EXACT TRIANGLES AS MODULES OVER AN A_∞-CATEGORY

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Abstract. This note describes a strictly-unital A_∞-category whose representations are exact triangles such that the three-fold symmetry on exact triangles is manifest on the A_∞-category.

This brief note answers the following exercise:

Exercise. *Present the notion of "exact triangle" as a representation of an algebraic object in a way that makes three-fold cyclicity manifest.*

Here is a more explicit version of the exercise. One of the most important tools with which to study a derived category $\mathcal{C}$ is the derived category $\mathcal{C}^\triangle$ of exact triangles in $\mathcal{C}$. It is for many purposes technically convenient to realize $\mathcal{C}^\triangle$ as the category of representations in $\mathcal{C}$ of some algebraic object. This is straightforward: the axioms of triangulated categories guarantee that every exact triangle $X \to Y \to Z \to X[1]$ in $\mathcal{C}$ is determined up to a contractible space of choices by the morphism $X \to Y$, and so $\mathcal{C}^\triangle$ is equivalent to the category $\mathcal{C}^\to$ of representations in $\mathcal{C}$ of the A_2 quiver $\mathcal{Q} = \{\bullet \to \bullet\}$. Thus, for example, if $\mathcal{C}$ is the derived category of modules for an algebra $\mathcal{A}$, then $\mathcal{C}^\to$, and hence $\mathcal{C}^\triangle$, is the derived category of modules for the algebra $\mathcal{A} \otimes \mathcal{Q}$, by which I mean the path algebra of $\mathcal{Q}$ with coefficients in $\mathcal{A}$.

The problem with thinking of $\mathcal{C}^\triangle$ in terms of the quiver $\mathcal{Q} = \{\bullet \to \bullet\}$ is that it obscures an important symmetry of $\mathcal{C}^\triangle$. The shift functor $X \mapsto X[1]$ on $\mathcal{C}$ has, when extended to $\mathcal{C}^\triangle$, a cube root given by rotating the triangle $X \to Y \to Z \to X[1]$ to the triangle $Y \to Z \to X[1] \to Y[1]$. The quiver $\mathcal{Q}$ has no such symmetry. Thus the exercise is to find an object $\mathcal{T}$ with a manifest three-fold symmetry whose representations in $\mathcal{C}$ are exact triangles.

One answer is described in [Kon09], in which the algebraic object is a strict quasifree dg category on 12 generating morphisms. (A dg algebraic object is *quasifree* if its underlying graded object is free, and its differential is well-behaved with respect to a well-ordering on the generators.) Here, instead, is a finite-dimensional (but neither strict nor quasifree) answer to the exercise; the same answer also appears in [Sei08, Nad13]:

Solution. *Let $\mathcal{T}$ denote the strictly-unital A_∞-category with three objects X, Y, Z and the following morphism structure. The hom spaces are*

$$\hom(X,X) = \mathbb{Z}\,\mathrm{id}_X, \quad \hom(Y,Y) = \mathbb{Z}\,\mathrm{id}_Y, \quad \hom(Z,Z) = \mathbb{Z}\,\mathrm{id}_Z$$
$$\hom(X,Y) = \mathbb{Z}\,f, \quad \hom(Y,Z) = \mathbb{Z}\,g, \quad \hom(Z,X) = \mathbb{Z}\,h$$
$$\hom(Y,X) = \hom(Z,Y) = \hom(X,Z) = 0$$

where f and g are of cohomological degree 0 and h is of cohomological degree $+1$. The differential vanishes as do the binary compositions $fg = gh = hf = 0$. (Here and throughout this note, I will write composition of morphisms in the "left to right" order, so that $fg : X \to Z$ is the composition $X \overset{f}{\to} Y \overset{g}{\to} Z$.) The unique non-trivial data in $\mathcal{T}$ are the ternary "associator" operations μ_3, which are set to be

$$\mu_3(f,g,h) = -\mathrm{id}_X, \quad \mu_3(g,h,f) = -\mathrm{id}_Y, \quad \mu_3(h,f,g) = -\mathrm{id}_Z.$$

Then representations of $\mathcal{T}$ are exact triangles.

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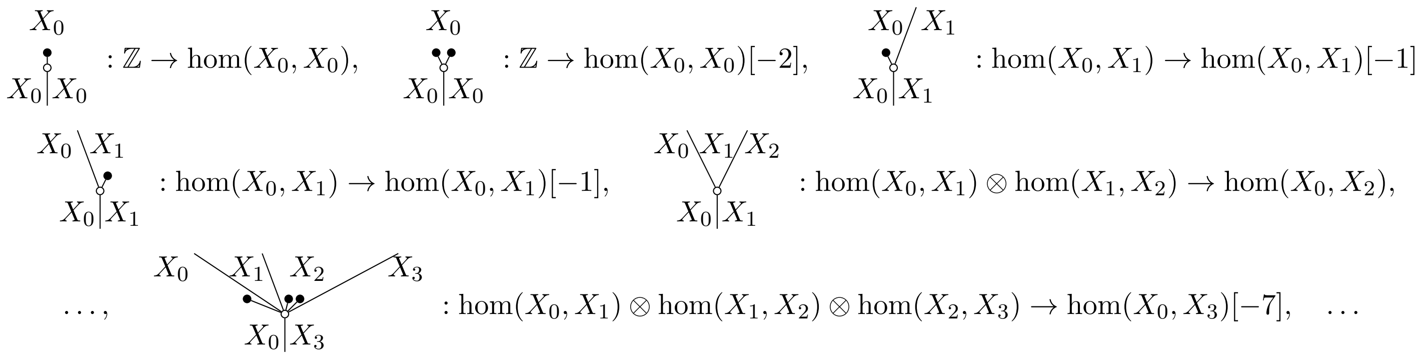
I will give the proof momentarily, and then describe two examples. First, four remarks are in order:

Remark 1. The notion of *(homotopy) unital A_∞-category* is due to [FOOO09] and consists of the following data:

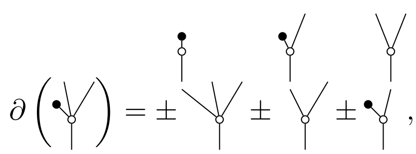
(1) A set of *objects* and for each pair of objects (X, Y) a cochain complex $\hom(X, Y)$ of *morphisms* from X to Y.

(2) Operations indexed by planar "tagged vertices" with two-dimensional regions labeled by objects (I will denote the vertex itself by an open circle and the tags by solid dots):

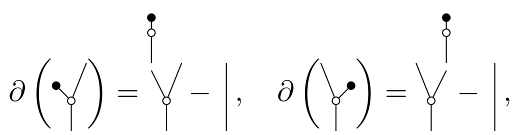


The rules are that, other than the operation , every vertex has at least two (tags + input edges), and that the operation indexed by a vertex with n "input edges" and m "tags" has domain $\hom(X_0, X_1) \otimes \hom(X_1, X_2) \otimes \cdots \otimes \hom(X_{n-1}, X_n)$ and codomain $\hom(X_0, X_n)$ and is of cohomological degree $2 - n - 2m$.

(3) The operations must satisfy a sequence of differential relations. For most of these, the relation is of the form $\partial(\text{operation}) = \sum \pm(\text{composition of two operations})$ where the signs depend on a choice of conventions and the composition ranges over ways to split the vertex into two. For example,



where the small gap denotes composition. In particular, the operations and , being unsplittable, are ∂-closed. The only exceptions to the above relation are the operations with precisely one tag and one input. These satisfy instead



where the unmarked line indicates the identity map on $\hom(X_0, X_1)$.

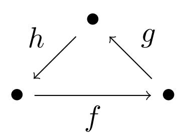
Restricting just to the untagged vertices $\mu_2 =$, $\mu_3 =$, etc., gives the structure of a usual (nonunital) A_∞-category. The zero-ary operation should be understood as the inclusion of a homotopy-unit; the operations and are right and left unitors; the remaining operations are homotopy coherences for these.

A unital A_∞-category is *strictly unital* if, other than , the operations with at least one tag vanish identically. More details are available in [HM12], where it is shown that the above notion of "unital A_∞-category" provides a homotopy-coherent version of the notion of "unital dg category" (their argument is worded in terms of algebras rather than categories, but extends verbatim to categories with a given set of objects). By definition, a *representation* of a unital A_∞-category $\mathcal{R}$ in some unital A_∞-category $\mathcal{V}$ is a unital A_∞-morphism (also called "∞-morphism") $\mathcal{R} \to \mathcal{V}$. $\diamond$

Remark 2. A simple "cohomological degree" argument verifies that no other unital A_∞ operations on $\mathcal{T}$ can be non-zero. $\diamond$

Remark 3. The cohomological degrees of the morphisms f, g, h are arbitrary except for the condition that their sum is $\deg(f) + \deg(g) + \deg(h) = 1$. Indeed, any other assignment of degrees satisfying this condition will produce an A_∞-category canonically Morita-equivalent to $\mathcal{T}$. Thus rotating $X \mapsto Y \mapsto Z \mapsto X$ and $f \mapsto g \mapsto h \mapsto f$ is a three-fold symmetry of $\mathcal{T}$ up to shuffling of degrees. $\diamond$

Remark 4. I think of the A_∞-category $\mathcal{T}$ as a "homotopical enhancement" of the $\tilde{A}_2$ quiver



with potential $W = fgh$. In usual quivers-with-potential, the partial derivatives of the potential provide the relations in the quiver, thereby giving a noncommutative version of critical loci [DWZ08]. In $\mathcal{T}$, the potential, of total cohomological degree $+1$, provides both the relations and also the associator μ_3. The usual quiver-with-potential $(\tilde{A}_2, fgh)$, without this homotopical enhancement, is the strict category $\mathrm{H}^\bullet(\mathcal{T})$. I suggest that one should look more generally for homotopical enhancements of quivers with relations.

One place critical loci of potentials of degree 0 occur is in the Batalin–Vilkovisky formalism in quantum field theory; critical loci of potentials of degree 1 occur in the Batalin–Fradkin–Vilkovisky formalism for boundary conditions in quantum field theory [CMR14]. I don't know if this is pure coincidence or should be taken seriously. $\diamond$

I will now prove, as claimed in my Solution, that representations of $\mathcal{T}$ are exact triangles. The proof will also show that the rotation of exact triangles is implemented by the manifest three-fold symmetry of $\mathcal{T}$ from Remark 3.

Proof of Solution. Since the category $\mathcal{C}^\triangle$ of exact triangles in $\mathcal{C}$ is equivalent to the category $\mathcal{C}^\to$ of representations of the A_2-quiver $\mathcal{Q} = \{\bullet \to \bullet\}$, it suffices to show that $\mathcal{Q}$ and $\mathcal{T}$ are unital-A_∞-Morita equivalent. For this, it suffices to find a collection M of three compact $\mathcal{Q}$-modules (among abelian groups) that together generate the category of $\mathcal{Q}$-modules and such that the derived endomorphism category of M is equivalent to $\mathcal{T}$ [LH03, Section 7.6]. Here by "compact" I mean that the *derived* hom functor $\mathrm{Rhom}(M, -)$ commutes with *homotopy* filtered colimits. By "derived endomorphism category" I mean the category whose objects are the elements of M and whose morphism spaces are the derived hom spaces between elements of M.

I will take M to consist of the three basic indecomposable $\mathcal{Q}$-modules:

$$P = \{0 \overset{0}{\to} \mathbb{Z}\}, \quad E = \{\mathbb{Z} \overset{1}{\to} \mathbb{Z}\}, \quad I = \{\mathbb{Z} \overset{0}{\to} 0\}$$

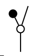
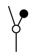
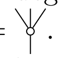
The module P is projective, the module I is injective, and the module E is both projective and injective and is an extension of I by P. Since P and E are projective, derived mapping spaces $\mathrm{Rhom}(P, -)$ and $\mathrm{Rhom}(E, -)$ are computed by the corresponding strict mapping spaces; this in particular shows that P and E are compact. A projective resolution of I is

$$\tilde{I} = \big\{\mathbb{Z} \overset{(0,1)}{\to} (\mathbb{Z}[1] \oplus \mathbb{Z}, \partial = 1)\big\},$$

where $(\mathbb{Z}[1] \oplus \mathbb{Z}, \partial = 1)$ denotes the graded $\mathbb{Z}$-module $\mathbb{Z}[1] \oplus \mathbb{Z}$ with differential the isomorphism $\mathbb{Z} \overset{\sim}{\to} \mathbb{Z}$ and $(0, 1)$ denotes inclusion into the degree-zero summand. Thus $\mathrm{Rhom}(I, -)$ can be computed as the strict dg mapping space from $\tilde{I}$, showing also that I is compact. It is clear that the objects P, E, and I together generate the category of $\mathcal{Q}$-modules, since for example just P and E together do.

Let $i : P \to E$ denote the inclusion and $p : E \to I$ the projection, and let $e \in \mathrm{Rhom}(I, P)$ denote the degree-1 element that classifies the extension. An easy calculation shows that there are quasiisomorphisms

$$\mathrm{Rhom}(P,P) \simeq \mathbb{Z}\,\mathrm{id}_P, \quad \mathrm{Rhom}(E,E) \simeq \mathbb{Z}\,\mathrm{id}_E, \quad \mathrm{Rhom}(I,I) = \mathbb{Z}\,\mathrm{id}_I,$$
$$\mathrm{Rhom}(P,E) \simeq \mathbb{Z}\,i, \quad \mathrm{Rhom}(E,I) \simeq \mathbb{Z}\,p, \quad \mathrm{Rhom}(I,P) \simeq \mathbb{Z}\,e,$$
$$\mathrm{Rhom}(E,P) \simeq \mathrm{Rhom}(I,E) \simeq \mathrm{Rhom}(P,I) \simeq 0.$$

What, therefore, can be the endomorphism A_∞-category of M? Clearly the units  are $\mathrm{id}_P, \mathrm{id}_E, \mathrm{id}_I$, and for degree reasons all unitors  and  must vanish. The only nonzero binary multiplications must be those involving units. For degree reasons all higher unital A_∞-operations much vanish except possibly the associators $\mu_3 =$ .

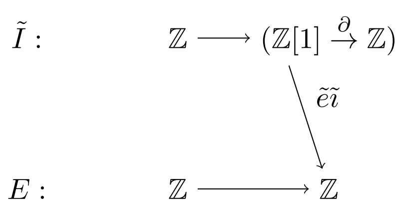
Thus to prove the claim it remains to calculate these associators. For degree reasons, the only possible nontrivial associators are $\mu_3(i,p,e) \in \mathrm{Rhom}(P,P)$, $\mu_3(p,e,i) \in \mathrm{Rhom}(E,E)$, and $\mu_3(e,i,p) \in \mathrm{Rhom}(I,I)$. Since $ip = pe = ei = 0$, the associators we're after are nothing but Massey products for the strict dg category of endomorphisms of the projective resolution $\tilde{M} = \{P, E, \tilde{I}\}$ of M [Sta70]. Let $\tilde{\imath} = i : P \to E$ denote the inclusion thought of as an endomorphism of $\tilde{M}$. Let $\tilde{p} : E \to \tilde{I}$ denote the inclusion; it represents p in cohomology. Let $\tilde{e} : \tilde{I} \to P$ denote the projection; it represents the class e.

Then $\tilde{p}\tilde{e} = 0 : E \to P$. Since the space of maps from P to itself is a copy of $\mathbb{Z}$, the unique primitive of $\tilde{p}\tilde{e}$ is $\partial^{-1}(\tilde{p}\tilde{e}) = 0$. On the other hand, $\tilde{\imath}\tilde{p} : P \to \tilde{I}$ is the degree-0 inclusion. It has a unique primitive $\partial^{-1}(\tilde{\imath}\tilde{p})$ in the cochain complex of maps from P to $\tilde{I}$, namely the inclusion of degree-(-1). By definition, the Massey product $\mu_3(i,p,e)$ is the class of the closed element $\tilde{\imath}\partial^{-1}(\tilde{p}\tilde{e}) - \partial^{-1}(\tilde{\imath}\tilde{p})\tilde{e}$; this element is closed because the endomorphism dg category of $\tilde{M}$ is strictly associative. But

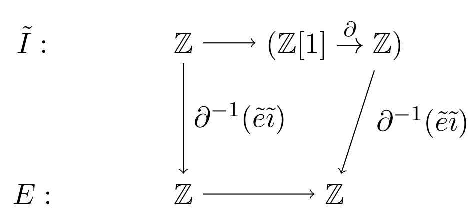
$$\tilde{\imath}\partial^{-1}(\tilde{p}\tilde{e}) - \partial^{-1}(\tilde{\imath}\tilde{p})\tilde{e} = -\partial^{-1}(\tilde{\imath}\tilde{p})\tilde{e} = -\mathrm{id}_P,$$

and so $\mu_3(i,p,e) = -\mathrm{id}_P$.

For the remaining two Massey brackets, note also that the composition $\tilde{e}\tilde{\imath} : \tilde{I} \to \tilde{E}$ is the degree-1 map



The unique primitive $\partial^{-1}(\tilde{e}\tilde{\imath})$ is the degree-0 map



Thus the remaining Massey products are the cohomology classes

$$\mu_3(p,e,i) = [-\tilde{p}\partial^{-1}(\tilde{e}\tilde{\imath}) + \partial^{-1}(\tilde{p}\tilde{e})\tilde{\imath}] = [-\tilde{p}\partial^{-1}(\tilde{e}\tilde{\imath})] = [-\mathrm{id}_E] = -\mathrm{id}_E,$$
$$\mu_3(e,i,p) = [-\tilde{e}\partial^{-1}(\tilde{\imath}\tilde{p}) - \partial^{-1}(\tilde{e}\tilde{\imath})\tilde{p}] = [-\mathrm{id}_{\tilde{I}}] = -\mathrm{id}_I.$$

My sign conventions for Massey products are those of [Val12], which provides a great survey of the relationship between Massey products and A_∞ operations. This completes the proof that the endomorphism category of $M = \{P,E,I\}$ is a copy of $\mathcal{T}$, proving that $\mathcal{T}$ and $\mathcal{Q}$ are Morita-equivalent.

Finally, note that rotating exact triangles in the category of $\mathcal{Q}$-modules in particular rotates the triangle $P \to E \to I \to P[1]$, and so acts on $\mathcal{T}$ via the manifest three-fold symmetry. □

I will end with two elementary examples:

Example 1. Let us fix a field $\mathbb{K}$ and look for $\mathbb{K}$-linear representations Φ of $\mathcal{T}$ in which the objects X, Y, Z all map to vector spaces $\Phi X, \Phi Y, \Phi Z$ in degree 0.

A full presentation of the notion of *unital A_∞-morphism* $\mathcal{T} \to \mathrm{VECT}_{\mathbb{K}}$ is available in [HM12]. It begins with an assignment to each morphism in $(T_1 \xrightarrow{t} T_2) \in \mathcal{T}$ of a morphism Φt between the image objects ΦT_1 and ΦT_2. The next piece of data is a homotopy between $\Phi \mathrm{id}_T$ and $\mathrm{id}_{\Phi T}$ for each object $T \in \mathcal{T}$. Since we are assuming that ΦX, ΦY, and ΦZ are all in degree zero, these homotopies must vanish, and $\Phi\mathrm{id}_T = \mathrm{id}_{\Phi T}$ for each $T = X, Y, Z$. Similarly, a unital A_∞-morphism includes higher homotopies interpolating between the image under Φ of a coherence relation in $\mathcal{T}$ and the corresponding coherence relation in the target category; in our case if such a homotopy involves units, i must also all vanish for degree reasons.

Thus the only nontrivial data in the values of Φ on morphisms consists of the linear maps $\Phi f : \Phi X \to \Phi Y$ and $\Phi g : \Phi Y \to \Phi Z$; the map Φh is of degree 1 and so must vanish. The representation Φ has a bit more data in it needed to impose coherent functoriality. First, for each pair (s,t) of composable morphisms in $\mathcal{T}$, we need a primitive $\Phi_2(s,t)$ of $\Phi(st) - \Phi s \circ \Phi t$. (As above, by "$\Phi s \circ \Phi t$" I mean the left-to-right composition $\bullet \xrightarrow{\Phi s} \bullet \xrightarrow{\Phi t} \bullet$.) Inspecting all possible composable pairs of morphisms in $\mathcal{T}$, we find only two for which $\Phi_2(s,t)$ is not forced to be zero for degree reasons: $\Phi_2(g,h) : \Phi Y \to \Phi X$ and $\Phi_2(h,f) : \Phi Z \to \Phi Y$.

Going further, for each triple (s,t,u) of composable morphisms in $\mathcal{T}$ the representation includes among its data a homotopy $\Phi_3(s,t,u)$ interpolating between various data already listed. Using the fact that we are mapping into a strict category, the homotopy $\Phi_3(s,t,u)$ must have boundary

$$\partial(\Phi_3(s,t,u)) = \Phi(\mu_3(s,t,u)) - (-1)^{\deg s + \deg t}\Phi s \circ \Phi_2(t,u) + (-1)^{\deg t}\Phi_2(s,t) \circ \Phi u.$$

On the other hand, for degree reasons all Φ_3s must vanish. Using the fact that $\Phi_2(f,g) = 0$ and that $\Phi\mathrm{id}_T = \mathrm{id}_{\Phi T}$, we are left with three nontrivial equations:

$$\partial(\Phi_3(f,g,h)) = 0 = -\mathrm{id}_{\Phi X} - \Phi f \circ \Phi_2(g,h) + 0$$
$$\partial(\Phi_3(g,h,f)) = 0 = -\mathrm{id}_{\Phi Y} + \Phi g \circ \Phi_2(h,f) - \Phi_2(g,h) \circ \Phi f$$
$$\partial(\Phi_3(h,f,g)) = 0 = -\mathrm{id}_{\Phi Z} + 0 + \Phi_2(h,f) \circ \Phi g$$

Equivalently, the inclusions $\Phi f : \Phi X \to \Phi Y$ and $\Phi_2(h,f) : \Phi Z \to \Phi Y$ and the projections $-\Phi_2(g,h) : \Phi Y \to \Phi X$ and $\Phi g : \Phi Y \to \Phi Z$ together write ΦY as a direct sum $\Phi X \oplus \Phi Z$. Thus representations of $\mathcal{T}$ among $\mathbb{K}$-vector spaces are precisely direct sums. ◇

Example 2. The first nontrivial extension met by most mathematicians is the extension of abelian groups $\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$. Let us find the corresponding $\mathcal{T}$-representation $\Psi : \mathcal{T} \to \mathrm{ABGP}$. Clearly

$$\Psi X = \mathbb{Z}/2, \quad \Psi Y = \mathbb{Z}/4, \quad \Psi Z = \mathbb{Z}/2,$$

and $\Psi f = i \in \mathrm{Rhom}^0(\mathbb{Z}/2,\mathbb{Z}/4) = \hom(\mathbb{Z}/2,\mathbb{Z}/4)$ and $\Psi g = p \in \mathrm{Rhom}^0(\mathbb{Z}/4,\mathbb{Z}/2) = \hom(\mathbb{Z}/4,\mathbb{Z}/2)$ are the obvious inclusion and projection.

A projective resolution of $\Psi Z = \mathbb{Z}/2$ is $(\mathbb{Z}[1] \oplus \mathbb{Z}, \partial = 2)$, by which I mean the complex $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ in cohomological degrees -1 and 0. From this it is easy to compute that $\mathrm{Rhom}^\bullet_{\mathrm{ABGP}}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2[-1]$. The extension at issue is classified by the non-zero element $e \in \mathrm{Rhom}^1(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$, which we take to be the value of $\Psi h = e$. Note that in cohomology, $pe = ei = 0$.

As in the previous example, the data of the representation Ψ includes various homotopies. Note that all $\mathrm{Rhom}^\bullet$ complexes between $\Psi X, \Psi Y, \Psi Z$ are supported in nonnegative degrees in cohomology, but most of the homotopies are in negative degrees. It follows that the only possible nonzero homotopies are the two degree-zero maps

$$\Psi_2(g,h) \in \mathrm{Rhom}^0(\Psi Y, \Psi X) = \mathbb{Z}/2, \quad \Psi_2(h,f) \in \mathrm{Rhom}^0(\Psi Z, \Psi Y) = \mathbb{Z}/2.$$

These, the maps $\Psi f, \Psi g, \Psi h$, and their Massey products must satisfy three relations coming from the degree-(-1) homotopies $\Psi_3(f,g,h)$, $\Psi_3(g,h,f)$, and $\Psi_3(h,f,g)$.

Ignoring signs, when s,t,u are all closed the general relation for $\Psi_3(s,t,u)$ is

$$\partial\Psi_3(s,t,u) = \Psi(\mu_3(s,t,u)) \pm \Psi_2(st,u) \pm \Psi_2(s,tu) \pm \Psi_2(s,t)\circ\Psi u \pm \Psi s \circ \Psi_2(t,u) \pm \mu_3(\Psi s, \Psi t, \Psi u)$$

For any of the three permutations (s,t,u) of (f,g,h), we have $st = tu = 0$, $\Psi(\mu_3(s,t,u)) = -\mathrm{id}$, and $\Psi_3(s,t,u) = 0$ for degree reasons. Also recalling $\Psi_2(f,g) = 0$, we have the following three equations:

$$(*) \qquad \begin{cases} 0 &= \mathrm{id}_{\mathbb{Z}/2} + i \circ \Psi_2(g,h) + \mu_3(i,p,e) \\ 0 &= -\mathrm{id}_{\mathbb{Z}/4} - \Psi_2(g,h)\circ i + p \circ \Psi_2(h,f) - \mu_3(p,e,i) \\ 0 &= \mathrm{id}_{\mathbb{Z}/2} + \Psi_2(h,f)\circ p + \mu_3(e,i,p) \end{cases}$$

The first and third equations take place in $\hom_{\mathrm{ABGP}}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$ and so I have dropped all signs there. The middle equation is in $\hom_{\mathrm{ABGP}}(\mathbb{Z}/4, \mathbb{Z}/4) = \mathbb{Z}/4$, and I have restored its signs. Note, however, that $\Psi_2(g,h)\circ i$ and $p \circ \Psi_2(h,f)$ are necessarily in the $\mathbb{Z}/2$ subgroup of $\mathrm{Rhom}^0(\mathbb{Z}/4, \mathbb{Z}/4) = \mathbb{Z}/4$, and so their signs are immaterial; only the relative sign between $\Psi(\mu_3(s,t,u))$ and $\mu_3(\Psi s, \Psi t, \Psi u)$ matters, and in any choice of sign conventions this relative sign is -1.

We should therefore calculate the Massey products $\mu_3(i,p,e)$, $\mu_3(p,e,i)$, and $\mu_3(e,i,p)$. To do so, it is convenient to resolve both $\Psi X = \Psi Z = \mathbb{Z}/2$ and $\Psi Y = \mathbb{Z}/4$ as complexes of the form $\mathbb{Z} \oplus \mathbb{Z}[1]$, with differential either multiplication by 2 or by 4. I will write operations between these complexes as matrices always with $\mathbb{Z}$-entries first and then $\mathbb{Z}[1]$ entries. In keeping with my convention to write composition from left to write, I will write matrices in terms of their right-multiplication action on row vectors. Thus the differentials are

$$\partial_X = \partial_Z = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad \partial_Y = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}.$$

Of course, when differentials act on morphisms, they act by the graded commutator $[\partial, -]$.

With these choices, we can lift the morphisms i,p,e to matrices

$$\tilde{\imath} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{p} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \tilde{e} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

None of the pairwise compositions are zero, but they are all exact:

$$\tilde{\imath}\tilde{p} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \left[\partial, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \partial_X \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \partial_Z$$

$$\tilde{p}\tilde{e} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = \left[\partial, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\right] = \partial_Y \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \partial_Z$$

$$\tilde{e}\tilde{\imath} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = \left[\partial, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \partial_Z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \partial_Y$$

Thus the Massey products have the following representatives:

$$\tilde{\mu}_3(i,p,e) = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix} - \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

$$\tilde{\mu}_3(p,e,i) = -\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

$$\tilde{\mu}_3(e,i,p) = -\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

Already the first and last of equations $(*)$ are satisfies: whether $\Psi_2(g,h)$ and $\Psi_2(h,f)$ are non-zero or not, their compositions with i and p respectively vanish. The middle equation holds provided either $\Psi_2(g,h)$ and $\Psi_2(h,f)$ both vanish or are both non-zero. These are not actually different options. Let Ψ and Ψ' be the representations in which $\Psi_2(g,h) = \Psi_2(h,f) = 0$ and $\Psi'_2(g,h) = \Psi'_2(h,f) = 1$. Then Ψ and Ψ' are homotopic via a homotopy η that acts trivially on f and g but takes h to the nontrivial class $\eta(h) \in 1 \in \mathrm{Rhom}^0(\mathbb{Z}/2,\mathbb{Z}/2)$. Under this homotopy, the values of $\Psi_2(g,h)$ and $\Psi_2(h,f)$ change to $\Psi'_2(g,h) = \Psi_2(g,h) + \eta(g) + \eta(h) = \Psi_2(g,h) + 1$ and $\Psi'_2(h,f) = \Psi_2(h,f) + 1$. ◇

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