Lecture 1: Variations on the Theme of Symmetry

Introduction

The ideas discussed here about symmetry in field theory represent joint work with Constantin Teleman and Greg Moore; large parts of these notes are adapted from a forthcoming paper.[1] We give a conceptual framework for some of the developments of the past few years around "global categorical symmetries". A few immediate comments to give some perspective.

Remark 1.1.

(1) The word 'global' can be dropped: we are discussing symmetries of a field theory which are analogous to symmetries of any other mathematical structure. The word 'global' is often used in contradistinction to 'gauge' symmetries, but gauge symmetries are not symmetries of theories: they are encoded in the (higher) groupoid structure of fields.

(2) The word 'categorical' can also be dropped. Mathematics is traditionally expressed in the language of sets and functions, and when mathematical objects have internal symmetries they organize into (higher) categories rather than sets. The symmetries we discuss often have this higher structure, so naturally involve categories.

(3) There is a large amount of work over the past few years on the topic of symmetry in quantum field theory, and in particular the application of "global categorical symmetries" to dynamics and other questions. We give some excerpts from this literature to illustrate how the framework we develop here applies. However, we emphasize that the topic of symmetry in quantum field theory is a large one with many facets, and the framework here does not apply to all of it.

(4) We restrict to the analog of *finite* group symmetry, including homotopical versions;[2] it will be interesting to generalize to the analog of Lie group symmetry. It is more natural in quantum theory to have *algebras* of symmetries, rather than *groups* of symmetries, and so in particular we encounter non-invertible symmetries.

(1.2) *Main idea.* The motivating thought is simple: we separate out the abstract structure of symmetry from its concrete manifestation acting in a particular situation. Historically, the concept of an abstract group was introduced to synthesize and further develop diverse instances of group symmetry in geometry, in algebra (Galois), in number theory (Gauss), etc. Perhaps it is Arthur Cayley in 1854 who first articulated the definition of an abstract group—I'm no historian—and now every student of mathematics learns this concept early on. The structure of groups is then used to study representations—linear and nonlinear. Similar comments apply to algebras. The elements of an algebra act as linear operators on any module. In the context of field theory, the analog of an algebra of symmetries is a *topological* field theory together with a boundary theory. The analog of

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[1]The paper will include many references; these notes only give minimal direction for further exploration.

[2]which include "higher form symmetries" (though we do not use this term: there are no differential forms in the finite case) and "2-group symmetries"

elements of an algebra are defects in the topological field theory, which act on any quantum field theory which is a "module" over the topological field theory.

In this lecture we begin with a brief discussion of groups and algebras before turning to symmetry in field theory. We return at the end to further aspects of groups and algebras whose analogs in field theory play an important role in later lectures.

Groups

(1.3) *Taxonomy.* The simplest dichotomy is between (1) discrete groups, and (2) groups which have a nontrivial topology. For the former we distinguish according to the trichotomy of cardinalities: finite, countable, uncountable. The nature of finite groups is quite different from that of infinite discrete groups; the phrase 'discrete group' evokes very different images from the phrase 'finite group', even though finite groups are discrete. For example, linear representations of a finite group are rigid—they do not deform—whereas, for example, the infinite cyclic group $\mathbb{Z}$ has a continuous family of 1-dimensional complex unitary representations $n \mapsto e^{ixn}$ parametrized by $x \in \mathbb{R}$. Among topological groups, the nicest are Lie groups. Here there is a dichotomy: compact vs. noncompact. Compact Lie groups, which include finite groups, have a well-established structure theory and representation theory; again, representations are rigid. Noncompact Lie groups, which include countable discrete groups, also enjoy a robust structure and representation theory, but of a very different nature. Moving on, there are infinite dimensional Lie groups as well as topological groups which do not admit a manifold structure. In a different direction, there are homotopical groups—finite and infinite. Namely, if $\mathcal{X}$ is any pointed topological space, then the space $\Omega\mathcal{X}$ of loops at the basepoint has the composition law of concatenation of loops, and this makes $\Omega\mathcal{X}$ a group up to homotopy. If $\mathcal{X}$ is connected, then its "base" has the form BG for a group G, and there is a version in which G is a Lie group.

The field theory symmetry structure we study is analogous to that of a *finite* group, or, more generally, to a homotopical group that is π*-finite* in the sense that it has only finitely many nonzero homotopy groups, each of which is finite.

(1.4) *Fibering over BG.* Let G be a finite group. A *classifying space* BG is derived from a contractible topological space EG equipped with a free G-action by taking the quotient; the homotopy type of BG is independent of choices. If X is a topological space equipped with a G-action, then the *Borel construction* is the total space of a fiber bundle

$$\begin{tikzcd} X_G = EG \times_G X \arrow[d, "\pi"] \\ BG \end{tikzcd} \tag{1.5}$$

with fiber X. If $* \in BG$ is a chosen point, and we choose a basepoint in the G-orbit in EG labeled by $*$, then the fiber $\pi^{-1}(*)$ is canonically identified with X. We say the *abstract symmetry data* is the pair $(BG,*)$, and a *realization* of the symmetry $(BG,*)$ on X is a pair consisting of a fiber bundle (1.5) over BG together with an identification of the fiber over $* \in BG$ with X.

Remark 1.6. We are already moving to homotopy theory, and it is more natural to take the *homotopy fiber*, which is a special case of a *homotopy fiber product*. For continuous maps f,g we can realize the homotopy fiber product as the space F and dotted maps indicated in the diagram

(1.7)
$$\begin{tikzcd} & F \arrow[dl, dashed] \arrow[dr, dashed] & \\ X \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

A point of F is a triple (x,z,γ) in which $x \in X$, $z \in z$, and γ is a path in Y from $f(x)$ to $g(z)$. The homotopy fiber Z over the basepoint in (1.5) is the homotopy fiber product

(1.8)
$$\begin{tikzcd} & Z \arrow[dl, dashed] \arrow[dr, dashed] & \\ * \arrow[dr] & & X_G \arrow[dl] \\ & BG & \end{tikzcd}$$

Exercise 1.9. Construct a homotopy equivalence $X \xrightarrow{\simeq} Z$. (You may want to know the *homotopy lifting property* for the fiber bundle $X_G \to BG$.)

(1.10) *Homotopical groups*[3]. A pair $(\mathcal{X},*)$ consisting of a π-finite topological space $\mathcal{X}$ and a basepoint $* \in \mathcal{X}$ is a generalization of $(BG,*)$. Here is the formal definition.

Definition 1.11.

(1) A topological space $\mathcal{X}$ is π-*finite* if (i) $\pi_0\mathcal{X}$ is a finite set, (ii) for all $x \in \mathcal{X}$, the homotopy group $\pi_q(\mathcal{X},x)$, $q \geqslant 1$, is finite, and (iii) there exists $Q \in \mathbb{Z}^{>0}$ such that $\pi_q(\mathcal{X},x) = 0$ for all $q > Q$, $x \in \mathcal{X}$. (For a fixed bound Q we say that $\mathcal{X}$ is Q-finite.)

(2) A continuous map $f\colon \mathcal{Y} \to \mathcal{Z}$ of topological spaces is π-*finite* if for all $z \in \mathcal{Z}$ the homotopy fiber[4] over z is a π-finite space.

(3) A spectrum[5] E is π-*finite* if each space in the spectrum is a π-finite space.

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[3]By 'homotopical group' we mean an *H-group*. This nomenclature pertains if $\mathcal{X}$ is path connected; see Remark 1.14 below. We allow disconnected spaces.

[4]As in (1.8), the homotopy fiber over $z \in \mathcal{Z}$ consists of pairs (y,γ) of a point $y \in \mathcal{Y}$ and a path γ in $\mathcal{Z}$ from z to $f(y)$.

[5]A spectrum is a sequence of pointed topological spaces $\{E_q\}_{q\in\mathbb{Z}}$ and maps $\Sigma E_q \to E_{q+1}$.

Example 1.12. An Eilenberg-MacLane space $K(\pi,q)$ is π-finite if π is a finite group. We use notation which emphasizes the role of $\mathcal{X}$ as a classifying space: if $q=1$ we denote $K(\pi,1)$ by $B\pi$, and if $q\geqslant 1$ and A is a finite abelian group, we denote $K(A,q)$ by B^qA. Just as there are group extensions of ordinary groups, so too there are group extensions of homotopical groups. These are often Postnikov towers. For example, let G be a finite group and let A be a finite abelian group. Then extensions of the form

$$1 \longrightarrow B^2A \longrightarrow \mathcal{X} \longrightarrow BG \longrightarrow 1 \tag{1.13}$$

are classified by group actions of G on A together with a cohomology class in $H^3(BG;A)$, where the coefficients A are twisted by the group action. Thus $\mathcal{X}$ is a topological space with only two nonzero homotopy groups: $\pi_1\mathcal{X}=G$, $\pi_2\mathcal{X}=A$. This class of spaces was studied long ago by George Whitehead. Nowadays we might say that $\mathcal{X}$ is the classifying space of a *2-group*.

Remark 1.14. If $\mathcal{X}$ is a path connected topological space with basepoint $x\in\mathcal{X}$, then $\mathcal{X}$ is the classifying space of its based loop space $\Omega\mathcal{X}$, where the latter is a H-group by composition of based loops.

Remark 1.15.

(1) A topological space $\mathcal{X}$ gives rise to a sequence of higher groupoids $\pi_0\mathcal{X}$, $\pi_{\leqslant 1}\mathcal{X}$, $\pi_{\leqslant 2}\mathcal{X}$,..., or indeed to an ∞-groupoid. There is a classifying space construction which passes in the opposite direction from higher groupoids to topological spaces. An ∞-groupoid is π-finite if it satisfies the conditions in Definition 1.11(1), which hold iff the corresponding topological space is π-finite.
(2) In a similar way, one can define π-finiteness for a simplicial set.

Algebras

(1.16) *The sandwich.* Let A be an algebra, and for definiteness suppose that the ground field is $\mathbb{C}$. Partly for simplicity, and partly by the analogy with *finite* groups, assume that A and the modules that follow are finite dimensional. Let R be the *regular* right A-module, i.e., the vector space A furnished with the right action of A by multiplication. The pair (A,R) is *abstract symmetry data*: the *realization* of (A,R) on a vector space V is a pair (L,θ) consisting of a left A-module L together with an isomorphism of vector spaces

$$\theta\colon R\otimes_A L \xrightarrow{\cong} V. \tag{1.17}$$

The tensor product in (1.17)—an algebra sandwiched between a right and left module—is a general structure that recurs in these lectures.

Remark 1.18. It may seem pedantic to introduce the module R here; one usually simply talks about a left module over A. But I want to emphasize the distinction between the abstract symmetry structure and its concrete action on a vector space, and for this we need to be able to recover the vector space from the left module.

Observe that the right regular module satisfies the algebra isomorphism

$$\mathrm{End}_A(R) \cong A, \tag{1.19}$$

where the left hand side is the algebra of linear maps $R \to R$ that commute with the right A-action.

(1.20) *Left vs. right.* The choice of left vs. right in a given situation is made by choice or convention. My convention puts *structural* actions on the right and *geometric* actions on the left. Here the right module R is part of the symmetry structure, and the left action is the geometric action on the vector space. As another example, if V is a finite dimensional real vector space of dimension n, then the space of bases $\mathcal{B}(V)$—the set of isomorphisms $\mathbb{R}^n \to V$—carries a right structural action of $\mathrm{GL}_n\,\mathbb{R}$ and a left geometric action of $\mathrm{Aut}(V)$.

(1.21) *The group algebra.* Let G be a finite group. The *group algebra* $A = \mathbb{C}[G]$ is the free vector space on the set G, which is then a linear basis of A; multiply basis elements according to the group law in G. A left A-module L is a linear representation of G. The tensor product in (1.17) recovers the vector space which underlies the representation. In the setup of **(1.4)**, take $X = L$ to construct a vector bundle $L_G \to BG$ whose fiber over $* \in BG$ is L.

There is an inclusion $G \subset \mathbb{C}[G]$ whose image consists of *units*, i.e., of invertible elements in the algebra. But the typical element of $\mathbb{C}[G]$ is not invertible. For example, the sum $g_1 + \cdots + g_k$ over a conjugacy class in G is not invertible unless $k = 1$. In general it is a central element. In fact, the center of $\mathbb{C}[G]$ is generated by these elements.

Noninvertible elements in $\mathbb{C}[G]$ play an important role in the study of G-symmetry. For example, when G is the symmetric group on n letters, then the theory of irreducible representations and their associated Young tableaux is developed in terms of certain projectors in $\mathbb{C}[G]$.

Example 1.22. Consider the Lie algebra $\mathfrak{su}_3$, and let $A = U(\mathfrak{su}_3)$ be its universal enveloping algebra (over $\mathbb{C}$). The center of A is isomorphic to a polynomial algebra in 2 variables; it is generated by the Casimir elements $x_2, x_3 \in A$. These Casimirs act as linear operators on any A-module—i.e., on a representation of SU_3—and by Schur's lemma they act by a scalar if the module is irreducible. So these operators can be used to decompose an arbitrary A-module into a direct sum of isotypical submodules. This is simply another illustration of: (1) the importance of noninvertible elements in an algebra, and (2) the use particular elements in an abstract algebras (here central elements) in concrete realizations.

(1.23) *Higher algebra.* The higher versions of finite groups in **(1.10)** have an analog in algebras as well. For example, a *fusion category* $\mathcal{A}$ is a "once higher" version of a finite dimensional semisimple algebra, and there is a well-developed theory of modules over a fusion category. In particular, $\mathcal{A}$ is a right module over itself, the right regular module. A finite group G gives rise to the fusion category $\mathcal{A} = \mathrm{Vect}[G]$ of finite rank vector bundles over G with convolution product.

Main Definitions

These are Definition 1.34 and Definition 1.37 below.

(1.24) *Remarks about field theory.* We begin with a few general remarks, deferring to Lecture 2 a more in-depth discussion.

Perhaps the first point to make is the metaphor of a field theory as a representation of a Lie group, or better

$$\text{field theory} \sim \text{module over an algebra} \tag{1.25}$$

This is of course only a very rough analogy, but nonetheless it provides useful guidance and language. (Our language sometimes seems to assume the module has an algebra structure, but that is not an assumption we make.) We do always work in the Wick-rotated context, so for a quantum field theory we work on Riemannian manifolds rather than Lorentz manifolds. As pioneered by Graeme Segal, a Wick-rotated theory is a linear representation of a bordism category; it is the bordism category which plays the role of the algebra in (1.25).

The field theories which encode finite symmetries are *topological*, and we bring to bear the mathematical development of topological field theory. In particular, we work with *fully local* (or *fully extended*) topological field theories. In the axioms this is realized by having the theory defined on a higher bordism category of manifolds with corners of all codimension. The field theories on which the symmetry acts are typically not topological, and for general quantum field theories the fully local aspect has yet to be fully developed. Nonetheless, our exposition often implicitly assumes full locality.

Remark 1.26. Just as one specifies a Lie group to talk about its representations, one must specify a bordism category to talk about its representations: field theories. There are two sorts of "discrete parameters". First, there is a dimension n, which in the physical anti-Wick-rotated theory is the dimension of spacetime. Second there is a collection $\mathcal{F}$ of background fields. We may use the terminology 'n-dimensional field theory on $\mathcal{F}$' or 'n-dimensional field theory over $\mathcal{F}$'. We define background fields in the next lecture; for today's lecture they remain in the deep background. We often work in shorthand, illustrated by the following for a gauged nonlinear σ-model:

$$\mathcal{F} = \{\text{orientation, Riemannian metric, } \mathrm{SO}_3\text{-connection, section of twisted } S^2\text{-bundle}\} \tag{1.27}$$

(1.28) *Domain walls and boundary theories.* Let σ_1, σ_2 be $(n+1)$-dimensional theories on background fields $\mathcal{F}_1, \mathcal{F}_2$. A *domain wall* $\delta\colon \sigma_1 \to \sigma_2$ is the analog[6] of a "(σ_2,σ_1)-bimodule"; see Figure 1 for a depiction. We remove the scare quotes and use the convenient terminology '(σ_2,σ_1)-bimodule' for a domain wall. The triple $(\sigma_1,\sigma_2,\delta)$ is formally a functor with domain a bordism category of

[6]We emphasize that σ_1 and σ_2 are not assumed to be algebra objects in the symmetric monoidal category of field theories.

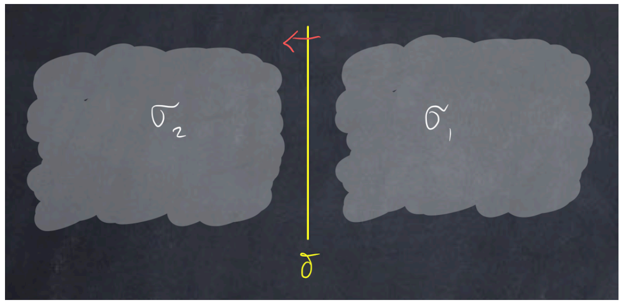

FIGURE 1. A domain wall $\delta\colon \sigma_1 \to \sigma_2$

smooth $(n+1)$-dimensional manifolds with corners which are equipped with a partition into regions labeled '1' and '2' separated by a cooriented codimension one submanifold (with corners) which is "δ-colored". This is illustrated in Figure 2. As a special case, a domain wall from the tensor unit theory $\mathbb{1}$ to itself is an n-dimensional (absolute, standalone) theory. More generally, we can tensor any domain wall $\delta\colon \sigma_1 \to \sigma_2$ with an n-dimensional theory to obtain a new domain wall. There is a composition law on topological domain walls which are "parallel" (in the sense that they have trivial normal bundles and one is the image of a nonzero section of the normal bundle of the other, using a tubular neighborhood):

$$\begin{tikzcd} \sigma_1 \arrow[r, "\delta'"] \arrow[rr, bend right, "\delta''\circ\delta'"'] & \sigma_2 \arrow[r, "\delta''"] & \sigma_3 \end{tikzcd} \tag{1.29}$$

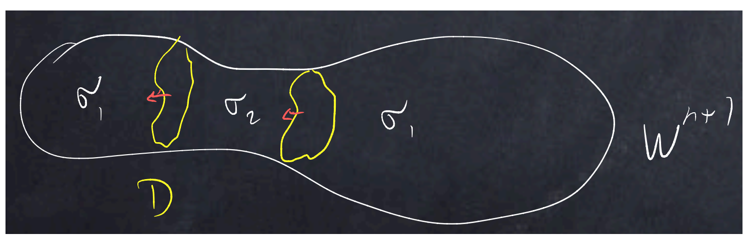

FIGURE 2. Domain walls in the manifold W

(1.30) *Boundary theories.* Following the metaphor of domain wall as bimodule, there are special cases of right or left modules. For field theory these are called *right boundary theories* or *left boundary theories*, as depicted in Figure 3. (Normally, we omit the region labeled '$\mathbb{1}$' in the drawings: it is transparent.) A right boundary theory of σ is a domain wall $\sigma \to \mathbb{1}$; a left boundary theory is a domain wall $\mathbb{1} \to \sigma$.

Remark 1.31. The nomenclature of right vs. left may at first be confusing; it does follow standard usage for modules over an algebra—the direction (right or left) is that of the action of the algebra on the module. In fact, following our general usage for domain walls, we use the terms 'right σ-module' and 'left σ-module' for right and left boundary theories. But a right boundary theory appears on the left in drawings, just as a right module R over an algebra A appears to the left of the algebra in the expression 'R_A'.

FIGURE 3. A right boundary theory and a left boundary theory

(1.32) *Abstract finite symmetry in field theory.* We turn now to the central concept in these lectures.

Definition 1.33. Fix $n \in \mathbb{Z}^{\geqslant 0}$. Then *finite field-theoretic symmetry data of dimension* n is a pair (σ, ρ) in which σ is an $(n+1)$-dimensional topological field theory and ρ is a right σ-module.

The dimension n pertains to the theories on which (σ, ρ) acts, not to the dimension of the field theory σ. We do not insist on the burdensome nomenclature 'finite field-theoretic symmetry data of dimension n', but simply refer to the pair of a topological field theory and a right boundary theory. One might want to assume that ρ is nonzero, which is true for the particular boundaries in Definition 1.34 below. Our statement of Definition 1.33 has not made explicit the background fields, but they are there in the background (and there are issues to address concerning them).

This definition is extremely general. The following singles out a class of boundary theories which more closely models the discussions in **(1.4)** and **(1.16)**. We freely use the language and setting of fully local topological field theory. Recall that if $\mathcal{C}'$ is a symmetric monoidal n-category, then there is a symmetric monoidal Morita $(n+1)$-category $\mathrm{Alg}(\mathcal{C}')$ whose objects are objects in $\mathcal{C}'$ equipped with an algebra structure and whose 1-morphisms $A_0 \to A_1$ are (A_1, A_0)-bimodules.

Definition 1.34. Suppose $\mathcal{C}'$ is a symmetric monoidal n-category and σ is an $(n+1)$-dimensional topological field theory with codomain $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$. Let $A = \sigma(\mathrm{pt})$. Then A is an algebra in $\mathcal{C}'$ which, as an object in $\mathcal{C}$, is $(n+1)$-dualizable. Assume that the right regular module A_A is n-dualizable as a 1-morphism in $\mathcal{C}$. Then the boundary theory ρ determined by A_A is the *right regular boundary theory* of σ, or the *right regular σ-module.*

We use an extension of the cobordism hypothesis to generate the boundary theory ρ from the right regular module A_A. Observe that A_A is the value of the pair (σ, ρ) on the bordism depicted in Figure 4; the white point is incoming, so the depicted bordism maps $\mathrm{pt} \to \emptyset$.

FIGURE 4. The bordism which computes A_A

Remark 1.35.

(1) The right regular σ-module ρ satisfies $\operatorname{End}_\sigma(\rho) \cong \rho$; compare (1.19).
(2) The regular boundary theory is also called a *Dirichlet boundary theory.*
(3) Not every topological field theory σ can appear in Definition 1.33. For example, the main theorem in [7] asserts that "most" 3-dimensional Reshetikhin-Turaev theories do not admit any nonzero topological boundary theory, hence they cannot act as symmetries of a 2-dimensional field theory. On the other hand, the Turaev-Viro theory σ_Φ formed from a (spherical) fusion category Φ takes values in the 3-category Alg(Cat) for a suitable 2-category Cat of linear categories. Thus σ_Φ admits the right regular σ-module defined by the right regular module Φ_Φ.
(4) Let G be a finite group. Then G-symmetry in an n-dimensional quantum field theory is realized via $(n+1)$-dimensional finite gauge theory. The partition function counts principal G-bundles, weighted by the reciprocal of the order of the automorphism group. The regular boundary theory has an additional fluctuating field: a section of the principal G-bundle.

(1.36) *Concrete realization of finite symmetry in field theory.* Let (σ, ρ) be a topological field theory together with a right σ-module. We now define a concrete realization of (σ, ρ) as symmetries of a quantum field theory.

Definition 1.37. Let σ be an $(n+1)$-dimensional topological field theory and let ρ be a right σ-module. Let F be an n-dimensional field theory. A (σ, ρ)-*module structure* on F is a pair $(\widetilde{F}, \theta)$ in which $\widetilde{F}$ is a left σ-module and θ is an isomorphism

$$\theta\colon \rho \otimes_\sigma \widetilde{F} \xrightarrow{\;\cong\;} F \tag{1.38}$$

of absolute n-dimensional theories.

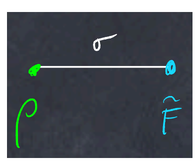

FIGURE 5. The "sandwich"

Here '$\rho \otimes_\sigma \widetilde{F}$' notates the dimensional reduction of σ along the closed interval with boundaries colored with ρ and $\widetilde{F}$; see Figure 5. The bulk theory σ with its right and left boundary theories ρ and $\widetilde{F}$ is sometimes called a "sandwich".

Remark 1.39.

(1) The theory F and so the boundary theory $\widetilde{F}$ may be topological or nontopological, and we allow it to be not fully extended (in which case we use truncations of σ and ρ).

[7] Daniel S. Freed and Constantin Teleman, *Gapped boundary theories in three dimensions*, Comm. Math. Phys. **388** (2021), no. 2, 845–892, arXiv:2006.10200.

(2) The sandwich picture of F as $\rho \otimes_\sigma \widetilde{F}$ separates out the topological part (σ, ρ) of the theory from the potentially nontopological part $\widetilde{F}$ of the theory. This is advantageous, for example in the study of defects. It allows general computations in the abstract symmetry data which apply to every realization as a symmetry of a field theory.
(3) Typically, symmetry persists under renormalization group flow, hence a low energy approximation to F should also be a (σ, ρ)-module. If F is gapped, then at low energies we expect a topological theory (up to an invertible theory), so we can bring to bear powerful methods and theorems in topological field theory to investigate *topological* left σ-modules. This leads to dynamical predictions.

Examples

Example 1.40 (quantum mechanics $n = 1$)**.** Consider a quantum mechanical system defined by a Hilbert space $\mathcal{H}$ and a Hamiltonian H. The Wick-rotated theory F is regarded as a map with domain $\mathrm{Bord}_{\langle 0,1\rangle}(\mathcal{F})$ for

$$\mathcal{F} = \{\text{orientation, Riemannian metric}\}. \tag{1.41}$$

Roughly speaking, $F(\mathrm{pt}_+) = \mathcal{H}$ and $F(X) = e^{-\tau H/\hbar}$ for $\tau \in \mathbb{R}^{>0}$ and $X = [0, \tau]$ with the standard orientation and Riemannian metric. We refer to a recent preprint of Kontsevich-Segal [8] for more precise statements.

Now suppose G is a finite group equipped with a unitary representation $S\colon G \to U(\mathcal{H})$, and assume that the G-action commutes with the Hamiltonian H. To express this symmetry in terms of Definition 1.33 and Definition 1.37, let σ be the 2-dimensional finite gauge theory with gauge group G. If we were only concerned with σ we might set the codomain of σ to be $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$ for $\mathcal{C}'$ the category of finite dimensional complex vector spaces and linear maps. But to accommodate the boundary theory $\widetilde{F}$ for quantum mechanics, we let $\mathcal{C}'$ be a suitable category of topological vector spaces. The pair (σ, ρ) is defined on $\mathrm{Bord}_2 = \mathrm{Bord}_{\langle 0,1,2\rangle}$ with no background fields. Then $\sigma(\mathrm{pt}) = \mathbb{C}[G]$ is the complex group algebra of G, and $\rho(\mathrm{pt})$ is its right regular module.

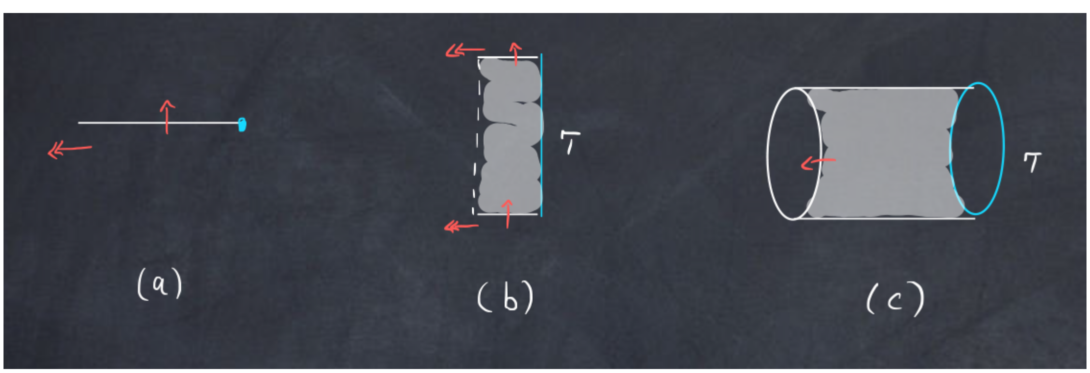

FIGURE 6. Three bordisms evaluated in (1.42) in the theory $(\sigma, \widetilde{F})$

[8]Maxim Kontsevich and Graeme Segal, *Wick rotation and the positivity of energy in quantum field theory*, arXiv:2105.10161.

Now we describe the left boundary theory $\widetilde{F}$, which has as background fields (1.41), as does the (absolute) quantum mechanical theory F. Observe that by cutting out a collar neighborhood it suffices to define $\widetilde{F}$ on cylinders (products with $[0,1]$) over $\widetilde{F}$-colored boundaries. The bordisms in Figure 6 do not have a well-defined width since there is a Riemannian metric only on the colored boundary. That boundary has a well-defined length τ in (b) and (c). We refer to §2.1.1 of [7] for the conventions about arrows of time. Evaluation of these bordisms under $(\sigma, \widetilde{F})$ gives:

$$\begin{aligned} &\text{(a) the left module } {}_{\mathbb{C}[G]}\mathcal{H} \\ &\text{(b) } e^{-\tau H/\hbar}\colon {}_{\mathbb{C}[G]}\mathcal{H} \longrightarrow {}_{\mathbb{C}[G]}\mathcal{H} \\ &\text{(c) the central function } g \longmapsto \operatorname{Tr}_{\mathcal{H}}\bigl(S(g)e^{-\tau H/\hbar}\bigr) \text{ on } G \end{aligned} \tag{1.42}$$

Of course, this is not a complete construction of the nontopological σ-module $\widetilde{F}$, but it gives some intuition for that theory.

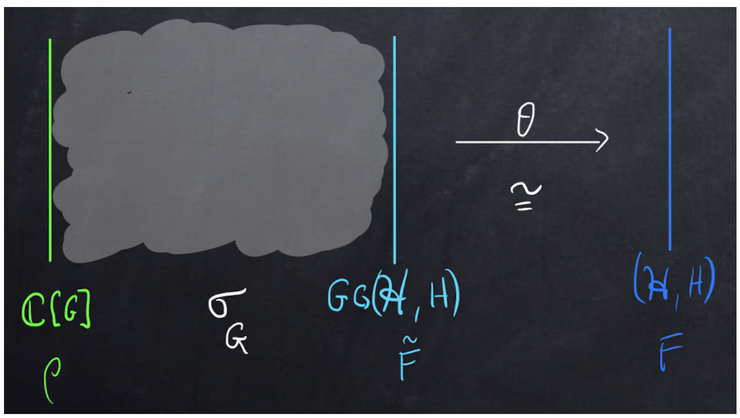

Figure 7. Quantum mechanics with G-symmetry

Figure 7 illustrates the G-action on the quantum mechanics theory F. Although we have not discussed defects yet, we illustrate how they work in this basic example. A point defect in F is what is usually called an *observable*. Think of time running up vertically, and then a point defect δ is the insertion of an observable, or operator, at a given time, as depicted in Figure 8. Also shown is the *link* of the point, which is a 0-sphere. The possible defects on the point are the elements of the topological vector space

$$\varprojlim_{\epsilon\to 0} \operatorname{Hom}\bigl(1, F(S^0_\epsilon)\bigr). \tag{1.43}$$

Here ϵ measures the size of the linking 0-sphere, and one takes an inverse limit as this size shrinks to zero. That inverse limit is a space of singular operators on $\mathcal{H}$; see [8]. In these lectures we mostly compute with defects in topological theories, and for these there is no need to take a limit. Here, for ease of notation and because we are after more formal points, we denote this space of operators as 'End($\mathcal{H}$)', even though the notation suggests bounded operators.

Now we look at defects in the sandwich picture in Figure 7 and transport to point defects in the theory F. First consider a point defect on the $\widetilde{F}$-colored boundary, as in Figure 9.

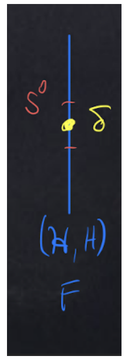

FIGURE 8. A point observable in quantum mechanics

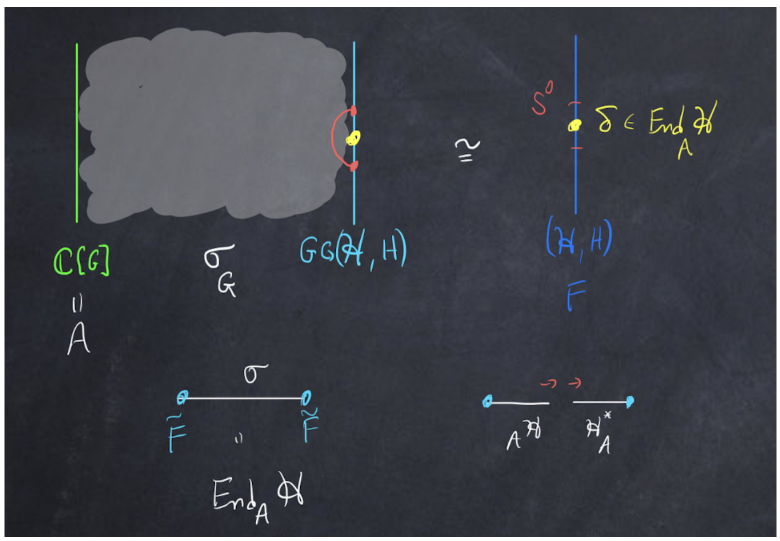

FIGURE 9. A point defect on the $\widetilde{F}$-boundary

The link of the point is depicted, and its value under the pair $(\sigma, \mathcal{F})$ is computed at the bottom of the figure. (There should be an inverse limit, which is omitted.) The result is that such a defect is an operator on $\mathcal{H}$ which commutes with the G-action. Of course, such an operator need not be invertible. Also, since $\widetilde{F}$ is not topological, this is not a topological defect.

At the other extreme is a point defect on the ρ-colored boundary, which we call a *point ρ-defect*; see Figure 10. Now the link is in the topological field theory (σ, ρ); that is, the bulk topological theory σ with topological boundary theory ρ. The value of the link is the vector space which underlies the group algebra $A = \mathbb{C}[G]$. Hence the point defect is labeled by an element of A. This may be an element of the group, which is a unit in A, or it may be a nonunit (noninvertible element) in A. This is a topological defect, as it is a defect in the topological field theory. Now imagine having both of these point defects. Since the point ρ-defect is topological, it can be moved in time without changing any correlation functions. Visibly it commutes with the point defect on the $\widetilde{F}$-boundary, which recall is an operator that commutes with A.

One can have a point defect in the bulk theory, as in Figure 11. The link of this point is a circle, and the value $\sigma(S^1)$ of the finite gauge theory on the circle is a vector space which may be identified with the center of the group algebra $\mathbb{C}[G]$. As stated earlier, it has a basis labeled by

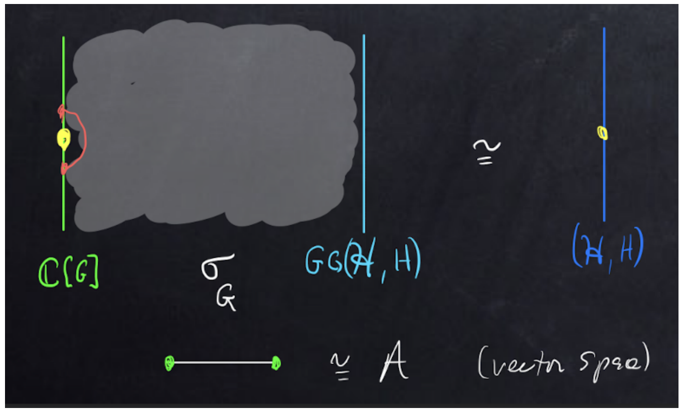

FIGURE 10. A point ρ-defect

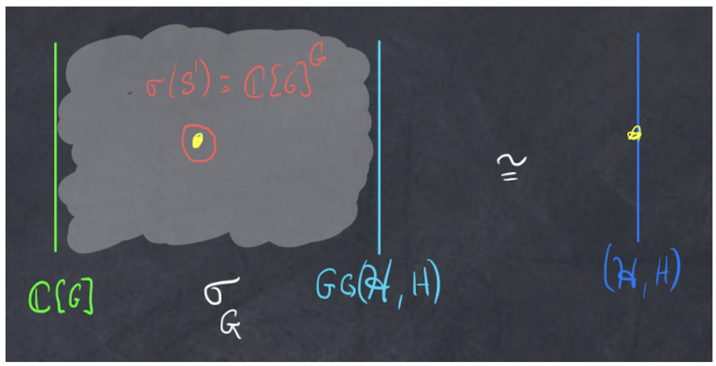

FIGURE 11. A central point defect

conjugacy classes: the central element of $\mathbb{C}[G]$ is the sum of elements in the conjugacy class. These topological defects commute with the point defects in Figures 9 and 10.

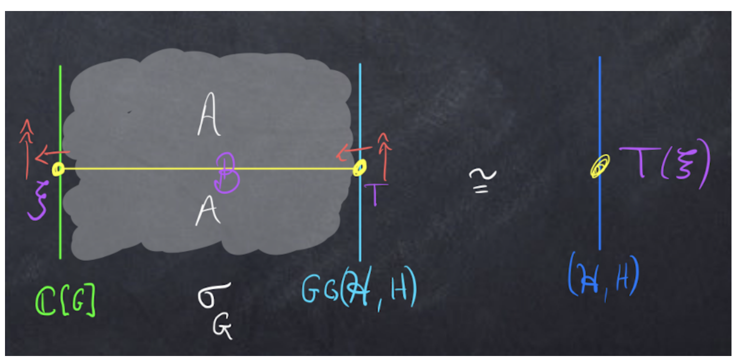

FIGURE 12. A general point defect

The general point defect in F can be realized by a defect on the closed interval depicted in Figure 12. When working on a stratified manifold, we evaluate on links working from higher dimensional strata to lower dimensional strata. The space of labels on a given stratum may depend on the labels chosen on higher dimensional strata, as they do at the endpoints in this case. We do not explain the evaluation of the links in detail here, but simply report that: (1) the label in the interior of the defect is an (A, A)-bimodule B; (2) the label at the endpoint on the ρ-colored boundary is a vector $\xi \in B$; and (3) the label at the endpoint on the $\widetilde{F}$-colored boundary is an (A, A)-bimodule map $B \to \mathrm{End}(\mathcal{H})$. Under the isomorphism θ, this maps to the point defect labeled by $T(\xi)$ in the theory F.

Exercise 1.44. What happens if $B = A$? (This is the transparent defect in the interior.)

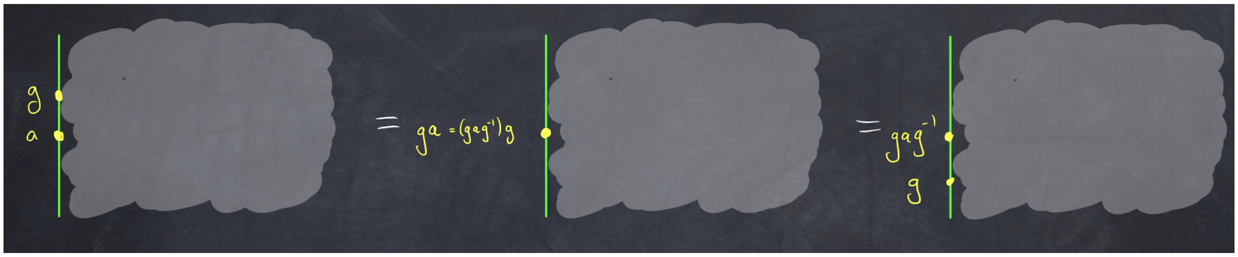

FIGURE 13. Commuting point ρ-defects

We compute the action of the point ρ-defect labeled by $g \in G \subset \mathbb{C}[G]$ on a general point defect in F. This is a computation in the topological field theory (σ, ρ); it applies universally to any left (σ, ρ)-module. First, consider the special case of a topological point defect labeled by $a \in A = \mathbb{C}[G]$, as illustrated in Figure 13. Passing from the first picture to the second is the fusion of point defects, which we will compute later is multiplication in A. The same fusion applies when passing from the third picture to the second. So the effect of moving the g-defect past the a-defect is conjugation of a. In Figure 14 we illustrate how the g-defect moves past a general (nontopological) point defect. Now the label at the ρ-colored boundary is a vector $\xi \in B$, where B is an (A, A)-bimodule, and again the effect is to conjugate ξ by g. Applying the isomorphism θ from the sandwich theory to the theory F, we pass from $T(\xi)$ to $T(g\xi g^{-1}) = gT(\xi)g^{-1}$, since T is an (A, A)-bimodule map. This is the expected action on point defects.

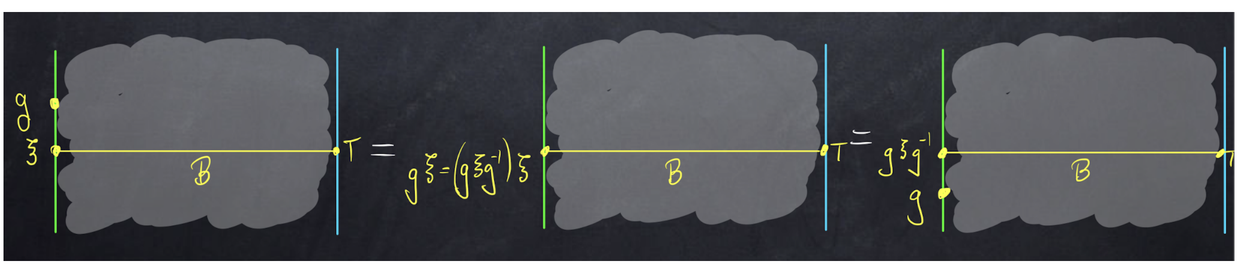

FIGURE 14. Action of a topological point defect on a general point defect

Remark 1.45. The finite gauge theory σ can be constructed via a finite path integral from the π-finite space BG; we discuss finite path integrals in the next lecture. Similarly, the boundary theory ρ can be constructed from a basepoint $* \to BG$: the principal G-bundles are equipped with a trivialization on ρ-colored boundaries. A traditional picture of the G-symmetry of the theory F uses this *classical* picture: the sheaf of background fields $\mathcal{F}$ is augmented to the sheaf $\widetilde{\mathcal{F}} = \{\text{orientation, Riemannian metric, } G\text{-bundle}\}$, which fibers over the sheaf $B_\bullet G = \{G\text{-bundle}\}$, so in that sense "spreads over BG" as in §(**1.4**). There is an absolute field theory on $\widetilde{\mathcal{F}}$ which is the "coupling of F to a background gauge field" for the symmetry group G. The framework we are advocating here of F as a (σ,ρ)-module uses the *quantum* finite gauge theory σ: we sum over principal G-bundles.

Remark 1.46. The finite path integral construction of the regular (Dirichlet) boundary theory makes the isomorphism θ in (1.38) apparent. Namely, to evaluate (σ,ρ) we sum over G-bundles equipped with a trivialization on ρ-colored boundaries. Since the trivialization propagates across an interval, the sandwich theory (Figure 5) is the original theory F without the explicit G-symmetry.

Example 1.47 (a homotopical symmetry)**.** Let H be a connected compact Lie group, and suppose $A \subset H$ is a finite subgroup of the center of H. Let $\overline{H} = H/A$. Then a H-gauge theory in, say, 4 dimensions—for example, pure Yang-Mills theory—has a BA symmetry. In this case we take $\sigma = \sigma^{(3)}_{B^2A}$ to be the A-gerbe theory based on the π-finite space B^2A, and we take ρ to be the regular boundary theory constructed from a basepoint $* \to B^2A$. The left σ-module $\widetilde{F}$ is a $\overline{H}$-gauge theory. (Aspects of this example are discussed in more detail in [9].)

Quotients

For the remainder of this lecture we turn back to a general discussion of symmetry to remind about two aspects that we will take up in field theory in subsequent lectures: quotients and projective symmetries.

If X is a set equipped with the action of a group G, then there is a quotient set X/G: a point of X/G is a G-orbit in X. We now give analogs in the homotopy and algebra settings.

(1.48) *The homotopy quotient.* In the topological setting of (**1.4**), the total space X_G of the Borel construction plays the role of the quotient space X/G. Indeed, if G acts freely on X, then there is a homotopy equivalence $X_G \simeq X/G$; in general, X_G is the *homotopy quotient*.

For any map $f\colon Y \to BG$ of topological spaces we form the *homotopy pullback* (see (1.7))

(1.49)
$$\begin{tikzcd} & Z \arrow[dl, dashed] \arrow[dr, dashed] & \\ Y \arrow[dr, "f"'] & & X_G \arrow[dl, "\pi"] \\ & BG & \end{tikzcd}$$

If Y is path connected and pointed, then there is a homotopy equivalence $Y \simeq B(\Omega Y)$. If BG also has a basepoint, and if the map $f\colon Y \to BG$ is basepoint-preserving, then f is the classifying map of a homomorphism $\Omega Y \to G = \Omega BG$, at least in the homotopical sense. In this case Z is the homotopy quotient of X by the action of ΩY. As a special case, if $G' \subset G$ is a subgroup, and $Y = BG' \to BG$ is the classifying map of the inclusion, then Z is homotopy equivalent to the total space of the Borel construction $X_{G'}$. Hence (1.49) is a generalized quotient construction. For $G' = \{e\}$ we have $Y = *$ and we recover $Z = X$, as in (1.8): no quotient at all.

[9] Daniel S. Freed and Constantin Teleman, *Relative quantum field theory*, Comm. Math. Phys. **326** (2014), no. 2, 459–476, arXiv:1212.1692.

(1.50) *Augmentations of algebras.* There is an analogous story in the setting (**1.16**) of algebras. An *augmentation* of an algebra A is an algebra homomorphism $\epsilon\colon A \to \mathbb{C}$. Use ϵ to endow the scalars $\mathbb{C}$ with a right A-module structure: set $\lambda \cdot a = \lambda\epsilon(a)$ for $\lambda \in \mathbb{C}, a \in A$. If L is a left A-module, the vector space

$$Q = \mathbb{C} \otimes_A L = \mathbb{C} \otimes_\epsilon L \tag{1.51}$$

plays the role of the "quotient" of L by A.

Example 1.52. For the group algebra $\mathbb{C}[G]$ of a finite group G, there is a natural augmentation

$$\begin{aligned} \epsilon\colon \mathbb{C}[G] &\longrightarrow \mathbb{C} \\ \sum_{g\in G} \lambda_g g &\longmapsto \sum_{g \in G} \lambda_g \end{aligned} \tag{1.53}$$

where $\lambda_g \in \mathbb{C}$. The augmentation is the pushforward on functions under the map $G \to *$. If L is a representation of G, extended to a left $\mathbb{C}[G]$-module, then the tensor product (1.51) is the vector space of *coinvariants*:

$$1 \otimes \ell = 1 \otimes g \cdot \ell = 1 \otimes g' \cdot \ell, \qquad \ell \in L, \quad g, g' \in G, \tag{1.54}$$

in the tensor product with the augmentation. As a particular case, let S be a finite set equipped with a left G-action, and let $L = \mathbb{C}\langle S\rangle$ be the free vector space generated by S. Then $\mathbb{C} \otimes_\epsilon L$ can be identified with $\mathbb{C}\langle S/G\rangle$, the free vector space on the quotient set.

Exercise 1.55. Prove this last assertion.

Exercise 1.56. For the finite group G acting on the finite set S, consider the Borel construction $S_G \to BG$. Construct an isomorphism $S/G \to \pi_0(S_G)$. Compare the information content of S_G and $\mathbb{C}\otimes_\epsilon\mathbb{C}\langle S\rangle$. Which has more information? How can you alter the other to recover more information?

Remark 1.57. Recall the fusion category $\mathcal{A}$ in (**1.23**). The analog of an augmentation is a *fiber functor* on $\mathcal{A}$: a homomorphism $\mathcal{A} \to \mathrm{Vect}$. For $\mathcal{A} = \mathrm{Vect}[G]$ the natural choice is pushforward under the map $G \to *$ to a point.

(1.58) *Quotient by a subgroup.* We can form the "sandwich" (1.51) with any right A-module in place of the augmentation. For $A = \mathbb{C}[G]$, if $G' \subset G$ is a subgroup, then $\mathbb{C}\langle G'\backslash G\rangle$ is a right G-module; for $G' = G$ it reduces to the augmentation module (1.53). If L is a G-representation, then the tensor product

$$\mathbb{C}\langle G'\backslash G\rangle \otimes_{\mathbb{C}[G]} L \cong \mathbb{C} \otimes_{\mathbb{C}[G']} L \tag{1.59}$$

is the vector space of coinvariants of the restricted G'-representation. This represents the quotient by the subgroup G'.

Central extensions and projective actions

(1.60) *Projective representations.* There are many situations in which one encounters projective representations of groups. For example, suppose A is an algebra and L is an *irreducible* left module. Let G be a finite group that acts on A by algebra automorphisms, i.e., via a group homomorphism $\alpha\colon G \to \operatorname{Aut} A$. They, typically, we can implement these symmetries on the module L: if $g \in G$ then we can find a linear automorphism $t\colon L \to L$ such that

$$t(a\xi) = \big(\alpha(g)a\big)t(\xi), \qquad a \in A, \quad \xi \in L. \tag{1.61}$$

The map t exists if the twisted A-module L^α is isomorphic to L, and by Schur's lemma t is determined up to a scalar. In other words, each $g \in G$ determines a $\mathbb{C}^\times$-torsor T_g, and the torsors depend multiplicatively on G. They fit together into a group G^τ which is a central extension of G by $\mathbb{C}^\times$:

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow G^\tau \longrightarrow G \longrightarrow 1 \tag{1.62}$$

A familiar example[10] has A a Clifford algebra, L an irreducible module, and G is the orthogonal group. Then G acts projectively on the Clifford module, and one obtains the (s)pin central extension of the orthogonal group.

Remark 1.63. For a group extension (1.62) one considers representations $\rho\colon G^\tau \to \operatorname{Aut}(V)$ for which $\rho\big|_{\mathbb{C}^\times}$ is scalar multiplication.

(1.64) *The twisted group algebra.* Suppose G in (1.62) is a finite group. Let $L^\tau \to G$ be the complex line bundle associated to the principal $\mathbb{C}^\times$-bundle (1.62). Define the *twisted group algebra*

$$A^\tau = \bigoplus_{g\in G} L^\tau_g. \tag{1.65}$$

Then A^τ inherits an algebra structure from the group structure of G. Furthermore, $G^\tau \subset A^\tau$ is the group of units. An A^τ-module restricts to a linear representation of G^τ on which the center $\mathbb{C}^\times$ acts by scalar multiplication, and vice versa. Observe that there is no analog of the augmentation (1.53) unless we are given a splitting of the central extension (1.62). More generally, if $H \subset G$ is a subgroup, then a splitting of the restriction of (1.62) over H induces an A^τ-module structure on $\mathbb{C}\langle H\backslash G\rangle$, and we can use this to define the quotient by H, as in (1.59). Absent the splitting, the projectivity obstructs the quotient construction.

Exercise 1.66. Given an algebra homomorphism $\epsilon\colon A^\tau \to \mathbb{C}$, construct a splitting of the central extension (1.62).

Remark 1.67. In field theory, the analog of an action by the central extension of a group is called an ('t Hooft) *anomalous symmetry*, and the central extension (1.62) is called the *anomaly*. In that context too, the central extension obstructs the quotient construction (often called *gauging*).

[10] In this example one uses $\mathbb{Z}/2\mathbb{Z}$-gradings everywhere.