0d QFT and Feynman diagrams

"QFT for Mathematicians" Perimeter Institute

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Phil asked me to give an introduction to Feynman diagrams, and I thought I might begin with some history. The story really begins with Dirac. Dirac knew a lot of symplectic geometry. In particular, he knew there was an analogy

Classical Mech on X	QM on X
T^*X	$L^2(X)$
$S_t(x_0, x_1)$	$U_t(x_0, x_1)$

Consider the "flow by time t" operator. On the CM side, it is a symplectomorphism $T^*X \to T^*X$, so its graph is Lagrangian $\subseteq T^*(X\times X)$ and is the differential of a function $S_t(x_0, x_1)$, called the generating function, and on the QM side, it is the unitary operator $(U_t\psi)(x_1) = \int U_t(x_0, x_1)\,\psi(x_0)\,dx_0$. Furthermore, Dirac knew that these were "the same":

$$U_t(x_0, x_1) \underset{t\to 0}{\sim} \exp\left(\tfrac{i}{\hbar} S_t(x_0, x_1)\right) \qquad (*)$$

Exercise: Show from Ham-Jacobi eqn that S satisfies a nonlinear ODE. Compare to Schrodinger operator acting on U.

Furthermore, the least action principle gives a formula for S:

$$S_t(x_0, x_1) = \min_{\varphi} (S(\varphi))$$

Dirac, 1932, "The Lagrangian in QM"

where Φ is the space of all paths $[0,t] \xrightarrow{\varphi} X$ s.t. $\varphi(0)=x_0$, $\varphi(t)=x_1$, and $S(\varphi)=\int_0^t \mathcal{L}$ is the <u>action</u> of the path.

Dirac asked: what is the "principle of least action" for QM? It's not too hard to guess:

$$S_{t_1+t_2}(x_0,x_2) = \min_{x_1}\left(S_{t_1}(x_0,x_1) + S_{t_2}(x_1,x_2)\right)$$

$$\mathcal{U}_{t_1+t_2}(x_0,x_2) = \int dx_1\, \mathcal{U}_{t_1}(x_0,x_1)\, \mathcal{U}_{t_2}(x_1,x_2)$$

So min | $\int$

and equation (*) is a "stationary phase approximation".

Feynman finished developing this line of reasoning in his thesis (1938):

$$\mathcal{U}_t(x_0,x_1) = \int_\Phi d\varphi \, \exp\left(\frac{\sqrt{-1}}{\hbar} S(\varphi)\right)$$

Then theoretical physics got interrupted by war and A-bombs.

When Theory resumed, Schwinger and (separately) Tomonaga quantized electromagnetism by treating it as ∞-dim QM. At a 1948 conference, Schwinger wowed the audience. Then Feynman spoke and bombed: no one liked his diagrams. After a long car ride w/ Dyson, the latter wrote up Feynman diagrams, proved equivalence to Schwinger–Tomonaga, and implemented renormalization.

So what did Dyson do? Let's focus on computing an expectation value $\langle \mathcal{O} \rangle$ for some operator:

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}\varphi \; \mathcal{O}(\varphi) \exp\left(\frac{\sqrt{-1}}{\hbar} s(\varphi)\right)}{\int \mathcal{D}\varphi \; \exp\left(\frac{\sqrt{-1}}{\hbar} s(\varphi)\right) =: Z}$$

For definiteness, let's pretend $\Phi = \mathbb{R}^n$ is finite dim, $\mathcal{O}(\varphi) = f(x_1, \dots, x_n)$, and ~~[illegible]~~ $\min(S)$ occurs at $x = 0$. Then "stationary phase" says that ~~a~~ if $\hbar \ll 1$, the $\int$ is supported in a formal nbhd of 0, so ~~of~~, $s \in \mathbb{C}[[x_1, \dots, x_n]]$. Further, let's suppose the minimum is nondegen:

$$s(x) = \underbrace{\tfrac{1}{2} a_{ij} x_i x_j}_{\text{"free theory"}} + \underbrace{b(x)}_{\text{cubic + higher "interactions"}}$$

$$b(x) = \tfrac{1}{3} b_{ijk} x_i x_j x_k + \dots$$

Of course, sum over repeated indices. a_{ij}, b_{ijk}, etc. are symmetric tensors, and a_{ij} is pos. def.

There aren't very many ways to compute integrals. You can try to cleverly choose coordinates ("u-sub"), but otherwise pretty much your only option is integration by parts. So that's what we will do. ~~Suppose $f = f(x_1 \dots x_n)$ is homogeneous of degree~~

How does integration by parts go? The idea is that if you have a vector $g_i(x)$ worth of functions (i.e. a vector field), then

$$\frac{1}{Z} \cdot \int \partial_i \left(g_i(x) \exp\left(-\frac{\sqrt{-1}}{\hbar} a \frac{x^2}{2} + \dots\right)\right) = 0.$$

$$\shortparallel$$

$$\frac{1}{Z} \cdot \int \left(\partial_i g_i(x)\right) \exp(\dots) + g_i(x)\, \partial_i \exp(\dots)$$

$$\shortparallel$$

(**) $$\langle \partial_i g_i(x) \rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} a_{ij} x_j \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} \partial_i b(x) \right\rangle$$

Now the idea is to cleverly choose $g(x)$. What is the largest term in (**)? $x, \hbar \ll 1$, and $\partial_i b = O(x^2)$, so the largest term is the middle one. So let's choose g s.t.

$$g_i(x) \frac{\sqrt{-1}}{\hbar} a_{ij} x_j = f(x)$$

i.e. $$g_i(x) = \frac{\hbar}{\sqrt{-1}} \frac{1}{\deg(f)} a^{-1}_{ij} \partial_j f(x)$$

assuming f is homogeneous of degree $\deg(f) \neq 0$. Of course, if $\deg(f) = 0$, then $\langle f \rangle = f$ is a constant, and there was nothing to do.

Plugging into (**), we find

If f is homogeneous of degree $N \neq 0$, then

$$\boxed{\langle f(x) \rangle = -\frac{1}{N}\frac{\hbar}{\sqrt{-1}}\left\langle a^{-1}_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}\right\rangle - \frac{1}{N}\left\langle a^{-1}_{ij}\frac{\partial f}{\partial x_i}\frac{\partial b}{\partial x_j}\right\rangle}$$

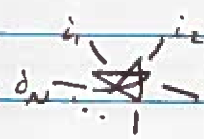
In general, $\langle f \rangle = f(0) + \underbrace{\langle f - f(0) \rangle}_{\text{sum of terms of nonzero degrees.}}$

And the point is that both terms on the RHS are $\ll f$, because they are strictly higher in degree in either $\hbar$ or x.

So by repeatedly subtracting constant terms and then applying the boxed equation, we can find
$\langle f \rangle \in \mathbb{C}[[\hbar]]$.

Note the importance of inverting a_{ij}. This matrix a^{-1} is called the propagator.

Fine, so where are the diagrams? In this telling, they arise from the combinatorics of Taylor series. Take f homogeneous of degree N, and draw

$f_{i_1 \ldots i_N}$ = 

where $f(x) = \frac{1}{N!} f_{i_1 \ldots i_N} x_{i_1} \cdots x_{i_N}$.
It is symmetric, and the "$N!$" counts its symmetries.

Think of ☆ as a Hydra, the many-headed monster that Hercules has to slay.

Also expand

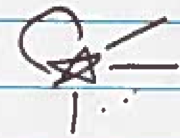
$$b(x) = \sum_{m \geq 3} \frac{1}{m!} b^{(m)}_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}$$

$b_{i_1 \cdots i_m} =$ 

and i ——— $j = a^{-1}_{ij}$

↑ "long" aka "internal"

So for instance, up to a combinatorial factor,

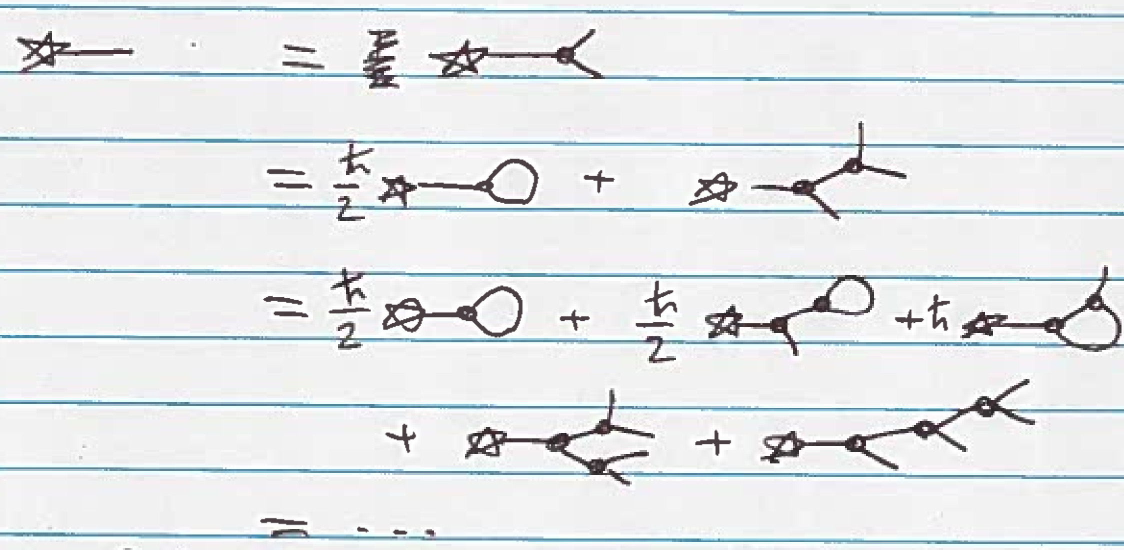
$a^{-1}_{ij}\, \partial^2_{ij} f =$ 

$a^{-1}_{ij}\, \partial_i f\, \partial_j b =$ 

Then the boxed equation says

up to combinatorial factors. In terms of Hercules, you try to clip of a head, and

it sprouts two-or-more new ones, <u>or</u> you seal two heads together with wax.
You have to continue infinitely many times, but the series converges because $\hbar, x \ll 1$.
It converges to a sum over all closed diagrams (since an open diagram would have a head to chop off) built from one ✯ and many •s. You get a factor of $\hbar$ for each loop, some $\sqrt{-1}$'s I won't work out, and a combinatorial term which ends up being the # of automorphisms of the diagram.
For instance, if $f = f_i x_i$ is linear and b is cubic, then



One final remark. In the QFT case, the index "i" ranges over all points in spacetime, and tensor contraction is (f.d.) integration. The propagator a^{-1} is a Green's function, because $a_{ij} = \text{"}\frac{\partial^2}{\partial i \partial j}\text{"}$, and the diagrams are pictures of interactions!