Reciprocally, according to Deligne (1990), 8.19, if ω is a super fiber functor of the k-tensorial category $\mathcal{A}$ over k, while G is the super scheme in groups of $\otimes$-automorphisms of ω and ϵ the parity automorphism of ω, then ω induces an equivalence of $\mathcal{A}$ with $\mathrm{Rep}(G, \epsilon)$.

Our strategy will be to show that if the objects of $\mathcal{A}$ verify the condition 0.5 (i) (a), then $\mathcal{A}$ admits a super fiber functor over k. In section 2, we show that $\mathcal{A}$ admits a super fiber functor over a suitable super k-algebra $R \neq 0$. In section 3, we generalize to the "super" case one part of the theory of fiber functors. So that the question of signs stays hidden in the commatuvity of the tensor product, it will be convenient to generalize more, and replace the k-tensorial category $(\mathrm{s} - \mathrm{Vect})$ of super vector spaces of finite dimension with an arbitrary k-tensorial category verifying the finitude property of (2.1.1). As an application, we verify in section 4 that if $\mathcal{A}$ is of finite $\otimes$-generation and ω is a super fiber functor over R, then there exists a super subalgebra R' of R, of finite type over k, such that ω provides by extension of scalars (3.1) a super fiber functor ω' over R'. Because k is algebraically closed, if R and therefore R' are not trivial, there exists a homomorphism $\chi : R' \to k$. Extending scalars by χ, we obtain a super fiber functor over k. We conclude section 4 with the ends of the proofs of 0.5 and 0.6, and with the proofs of 0.7 and 0.8.

1. Preliminaries

Proposition 1.1. *Given a k-tensorial category all of whose objects are of finite length, its* $\mathrm{Hom}(X, Y)$*s are finite dimensional.*

Proof. The identities (0.1.4) between δ and ev are equivalent to the functor $Y \mapsto Y \otimes X^\vee$ being a right adjoint to the functor $Y \mapsto Y \otimes X$, for the morphisms of adjunction $Y \mapsto Y \otimes X \otimes X^\vee$ and $Y \otimes X^\vee \otimes X \mapsto Y$ are derived from δ and ev. In other terms, $\mathcal{H}om(X, Y) := Y \otimes X^\vee$ is an internal Hom: we have a functorial isomorphism

$$\mathrm{Hom}(Z, \mathcal{H}om(X, Y)) = \mathrm{Hom}(Z \otimes X, Y).$$

For $Z = 1$, we obtain

$$\mathrm{Hom}(1, \mathcal{H}om(X, Y)) = \mathrm{Hom}(X, Y) \tag{1.1.1}$$

The object 1 being simple (Deligne–Milne 1982, 1.17), there results from (0.1.5) and (1.1.1) that n linearly independent morphisms $f_i : X \to Y$ define a morphism from 1^n in $\mathcal{H}om(X, Y)$. If $\mathcal{H}om(X, Y)$ is of finite length, there is a largest such n and $\mathrm{Hom}(X, Y)$ is of finite dimension. □

1.2. We will use categories with tensor product $(\mathcal{A}, \otimes, \text{some auxiliaries})$ that are more general than those from 0.1, checking only the following conditions.

(1.2.1) $\mathcal{A}$ is additive, k-linear and karoubian: all idempotent endomorphisms are split. An idempotent endomoprhism e of X has then an image, and X is decomposed as a direct sum of $\mathrm{Im}(e)$ and $\mathrm{Ker}(e)$.

(1.2.2) The tensor product is additive and k-linear in each variable.

(1.2.3) it is under the constraints of additivity, commutativity, and unity.

1.3. The hypothesis (1.2.1) permits the following construction:

For X in $\mathcal{A}$ and V a vector space of finite dimension over k, we define the objects $V \otimes X$ and $\mathcal{H}om(V, X)$ of $\mathcal{A}$ by

$$\mathrm{Hom}(V \otimes X, Y) = \mathrm{Hom}(V, \mathrm{Hom}(X, Y)) \quad \text{and} \tag{1.3.1}$$

$$\mathrm{Hom}(Y, \mathcal{H}om(V, X)) = \mathrm{Hom}(V \otimes Y, X). \tag{1.3.2}$$

The choice of a basis $(e_i)_{i \in I}$ of V identifies $V \otimes X$ with the sum of a family of copies of X indexed by I, and $\mathcal{H}om(V, X)$ is canonically isomoprhic to $V^\vee \otimes X$.

If a finite group S acts on X, the endomorphism $e := \frac{1}{|S|} \sum s$ of X is idempotent. The direct factor $\mathrm{Im}(e)$ of X, seen as a subobject of X, is denoted X^S (invariants). Seen as a quotient of X by $\mathrm{Ker}(e)$, it is denoted X_S (coinvariants). If V is a linear representation of S, S acts on $\mathcal{H}om(V, X)$, and we have

$$\mathcal{H}om_S(V, X) := \mathcal{H}om(V, X)^S. \tag{1.3.3}$$

If we choose a representative V_λ of each isomorphism class of irreducible linear representations of S, it results formally that $k[S] \xrightarrow{\sim} \prod \mathrm{End}_k(V_\lambda)$ and the application

$$\bigoplus V_\lambda \otimes \mathcal{H}om_S(V_\lambda, X) \to X \tag{1.3.4}$$

is an isomorphism.

1.4. For $\mathcal{A}$ as in 1.2 and X an object of $\mathcal{A}$, the symmetric group S_n acts on the tensor powers $X^{\otimes n}$. We identify the isomorphism classes of irreducible representations of S_n with partitions of n, and four each partition λ we choose V_λ of class λ. The *Schur functor* S_λ is

$$S_\lambda(X) := \mathrm{Hom}_{S_n}(V_\lambda, X^{\otimes n}), \tag{1.4.1}$$

and (1.3.4) specializes to

$$\bigoplus V_\lambda \otimes S_\lambda(X) \xrightarrow{\sim} X^{\otimes n} \tag{1.4.2}$$

(sum over the partitions of n).

Notations. a partition λ is a sequence $(\lambda_1, \ldots, \lambda_r)$ of integers $\lambda_1 \geq \cdots \geq \lambda_r > 0$; for $s > r$, we have $\lambda_s := 0$; we define $|\lambda| = \sum \lambda_i$ and we say that λ is a partition of $|\lambda|$. The *diagram* $[\lambda]$ of λ is the collection of pairs (i, j) of integers ≥ 1 such that $j \leq \lambda_i$. For example, if λ is the partition $(3, 1)$ of 4, $[\lambda]$ is

$$\begin{matrix} (1,1) & (1,2) & (1,3) \\ (2,1) & & \end{matrix}$$

Note the matrix, rather than cartesian, arrangement of the (i, j)s. If to the diagram of λ we apply the involution $(i, j) \mapsto (j, i)$, we obtain the diagram of the *transposed partition* λ^t.

Examples. For the partition (n) (resp. $(n)^t = (1^n) = (1, \dots, 1)$) of n, we have $V_\lambda = k$, under the action of S_n by the trivial (resp. sign) character. We moreover have $S_{(n)}(X) = \mathrm{Sym}^n(X)$ and $S_{(n)^t}(X) = \bigwedge^n X$.

1.5. Let $n_1, \dots, n_r$ be integers that sum to n; mapping the product of S_{n_i}s into S_n identifies $\{1, \dots, n\}$ with the disjoint sum of $\{1, \dots, n_i\}$s. If, for each i, μ_i is a partition of n_i, the tensor product of V_{μ_i}s is an irreducible representation of the product $\prod S_{n_i}$ of the S_{n_i}s. If λ is a partition of n, we denote by $[\lambda : \mu_1, \dots, \mu_r]$ the multiplicity of this representation in the restriction of V_λ to $\prod S_{n_i}$. By Frobenius reciprocity, this is also the multiplicity of V_λ in the induction of $\otimes V_{\mu_i}$ to S_n:

$$[\lambda : \mu_1, \dots, \mu_r] = [V_\lambda; \bigotimes V_{\mu_i}] = [\mathrm{Ind}_{\prod S_{n_i}}^{S_n}(\bigotimes V_{\mu_i}) : V_\lambda].$$

These multiplicites are given by the Littlewood–Richardson rule therefore we use the following consequences:

(1.5.1) If $|\lambda| = |\mu| + 1$, we have $[\lambda : \mu, (1)] = 1$ if $[\mu] \subset [\lambda]$, 0 else. Using the formula

$$[\lambda : \mu_1, \mu_2, \mu_3] = \sum_{\lambda'} [\lambda : \lambda', \mu_3][\lambda' : \mu_1, \mu_2]$$

that expresses the transitivity of restriction to a subgroup, we deduce that four $|\mu| \leq |\lambda|$, the following conditions are equivalent: $[\mu] \subset [\lambda]$; there exists a partition ν of $|\lambda| - |\mu|$ such that $[\lambda : \mu, \nu] \neq 0$; $[\lambda : \mu, (1), \dots, (1)] \neq 0]$.

(1.5.2) Fix r and a partition λ of n. For there to exist $n_1, \dots, n_r$ of sum n such that $[\lambda : (n_1), \dots, (n_r)] \neq 0$, it is necessary and sufficient that $[\lambda]$ has at least r rows.

(1.5.3) Fix r, s and a partition λ of n. For there to exist $n_1, \dots, n_r, m_1, \dots, m_s$ of sum n such that $[\lambda : (n_1), \dots, (n_r), (m_1)^t, \dots, (m_s)^t] \neq 0$, it is necessary and sufficient that

$$[\lambda] \subseteq \{(i, j) : i \leq r \text{ or } j \leq s\},$$

i.e. $(r + 1, s + 1) \notin [\lambda]$.

As classically, and with the same proof, the Schur functors obey 1.6, 1.8, 1.11, and 1.15 below.

Proposition 1.6. *$S_\mu(X) \otimes S_\nu(X) \sim \bigoplus S_\lambda(X)^{[\lambda:\mu,\nu]}$, the sum running over the partitions λ of $n = |\mu| + |\nu|$.*

Proof. We have

$$S_\mu(X) \otimes S_\nu(X) = \mathcal{H}om_{S_{|\mu|} \times S_{|\nu|}}\big(V_\mu \otimes V_\nu, X^{\otimes|\mu|} \otimes X^{\otimes|\nu|}\big).$$

By 1.4.2, we have

$$X^{\otimes|\mu|} \otimes X^{\otimes|\nu|} = X^{\otimes n} = \bigoplus_\lambda V_\lambda \otimes S_\lambda(X)$$

and 1.6 follows. □

Applying (1.5.1), we conclude from 1.6 the

Corollary 1.7. *If $S_\mu(X) = 0$, then $S_\lambda(X) = 0$ for all partitions λ such that $[\mu] \subset [\lambda]$.*

The tensor power $(X \oplus Y)^{\otimes n}$ is the sum over $p + q = n$ of inductions

$$\mathrm{Ind}_{S_p \times S_q}^{S_n} (X^{\otimes p} \otimes X^{\otimes n}).$$

It then results that:

Proposition 1.8. *If λ is a partition of n, we have*

$$S_\lambda(X \oplus Y) \sim \bigoplus (S_\mu(X) \otimes S_\nu(Y))^{[\lambda:\mu,\nu]}, \tag{1.8.1}$$

the sum running over the partitions μ, ν such that $|\mu| + |\nu| = n$.

Corollary 1.9. *In the k-tensorial category of super vector spaces of finite dimension, if X is of super dimension $p|q$, i.e. if $\dim X^0 = p$ and $\dim X^1 = q$, in order for $S_\lambda(X) \neq 0$, it is necessary and sufficient that*

$$[\lambda] \subset \{(i,j) : i \leq p \text{ or } j \leq q\}. \tag{1.9.1}$$

Proof. If Y is purely odd, with underlying vector space $|Y|$, the underlying vector space of $Y^{\otimes n}$ is $|Y|^{\otimes n}$, and the action of $\sigma \in S_n$ on $|Y^{\otimes n}|$ is $\mathrm{sgn}(\sigma)$ times the natural action on $|Y|^{\otimes n}$. If ν^t is the transposed partition of the partition ν of n, we have $V_{\nu^t} \sim \mathrm{sgn} \otimes V_\nu$ and

$$|S_\nu(Y)| \sim S_{\nu^t}(|Y|) \quad (\text{for } Y \text{ odd}). \tag{1.9.2}$$

We decompose X into even and odd parts X^0 and X^1. According to 1.8, and 1.9.2, we have

$$|S_\lambda(X)| = \bigoplus_{|\mu|+|\nu|=|\lambda|} (S_\mu(|X^0|) \otimes S_{\nu^t}(|X^1|))^{[\lambda:\mu+\nu]}.$$

In order for $S_\lambda(X) \neq 0$, it is necessary and sufficient therefore for the there to exist partitions μ and ν such that $[\mu]$ has at least p rows, $[\nu]$ has at least q columns and that $[\lambda : \mu, \nu] \neq 0$. By (1.5.2) and (1.5.3), such is the case if and only if we have (1.9.1). □

Corollary 1.10. *Let $p, q, r, s \geq 0$, and λ, μ, ν three partitions verifying $|\lambda| = |\mu| + |\nu|$. If $(p + r + 1, q + s + 1) \in [\lambda]$ and $[\lambda : \mu, \nu] \neq 0$, then $(p + 1, q + 1) \in [\mu]$ or $(r + 1, s + 1) \in [\nu]$.*

Proof. We apply 1.8, in the category of super vector spaces, to X of dimension $p|q$ and Y of dimension $r|s$. According to 1.9, we have $S_\lambda(X \oplus Y) = 0$. According to 1.8 and 1.9, the conclusion expressed the vanishing of the right-hand side of (1.8.1). □

Proposition 1.11. *For λ a partition of n, we have*

$$S_\lambda(X \otimes Y) \sim \bigoplus (S_\mu(X) \otimes S_\nu(Y))^{[V_\mu \otimes V_\nu : V_\lambda]} \tag{1.11.1}$$

(sum over μ, ν partitions of n).

Proof. We use that

$$\begin{aligned}(X \otimes Y)^{\otimes n} = X^{\otimes n} \otimes Y^{\otimes n} &= \big(\bigoplus V_\mu \otimes S_\mu(X)\big) \otimes \big(\bigoplus V_\nu \otimes S_\nu(Y)\big) \\ &= \bigoplus V_\mu \otimes V_\nu \otimes (S_\mu(X) \otimes S_\nu(Y)).\end{aligned}$$

□

Corollary 1.12. *Let $p, q, r, s \geq 0$ and λ, μ, ν three partitions of n. If $(pq + rs + 1, ps + qr + 1) \in [\lambda]$ and $[V_\mu \otimes V_\nu : V_\lambda] \neq 0$, then $(p + 1, q + 1) \in [\mu]$ or $(r + 1, s + 1) \in [\nu]$.*

The proof is parallel to that of 1.10, with $\oplus$ replaced by $\otimes$ and 1.8 by 1.11.

Corollary 1.13. *For $\mathcal{A}$ as in 1.2, the collection of objects of $\mathcal{A}$ annihilated by at least one functor of Schur is stable under direct sums and tensor products.*

Proof. We suppose that $S_\mu(X) = S_\nu(Y) = 0$. Let $p, q, r, s \geq 0$ such that

$$[\mu] \subset [1, p+1] \times [1, q+1] \quad \text{and} \quad [\nu] \subset [1, r+1] \times [1, s+1],$$

of type that if $(p+1, q+1) \in [\mu']$ (resp. $(r+1, s+1) \in [\nu']$), we have $[\mu] \subset [\mu']$ (resp. $[\nu] \subset [\nu']$). If λ is such that

$$(p+r+1, q+s+1) \in [\lambda] \quad (\text{resp. } (pr+qs+1, ps+qr+1) \in [\lambda]),$$

it results from 1.7, 1.8, and 1.10 (resp. 1.7, 1.11, and 1.12) that $S_\lambda(X \oplus Y)$ (resp. $S_\lambda(X \otimes Y)$) is zero. □

1.14. If X is dualizable, with dual $X^\vee$, we verify as in 1.1 that the functor $Y \mapsto Y \otimes X^\vee$ is right adjoint to the functor $Y \mapsto Y \otimes X$. The morphisms $1 \to X^\vee \otimes X$ and $X \otimes X^\vee \to 1$ derived from δ and ev by the symmetry of the tensor product mave X into a dual to $X^\vee$. The functor $Y \mapsto Y \otimes X^\vee$ is therefore also left adjoint to $Y \mapsto Y \otimes X$. If $\mathcal{A}$ is abelian, the functor $Y \mapsto Y \otimes X$ is therefore exact, since it is both a right and a left adjoint.

1.15. We leave it to the reader to verify that if X and Y are dualizable, so also $X \otimes Y$ (resp. $X \oplus Y$) is dualizable, with dual $X^\vee \otimes Y^\vee$ (resp. $X^\vee \oplus Y^\vee$), the morphisms δ and ev being the tensor product (resp. direct sum) of δ and ev for X and Y.

A direct factor A of a dualizable object X is moreover dualizable. If $X = A \oplus B$ and e is the idempotent endomorphism "projection onto A" of X, A admits for its dual the image $A^\vee$ of the idempotent endomorphism of $X^\vee$ given by the transpose of e. If A and $A^\vee$ are seen as subobjects (resp. quotients) of X and $X^\vee$, ev (resp. δ) for A is derived from ev (resp. δ) for X.

Particular case 1.15.1. Suppose that a finite group S acts on X dualizable, and act by S on $X^\vee$ by the contragradient action. The direct factors X_S and $X_S^\vee$ of X and $X^\vee$ are then in duality. The morphism $\delta : 1 \to X_S \otimes X_S^\vee$ is the composition $1 \to X \otimes X^\vee \to X_S \otimes X_S^\vee$.

Notation 1.16. If X is dualizable, an endomorphism f of X corresponds by 1.1.1 to a morphism $\delta(f) : 1 \to X \otimes X^\vee$. We define the trace $\mathrm{Tr}(f) \in \mathrm{End}(1)$ of f as the composition

$$\mathrm{ev} \circ \delta(f) : 1 \to X \otimes X^\vee = X^\vee \otimes X \to 1,$$

and $\dim(X) := \mathrm{Tr}(\mathrm{Id}_X) = \mathrm{ev} \circ \delta$.

A $\otimes$-functor transforms dualizable objects into dualizable objects and preserves traces and dimensions.

By induction in n we prove:

Lemma 1.17. *If X is dualizable and $X^{\otimes n} = 0$, then $X = 0$.*

Proof. We can suppose $n \geq 2$. We tensor the composition

$$\mathrm{Id}_X : X \to X \otimes X^\vee \otimes X \to X$$

with $X^{\otimes n-2}$. We obtain that the identity on $X^{\otimes n-1}$ factors through $X^{\otimes n} \otimes X^\vee = 0$: we have $X^{\otimes(n-1)} = 0$ and we conclude by recurrence. □

Proposition 1.18. *If X is dualizable, $S_\lambda(X)$ is dualizable, with dual isomorphic to $S_\lambda(X^\vee)$. In particular, if $S_\lambda(X) = 0$, we have $S_\lambda(X^\vee) = 0$.*

Proof. The representations of S_n are self-dual, as results from 1.4.2. $\square$

Proposition 1.19. *In a k-tensorial category, each object annihilated by some Schur functor is of finite length, and the collection of such objects is stable under subquotients and extensions.*

Proof. If Y is a subobject of X, we have by the exactness of the tensor product $Y^{\otimes n} \hookrightarrow X^{\otimes n}$, and for λ a partition of n

$$\mathcal{H}om(V_\lambda, Y^{\otimes n}) \hookrightarrow \mathcal{H}om(V_\lambda, X^{\otimes n}).$$

By exactness of the functor "S_n invariants", $S_\lambda(Y)$ is a subobject of $S_\lambda(X)$. Dually, if Y is a quotient of X, $S_\lambda(Y)$ is a quotient of $S_\lambda(X)$. This proves the stability for subquotients.

If X is an extension of X' by X'', the exactness of the tensor product provides an S_n-equivariant filtration of $X^{\otimes n}$ with associated graded $(X' \oplus X'')^{\otimes n}$. This induces a filtration of $S_\lambda(X)$ with associated graded $S_\lambda(X' \oplus X'')$ and the stability for extensions results from the stability for direct summs 1.13.

More generally, a finite filtration F of X induces a filtration of $S_\lambda(X)$ with associated graded $S_\lambda(\mathrm{Gr}_F(X))$. A tensor product of non-zero objects is non-zero, since "Y is non-zero" is equivalent to "$\mathrm{ev} : Y \otimes Y^\vee \to 1$ is an epimorphism", a stable condition for tensor product. If $\mathrm{Gr}^i_F(X)$ is non-zero for n values of i, $S_\lambda(\mathrm{Gr}_F(X))$ contains the tensor product of these $\mathrm{Gr}^i_F(X)$s, and $S_\lambda(X) \neq 0$. If $S_\lambda(X) \neq 0$, X is therefore of length $< n$. $\square$

1.20. This verifies that in any k-tensorial category, condition (b) of 0.5 (i) implies (a). If the $S_\lambda(X)$s are all non-zero, (1.4.2) gives

$$\mathrm{length}(X^{\otimes n}) \geq \sum_{|\lambda|=n} \dim(V_\lambda) \geq \left(\sum \dim(V_\lambda)^2\right)^{1/2} = (n!)^{1/2}$$

and $(n!)^{1/2}$ grows more quickly than any geometric progression.

1.21. The objects of a category $\mathrm{Rep}(G, \epsilon)$ verify conditions (a) and (b) of 0.5 (i): (a) results from 1.9, and (b) from this because if a super representation X of G is of super dimension $p|q$, the length of $X^{\otimes n}$ is at most the dimension $(p+q)^n$ of the underlying vector space.

2. Existence of super fiber functors

The key result of the article is the following.

Proposition 2.1. *If for each object of a k-tensorial category $\mathcal{A}$ there exists a Schur functor that annihilates it, then there exists a super fiber functor over a non-zero super commutative k-algebra.*

If the hypotheses of 2.1 are verified, it results from 1.19 that

(2.1.1) every object of $\mathcal{A}$ is of finite length, and by 1.1, the $\mathrm{Hom}(X, Y)$s are therefore of finite dimension over k.

For an example of a k-tensorial category that does not verify (2.1.1), see Deligne (1990), 2.19.

2.2. Let $\mathcal{A}$ be a k-tensorial category verifying (2.1.1) and $\operatorname{Ind}\mathcal{A}$ the category of its Ind-objects. This is an abelian category of which $\mathcal{A}$ is a plain subcategory. The hypothesis (2.1.1) assures that the subcategory $\mathcal{A}$ of $\operatorname{Ind}\mathcal{A}$ is stable under subquotients and that every object of $\operatorname{Ind}\mathcal{A}$ is the filtered inductive limit of its sub-objects.

Model: (finite dimensional vector spaces) $\subset$ (vector spaces)

The tensor product of $\mathcal{A}$ provides $\operatorname{Ind}\mathcal{A}$ with a tensor product defined by

$$(\operatorname{colim} X_i) \otimes (\operatorname{colim} Y_j) = \operatorname{colim}(X_i \otimes Y_j)$$

for filtered inductive systems (X_i) and (Y_i) in $\mathcal{A}$.

The category $\operatorname{Ind}\mathcal{A}$, under this tensor product, is as in 0.1, except for the smallness (0.1.0) and the existence of duals (0.1.4). An Ind-object is not dualizable if it is not in $\mathcal{A}$. Indeed, if X is dualizable with dual $X^\vee$, there exists a subobject X' of X that is in $\mathcal{A}$ and such that $\delta : 1 \to X \otimes X^\vee$ factors through $X' \otimes X^\vee$. The commutative diagram

$$\begin{tikzcd}
\mathrm{Id}_X : X \arrow[r] \arrow[dr] & X \otimes X^\vee \otimes X \arrow[r] & X \\
& \cup & \cup \\
& X' \otimes X^\vee \otimes X \arrow[r] & X'
\end{tikzcd}$$

shows that $X' = X$.

That $(\operatorname{Ind}\mathcal{A}, \otimes)$ is as in 1.2 suffices to define, in $\operatorname{Ind}\mathcal{A}$, the notions of associative and commutative algebra with unit (we say simply "*algebra*") and of *module* over such an algebra. A *homomorphism* of algebras $f : A \to B$ is supposed to transform the unit $1 \to A$ of A into that of B.

2.3. If A is an algebra in $\operatorname{Ind}\mathcal{A}$, the A-modules form an abelian category Mod_A. Under the tensor product

$$M \otimes_A N := \operatorname{coker}(M \otimes A \otimes N \rightrightarrows M \otimes N),$$

this is of the type considered in 1.2. The unit object 1_A is the A-module A. The tensor product is right exact. An A-module P is called *flat* (resp. *faithfully flat*) if the functor $M \mapsto M \otimes_A P$ is exact (resp. exact and faithful). For X in $\operatorname{Ind}\mathcal{A}$, the A-module $A \otimes X$ is flat, since $M \otimes_A (A \otimes X) = M \otimes X$. Each A-module M is a quotient of a flat A-module, for example the A-module $A \otimes M$, and we can therefore formulate the usual Tor functors. In particular, if an exact sequence $0 \to M \to N \to P \to 0$ with P flat is tensored with an A-module, the result is exact.

An A-*algebra* is an algebra B along with a homomorphism $f : A \to B$. If B is an A-algebra, and M a B-module, the morphism $\text{product} \circ (f \otimes M) : A \otimes M \to B \otimes M \to M$ makes M into an A-module. This functor of restriction of scalars is left adjoint to the functor of extension of scalars

$$M \mapsto M_B := B \otimes AM$$

from Mod_A to Mod_B. For M in Mod_A and N in Mod_B, the isomorphism $\operatorname{Hom}_B(M_B, N) \to \operatorname{Hom}_A(M, \operatorname{restriction}(N))$ and its inverse are

$$\begin{aligned}
u : M_B \to N \quad &\mapsto \quad M = A \otimes_A M \to B \otimes_A M \xrightarrow{u} N \\
v : M \to N \quad &\mapsto \quad M_B = B \otimes_A M \xrightarrow{v} B \otimes_A N \to N
\end{aligned}$$

The functor of extension of scalars is a $\otimes$-functor.

Particular case. the isomorphism $1 \otimes 1 \to 1$ makes 1 an algebra. Every object of $\mathrm{Ind}\,\mathcal{A}$ admits a unique structure as a module over this algebra via the isomorphism $1 \otimes M \to M$. This defines an equivalence $\mathrm{Mod}_1 \sim \mathrm{Ind}\,\mathcal{A}$. The unit $1 \to A$ of an algebra is a homomorphism of algberas. It is either injective, or zero, in which case $A = 0$. The functor of extension of scalars for $1 \to A$, from $\mathrm{Ind}\,\mathcal{A}$ to Mod_A, is the exact functor $M \mapsto A \otimes M$. It if faithful if $A \neq 0$.

2.4. Notation. We denote by Γ the functor

$$\Gamma(X) := \mathrm{Hom}(1, X)$$

from $\mathrm{Ind}\,\mathcal{A}$ to k-vector spaces. For A an algebra of $\mathrm{Ind}\,\mathcal{A}$, $\Gamma(A)$ is a k-algebra (commutative). By the adjunction,

$$\mathrm{Hom}_A(1_A, 1_A) = \mathrm{Hom}(1, A) = \Gamma(A).$$

The dimension (1.16) of a dualizable A-module is an element of $\Gamma(A)$.

Definition 2.5. We say that a system of objects and morphisms in $\mathrm{Ind}\,\mathcal{A}$ has *locally* a property if this property remains true after extension of scalars to a convenient non-zero algebra.

For example, X is *locally isomorphic* to Y if there exists a non-zero algebra A such that the A-modules X_A and Y_A are isomoprhic. If $A \neq 0$, k injects into $\Gamma(A)$. It results therefore from 1.16 that two locally isomorphic objects of $\mathcal{A}$ are of the same dimension.

The terminology "locally" is inspired by the usable from the topology fpqc (faithfully flat quasi-compact) in algebraic geometry.

Example 2.6. In a k-tensorial category $\mathrm{Rep}(G)$, two objects having the same dimension are locally isomorphic. Indeed, the algebras of $\mathrm{Ind}\,\mathrm{Rep}(G)$ are identified by $A \mapsto \mathrm{Spec}(A)$ with affine schemes S over k along with an action of G, and two representations X, Y of G are locally isomorphic if and only if their inverse images over a convenient nonempty S are isomorphic equivariant vector bundles. If X and Y are of the same dimension n, we can set $S = \mathcal{I}som(X, Y)$ with the action $g(f) = gfg^{-1}$ of G. The scheme $\mathcal{I}som(X, Y)$ is affine and nonempty because it is isomorphic to $\mathrm{GL}(n)$.

Example 2.7. In a k-tensorial category $\mathrm{Rep}(G, \epsilon)$, two objects X and Y are locally isomorphic if and only if they are of the same super dimension (1.9).

Lemma 2.8. *Let M be a dualizable module over a non-zero algebra A. For there to exist a non-zero A-algebra B such that the B-module 1_B is a direct factor of M_B, it is necessary and sufficient that* $\mathrm{Sym}^n_A(M) \neq 0$ *for all n.*

Proof. For each A-algebra B, we let $\mathrm{Fact}(B)$ denote the collection of pairs

$$\alpha : 1_B \to M_B, \quad \beta : M_B \to 1_B$$

such that $\beta\alpha = 1$, i.e. that make 1_B into a direct factor of M_B. This is a covariant functor in B. We will construct the universal (B_0, α_0, β_0), i.e. that corepresents the functor Fact.

(a) The data of $\beta : M_B \to 1_B$ is equivalent by adjuntion to that of a morphism of A-modules $M \to 1_B = B$: to

$$v : M \to B$$

corresponds the composition

$$\mathrm{product} \circ (B \otimes u) : B \otimes M \to B \otimes B \to B.$$

The data of v is equivalent in turn to that of a morphism of A-algebras

$$v_{\mathrm{alg}} : \mathrm{Sym}_A(M) = \bigoplus \mathrm{Sym}_A^n(M) \to B$$

(b) The data of $\alpha : 1_B \to M_B$ is equivalent to that of a morphism of A-modules $A \to B \otimes M$, and to that of $u : M^\vee \to B$: to

$$u : M^\vee \to B$$

corresponds the composition

$$(u \otimes M) \circ \delta : 1_A \xrightarrow{\delta} M \otimes M^\vee = M^\vee \otimes M \to B \otimes M.$$

In turn, u corresponds to a morphism of A-algebras

$$u_{\mathrm{alg}} : \mathrm{Sym}_A(M^\vee) \to B$$

(c) Let α and β give u and v. In order for $\beta\alpha : 1_B \to M_B \to 1_B$ to be the identity, it is necessary and sufficient for its restriction to 1_A to be the natural morphism $1_A \to 1_B$. This restriction is the composition

$$1_A \xrightarrow{\delta} M \otimes M^\vee = M^\vee \otimes M \xrightarrow{u} B \otimes M \xrightarrow{v} B \otimes B \to B,$$

i.e. it is obtained by applying to δ the product $u.v$ of u and v: $M \otimes M^\vee \to B \otimes B \to B$.

In total, the data over B of α, β making 1_B into a direct factor of M_B is equivalent to that of a homomorphism

$$\mathrm{Sym}_A(M) \otimes_A \mathrm{Sym}_A(M^\vee) = \bigoplus \mathrm{Sym}_A^p(M) \otimes \mathrm{Sym}_A^q(M) \to B$$

such that the unit $1_A \xrightarrow{\sim} \mathrm{Sym}_A^0(M) \otimes \mathrm{Sym}_A^0(M^\vee)$ and $\delta : 1_A \to M \otimes_A M^\vee$ have the same image.

For x a morphism of A-modules from 1_A to an A-algebra C, the ideal (x) *created* by x is the image of the multiplication by x:

$$C = 1_A \otimes_A C \xrightarrow{x} C \otimes_A C \to C.$$

The quotient $C/(x)$ is the universal quotient of the algebra C in which x is killed. With this terminology, the universal (B_0, α_0, β_0) that we are looking for is the quotient of the algebra $\mathrm{Sym}_A(M) \otimes_A \mathrm{Sym}_A(M^\vee)$ by the ideal $(\delta - 1)$, and M admits locally 1 as a direct factor if and only if $B_0 = 0$, or, what is the same, if the unit morphism $1 \to B_0$ is zero.

The multiplication by δ is a morphism

$$\mathrm{Sym}_A^p(M) \otimes \mathrm{Sym}_A^q(M^\vee) \to \mathrm{Sym}_A^{p+1}(M) \otimes \mathrm{Sym}_A^{q+1}(M^\vee) \tag{2.8.1}$$

and

$$B_0 = \bigoplus_{a \in \mathbb{Z}} \operatorname*{colim}_n \mathrm{Sym}_A^n(M) \otimes \mathrm{Sym}_A^{n+a}(M^\vee),$$

the transition morphisms in the inductive limit being given by (2.8.1).

The unit of B_0 is the inductive limit of morphisms

$$\delta^n : 1 \to \mathrm{Sym}_A^n(M) \otimes \mathrm{Sym}_A^n(M^\vee).$$

The functor $\mathrm{Hom}(1, \ )$ commutes with filtered inductive limits. In order for the unit of B_0 to be zero, it is necessary and sufficient therefore that for some n we have $\delta^n = 0$.

The symmetric algebra $\mathrm{Sym}_A(M)$ is a quotient of the tensor algebra (noncommutative) $\bigoplus M^{\otimes n}$, and δ^n is the image of

$$\delta^{\otimes n} : 1_A \to M^{\otimes n} \otimes M^\vee \otimes n.$$

This morphisms if the morphism δ for a duality between $M^{\otimes n}$ and $(M^\vee)^{\otimes n}$. According to (1.15.1), δ^n is therefore the morphism δ for a duality between $\mathrm{Sym}^n M$ and $\mathrm{Sym}^n M^\vee$. It is zero if and only if $\mathrm{Sym}^n M = 0$. □

Proposition 2.9. *Suppose we are given in a k-tensorial category $\mathcal{A}$ verifying (2.1.1) an object $\bar{1}$ such that $\bar{1}\otimes\bar{1}$ is isomorphic to 1 and the commutativity automorphism of the tensor product $\bar{1}\otimes\bar{1} \to \bar{1}\otimes\bar{1}$ is the multiplication by -1. For X in $\mathcal{A}$, the following conditions are equivalent.*

(i) *There exists p and q such that X is locally isomorphic to $1^p \oplus \bar{1}^q$.*
(ii) *There exists a Schur functor S_λ such that $S_\lambda(X) = 0$.*

For V a super vector space of finite dimension, we set

$$F(V) : V^0 \otimes 1 \oplus V^1 \otimes \bar{1}.$$

The coice of the isomorphism $\bar{1}\otimes\bar{1} \to 1$ provides a functorial isomorphism

$$F(V) \otimes F(W) \to F(V \otimes W)$$

and the hypotheses made on $\bar{1}$ assure that F is a $\otimes$-equivalence of the k-tensorial category (s-Vect) of super vector spaces of finite dimension with the plain subcategory $\langle 1, \bar{1}\rangle$ of $\mathcal{A}$ whose objects are sums of copies of 1 and $\bar{1}$.

Proof of (i) ⇒ (ii). If X_A is isomorphic to $(1^p\oplus\bar{1}^q)_A$, $S_\lambda(X)_A$ is isomorphic to $S_\lambda(1^p\oplus\bar{1}^q)_A$. According to 1.9, there exists λ such that $S_\lambda(1^p \oplus \bar{1}^q) =$. If $A \neq 0$, that $S_\lambda(X)_A = 0$ implies that $S_\lambda(X) = 0$. □

Proof of (ii) ⇒ (i). Suppose that after extension of scalars to a non-zero algebra A, $1^r \oplus \bar{1}^s$ becomes a direct factor of X:

$$X_A = 1_A^r \oplus \bar{1}_A^s \oplus R.$$

The A-module R is dualizable being a direct factor of the dualizable A-module X_A (1.15). We distinguish three cases.

a. All the $\mathrm{Sym}^n_A(R)$ are non-zero. According to 2.8, there exists then a non-zero A-algebra B such that R_B admits 1_B as a direct factor, hence a decomposition

$$X_B = 1_B^{r+1} \oplus \bar{1}_B^s \oplus R'.$$

b. All the $\mathrm{Sym}^n_A(\bar{1}\otimes R)$ are non-zero. Because $\mathrm{Sym}^n_A(\bar{1}\otimes R) \sim \bar{1}^{\otimes n} \otimes \bigwedge^n_A R$, this is equivalent to the non-nullity of all the $\bigwedge^n_A R$. Over a convenient non-zero B, we can then extract a factor 1_B of $\bar{1}\otimes R_B$, which returns a factor $\bar{1}_B$ of R_B, hence a decomposition

$$X_B = 1_B^r \oplus \bar{1}^{s+1} \oplus R'.$$

c. If neither a. nor b. are applicable, there exists n and m such that $\mathrm{Sym}^{n+1}_A R = \bigwedge^{m+1}_A R = 0$. Let k be an integer $> nm$. According to 1.7, for all partitions λ of k, we have $S_\lambda(R) = 0$. By 1.4.2, we have $R^{\otimes k} = 0$, hence the result that $R = 0$ (1.17).

Thus from $A = 1$, $r = s = 0$ and $R = X$ and iteratively applying the constructions a. or b. we obtain that either X is locally isomorphic to $1^p \oplus \bar{1}^q$ for some p and q, or X admits locally a direct factor $1^p \oplus \bar{1}^q$ with $p + q$ arbitrarily large. In the second case, contrary to our hypothesis, the $S_\lambda(X)$ are all non-zero: if λ is a partition of n, we can choose p and q such that $n < (p+1)(q+1)$, and $S_\lambda(1^p \oplus \bar{1}^q)$, non-zero according to 1.9, is locally a direct factor of $S_\lambda(X)$ that therefore is non-zero. □

Recollection 2.10. *In a k-tensorial category verifying (2.1.1), all short exact sequences are locally split.*

This is Deligne (1990), 7.14. We begin by reducing to the case of short exact sequences of the form $0 \to X \xrightarrow{a} Y \xrightarrow{b} 1 \to 0$: replace the short exact sequence $0 \to A \to B \to C \to 0$ by the corresponding short exact sequences $0 \to \mathrm{Hom}(C, A) \to E \to 1 \to 0$. This being done, the proof is analogous to that of 2.8, and even simpler. If $b^t : 1 \hookrightarrow Y^\vee$ is the transpose of b, we are left to verify the non-nullity of the algebra

$$\mathrm{Sym}(Y^\vee)/(b^t - 1) = \operatorname{colim} \mathrm{Sym}^n(Y^\vee)$$

(transition morphisms: the multiplication by b^t).

2.11. Proof of 2.1. Assume first that $\mathcal{A}$ contains an object $\bar{1}$ as in 2.9. For each isomorphism class of objects of $\mathcal{A}$, represented by X, choose a non-zero algebra B such that X_B is isomorphic to a sum of copies of 1_B and $\bar{1}_B$. This is possible by 2.9. For each isomorphism class of short exact sequences in $\mathcal{A}$, represented by Σ, choose a non-zero algebra C such that Σ_C is split. This is possible by 2.10. Let A be the tensor product of these algebras: the inductive limit of tensor products of a finite number of them. The algebra A is non-zero and, after extension of scalars to A, each object of $\mathcal{A}$ becomes isomorphic to a sum of copies of 1_A and $\bar{1}_A$ and each exact sequence of $\mathcal{A}$ becomes split.

We identify as in 2.9 the k-tensorial category (s-Vect) with the subcategory $\langle 1, \bar{1} \rangle$ of $\mathcal{A}$, and we identify $\mathrm{Ind}\langle 1, \bar{1} \rangle \subset \mathrm{Ind}\,\mathcal{A}$ with the category of all super vector spaces. For M in $\mathrm{Ind}\,\mathcal{A}$, denote by $\rho(M)$ the largest subobject of M in $\mathrm{Ind}\langle 1, \bar{1} \rangle$. It is identified with the super vector space with even part and odd part $\mathrm{Hom}(1, M)$ and $\mathrm{Hom}(\bar{1}, M)$.

The multiplication $A \otimes A \to A$ induces on $\rho(A)$ the structure of an algebra, i.e. it makes $\rho(A)$ into a k-super algebra. For M an A-module, the multiplication $A \otimes M \to M$ induces similarly a morphism $\rho(A) \otimes \rho(M) \to \rho(M)$ making $\rho(M)$ into a $\rho(A)$-module. For M and N two A-modules, taking the cokernels of the double arrows in

$$\begin{tikzcd} \rho(M) \otimes \rho(A) \otimes \rho(N) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \rho(M) \otimes \rho(N) \arrow[d] \\ M \otimes A \otimes N \arrow[r, shift left] \arrow[r, shift right] & M \otimes N \end{tikzcd}$$

we obtain a morphism from $\rho(M) \otimes_{\rho(A)} \rho(N)$ to $M \otimes_A N$. This induces a morphism

$$\rho(M) \otimes_{\rho(A)} \rho(N) \to \rho(M \otimes_A N). \tag{2.11.1}$$

If M is of the form $A \otimes M_0$, with M_0 in $\langle 1, \bar{1} \rangle$, we have

$$\rho(A) \otimes M_0 \xrightarrow{\sim} \rho(M). \tag{2.11.2}$$

If another N is of the form $A \otimes N_0$, with N_0 in $\langle 1, \bar{1} \rangle$, we have

$$M \otimes_A N = A \otimes (M_0 \otimes N_0)$$

and the commutative diagram

$$\begin{tikzcd} (\rho(A) \otimes M_0) \otimes_{\rho(A)} (\rho(A) \otimes N_0) \arrow[r, "\sim"] \arrow[d, "\wr"] & \rho(A) \otimes (M_0 \otimes N_0) \arrow[d, "\wr"] \\ \rho(M) \otimes_{\rho(A)} \rho(N) \arrow[r] & \rho(M \otimes_A N) \end{tikzcd}$$

shows that (2.11.1) is an isomorphism.

Set $R := \rho(A)$ and for X in $\mathcal{A}$ let $\omega(X)$ be the R-module $\rho(X_A)$. By the construction of A, each X_A is of the form $A \otimes M_0$ with M_0 in $\langle 1, \bar{1} \rangle$. The morphism (2.11.1) : $\rho(X) \otimes_R \rho(Y) \to \rho(X \otimes Y)$ is therefore an isomorphism. For each short exact sequences Σ of $\mathcal{A}$ the sequence Σ_A is split. The sequence $\rho(\Sigma)$ therefore is too and in particular it is exact. The functor ω is the super fibered functor that was promised.

We move to the general case. Let $\mathcal{A}_1$ be the k-tensorial category of $\mathbb{Z}/2$-graded objects of $\mathcal{A}$, with the commutativity constraint $X \otimes Y \to Y \otimes X$ being, for X and Y homogeneous of degree n and m, that of $\mathcal{A}$ multiplied by $(-1)^{nm}$. By 0.5 (ii), the k-tensorial category $\mathcal{A}_1$ verifies also the hypotheses of 2.1. The object 1 in odd degree is an object $\bar{1}$ as in 2.9. The category $\mathcal{A}_1$ admits therefore a super fiber functor of the type seen. The rest can be had by restriction to $\mathcal{A}$, identifying it with the subcategory of even objects in $\mathcal{A}_1$. □

3. Formalism of super fiber functors

3.1. Let $\mathcal{A}$ and $\mathcal{T}$ be two k-tensorial categories verifying (2.1.1): all objects are of finite length. Let R be an algebra of $\operatorname{Ind}\mathcal{T}$. Recall that our algebras are supposed to be commutative and unital (2.2). A *fiber functor* ω of $\mathcal{A}$ over R is an exact $\otimes$-functor from $\mathcal{A}$ to the category with tensor product $\operatorname{Mod}R$ of R-modules. For $\mathcal{T} = (s\text{-Vect})$ (0.9), we recover the super fiber functors from 0.9.

According to 1.16, the $\omega(A)$s are dualizable R-modules. According to 1.14, they are therefore flat (2.3) and for each R-algebra R', the $\otimes$-functor $A \mapsto \omega'(A) := \omega(A) \otimes_R R'$ is exact (c.f. 2.3): it is a fiber functor over R'.

A *morphism* $f : F' \to F''$ of $\otimes$-functors is a morphism of functors making commutative the diagrams

$$\begin{tikzcd} F'(X) \otimes F'(Y) \arrow[r] \arrow[d, "\wr"] & F''(X) \otimes F''(Y) \arrow[d, "\wr"] \\ F'(X \otimes Y) \arrow[r] & F''(X \otimes Y) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F'(X) \otimes F'(Y) \arrow[r, equal] \arrow[d, "\wr"] & F''(X) \otimes F''(Y) \arrow[d, "\wr"] \\ F'(X \otimes Y) \arrow[r] & F''(X \otimes Y) \end{tikzcd}$$

Lemma 3.2. *Each morphism of $\otimes$-functors between fiber functors over R is an isomorphism.*

Proof. If $f : \omega' \to \omega''$ is a morphism,

$$f_X : \omega'(X) \to \omega''(X) \ \text{ and } \ f_{X^\vee} : \omega'(X^\vee) \to \omega''(X^\vee)$$

are contragradient and we apply Deligne (1990), 2.4. □

For ω a fiber functor from $\mathcal{A}$ over R, we continue to denote by ω the extension of ω to $\operatorname{Ind}\mathcal{A}$ that commutes with inductive limits. This is also a $\otimes$-functor.

Lemma 3.3. *For every X in $\operatorname{Ind}\mathcal{A}$, $\omega(X)$ is flat, and faithfully flat if $X \neq 0$.*

Proof. The module $\omega(X)$ is flat as it is a filtered inductive limit of dualizable and hence flat modules. If $X \neq 0$, X admits a non-zero subobject A in $\mathcal{A}$. Because $\omega(X/A)$ is flat, the short exact sequence $0 \to \omega(A) \to \omega(X) \to \omega(X/A) \to 0$ shows that the faithful flatness of X results from that of A. Because $A \neq 0$, $\mathrm{ev} : A^\vee \otimes A \to 1$ is an epimorphism and if $\omega(A) \otimes_R M = 0$, M is zero as it is a quotient of $\omega(A^\vee) \otimes_R \omega(A) \otimes_R M = \omega(A^\vee \otimes A) \otimes_R M$. □

3.4. For α and β two fiber functors from a category $\mathcal{C}$ into the category of dualizable R-modules, let $\Lambda(\alpha,\beta)$ be the coend of the following contravariant in X and covariant in Y bifunctor: $X, Y \mapsto \alpha(X)^\vee \otimes_R \beta(Y)$. By definition of "coend", we have morphisms

$$\alpha(X)^\vee \otimes_R \beta(X) \to \Lambda(\alpha,\beta). \tag{3.4.1}$$

For all morphisms $f : X \to Y$, the diagram

$$\begin{tikzcd} \alpha(Y)^\vee \otimes \beta(X) \arrow[r, "\beta(f)"] \arrow[d, "\alpha(f)^t"] & \alpha(Y)^\vee \otimes \beta(Y) \arrow[d, "(3.4.1)"] \\ \alpha(X)^\vee \otimes \beta(X) \arrow[r, "(3.4.1)"] & \Lambda(\alpha,\beta) \end{tikzcd} \tag{3.4.2}$$

is commutative and $\Lambda(\alpha,\beta)$ is universal for these properties. The data of the morphisms (3.4.1) is equivalent to that of morphisms

$$\beta(X) \to \alpha(X) \otimes_R \Lambda(\alpha,\beta) \tag{3.4.3}$$

and the commutativity of 3.4.2 is equivalent to the functoriality in X of (3.4.3).

For $\mathcal{C} = \mathcal{C}' \times \mathcal{C}''$, $\alpha = \alpha' \otimes_R \alpha''$ and $\beta = \beta' \otimes_R \beta''$, we have

$$\Lambda(\alpha' \otimes \alpha'', \beta' \otimes \beta'') = \Lambda(\alpha',\beta') \otimes_R \Lambda(\alpha'',\beta'').$$

For T a functor $\mathcal{D} \to \mathcal{C}$, we have a featured morphism

$$\lambda(\alpha \circ T, \beta \circ T) \to \Lambda(\alpha,\beta).$$

In particular, for $\mathcal{C}$ equipped with a bifunctor $T : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ and α,β functorial isomorphisms

$$\alpha(T(X,Y)) = \alpha(X) \otimes_R \alpha(Y) \tag{3.4.4}$$

and the same for β, we discover a product

$$\Lambda(\alpha,\beta) \otimes_R \Lambda(\alpha,\beta) \to \Lambda(\alpha,\beta). \tag{3.4.5}$$

If T is under compatible associativity, commutativity, and unity constraints (3.4.4), this product is associative, commutative, and unital.

Lemma 3.5. *If α and β are two fibter functors from $\mathcal{A}$ over R, the R-module $\Lambda(\alpha,\beta)$ is faithfully flat.*

Proof. The category $\mathcal{A} \otimes_k \mathcal{A}$ from Deligne (1990), 5.1 and 5.13 is k-tensorial and verifies also (2.1.1) (ibid. 5.17). Denote by $\otimes_k$ the structural functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A} \otimes_k \mathcal{A}$. The functor $\alpha(X) \otimes_R \beta(Y)$ is exact in each variable, because $\alpha(X)$ and $\beta(Y)$ are flat. It defines therefore an exact functor (ibid. 5.17 (vi)) from $\mathcal{A} \otimes_k \mathcal{A}$ to R-modules, which will be denoted $\alpha \boxtimes_R \beta$, characterized by

$$\alpha \boxtimes_R \beta(X \otimes_k Y) = \alpha(X) \otimes_R \beta(Y).$$

The structure of $\otimes$-functor of α and β furnishes one over $\alpha \boxtimes_R \beta$, extending the isomorphism

$$\begin{aligned} (\alpha \boxtimes_R \beta)(X' \otimes_k Y') \otimes (\alpha \boxtimes_R \beta)(X'' \otimes Y'') &= \alpha(X') \otimes_R \beta(Y') \otimes_R \alpha(A'') \otimes_R \beta(Y'') \\ &= \alpha(X') \otimes_R \alpha(X'') \otimes_R \beta(Y') \otimes_R \beta(Y'') = \alpha(X' \otimes X'') \otimes_R \beta(Y' \otimes Y'') \\ &= \alpha \boxtimes_R \beta((X' \otimes X'') \otimes_k (Y' \otimes Y'')) = \alpha \boxtimes_R \beta((X' \otimes_k Y') \otimes (X'' \otimes_k Y'')), \end{aligned}$$

and makes $\alpha \boxtimes_R \beta$ a fiber functor from $\mathcal{A} \otimes_k \mathcal{A}$ over R. $\square$

Let inj_1 and inj_2 be the functors $X \mapsto X \otimes_k 1$ and $X \mapsto 1 \otimes_k X$ from $\mathcal{A}$ to $\mathcal{A} \otimes_k \mathcal{A}$. These are fiber functors from $\mathcal{A}$ into $\mathcal{A} \otimes_k \mathcal{A}$, i.e. over the algebra 1 in $\mathcal{A} \otimes_k \mathcal{A}$. Set

$$\Lambda_0 := \Lambda(\mathrm{inj}_1, \mathrm{inj}_2) \tag{3.5.1}$$

This Λ_0 is universal in the sense that

$$\Lambda(\alpha, \beta) = \alpha \boxtimes_R \beta(\Lambda_0),$$

and according to 3.3 applied to $\alpha \boxtimes_R \beta$, 3.5 results in

Lemma 3.6. $\Lambda_0 \neq 0$.

Proof. Let T be the fiber functor of $\mathcal{A} \otimes_k \mathcal{A}$ over $\mathcal{A}$ such that $T(X \otimes_k Y) = X \otimes Y$. If we apply it to Λ_0, we obtain the coend $\Lambda(\mathrm{Id}_{\mathcal{A}}, \mathrm{Id}_{\mathcal{A}})$ of the bifunctor $X, Y \mapsto X^\vee \otimes Y$ and the morphisms $\mathrm{ev} : X^\vee \otimes X \to 1$ furnish

$$T\Lambda_0 \to 1. \tag{3.6.1}$$

The object 1 of $\mathcal{A}$ furnishes a morphism (3.4.1)

$$1 \to \Lambda_0 \tag{3.6.2}$$

and (3.6.1) $\circ\, T($(3.6.2)$)$ is the identity on 1. That $\Lambda_0 \neq 0$ follows. ☐

Remark 3.7. The construction 3.4, applied to α and β, make $\Lambda(\alpha, \beta)$ into an R-algebra. Applied to $\mathrm{inj}_1, \mathrm{inj}_2 : \mathcal{A} \to \mathcal{A} \otimes_k \mathcal{A}$, it makes $\Lambda_0 := \Lambda(\mathrm{inj}_1, \mathrm{inj}_2)$ into an algebra in $\mathrm{Ind}(\mathcal{A} \otimes_k \mathcal{A})$. The isomorphism $\alpha \boxtimes_R \beta(\Lambda) \xrightarrow{\sim} \Lambda(\alpha, \beta)$ is an isomoprhism of R-algebras.

3.8. The R-algebra structure on the R-module $\Lambda := \Lambda(\alpha, \beta)$ is characterized by the commtuativity of the diagram

$$\begin{tikzcd}
\beta(X) \otimes \beta(Y) \arrow[r] \arrow[dd, equal] & (\alpha(X) \otimes \Lambda) \otimes (\alpha(Y) \otimes \Lambda) \arrow[d, equal] \\
& \alpha(X \otimes Y) \otimes (\Lambda \otimes \Lambda) \arrow[d] \\
\beta(X \otimes Y) \arrow[r] & \alpha(X \otimes Y) \otimes \Lambda
\end{tikzcd} \tag{3.8.1}$$

If α_Λ and β_Λ are the fiber functors over Λ built from α and β by extension of scalars from R to Λ, (3.4.3) defines a functorial morphism of Λ-modules

$$\varphi : \beta_\Lambda(X) \to \alpha_\Lambda(X) \tag{3.8.2}$$

and (3.8.1) expresses that this is a morphism of $\otimes$-functors, therefore and isomorphism (3.2).

The R-algebra Λ and φ are universal: if Λ_1 is an R-algebra and $\varphi_1 : \beta_{\Lambda_1} \to \alpha_{\Lambda_1}$ is a morphism of functors, φ_1 defines a functorial morphism

$$\beta(X) \to \alpha(X) \otimes_R \Lambda_1,$$

necessarily built from $f : \Lambda \to \Lambda_1$, and f is a homomorphism of algebras if and only if φ_1 is a morphism of $\otimes$-functors.

3.9. We pass to the geometric language of Deligne (1990), 7.5. Point of departure: we define the category of *affine $\mathcal{T}$-schemes* as the dual to the category of algebras of $\mathrm{Ind}\,\mathcal{T}$, we denote by $\mathrm{Spec}(A)$ the affine $\mathcal{T}$-scheme corresponding to an algebra A, and we call A its *affine algbera*, we say *module* over $\mathrm{Spec}(A)$ for A-module, *fiber functor over* $\mathrm{Spec}(A)$ for fiber functor over A, $\mathrm{Spec}(A)$*-scheme* or scheme over $\mathrm{Spec}(A)$ for A-algebra and *inverse image* (of modules) for the extension of scalars.

A *groupoid* acting on a $\mathcal{T}$-scheme $S = \mathrm{Spec}(A)$ is a $\mathcal{T}$-scheme H provied with "source" and "target" functions $s, b : H \to S$ and an associative composition law $H \times_{s,S,b} H \to H$ admitting units and inverses. It is the same to say that for each $\mathcal{T}$-scheme T, s, b and the composition law make the collection $H(T) := \mathrm{Hom}(T, H)$ of T-points of H into a groupoid acting over $S(T)$. The groupoid H is called *transitive* if $(b, s) : H \to S \times S$ is faithfully flat, i.e. if the affine algebra of H is faithfully flat over that, $A \otimes A$, of $S \times S$.

If H is a groupoid acting over S, with $c, d \in S(T)$ and $h \in H(T)$, we write $h : c \to d$ for "h has source c and target d". If M is a module over S, the data of $\varphi : s^*M \to b^*M$ is equivalent to the data for each T and each $h : c \to d$ over T of $\varphi_h : c^*M \to d^*M$, compatible with base changes $T' \to T$. In this language, an *action* of H on M is a morphism $\varphi : s^*M \to b^*M$, such that for $\mathrm{id}(a) : a \to a$ the identity of a, $\varphi_{\mathrm{id}(a)}$ is the identity on a^*M, and for $c \xrightarrow{f} d \xrightarrow{g} e$, we have $\varphi_{gt} = \varphi_g \varphi_f$. It suffices to verify these conditions in the universal case, over S and $H \times_S H$ respectively.

3.10. Translating 3.8: the scheme $\mathrm{Spec}(\Lambda(\alpha, \beta))$ over $S = \mathrm{Spec}(R)$ represents the functor that to an S-scheme T associates the collection of isomorphisms of $\otimes$-functors $\beta_T \to \alpha_T$ between the inverse images of α and β over T. This justifies the notation $\mathcal{I}som^{\otimes}_S(\beta, \alpha) := \mathrm{Spec}(\Lambda(\alpha, \beta))$.

If α (resp. β) is a fiber functor over S (resp. T), α and β furnish by extension of scalars fiber functors $\mathrm{pr}_1^*\alpha$ and $\mathrm{pr}_2^*\beta$ over $S \times T$. We set

$$\mathcal{I}som^{\otimes}_{S\times T}(\beta, \alpha) := \mathcal{I}som^{\otimes}_{S\times T}(\mathrm{pr}_2^*\beta, \mathrm{pr}_1^*\alpha) \tag{3.10.1}$$

For three fiber functors α, β, γ over S, T, U, the composition of isomorphisms furnishes a morphism of $S \times U$-schemes:

$$\mathcal{I}som^{\otimes}_{S\times T}(\beta, \alpha) \times_T \mathcal{I}som^{\otimes}_{T\times U}(\gamma, \beta) \to \mathcal{I}som^{\otimes}_{S\times U}(\gamma, \alpha) \tag{3.10.2}$$

and for α, γ, δ over S, T, U, V the morphisms (3.10.2) verify an associativity.

Particular case: let ω be a fiber functor over S and set

$$(\alpha, S) = (\beta, T) = (\gamma, U) = (\delta, V) := (\omega, S).$$

The $S \times S$-scheme $q : \mathcal{I}som^{\otimes}_{S\times S}(\omega) \to S \times S$, equipped with source $s := \mathrm{pr}_2 q$ and target $b := \mathrm{pr}_1 q$ and with the composition law (3.10.2) is a groupoid acting over S. By construction, the morphism (3.8.2) for $\alpha = \mathrm{pr}_1^*\omega$ and $\beta = \mathrm{pr}_2^*\omega$,

$$\varphi : s^*\omega \to b^*\omega,$$

is an action of the groupoid $I := \mathcal{I}som^{\otimes}_{S\times S}(\omega, \omega)$ over $\omega(X)$, functorial in X and compatible with the tensor product. The groupoid I is universal for these properties.

4. Existence of super fiber functors over k

Proposition 4.1. *Let $\mathcal{A}$ be a k-tensorial category of finite $\otimes$-generation. If α and β are two super fiber functors from $\mathcal{A}$ over a super scheme S, the super scheme $\mathcal{I}som^{\otimes}_S(\alpha, \beta)$ is of finite presentation over S.*

Proof. We may suppose, and we will suppose, that S is affine non empty: $S = \operatorname{Spec}(R)$ with $R \neq 0$. Let m be a maximal ideal in R and S_0 the spectrum of the field $k_0 := R/m$. By extension of scalars of S to S_0, we deduce from α (or β, regardless) a fiber functor γ over S_0. If X is a $\otimes$-generator of $\mathcal{A}$, $\mathcal{I}som^{\otimes}_{S_0}(\gamma,\gamma)$ is a subscheme in groups of $\mathrm{GL}(\gamma(X))$, therefore is of finite type over S_0. Because S_0 is noetherian, this is the same as finite presentation over S_0.

The super schemes $\mathcal{I}som^{\otimes}_{S\times S_0}(\mathrm{pr}_1^*\alpha, \mathrm{pr}_2^*\gamma)$ and $\mathcal{I}som^{\otimes}_{S\times S_0}(\mathrm{pr}_1^*\beta, \mathrm{pr}_2^*\gamma)$ are faithfully flat over $S \times S_0$ (3.5). Let J be their fiber product. It is faithfully flat over $S \times S_0$ and therefore over S. The inverse images over J of α, β and γ are isomorphic, and the inverse image over J of the S-super scheme $\mathcal{I}som^{\otimes}_{S}(\alpha,\beta)$ is of finite presentaiton, being isomorphic to the inverse image over J of the S_0-super scheme of finite presentation $\mathcal{I}som^{\otimes}_{S_0}(\gamma,\gamma)$. By faithfully flat descent of finite presentation (cf. SGA1, VII, 3.4 and 1.10), $\mathcal{I}som^{\otimes}_{S}(\alpha,\beta)$ if of finite presentation over S_0. □

Remark 4.2. Let $\mathcal{T}_0$ be the tensorial category of super vector spaces of finite dimension over the field k_0. The proof of 4.1 makes use of the following property of $\mathcal{T}_0$:

(4.2.1) for X in $\mathcal{T}_0$, the algebra $\operatorname{Sym} X$ is noetherian: for each ideal of $\operatorname{Sym} X$, there exists N so as to be generated by its trace over $\bigoplus_{i\leq N} \operatorname{Sym}^i X$.

This property is not true for all k-tensorial categories $\mathcal{A}$ verifying (2.1.1). Let indeed Y be in $\mathcal{A}$ and E the $\mathcal{A}$-scheme that represents the functor $T \mapsto \operatorname{End}(Y_T)$. This is the spectrum of the algebra $\operatorname{Sym}((\mathcal{E}nd\, Y)^\vee)$. Let $\mathcal{A}_1$ be the category of $\mathbb{Z}/(2)$-graded objects of $\mathcal{A}$ considered at the end of 2.11. THe proof of 2.9 shows that if Y is not annihilated by any Schur functor, then, in $\mathcal{A}_1$, Y admits $1^p + \bar{1}^q$ locally as a direct factor, with $p+q$ arbitrarily large. It results that, regardless of n, Y admits locally an endomorphism u such that $\operatorname{Tr}(u^{n+1}) = 1$. This remains true after passing to $\mathcal{A}_1$: for $A = A^+ + A^-$ and algebra of $\mathcal{A}_1$, $u : Y_A \to Y_A$ comes from $u_0 : Y \to Y_A$ that, because Y is in $\mathcal{A}$, factors through Y_{A^+}: U comes by extension of scalars from an endomorphism of Y_{A^+}. Let F be the subsceme of E defined by the equations $\operatorname{Tr}(u^i) = 0$, $(i \geq 1)$. We have just seen that a finite number of these equations does not suffice to define it. The algebra $\operatorname{Sym}((\mathcal{E}nd\, Y)^\vee)$ therefore is not noetherian.

As an example of a k-tensorial category verifying (2.1.1) but not (4.2.1), we can therefore take, for t transcendental, the category $\operatorname{Rep}(\mathrm{GL}_t)$ of Deligne–Milne (1982), 1.27: its natural generator is not annihilated by any Schur functor.

By a standard argument of passage to the limit we deduce from 4.1

Corollary 4.3. *Let $\mathcal{A}$ be a k-tensorial category of finite $\otimes$-generation and ω a super fiber functor of $\mathcal{A}$ over the super commutative k-algebra R, a filtered inductive limit of R_αs. Set $S = \operatorname{Spec}(R)$, $S_\alpha = \operatorname{Spec}(R_\alpha)$. For α sufficiently large, the groupoid $I := \mathcal{I}som^{\otimes}_{S\times S}(\omega,\omega)$ acting over S comes from change of base from a groupoid I_α acting over S_α, of finite presentation over $S_\alpha \times S_\alpha$.*

Because I is transitive, the super analog of EGA 1V (3rd part) 11.2.6.1 (pass to the limit for flattness) shows that for convenient $\beta \geq \alpha$, the groupoid I_β acting over S_β as built from I_α by change of base is transitive. Below, a small detour will permit us to use EGA IV, 11.2.6.1 unchanged, without superising.

An argument of faithfully flat descent shows as in Deligne (1990), 3.5.1 that

Lemma 4.4. *Let $f : S \to T$ be a morphism of super schemes over k. If a groupoid I acting over S comes by base change from $S \times S \to T \times T$ of a transitive groupoid J acting over T, the base change of T to S induces an equivalence of the category of T-modules equipped with an action of J and that of S-modules equipped with an action of I.*

Proposition 4.5. *Let $\mathcal{A}$ be a k-tensorial category of finite $\otimes$-generation. If $\mathcal{A}$ admits a super fiber functor ω over a super commutative k-algebra $R \neq 0$, then $\mathcal{A}$ admits a fiber functor over k.*

Proof. Let N be the ideal of R generated by the odd part of R. This is a nilpotent ideal and therefore distinct from R if $R \neq 0$. Replacing R be R/n and ω by the fiber functor build from ω by extension of scalars, we can suppose, and we do suppose, that R is purely even.

The algebra R is the filtered inductive limit of its sub-k-algebras of finite type R_α. With the notations from 4.3, the groupoid I acting over $S = \mathrm{Spec}(R)$ comes for R_α sufficiently large from I_α acting over $S_\alpha = \mathrm{Spec}(R_\alpha)$ and of finite presentation over $S_\alpha \times S_\alpha$. For I_α to be flat over $S_\alpha \times S_\alpha$, it is necessary and sufficient that the even and odd components $\mathcal{O}(I_\alpha)^+$ and $\mathcal{O}(I_\alpha)^-$ of $\mathcal{O}(I_\alpha)$ are flat over $R_\alpha \otimes R_\alpha$. The first is a commutative $R_\alpha \otimes R_\alpha$-algebra of finite type, the second an $\mathcal{O}(I_\alpha)^+$-module of finite type. For I_α to be transitive, it is necessary and sufficient that moreover $\mathcal{O}(I_\alpha)^+$ is faithfully flat over $R_\alpha \otimes R_\alpha$.

According to EGA IV, 11.2.6.1, for convenient $R_\beta \supset R_\alpha$, the groupoid I_β acting over $S_\beta = \mathrm{Spec}(R_\beta)$ built from I_α by base change is transitive. According to 4.4, the functor of extension of scalars from R_β to R induces an equivalence of the category of super modules over S_β equipped with an action of I_β with that of super modules over S equipped with an action of I. Applying this to $\omega(X)$, for X in $\mathcal{A}$ we obtain that the fiber functor ω comes from a fiber functor ω_β over R_β. Because R_β is of finite type over k and non-zero, there exists a homomorphism χ from R_β to k and, $\omega_\beta \otimes_{R)\beta,\chi} k$ is a fiber functor over k. □

4.6. End of the proof of 0.5 and 0.6. If the k-tensorial category $\mathcal{A}$ verifies the condition (a) of 0.5 (i), it admits a super fiber functor over convenient R (2.1), therefore over k (4.5) and according to Deligne (1990), 8.19 is of the form $\mathrm{Rep}(G, \epsilon)$.

It remains to verify that the conditions (a) and (b) of 0.5 (i) are equivalent, and we already know that (b)⇒(a) (1.20). If the condition (a) is verified, for each object X of $\mathcal{A}$, the plain subcategory of $\mathcal{A}$ $\otimes$-generated by X also verifies (a) and is finitely $\otimes$-generated. It admits therefore a fiber functor ω. If $\omega(X)$ is of super dimenion $p|q$, that ω is exact and faithful ensures that $X^{\otimes d}$ is the length at most that, $(p+q)^d$, of $\omega(X^{\otimes d}) = \omega(X)^{\otimes d}$. This verifies (b).

4.7. Proof of 0.7 and 0.8. The assertion 0.7 results from

Lemma 4.8. *Let $\mathcal{A}$ be a k-tensorial category all of whose objects are of finite length. If $\mathcal{A}$ has a finite number of isomorphism classes of simple objects, then for each X in $\mathcal{A}$ there exists N such that*

$$\mathrm{length}(X^{\otimes n}) \leq N^n. \tag{4.8.1}$$

Proof. Let $K(\mathcal{A})$ be the Grothenieck group of the abelian category $\mathcal{A}$. If $(S_i)_{i \in I}$ is a system of representatives of the isomorphism classes of simple objects, $K(\mathcal{A})$ is the free $\mathbb{Z}$-module with basis the $[S_i]$s. Let $\ell : K(\mathcal{A}) \to \mathbb{Z}$ be the function "sum of the coordinates". For X in $\mathcal{A}$, we have

$$\mathrm{length}(X) = \ell([X]).$$

The tensor product of $\mathcal{A}$ makes $K(\mathcal{A})$ into a commutative ring, and (4.8.1) can be written as

$$\ell([X]^n) \leq N^n.$$

The matrix of multiplication by $[X]$ has coefficients integers ≥ 0. If they are $\leq a$, we have $\ell([X][Y]) \leq |I|\, a\, \ell(Y)$ and $\ell([X]^n) \leq (|I|\, a)^n$. □

Proof of 0.8. Suppose further that $\mathcal{A}$ is semisimple, and let ω by a fiber functor from $\mathcal{A}$ over k. Let X be the sum of the S_is. Each object of $\mathcal{A}$ is a subobject of a sum of copies of X, the canonical function from $\omega(X) \otimes \omega(X)^\vee$ to $\Lambda(\omega, \omega)$ is surjective, and $\Lambda(\omega, \omega)$ is therefore finite dimensional.

The k-tensorial category $\mathcal{A}$ is of the form $\mathrm{Rep}(G, \epsilon)$ with $\mathcal{O}(G) = \Lambda(\omega, \omega)$, of finite dimension. Divide the affine algebra $\mathcal{O}(G)$ of G by the ideal generated by its odd part. We obtain the affine algebra of the algebraic subgroup G_{red} of G, necessarily finite.

Let G^0 be the connected component of G. Because G^0_{red} is trivial, the Lie algebra of G^0 is purely odd. This determines $G^0 : \mathcal{O}(G^0)$ is the dual to the enveloping algebra of the commutative super Lie algebra $\mathrm{Lie}(G^0)$.

Because G^0 is invariant under G, the restriction to G^0 of a semisimple super representation of G is semisimple, therefore trivial: ω induces an equivalence of $\mathcal{A}$ with $\mathrm{Rep}(G/G^0, \epsilon)$. □

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