Lecture 2: Formal structures in field theory; finite homotopy theories

In the first part of this lecture we recall some general structures in field theory. Our framework is that introduced by Segal, adapted by Atiyah for topological theories, and more recently exposed for general quantum field theories by Kontsevich-Segal in [8]. This framework is sometimes referred to as "functorial field theory" (but see Remark 1.1(2), which strongly suggests dropping 'functorial'). In the literature what we say is developed most for *topological* field theories, though the structures should exist in some form for general field theories. In particular, the theory of fully local—or fully extended—field theories is only at its incipient stages in the nontopological case. We introduce a class of topological field theories called *finite homotopy theories*. These offer a kind of semiclassical calculus of defects which is quite convenient and powerful when it applies. These theories are also a nice playground for general ideas in field theory. In the last part of the lecture we return to finite symmetry in field theory and the central Definitions 1.33 and 1.37.

In the next lecture we take up one more general piece of structure—quotients—and then illustrate these ideas in many examples. You will notice that the notes have gotten ahead of the lectures. . .

Basics

(2.1) *A metaphor.* As with all analogies, not a perfect one:

$$\text{field theory} \sim \text{representation of a Lie group} \tag{2.2}$$

One imperfection can be improved if we take a module over an algebra instead of a representation of a group, but we use the word 'module' below in a different sense. Despite its drawbacks, this metaphor can guide us in some limited way.

(2.3) *Discrete parameters.* The discrete parameters on the right of (2.2) might be considered to be a dimension n and a Lie group G of dimension n. One would not just fix a dimension and, say, study representations of a Lie group of dimension 8! Surely, one would distinguish between different Lie groups of that dimension to fix the "type" of representation. In field theory, too, one can consider there to be two discrete parameters to fix the "type" of field theory: (1) the dimension n of *spacetime*, and (2) the collection $\mathcal{F}$ of *background fields*. We can then speak of an n-dimensional field theory on/over $\mathcal{F}$.

Remark 2.4. The word 'discrete parameter' is potentially confusing, since the choice of background fields may include non-discrete fields, such as a Riemannian metric. It is the *choice* of which background fields to include which is discrete. It may help to observe that discrete parameters are what we fix in geometry to construct moduli spaces, for example the moduli space of curves (of a fixed genus=discrete parameter).

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(2.5) *Background fields.* Informally, background fields of dimension n are sets assigned to n-dimensional manifolds X together with a pullback under local diffeomorphisms $f\colon X' \to X$; they are required to depend locally on X. However, we need more than set-valued fields, since fields—such as connections—may have internal symmetries. In the case of "B-fields" there are higher internal symmetries: automorphisms of automorphisms. Thus we could take higher groupoids as the codomain for fields. Instead we choose simplicial sets. Formally, let Man_n be the category whose objects are smooth n-manifolds and whose morphisms are local diffeomorphisms of n-manifolds. There is a Grothendieck topology of open covers.

Definition 2.6. A *field* in dimension n is a sheaf $\mathcal{F}\colon\ \mathrm{Man}_n \to \mathrm{Set}_\Delta$ with values in the category of simplicial sets.

We have not spelled out the sheaf condition, which is the encoding of locality. Here $\mathcal{F}$ could be a single field or a collection of fields; we do not attempt to define irreducibility here. A field on an n-manifold X is a 0-simplex in $\mathcal{F}(X)$. An example of a collection of fields is (1.27), which we repeat here for convenience:

$$\mathcal{F} = \{\text{orientation, Riemannian metric, } SO_3\text{-connection, section of twisted } S^2\text{-bundle}\} \tag{2.7}$$

Observe that the last two components are sheaves that extend to Man, the category of smooth manifolds and smooth maps, but the first two do not extend. See [11] for an exposition of simplicial sheaves on Man, and in particular a discussion of the sheaf condition.

Remark 2.8. One should think of $\mathcal{F}$ as a specification of type, not a choice of specific fields on a specific manifold. Rather, $\mathcal{F}(X)$ is the simplicial set of fields on a manifold X.

Remark 2.9. The sheaf on Man_n tells what a field on a *single* manifold is; one also needs to know what a smooth family of fields parametrized by an arbitrary smooth manifold is, as well as how to base change such families. In other words, we must sheafify over Man. We do not dwell on this point in these lectures.

Remark 2.10. The Axiom System we use for field theory does not include fluctuating fields, nor does it include lagrangians; it supposes that all quantization has already been executed. In that sense it is a purely quantum axiom system, as are the earlier axiom systems of Wightman and Haag-Kastler in Minkowski spacetime, or for that matter those of Dirac and von Neumann for quantum mechanics. We can, however, contemplate fiber bundles $\mathcal{F} \to \overline{\mathcal{F}}$ of fields and a quantization process which passes from a field theory over $\mathcal{F}$ to a field theory over $\overline{\mathcal{F}}$.

Exercise 2.11. Identify the background fields (and dimension) in familiar quantum field theories, such as:

(1) A 2-dimensional σ-model with target S^2
(2) Quantum mechanics of a particle on a ring
(3) 4-dimensional QCD and its low energy approximation, the pion theory

[11] Daniel S. Freed and Michael J. Hopkins, *Chern–Weil forms and abstract homotopy theory*, Bull. Amer. Math. Soc. (N.S.) **50** (2013), no. 3, 431–468, arXiv:1301.5959.

(4) 4-dimensional pure Yang-Mills theory
(5) Supersymmetric Yang-Mills theory
(6) Dijkgraaf-Witten theory

You may want to also identify the fields in the classical theory and the fiber bundle which maps background fields in the classical theory to those in the quantum theory.

(2.12) *Field theory.* These axioms capture a *Wick-rotated* field theory, formulated on compact Riemannian manifolds (if there is a Riemannian metric among the background fields). Fix $n \in \mathbb{Z}^{\geqslant 1}$ and a collection $\mathcal{F}$ of n-dimensional fields.

A field theory is expressed in the language of sets and functions, but because of the layers of structure it is rather in the language of categories and functors. The domain is a bordism category $\mathrm{Bord}_n(\mathcal{F})$ of n-dimensional smooth manifolds with corners equipped with a choice of fields from $\mathcal{F}$. In the literature one finds detailed constructions for the fully extended topological case, say in works of Lurie and Calaque-Scheimbauer;[12] and [8] for the nonextended general case. We assume that all *topological* theories are fully extended downward in dimension, in which case $\mathrm{Bord}_n(\mathcal{F})$ is a symmetric monoidal n-category. One hopes a similar strong locality is possible for *nontopological* theories, but that awaits further developments. In the nonextended case, we interpret '$\mathrm{Bord}_n(\mathcal{F})$' as a 1-category $\mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{F})$ whose objects are closed $(n-1)$-manifolds and whose morphisms are bordisms between them.

The codomain $\mathcal{C}$ of a field theory is a symmetric monoidal n-category.[13] For physical applications one takes $\mathcal{C}$ to be *complex linear*: the linearity from superposition in quantum mechanics, and the complex ground field from interference. Thus we usually have $\Omega^n\mathcal{C} = \mathbb{C}$ and $\Omega^{n-1}\mathcal{C}$ equivalent to the category Vect of vector spaces or to the category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces, discrete vector spaces for topological theories and topological vector spaces for nontopological theories.[14] These assumptions can be relaxed for applications outside of physics.

In the topological case (so $\mathcal{F}$ consists of "topological fields") we have the following.

Definition 2.13. A *topological field theory* of dimension n over $\mathcal{F}$ is a symmetric monoidal functor

$$F\colon\ \mathrm{Bord}_n(\mathcal{F}) \longrightarrow \mathcal{C}. \tag{2.14}$$

We would like to think that a suitable variation of this definition applies to all field theories. This is clearest in the nonextended nontopological case, in which we replace $\mathcal{C}$ by the 1-category $t\mathrm{Vect}$ of suitable complex topological vector spaces under tensor product, as in [8].

Remark 2.15.

[12] Jacob Lurie, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129–280. arXiv:0905.0465.
Damien Calaque and Claudia Scheimbauer, *A note on the (∞, n)-category of cobordisms*, Algebr. Geom. Topol. **19** (2019), no. 2, 533–655, arXiv:1509.08906

[13] One can replace 'n-category' with '(∞, n)-category' in our exposition. Most of our examples are quite finite and semisimple, but one should also use examples built on derived geometry.

[14] The *looping* $\Omega\mathcal{C}$ of an n-category is the $(n-1)$-category $\mathrm{Hom}_{\mathcal{C}}(1, 1)$. If $\mathcal{C}$ is symmetric monoidal, as it is in our case, then so too is $\mathrm{Hom}_{\mathcal{C}}(1, 1)$. Hence one can iterate.

(1) A field theory may be evaluated in a smooth family of manifolds and background fields parametrized by a smooth manifold S; the result is smooth and should behave well under base change. For example, typically correlation functions are written as smooth functions of parameters—it is these spaces of parameters which are the missing piece of structure. Therefore, (2.14) should be sheafified over Man, the site of smooth manifolds and smooth diffeomorphisms. (This point has been emphasized by Stolz and Teichner.[15]) We already remarked on the need to sheafify $\mathcal{F}$ over Man in Remark 2.9. We also need to sheafify the domain $\mathrm{Bord}_n(\mathcal{F})$ and the codomain $\mathcal{C}$ over Man.

(2) A topological field theory is constrained by strong finiteness properties that follow from Definition 2.13. In the metaphor (2.2), a *topological* field theory is analogous to a representation of a *finite* group.

(3) Definition 2.13 does not incorporate unitarity, or rather its Wick-rotated form: reflection positivity. One needs extra structure to do so, and it is an open problem to formulate *extended* reflection positivity.

(4) The notion of a *free* theory is not evident from Definition 2.13.

(5) The collection of field theories of a fixed dimension n on a fixed collection $\mathcal{F}$ of background fields has an associative composition law: juxtaposition of quantum systems with no interaction, sometimes called 'stacking'. There is a unit theory $\mathbb{1}$ for this operation. For example, if F_1, F_2 are theories, and Y is a closed $(n-1)$-manifold with background fields, then $(F_1 \otimes F_2)(Y) = F_1(Y) \otimes F_2(Y)$. The unit theory has $\mathbb{1}(Y) = \mathbb{C}$; there is a single state on every space. A unit for the composition law is called an *invertible field theory*.

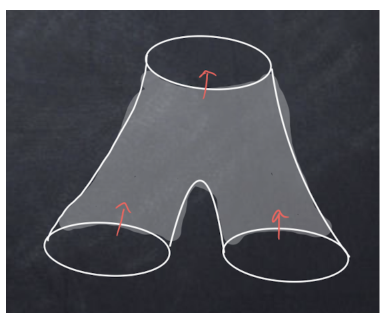

FIGURE 15. The pair of pants bordism

(2.16) *Product structures.* Let F be a field theory, as in (2.14). As with any homomorphism in algebra, structures and equations in the domain transport to the codomain. One example is the product structure on $F(S^k)$, $0 \leqslant k \leqslant n-1$. Assume first that F is a *topological* theory. To illustrate, let $n = 2$ and $k = 1$. Then the "pair of pants" bordism P in Figure 15 induces an algebra structure on the vector space $F(S^1)$. There is a diffeomorphism of P which exchanges the two incoming circles, and this implies that the multiplication is commutative. For a 3-dimensional theory F', the value $F'(S^1)$ of the theory on a circle is a linear category, say, and P induces a

[15] Stephan Stolz and Peter Teichner, *Supersymmetric field theories and generalized cohomology*, Mathematical foundations of quantum field theory and perturbative string theory, Proc. Sympos. Pure Math., vol. 83, Amer. Math. Soc., Providence, RI, 2011, pp. 279–340. arXiv:1108.0189.

monoidal structure. The aforementioned diffeomorphism induces *data*, not a *condition*, on $F'(S^1)$, namely the data of a *braiding* for the monoidal structure. As we go higher in category number, the data and conditions proliferate.

From a more sophisticated viewpoint, the sphere S^k is an E_{k+1}-*object* in the bordism category, and the functor (2.14) induces an E_{k+1} structure on its image.

In a nontopological theory one also has these product structures, but now they depend continuously on background fields, such as a conformal structure or Riemannian metric. So on the topological vector space attached to S^{n-1} in a limit of zero radius we obtain some version of an operator product expansion. In 2-dimensional conformal field theories this leads to vertex operator algebras.

Exercise 2.17. Draw the pictures which illustrate algebra structure on S^0 in the bordism category. Be sure to verify associativity and also to identify the unit. Include topological background fields, such as an orientation or spin structure.

Finite homotopy theories I

As mentioned in the introduction, this is a useful class of theories, both in applications and for general theory. One constructs them using a version of the Feynman path integral. It is simultaneously more special—because one has finite sums rather than integrals over infinite dimensional spaces—and more general—because one integrates in all codimensions, and in positive codimension one sums in a higher category rather than simply summing complex numbers.

This class of topological field theories was introduced by Kontsevich in 1988 and was picked up by Quinn a few years later. They are also the subject of a series of a series of papers by Turaev in the early 2000's. These *finite homotopy theories* lend themselves to explicit computation using topological techniques.

The "finite path integral" quantization in *extended* field theory was introduced[16] in 1992; see also §3 and §8 of [17]. The modern approach uses *ambidexterity* or *higher semiadditivity*, as introduced by Hopkins-Lurie[18]. We do not say much about quantization in these lectures; I believe Mike Hopkins will say more in his.

In these lectures we use finite homotopy theories as examples of $(n+1)$-dimensional topological field theories σ which act on n-dimensional quantum field theories. For that reason, in this section we denote the dimension by 'm' rather than 'n'.

[16] Daniel S. Freed, *Higher algebraic structures and quantization*, Comm. Math. Phys. **159** (1994), no. 2, 343–398, arXiv:hep-th/9212115.

[17] Daniel S. Freed, Michael J. Hopkins, Jacob Lurie, and Constantin Teleman, *Topological quantum field theories from compact Lie groups*, A celebration of the mathematical legacy of Raoul Bott, CRM Proc. Lecture Notes, vol. 50, Amer. Math. Soc., Providence, RI, 2010, pp. 367–403. arXiv:0905.0731.

[18] Michael J. Hopkins and Jacob Lurie, *Ambidexterity in K(n)-local stable homotopy theory*. https://www.math.ias.edu/~lurie/papers/Ambidexterity.pdf. preprint. See also Gijs Heuts and Jacob Lurie, *Ambidexterity*, Topology and Field Theories, Contemp. Math., vol. 613, Amer. Math. Soc., Providence, RI, 2014, pp. 79–110 as well as Yonatan Harpaz, *Ambidexterity and the universality of finite spans*, Proc. Lond. Math. Soc. (3) **121** (2020), no. 5, 1121–1170, arXiv:1703.09764 and Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski, *Ambidexterity in chromatic homotopy theory*, Invent. Math. **228** (2022), no. 3, 1145–1254, arXiv:1811.02057.

(2.18) *π-finite spaces, maps, simplicial sheaves, ...* We defined π-finite spaces and π-finite maps in Definition 1.11. I invite you to review it now. We will also use π-finite infinite loop spaces and π-finite infinite loop maps.

Remark 2.19.

(1) A topological space $\mathfrak{X}$ gives rise to a sequence of higher groupoids $\pi_0\mathfrak{X}$, $\pi_{\leqslant 1}\mathfrak{X}$, $\pi_{\leqslant 2}\mathfrak{X}, \dots$, or indeed to an ∞-groupoid. There is a classifying space construction which passes in the opposite direction from higher groupoids to topological spaces. An ∞-groupoid is π-finite if it satisfies the conditions in Definition 1.11(1), which hold iff the corresponding topological space is π-finite.
(2) In a similar way, one can define π-finiteness for a simplicial set.
(3) A simplicial sheaf is π-finite if its values are π-finite simplicial sets.
(4) These variations pertain to the relative cases of maps, as in Definition 1.11(2).

Example 2.20. Fix $m \in \mathbb{Z}^{\geqslant 1}$ and consider the simplicial sheaves of fields which assign to an m-manifold W:

$$\begin{aligned} \widetilde{\mathcal{F}}(W) &= \{\text{Riemannian metric, } \mathrm{SU}_2\text{-connection}\} \\ \mathcal{F}(W) &= \{\text{Riemannian metric, } \mathrm{SO}_3\text{-connection}\} \end{aligned} \tag{2.21}$$

There is a map $p\colon \widetilde{\mathcal{F}} \to \mathcal{F}$ which takes an SU_2-connection to the associated SO_3-connection. The map p is a fiber bundle of simplicial sheaves. Neither $\widetilde{\mathcal{F}}$ nor $\mathcal{F}$ is π-finite, but the map p is π-finite. The fiber over a principal SO_3-bundle $\overline{P} \to W$ is the groupoid of lifts to a principal SU_2-bundle $P \to W$. These form a torsor over the groupoid of double covers of W. The groupoid of double covers is the fundamental groupoid of the mapping space $\mathrm{Map}(W, B\mu\!\!\!/_2)$, where $\mu\!\!\!/_2 = \{\pm 1\}$ is the center of SU_2. Observe that $B\mu\!\!\!/_2 \simeq \mathbb{RP}^\infty$ is a π-finite space, in fact a π-finite infinite loop space. As a matter of notation, we write '$B_\nabla G$' for the sheaf of G-connections—here G is a Lie group—and the group extension $\mu\!\!\!/_2 \to \mathrm{SU}_2 \to \mathrm{SO}_3$ leads to the fiber bundle

$$\begin{tikzcd} B_\nabla \mathrm{SU}_2 \arrow[r] & B_\nabla \mathrm{SO}_3 \arrow[d] \\ & B\mu\!\!\!/_2 \end{tikzcd} \tag{2.22}$$

of simplicial sheaves on Man (which we restrict to Man_m).

(2.23) *Cocycles.* We abuse the term 'cocycle', which should be reserved for cohomology theories described in terms of cochain complexes. We use it for any 'geometric representative' of a cohomology class. Thus, if $\{E_q\}$ is a spectrum—a sequence of pointed infinite loop spaces that represent a cohomology theory—then a "cocycle" of degree q on a space X can be taken to be a continuous map $X \to E_q$. In any model there is a zero cocycle.

The data which determines a finite homotopy theory is a triple $(m, \mathcal{X}, \lambda)$ in which $m \in \mathbb{Z}^{\geqslant 1}$, the space $\mathcal{X}$ is π-finite, and λ is a cocycle on $\mathcal{X}$ of degree m. We denote the theory as '$\sigma^{(m)}_{(\mathcal{X},\lambda)}$' or '$\sigma^{(m)}_{\mathcal{X}}$', the latter in case $\lambda = 0$. A nonzero cocycle λ encodes an *'t Hooft anomaly* in case $\sigma^{(m)}_{(\mathcal{X},\lambda)}$ is part of symmetry data.

(2.24) *Examples.* I hope the following list helps to relate this account to familiar terrain.

(1) Let G be a finite group. Its classifying space $\mathcal{X} = BG$ is an Eilenberg-MacLane space, so is π-finite. Set $\lambda = 0$. Then for any m the triple $(m, BG, 0)$ gives rise to finite G-gauge theory in dimension m.

(2) Now suppose λ represents a cohomology class in $H^m(BG; \mathbb{C}^\times)$. Then $\sigma^{(m)}_{(BG,\lambda)}$ is a twisted finite G-gauge theory, a Dijkgraaf-Witten theory.

(3) Let A be a finite abelian group and $p \in \mathbb{Z}^{\geqslant 0}$. Set $\mathcal{X} = B^{p+1}A$, an Eilenberg-MacLane space $K(A, p+1)$. A map into $B^{p+1}A$ represents a higher A-gerbe, which is a background field for a "p-form" symmetry. More standardly, this is the symmetry group B^pA, which is a homotopical form of a group (called an *H-group*). For any $m \in \mathbb{Z}^{\geqslant 1}$, the theory $\sigma^{(m)}_{B^{p+1}A}$ counts these higher A-gerbes. In the context of Definitions 1.33 and 1.37 it encodes B^pA-symmetry.

(4) Example 1.12 describes the classifying space of a *2-group*, which is a (general) path-connected 2-finite space.

Exercise 2.25. Identify the triple $(m, \mathcal{X}, \lambda)$ for a spin Chern-Simons theory with finite gauge group.

Finite homotopy theories II

Fix $m \in \mathbb{Z}^{\geqslant 1}$ and suppose $p\colon \widetilde{\mathcal{F}} \to \mathcal{F}$ is a π-finite fiber bundle of simplicial sheaves $\mathrm{Man}_m \to \mathrm{Set}_\Delta$; see (2.22) for an example. The basic idea is that there is a finite process which takes an m-dimensional field theory $\widetilde{F}$ over $\widetilde{\mathcal{F}}$ as input and produces an m-dimensional field theory F over $\mathcal{F}$ as output. One obtains F from $\widetilde{F}$ by summing over the (fluctuating) fields in the fibers of p. Since p is π-finite, this is a finite sum—a finite version of the Feynman path integral.

Remark 2.26.

(1) It often happens that the theory $\widetilde{F}$ is[19] "classical", in which case it is an invertible field theory. Then F is its quantization.

(2) The framework is most developed for *topological* field theories, in which case we can work in *extended* field theory.

We do not give a systematic treatment of this quantization. Rather, we illustrate through an example, which brings in the semiclassical mapping spaces that are our focus. This example is relevant for many 4-dimensional gauge theories, in which case the abelian group A in the example is a subgroup of the center of the gauge group. (One can change the numbers to apply this example in any dimension.)

[19] One point of view, advocated by Nathan Seiberg, is: 'classical' field theory = invertible field theory.

Example 2.27. Let A be a finite abelian group and set $\mathcal{X} = B^2A$. For definiteness fix dimension $m = 5$. Our aim is to construct a 5-dimensional topological field theory $F = \sigma^{(5)}_{B^2A}$. In the terms above: $\widetilde{\mathcal{F}}$ is the simplicial sheaf on Man_5 which assigns to a 5-manifold W the 2-groupoid $\pi_{\leqslant 2}\operatorname{Map}(W, B^2A)$ (made into a simplicial set), $\widetilde{F}$ is the tensor unit theory, and $\mathcal{F}$ is the trivial simplicial sheaf which assigns a point to each 5-manifold W. (The theory F is unoriented: there are no background fields.) We have not specified the codomain $\mathcal{C}$ of the theory, and one has latitude in this choice. For our purposes we assume standard choices at the top three levels: $\Omega^3\mathcal{C} = \mathrm{Cat}$ is a linear 2-category of complex linear categories, from which it follows that $\Omega^4\mathcal{C} = \mathrm{Vect}$ is a linear 1-category of complex vector spaces and $\Omega^5\mathcal{C} = \mathbb{C}$.

Let M be a closed manifold. Then $F(M)$ is the quantization of the mapping space

$$\mathcal{X}^M = \operatorname{Map}(M, \mathcal{X}) \tag{2.28}$$

The nature of that quantization depends on $\dim M$.

$\dim M = 5$: The quantization is a (rational) number, a weighted sum over homotopy classes of maps $M \to \mathcal{X}$:

$$F(M) = \sum_{[\phi]\in\pi_0(\mathcal{X}^M)} \frac{\#\pi_2(\mathcal{X}^M, \phi)}{\#\pi_1(\mathcal{X}^M, \phi)} = \frac{\#H^0(M; A)}{\#H^1(M; A)}\, \#H^2(M; A). \tag{2.29}$$

$\dim M = 4$: The quantization is the vector space of locally constant complex-valued functions on $\mathcal{X}^M$:

$$F(M) = \operatorname{Fun}\bigl(\pi_0(\mathcal{X}^M)\bigr) = \operatorname{Fun}\bigl(H^2(M; A)\bigr). \tag{2.30}$$

$\dim M = 3$: The quantization is the linear category of flat vector bundles (local systems) over $\mathcal{X}^M$:

$$\begin{aligned} F(M) = \operatorname{Vect}\bigl(\pi_{\leqslant 1}(\mathcal{X}^M)\bigr) &= \operatorname{Vect}\bigl(H^2(M; A)\bigr) \times \operatorname{Rep}\bigl(H^1(M; A)\bigr) \\ &\simeq \operatorname{Vect}\bigl(H^2(M; A) \times H^1(M; A)^\vee\bigr), \end{aligned} \tag{2.31}$$

where $\mathcal{A}^\vee$ is the Pontrjagin dual group of characters of the finite abelian group $\mathcal{A}$. (If M is oriented, there is an isomorphism $H^1(M; A)^\vee \cong H^2(M; A^\vee)$.)

Remark 2.32. In this example $\mathcal{X}$ is an infinite loop space—an Eilenberg-MacLane space—which explains the cohomological translations in (2.29)–(2.31).

As a further illustration, we describe the quantization of a bordism of top dimension, which leads to a correspondence diagram of mapping spaces.

Example 2.33 ($\mathcal{X} = B^2A$ redux)**.** Suppose $M\colon N_0 \to N_1$ is a 5-dimensional bordism between closed 4-manifolds N_0, N_1. The restriction maps to incoming and outgoing boundaries

$$\begin{tikzcd} & \mathcal{X}^M \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ \mathcal{X}^{N_0} & & \mathcal{X}^{N_1} \end{tikzcd} \tag{2.34}$$

form a correspondence diagram of mapping spaces. The quantization $F(M)\colon F(N_0) \to F(N_1)$ maps a function $f \in \mathrm{Fun}\bigl(\pi_0(\mathcal{X}^{N_0})\bigr)$ to $(p_1)_*(p_0)^*f$, where the pushforward $(p_1)_*$ is the "weighted finite homotopy sum" or "finite path integral"

$$\bigl[(p_1)_*g\bigr](\psi) = \sum_{[\phi]\in\pi_0(p_1^{-1}\psi)} \frac{\#\pi_2(p_1^{-1}\psi,\phi)}{\#\pi_1(p_1^{-1}\psi,\phi)}, \qquad g \in \mathrm{Fun}\bigl(\pi_0(\mathcal{X}^M)\bigr), \quad \psi \in \mathcal{X}^{N_1}. \tag{2.35}$$

Remark 2.36. If $N_0 = \emptyset$, then $\mathcal{X}^{N_0} = *$ and we obtain an element of $\mathrm{Hom}\bigl(1, F(N_1)\bigr)$, i.e., a vector in the vector space $F(N_1)$.

In terms of the paradigm at the beginning of this subsection, the example so far has trivial $\widetilde{F}$. We now give an example in which $\widetilde{F}$ is a nontrivial invertible theory. The data which defines it is a pair $(\mathcal{X}, \lambda)$ consisting of a π-finite space $\mathcal{X}$ and a cocycle λ on $\mathcal{X}$. Typically we need a generalized orientation to integrate λ, depending on the generalized cohomology theory in which λ is a cocycle.

Example 2.37 (twisted $\mathcal{X} = B^2A$)**.** We continue with $\mathcal{X} = B^2A$, and now specialize to $A = \mathbb{Z}/2\mathbb{Z}$ the cyclic group of order 2. Then[20] $H^5(\mathcal{X};\mathbb{C}^\times) \cong H^6(\mathcal{X};\mathbb{Z})$ is cyclic of order 2. Let λ be a cocycle which represents this class. The quantizations in Example 2.27 are altered as follows. For $\dim M = 5$ weight the sum in (2.29) by $\langle \phi^*\lambda, [M]\rangle$, where $[M]$ is the fundamental class.[21] For $\dim M = 4$ the transgression of λ to $\mathcal{X}^M$ induces a flat complex line bundle (of order 2) $L \to \mathcal{X}^M$; now (2.30) becomes the space of flat sections of $L \to \mathcal{X}^M$. Similarly, for $\dim M = 3$ the cocycle λ transgresses to a twisting of K-theory, and the quantization is a category of twisted vector bundles. The quantization in Example 2.33 is also altered using transgressions of λ: they produce line bundles $L_0 \to \mathcal{X}^{N_0}$ and $L_1 \to \mathcal{X}^{N_1}$ as well as an isomorphism $p_0^*(L_0) \xrightarrow{\cong} p_1^*(L_1)$. Then $(p_1)_*(p_0)^*$ maps sections of $L_0 \to \mathcal{X}^{N_0}$ to sections of $L_1 \to \mathcal{X}^{N_1}$.

Domain walls and boundaries in finite homotopy theories

(2.38) *Semiclassical domain walls.* Fix $m \in \mathbb{Z}^{\geqslant 1}$ and let $(\mathcal{X}_1, \lambda_1)$, $(\mathcal{X}_2, \lambda_2)$ be pairs of π-finite spaces and degree m cocycles. In the following we use trivializations of cocycles. In a model with cochain complexes, a trivialization of a degree m cocycle λ is a cochain of degree $m-1$ whose differential is λ. In a model in which λ is a map to a space in a spectrum, then a trivialization is a null homotopy of the map, i.e., a homotopy to the constant map with value the basepoint.

Definition 2.39. A *semiclassical domain wall* from $(\mathcal{X}_1, \lambda_1)$ to $(\mathcal{X}_2, \lambda_2)$ is a pair $(\mathcal{Y}, \mu)$ consisting of a π-finite space $\mathcal{Y}$ equipped with a correspondence

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ \mathcal{X}_1 & & \mathcal{X}_2 \end{tikzcd} \tag{2.40}$$

[20]Let $\iota \in H^2(B^2\mathbb{Z}/2\mathbb{Z};\mathbb{Z}/2\mathbb{Z})$ be the tautological class. Then $\iota \smile \mathrm{Sq}^1\iota \in H^5(B^2\mathbb{Z}/2\mathbb{Z};\mathbb{Z}/2\mathbb{Z})$ becomes the nonzero class after extending coefficients $\mathbb{Z}/2\mathbb{Z} \to \mathbb{C}^\times$.

[21]Since λ is induced from a mod 2 class, orientations are not necessary—we can proceed in mod 2 cohomology.

and a trivialization μ of $f_2^*\lambda_2 - f_1^*\lambda_1$.

Remark 2.41.

(1) We have written (2.40) to conform to standard practice for a correspondence from $\mathcal{X}_1$ to $\mathcal{X}_2$, but to fit our right/left conventions, as illustrated in Figure 2, we might have swapped $\mathcal{X}_1$ and $\mathcal{X}_2$.

(2) If $(\mathcal{Y}', \mu')$ is a π-finite space and a degree $m-1$ cocycle, then there is a new semiclassical domain wall

$$\begin{tikzcd} & (\mathcal{Y}\times\mathcal{Y}', \mu+\mu') \arrow[dl] \arrow[dr] & \\ (\mathcal{X}_1, \lambda_1) & & (\mathcal{X}_2, \lambda_2) \end{tikzcd} \tag{2.42}$$

This corresponds to tensoring with the $(m-1)$-dimensional theory $(\mathcal{Y}', \mu')$ on the domain wall.

(2.43) *Quantization of a semiclassical domain wall.* To quantize a semiclassical domain wall, we use (2.40) to construct a mapping space. Let M be a closed manifold presented as a union

$$M = M_1 \cup_Z M_2 \tag{2.44}$$

of manifolds with boundary along the boundary Z; then $Z \subset M$ is a codimension 1 cooriented submanifold. Form the mapping space

$$\mathcal{M} = \bigl\{(\phi_1, \phi_2, \psi) : \phi_i\colon M_i \to \mathcal{X}_i,\ \psi\colon Z \to \mathcal{Y},\ f_i \circ \psi = \phi_i\big|_Z\bigr\}. \tag{2.45}$$

Now quantize $\mathcal{M}$ as illustrated in Example 2.27.

(2.46) *Composition.* Composition of semiclassical domain walls proceeds by homotopy fiber product. Suppose $(\mathcal{X}_1, \lambda_1)$, $(\mathcal{X}_2, \lambda_2)$, $(\mathcal{X}_3, \lambda_3)$ are π-finite spaces and degree m cocycles, and let

$$\begin{aligned} (\mathcal{Y}', \mu')\colon (\mathcal{X}_1, \lambda_1) &\longrightarrow (\mathcal{X}_2, \lambda_2) \\ (\mathcal{Y}'', \mu'')\colon (\mathcal{X}_2, \lambda_2) &\longrightarrow (\mathcal{X}_3, \lambda_3) \end{aligned} \tag{2.47}$$

be semiclassical domain walls. Their composition

$$(\mathcal{Y}, \mu)\colon (\mathcal{X}_1, \lambda_1) \longrightarrow (\mathcal{X}_3, \lambda_3) \tag{2.48}$$

is constructed via the homotopy fiber product

$$\begin{tikzcd}[column sep=small] & & \mathcal{Y} \arrow[dl, dashed] \arrow[dr, dashed] & & \\ & \mathcal{Y}' \arrow[dl] \arrow[dr] & & \mathcal{Y}'' \arrow[dl] \arrow[dr] & \\ \mathcal{X}_1 & & \mathcal{X}_2 & & \mathcal{X}_3 \end{tikzcd} \tag{2.49}$$

which is the composition of correspondence diagrams (in the homotopy category); the trivialization μ of $\lambda_3 - \lambda_1$ is the sum $\mu_1 + \mu_2$. (For ease of reading, we omitted pullbacks in the previous clause.) We write (2.49) with cocycles and trivializations as follows:

$$\begin{tikzcd}[column sep=small] & & (\mathcal{Y}, \mu' + \mu'') \arrow[dl, dashed] \arrow[dr, dashed] & & \\ & (\mathcal{Y}', \mu') \arrow[dl] \arrow[dr] & & (\mathcal{Y}'', \mu'') \arrow[dl] \arrow[dr] & \\ (\mathcal{X}_1, \lambda_1) & & (\mathcal{X}_2, \lambda_2) & & (\mathcal{X}_3, \lambda_3) \end{tikzcd} \tag{2.50}$$

(2.51) *Boundaries.* As in **(1.30)** we specialize domain walls to boundary theories, here in the semiclassical world of finite homotopy theories.

Definition 2.52. Let $\mathcal{X}$ be a π-finite space and suppose λ is a cocycle of degree m on $\mathcal{X}$.

(1) A *right semiclassical boundary theory* of $(\mathcal{X}, \lambda)$ is a pair $(\mathcal{Y}, \mu)$ consisting of a π-finite space $\mathcal{Y}$, a map $f : \mathcal{Y} \to \mathcal{X}$, and a trivialization μ of $-f^*\lambda$.
(2) A *left semiclassical boundary theory* of $(\mathcal{X}, \lambda)$ is a pair $(\mathcal{Y}, \mu)$ consisting of a π-finite space $\mathcal{Y}$, a map $f : \mathcal{Y} \to \mathcal{X}$, and a trivialization μ of $f^*\lambda$.

The mapping spaces used for quantization specialize (2.45).

In this finite homotopy context there is a special form for a regular boundary theory.

Definition 2.53. Let $\mathcal{X}$ be a π-finite space and suppose λ is a cocycle of degree m on $\mathcal{X}$. A *semiclassical right regular boundary theory* of $(\mathcal{X}, \lambda)$ is a basepoint $f : * \to \mathcal{X}$ and a trivialization μ of $-f^*\lambda$.

In terms of Definition 2.52, the semiclassical right regular boundary theory is $(*, \mu)$. If the cocycle has positive degree in an ordinary (Eilenberg-MacLane) cohomology theory, then it vanishes on a point so we can and do take $\mu = 0$.

Remark 2.54. The regular boundary condition amounts to an extra semiclassical (fluctuating) field on the boundary which is a trivialization of the bulk field (map to $\mathcal{X}$).

Exercise 2.55. Define a domain wall between boundary theories. For any $(m, \mathcal{X}, 0)$ consider the right semiclassical boundary theory D given by a basepoint $* \to \mathcal{X}$ and the right semiclassical boundary theory D given by the identity map $\mathrm{id}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$. Prove that there is a unique domain wall $\mathrm{D} \to \mathrm{N}$ and a unique domain wall $\mathrm{N} \to \mathrm{D}$. What is their composition (in both orders)?

Example 2.56. Let $m = 2$. Fix a finite group G and let $\mathcal{X} = BG$ with basepoint $* \to BG$. The quantization of the interval depicted in Figure 4 is the quantization of the restriction map to the right endpoint

$$\mathrm{Map}\big(([0,1],\{0\}),(BG,*)\big) \longrightarrow \mathrm{Map}\big(\{*\},BG\big), \tag{2.57}$$

which up to homotopy is the map $* \to BG$. Choose the codomain $\mathcal{C} = \mathrm{Cat}$ so that, as in (2.31), the quantization of $\mathrm{Map}(*, BG)$ is the category $\mathrm{Vect}(BG) \simeq \mathrm{Rep}(G)$. Then the quantization of the map $* \to BG$, or better of the correspondence

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ * & & BG \end{tikzcd} \tag{2.58}$$

of mapping spaces derived from Figure 4, is the pushforward of the trivial bundle over $*$ with fiber $\mathbb{C}$ (the tensor unit). This is the regular representation of G in $\mathrm{Rep}(G)$. If, instead, we choose $\mathcal{C} = \mathrm{Alg}(\mathrm{Vect})$, then BG quantizes to the group algebra $\mathbb{C}[G]$ and $* \to BG$ quantizes to the right regular module.

Exercise 2.59. Incorporate a nonzero cocycle in the form of a central extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow G^\tau \longrightarrow G \longrightarrow 1 \tag{2.60}$$

(2.61) *A special sandwich.* Let $(\mathcal{X}, \lambda)$ be given and suppose $(\mathcal{Y}', \mu')$ and $(\mathcal{Y}'', \mu'')$ are right and left semiclassical boundary theories for $(\mathcal{X}, \lambda)$. Then, as a special case of the composition (2.50), the $(m-1)$-dimensional semiclassical sandwich of $(\mathcal{X}, \lambda)$ between $(\mathcal{Y}', \mu')$ and $(\mathcal{Y}'', \mu'')$ has as its semiclassical data the pair $(\mathcal{Y}' \overset{h}{\times}_{\mathcal{X}} \mathcal{Y}'', \mu' + \mu'')$, where $\mathcal{Y}' \overset{h}{\times}_{\mathcal{X}} \mathcal{Y}''$ is the homotopy fiber product; observe that $\mu' + \mu''$ is a cocycle of degree $m - 1$.

Defects

Domain walls and boundaries are special cases of the general notion of a *defect* in a field theory. Our discussion here is specifically for *topological* theories, though with modifications it applies more generally (see Remark 2.68(7) below).

(2.62) *Preliminary: the category* $\mathrm{Hom}(1, x)$. We will encounter the expression '$\mathrm{Hom}(1, x)$' in a higher symmetric monoidal category, so we begin by elucidating its meaning. Suppose Vect is the symmetric monoidal category of vector spaces, and $V \in$ Vect is an object, i.e., a vector space. The tensor unit 1 in Vect is the vector space $\mathbb{C}$ of scalars. We usually identify a linear map $T \in \mathrm{Hom}(\mathbb{C}, V)$ with $T(1) \in V$, so in this case $\mathrm{Hom}(1, V)$ is the space of vectors in V. Similarly, if $C \in$ Cat is a category, then $\mathrm{Hom}(1, C)$ can be identified with the objects in C. On the other hand, for the Morita 2-category Alg(Vect) of complex algebras, the tensor unit 1 is the algebra $\mathbb{C}$ and for any algebra A, the 1-category $\mathrm{Hom}(1, A)$ is the category of left A-modules.

Let $\mathcal{C}$ be a symmetric monoidal n-category. If $x \in \mathcal{C}$, then $\mathrm{Hom}(1, x)$ is an $(n-1)$-category. It is possible that it is empty, or in our usual linear situation, that it only contains the zero object. We also use '$\mathrm{Hom}(1, x)$' when $x \in \Omega^\ell\mathcal{C}$ is in some looping of $\mathcal{C}$. Then the homs are taken in $\Omega^\ell\mathcal{C}$.

(2.63) *Definition of a defect in a topological theory.* Suppose m is a positive integer, $\mathcal{F}$ is a collection of background fields, and

$$\sigma\colon \mathrm{Bord}_m(\mathcal{F}) \longrightarrow \mathcal{C} \tag{2.64}$$

is a topological field theory with values in a symmetric monoidal m-category $\mathcal{C}$. We describe defects of *codimension* ℓ in a k-dimensional manifold M, where $k \in \{0, 1, \dots, m\}$, $\ell \in \{1, \dots, m\}$, and $\ell \leqslant k$. Let $Z \subset M$ be a submanifold of codimension ℓ, and let $\nu \subset M$ be an open tubular neighborhood of $Z \subset M$; assume the closure $\bar{\nu}$ is the total space of a fiber bundle $\bar{\nu} \to Z$ with fiber the closed ℓ-dimensional disk. The fiber over $p \in Z$ is denoted $\bar{\nu}_p$; its boundary $\partial\bar{\nu}_p$ is diffeomorphic to the ℓ-dimensional sphere $S^{\ell-1}$. It is the *link* of $Z \subset M$ at p; see Figure 16.

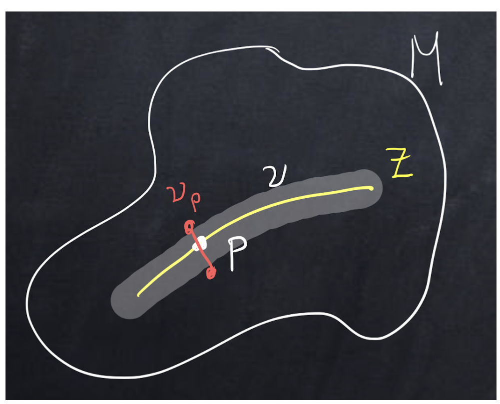

FIGURE 16. The tubular neighborhood and link of a submanifold

Definition 2.65. Assume that M is a closed manifold and $Z \subset M$ is a closed submanifold.

(1) A *local defect* at $p \in Z$ is an element

$$\delta_p \in \mathrm{Hom}\big(1, \sigma(\partial\bar{\nu}_p)\big). \tag{2.66}$$

(2) The *transparent (local) defect* is $\delta_p = \sigma(\bar\nu_p)$.
(3) A *global defect* on Z is a vector

$$\delta_Z \in \operatorname{Hom}\bigl(1, \sigma(\partial\bar\nu)\bigr). \tag{2.67}$$

(4) The *transparent (global) defect* is $\delta_Z = \sigma(\bar\nu)$.

The transparent defects can safely be erased.

Remark 2.68. We make several comments about this definition.

(1) $M\backslash\nu$ is a compact manifold with boundary $\partial\bar\nu$. Define the bordism $W\colon \partial\bar\nu \to \emptyset$ by letting the boundary be incoming. If δ_Z is a global defect, we evaluate the theory on (M, Z, δ_Z) as $\sigma(W)(\delta_Z)$. This is of the same type as the value $\sigma(M)$ on the closed manifold M: a complex number if $\dim M = m$, a complex vector space if $\dim M = n-1$, etc.
(2) A normal framing of $Z \subset W$ identifies each link $\partial\bar\nu_p$, $p \in Z$, with the standard sphere $S^{\ell-1}$. Assuming enough finiteness, $\sigma(S^{\ell-1}) \in \Omega^{\ell-1}\mathcal{C}$ defines[22] an $(m-\ell+1)$-dimensional field theory $\sigma^{(\ell-1)}$—the dimensional reduction of σ along $S^{\ell-1}$—and a local defect δ_p determines a left boundary theory $\delta^{(\ell-1)}$ for $\sigma^{(\ell-1)}$, again assuming sufficient finiteness. Using the normal framing it makes sense to assign a single local defect

$$\delta \in \operatorname{Hom}\bigl(1, \sigma(S^{\ell-1})\bigr) \tag{2.69}$$

to Z. The cobordism hypothesis computes an associated global defect

$$\delta_Z \in \operatorname{Hom}\bigl(1, \sigma^{(\ell-1)}(Z)\bigr) = \operatorname{Hom}\bigl(1, \sigma(Z \times S^{\ell-1})\bigr) \tag{2.70}$$

which we can use to evaluate the theory σ on (M, Z, δ) as in (1). Absent the normal framing, we can use a flat family of local defects and some twisted dimensional reduction. We can also use objects δ which are invariant under a subgroup of the orthogonal group. For example, see §8.1 of [23] for a discussion of topological defects using orientation in place of normal framing.
(3) A defect on Z may be tensored with a standalone field theory on Z to obtain a new defect. This corresponds to composing with an element of $\operatorname{Hom}(1,1)$ in (2.66) or (2.67). We illustrated that in a special case in Remark 2.41(2).
(4) The sheaf of background fields on a defect need not agree with the sheaf of background fields in the bulk; the former need only map to the latter. Thus we can have a spin defect in an theory of oriented manifolds.
(5) Defects are also defined for manifolds M with boundaries and corners, and we also allow boundaries, corners, and singularities in Z. In short, we allow Z to be a stratified manifold. Then different strata of Z have different links, and we compute them and assign (local) defects working from the lowest codimension to the highest. We give several illustrations in these lectures.

[22] The looping $\Omega\mathcal{C}$ of the symmetric monoidal n-category $\mathcal{C}$ is the symmetric monoidal $(n-1)$-category $\operatorname{Hom}(1,1)$ of endomorphisms of the tensor unit. We can iterate this construction. The theory $\sigma^{\ell-1}$ takes values in $\Omega^{\ell-1}\mathcal{C}$. The cobordism hypothesis, both with and without singularities, is used to define the theory and its boundary theory.

[23] Daniel S. Freed and Constantin Teleman, *Topological dualities in the Ising model*, arXiv:1806.00008.

(6) If M is closed with no boundary or corners, then $V = \sigma(M)$ is a vector space and $\sigma\bigl([0,1]\times M\bigr)$ is the identity map id_V. A defect supported in the interior of $[0,1]\times M$ evaluates under σ to a linear operator on V. In this situation the terms 'operator' and 'observable' are often used in place of with 'defect'.

(7) There are also (nontopological) defects in nontopological theories, but then unless we are in maximal codimension an extension beyond a two-tier theory is perhaps implicit. In nontopological theories local defects take values in a limit as the radius of the linking sphere shrinks to zero. For $\dim M = m$ and $\dim Z = 0$ the resulting *point defects* are often called 'local operators'. For $\dim Z = 1$ they are *line defects*.[24] It is increasingly common to work with a *1-category* of line defects; higher dimensional defects and higher categories of defects also appear in the physics literature.

(2.71) *Composition law on local defects.* The value of a topological field theory σ on $S^{\ell-1}$ is an E_ℓ-algebra. By **(2.16)**, this leads to a composition law on defects, either for local defects (2.66), (2.69) or global defects (2.67), (2.70). If Z is normally framed, one can consider two parallel copies Z', Z'', and then a normal slice of the complement of open tubular neighborhoods of Z', Z'' inside a closed tubular neighborhood of Z is the "pair of pants" which defines the composition law of the E_ℓ-structure. The composition law on topological point defects is a topological version of the usual operator product expansion. The composition law gives rise to the dichotomy between *invertible defects* and *noninvertible defects*.

Semiclassical defects in finite homotopy theories

Let F be an m-dimensional finite homotopy theory based on a π-finite space $\mathcal{X}$ and a cocycle λ of degree m on $\mathcal{X}$.

(2.72) *Preliminary on transgression.* Fix $\ell \in \{2,\dots,m\}$. Define the iterated free loop space

$$\mathcal{L}^{\ell-1}\mathcal{X} = \mathrm{Map}(S^{\ell-1},\mathcal{X}). \tag{2.73}$$

Consider the diagram

$$\begin{tikzcd} \mathcal{L}^{\ell-1}\mathcal{X}\times S^{\ell-1} \arrow[r, "e"] \arrow[d, "\pi_1"'] & \mathcal{X} \\ \mathcal{L}^{\ell-1}\mathcal{X} & \end{tikzcd} \tag{2.74}$$

in which e is evaluation and π_1 is projection onto the first factor. The composition $(\pi_1)_*(e)^*\lambda$ is a cocycle of degree $m-\ell+1$ on $\mathcal{L}^{\ell-1}\mathcal{X}$ called the *transgression* of λ. Note that the stable framing of the sphere is used to execute the pushforward.

[24]And for $\dim Z = 2$ they are called *surface defects*. The progression point-line-surface is an uncomfortable mishmash of point-line-plane (affine geometry) and point-curve-surface (differential geometry).

(2.75) *Semiclassical local defects.* For a defect on a submanifold of codimension $\ell \in \mathbb{Z}^{\geqslant 1}$, the link is $S^{\ell-1}$—canonically if the normal bundle is framed—and so the mapping space on the link is[25] $\mathcal{L}^{\ell-1}\mathcal{X}$, the iterated free loop space of $\mathcal{X}$ defined in (2.73). By **(2.72)** the cocycle λ transgresses to a cocycle $\tau^{\ell-1}\lambda$ on $\mathcal{L}^{\ell-1}\mathcal{X}$ with a drop of degree by $\ell - 1$. Recall the definition of a local defect in Definition 2.65(1) .

Definition 2.76. Fix $m, \ell \in \mathbb{Z}^{\geqslant 2}$ with $\ell \leqslant m$. Let $\mathcal{X}$ be a π-finite space and suppose λ is a cocycle of degree m on $\mathcal{X}$. A *semiclassical local defect* of codimension ℓ for $(\mathcal{X}, \lambda)$ is a π-finite map

$$\delta\colon \mathcal{Y} \longrightarrow \mathcal{L}^{\ell-1}\mathcal{X} \tag{2.77}$$

and a trivialization μ of $\delta^*(\tau^{\ell-1}\lambda)$.

Since $\mathcal{L}^{\ell-1}\mathcal{X}$ is π-finite, (2.77) amounts to a π-finite space $\mathcal{Y}$ and a continuous map δ. The local quantum defect in $\operatorname{Hom}\bigl(1, F(S^{\ell-1})\bigr)$ is the quantization of the map (2.77); see Remark 2.36 for an analogous quantization.

(2.78) *Semiclassical global defects.* To pass from local to global we use a tangential structure. As an example, if M is a closed manifold and $Z \subset M$ is a *normally framed* codimension ℓ submanifold on which the defect (2.77) is placed, the value of the theory F on M with the defect on Z is the quantization of the mapping space

$$\operatorname{Map}\bigl((M, Z), (\mathcal{X}, \mathcal{Y})\bigr) \tag{2.79}$$

consisting of pairs of maps $\phi\colon M \to \mathcal{X}$ and $\psi\colon Z \to \mathcal{Y}$ which satisfy a compatibility condition: if $Z \times S^{\ell-1} \hookrightarrow M$ is the inclusion of the boundary of a tubular neighborhood of $Z \subset M$, and $\phi'\colon Z \to \mathcal{L}^{\ell-1}\mathcal{X}$ is the transpose of the composition

$$Z \times S^{\ell-1} \hookrightarrow M \xrightarrow{\ \phi\ } \mathcal{X}, \tag{2.80}$$

then the diagram

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "\delta"] \\ Z \arrow[ur, "\psi"] \arrow[r, "\phi'"] & \mathcal{L}^{\ell-1}\mathcal{X} \end{tikzcd} \tag{2.81}$$

is required to commute.

Remark 2.82.

[25]The iterated free loop space notation holds for $\ell \geqslant 2$. The case $\ell = 1$ is a domain wall; see **(2.38)**

(1) One should use instead a mapping space of triples (ϕ,ψ,γ) where instead of demanding that (2.81) commute we specify a homotopy $\gamma\colon \delta\circ\psi\to\phi'$. This sort of derived mapping space should in principle replace all of the strict mapping spaces we write throughout the paper. However, the homotopy can be incorporated into a tubular neighborhood of Z, so in fact nothing is lost by using the strict mapping space.

(2) There are many variations of this basic scenario. The defect may have support on a manifold with boundary or corners, or more generally on a stratified manifold. Such is the case for the ρ-defects in Definition 2.92 below; a further example is in Figure 22.

(2.83) *Composition of semiclassical local defects.* The general composition law on local defects is constructed using the higher dimensional pair of pants or, in the case of ρ-defects as in Figure 18, using the higher dimensional pair of chaps. Here we state the semiclassical version of the first.

Resume the setup of Definition 2.76: $m,\ell\in\mathbb{Z}^{\geqslant 2}$ are integers with $\ell\leqslant m$, and $(\mathcal{X},\lambda)$ is the finite homotopy data for an m-dimensional theory F. Let P be the ℓ-dimensional pair of pants: as a manifold with boundary,

$$P = D^\ell\setminus B^\ell\amalg B^\ell, \tag{2.84}$$

where $B^\ell\amalg B^\ell$ are embedded balls in the interior of D^ℓ. As a bordism,

$$P\colon S^{\ell-1}\amalg S^{\ell-1}\longrightarrow S^{\ell-1}, \tag{2.85}$$

where the domain spheres are the inner boundaries of P and the codomain sphere is the outer boundary. The cocycle λ on $\mathcal{X}$ transgresses to an isomorphism

$$\mu\colon \pi_1^*(\tau^{\ell-1}\lambda)+\pi_2^*(\tau^{\ell-1}\lambda)\longrightarrow\tau^{\ell-1}\lambda \tag{2.86}$$

of cocycles on $\mathcal{X}^P$. Here $\pi_i\colon \mathcal{L}^{\ell-1}\mathcal{X}\times\mathcal{L}^{\ell-1}\mathcal{X}\to\mathcal{L}^{\ell-1}\mathcal{X}$ is projection onto the i^{th} factor, and in (2.86) we omit pullbacks under the source and target maps in the correspondence (2.87) below. Then the composition law on $F(S^{\ell-1})$ is the quantization of the correspondence

$$\begin{tikzcd}[column sep=small] & (\mathcal{X}^P,\,\mu) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\ \bigl(\mathcal{L}^{\ell-1}\mathcal{X}\times\mathcal{L}^{\ell-1}\mathcal{X},\,\pi_1^*(\tau^{\ell-1}\lambda)+\pi_2^*(\tau^{\ell-1}\lambda)\bigr) & & \bigl(\mathcal{L}^{\ell-1}\mathcal{X},\,\tau^{\ell-1}\lambda\bigr)\end{tikzcd} \tag{2.87}$$

The composition law on $F(S^{\ell-1})$ induces the composition law—the fusion product—on $\operatorname{Hom}\bigl(1,F(S^{\ell-1})\bigr)$, the higher category of local codimension ℓ defects. Suppose given $(\mathcal{Y}_1,\mu_1)$ and $(\mathcal{Y}_2,\mu_2)$ semiclassical local defects of codimension ℓ, as in Definition 2.76. Then their product in $\operatorname{Hom}\bigl(1, F(S^{\ell-1})\bigr)$ is the quantization of the composition $r_1 \circ g$ in the homotopy fiber product

$$\begin{tikzcd} & \bigl(\mathcal{Y},\, \pi_1^*\mu_1 + \pi_2^*\mu_2 + \mu\bigr) \arrow[dl, dashed] \arrow[d, dashed, "g"] & \\ \bigl(\mathcal{Y}_1 \times \mathcal{Y}_2,\, \pi_1^*\mu_1 + \pi_2^*\mu_2\bigr) \arrow[d] & \bigl(\mathcal{X}^P,\, \mu\bigr) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\ \bigl(\mathcal{L}^{\ell-1}\mathcal{X} \times \mathcal{L}^{\ell-1}\mathcal{X},\, \pi_1^*(\tau^{\ell-1}\lambda) + \pi_2^*(\tau^{\ell-1}\lambda)\bigr) & & \bigl(\mathcal{L}^{\ell-1}\mathcal{X},\, \tau^{\ell-1}\lambda\bigr) \end{tikzcd} \tag{2.88}$$

This diagram is the general semiclassical composition law on semiclassical defects.

Remark 2.89. The identity object—the tensor unit—in $\operatorname{Hom}\bigl(1, F(S^{\ell-1})\bigr)$ is the quantization of the semiclassical defect

$$\mathcal{X}^{D^\ell} \longrightarrow \mathcal{L}^{\ell-1}\mathcal{X} \tag{2.90}$$

given by the restriction from maps out of D^ℓ to maps out of its boundary $S^{\ell-1}$.

Defects and symmetry

Now, finally, we return to the setup for finite symmetry in field theory, as in Definitions 1.33 and 1.37.

(2.91) (σ, ρ)-*defects.* Fix a positive integer n. Suppose (σ, ρ) is n-dimensional symmetry data.

Definition 2.92. A (σ, ρ)-*defect* is a topological defect in the topological field theory (σ, ρ). We call it a ρ-*defect* if its support lies entirely in a ρ-colored boundary.

Figure 17 depicts some (σ, ρ)-defects. These are defects in the abstract symmetry theory. If F is a quantum field theory equipped with an (σ, ρ)-module structure $(\widetilde{F}, \theta)$, then a (σ, ρ)-defect induces a defect in the theory $(\sigma, \rho, \widetilde{F})$, and then θ maps it to a defect in the theory F. Since the defect in the sandwich picture is supported away from $\widetilde{F}$-colored boundaries, it is a *topological* defect in the theory F.

Remark 2.93. Computations with (σ, ρ)-defects, such as compositions, are carried out in the topological field theory (σ, ρ). They apply to the induced defects in any (σ, ρ)-module.

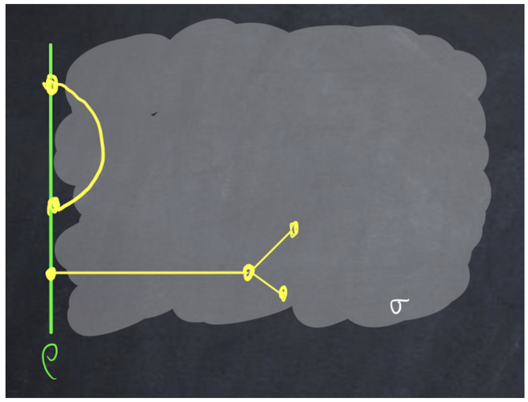

FIGURE 17. (σ, ρ)-defects

Remark 2.94. In higher dimensions, pictures such as Figure 17 are interpreted as a schematic for a tubular neighborhood of the support $Z \subset M$ of a defect on a manifold M (and its Cartesian product with $[0, 1]$). Also, unless otherwise stated, for ease of exposition we often implicitly assume a normal framing to Z so that its link may be identified with a standard sphere.

The image in F of a defect in the $(\sigma, \rho, \widetilde{F})$-theory may not be apparent; this is a significant advantage of the sandwich picture of F.

Example 2.95. Let $n = 3$ and consider a 3-dimensional quantum field theory F on S^3, and assume F has an (σ, ρ)-module structure. In the corresponding $(\sigma, \rho, \widetilde{F})$-theory we can contemplate a defect supported on a 2-disk D in $[0, 1) \times S^3$ whose boundary $K = \partial D \subset \{0\} \times S^3$ is a knot in the Dirichlet boundary. (Such a knot is termed 'slice'.) It is possible that K does not bound a disk in S^3—its Seifert genus may be positive. In this case the projection of the slice disk D to a defect in the theory F on S^3 is at best an immersed disk with boundary K, and it appears that such a topological defect is difficult to describe directly in the theory F.

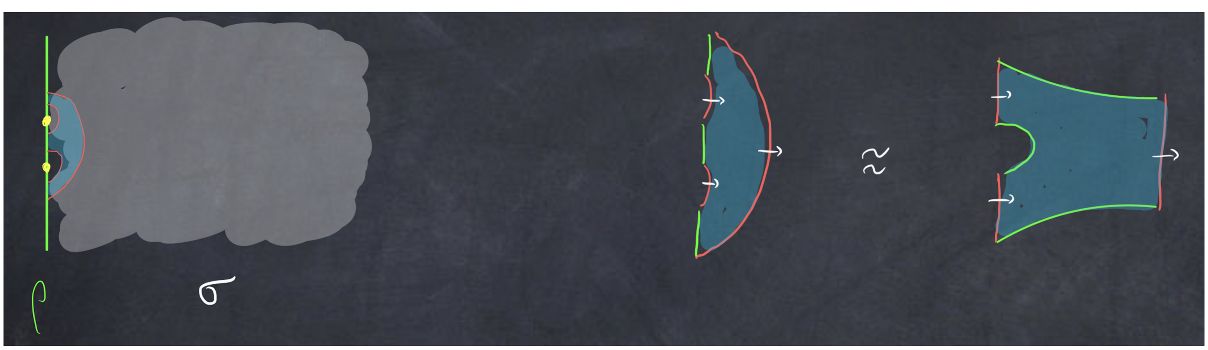

FIGURE 18. The composition law by evaluation on a pair of chaps

(2.96) *Composition of ρ-defects.* (What we say here also applies to more general (σ, ρ)-defects.) Since the labels on ρ-defects come from the topological field theory (σ, ρ) evaluated on the links, we compute the composition law by applying (σ, ρ) to a bordism whose boundary consists of links.

For concreteness, we again take up the quantum mechanical Example 1.40 from Lecture 1. The theory $\sigma = \sigma^{(2)}_{BG}$ is the 2-dimensional finite gauge theory with gauge group G a finite group which acts as symmetries of a quantum mechanical system $(\mathcal{H}, H)$. Figure 18 depicts the composition of two point ρ-defects. The link of such a defect evaluates under (σ, ρ) to the vector space which underlies the group algebra $A = \mathbb{C}[G]$. The composition law on point ρ-defects is computed by evaluating the[26] "pair of chaps" on the right in Figure 18. Picture vertical cross sections of this bordism as links of the two points as the move together and merge into a single point. The (σ, ρ)-value of the pair of chaps works out to be the multiplication map $A \otimes A \to A$ of the group algebra. In particular, on "classical labels" in $G \subset A$ it restricts to the group product $G \times G \to G$.

Exercise 2.97. Evaluate all of the bordisms in the previous paragraph using the finite path integral, as described in (**2.78**).

Remark 2.98. We make several observations that apply far beyond this particular example.

(1) This is a hint for Exercise 2.97! The mapping space of the link of a point ρ-defect is

$$\mathrm{Map}\big(([0,1], \{0,1\})\,,\,(BG, *)\big) \simeq \Omega BG \simeq G. \tag{2.99}$$

The mapping space of the pair of chaps C in Figure 18 fits into the correspondence diagram

$$\begin{tikzcd} & \mathrm{Map}\big((C, \partial C_\rho), (BG, *)\big) \arrow[dl] \arrow[dr] & \\ \Omega BG \times \Omega BG & & \Omega BG \end{tikzcd} \tag{2.100}$$

that encodes restriction to the incoming and outgoing boundaries. Here ∂C_ρ is the ρ-colored portion of ∂C. The left arrow in (2.100) is a homotopy equivalence and the right arrow is composition of loops.

(2) The computation in (2.100) generalizes to any pointed π-finite space $(\mathcal{X}, *)$ in place of $(BG, *)$. Then the correspondence is multiplication on the group $\Omega\mathcal{X}$, and the quantization is pushforward under multiplication, i.e., a convolution product. If the codomain of σ has the form $\mathrm{Alg}(\mathcal{C}')$, then compute $\sigma(\mathrm{pt})$ as follows: (1) quantize $\Omega\mathcal{X}$ to an object in $\mathcal{C}'$, and (2) induce the algebra structure from pushforward under multiplication $\Omega\mathcal{X} \times \Omega\mathcal{X} \to \Omega\mathcal{X}$.

(3) Even if we begin with a group symmetry, as in this example, there are noninvertible topological (σ, ρ)-defects. In this example, elements of the group algebra $\mathbb{C}[G]$ label point defects on the ρ-colored boundary, and the algebra $\mathbb{C}[G]$ contains noninvertible elements. This fits general quantum theory, which produces algebras rather than groups.

(4) (σ, ρ)-defects give rise to structure in any (σ, ρ)-module: linear operators on vector spaces of point defects and on state spaces, endofunctors on categories of line defects and categories of superselection sectors, etc. These can be used to explore dynamics.

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[26]This particular bordism is also known as *Gumby*: 

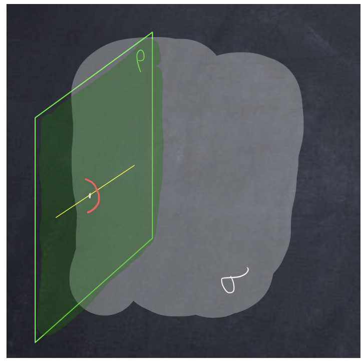

FIGURE 19. A line defect supported on the ρ-colored boundary

(2.101) *2-dimensional theories with finite symmetry.* Let G be a finite group and let $\sigma = \sigma^{(3)}_{BG}$ be finite pure 3-dimensional G-gauge theory. As an extended field theory, σ can take values in $\mathrm{Alg(Cat)}$, a suitable 3-category of tensor categories, in which case $\sigma(\mathrm{pt})$ is the fusion category $\mathcal{A} = \mathrm{Vect}[G]$ introduced in **(1.23)**. The right regular boundary theory ρ is constructed using the right regular module $\mathcal{A}_{\mathcal{A}}$. Alternatively, in terms of Definition 2.53, choose a basepoint in BG. There are no background fields for σ or ρ: (σ, ρ) is an unoriented theory.

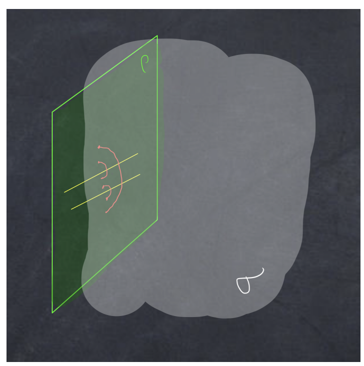

FIGURE 20. Fusion of line defects

The most familiar (σ, ρ)-defects are the codimension 1 defects supported on the ρ-colored boundary, as depicted in Figure 19. The link maps under (σ, ρ) to the quantization of the mapping space (2.99). (It is the same mapping space for the link of a codimension 1 defect in finite gauge theory of *any* dimension.) That quantization in this dimension is a linear category, the category $\mathrm{Vect}(G)$ of vector bundles over G; it is the linear category which underlies the fusion category $\mathcal{A}$. The fusion product—computed from the link in Figure 20, which is the same as the link in Figure 18—is derived from the correspondence (2.100) and is the fusion product of $\mathcal{A}$; see Remark 2.98(2). Each $g \in G$ gives rise to an *invertible* defect, labeled by the vector bundle over G whose fiber is $\mathbb{C}$ at g and is the zero vector space away from g.

Now consider a line defect supported in the bulk, as in Figure 21. The link is a circle, and so a local defect is an object in the category $\sigma(S^1) = \mathrm{Vect}_G(G)$ of G-equivariant vector bundles over G. (Here G acts on itself via conjugation.) This is the (Drinfeld) center of $\mathcal{A}$. Note that unlike the quantum mechanical situation in Figure 11, the center here is "larger" than the algebra $\mathcal{A}$. The

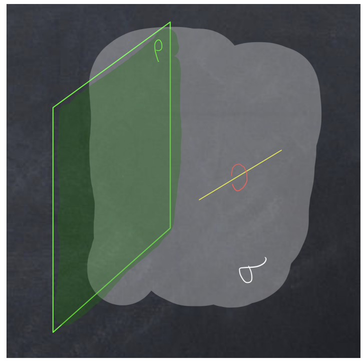

Figure 21. A line defect supported in the bulk

simple objects of the center are labeled by a pair consisting of a conjugacy class and an irreducible representation of the centralizer of an element in the conjugacy class. The corresponding defect is invertible iff the representation is 1-dimensional. Among these defects are the Wilson and 't Hooft lines of the 3-dimensional G-gauge theory. There is a rich set of topological defects that goes beyond those labeled by group elements.

Remark 2.102. So, even if we begin with the invertible G-symmetry, we are inexorably led to "non-invertible symmetries".

Exercise 2.103. Verify that there are no non-transparent point defects, either on the ρ-colored boundary or in the bulk.

Exercise 2.104. Evaluate $\sigma(S^1)$ using the finite path integral.

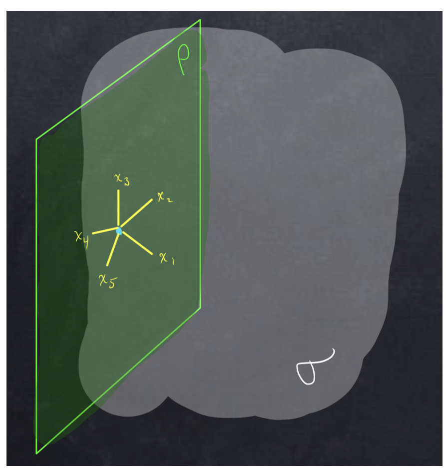

Figure 22. A stratified ρ-defect

(2.105) *Turaev-Viro theories.* The example of finite G-gauge theory generalizes to arbitrary Turaev-Viro theories. Let Φ be a spherical fusion category, let σ be the induced 3-dimensional topological field theory (of oriented bordisms) with $\sigma(\mathrm{pt}) = \Phi$, and define the regular boundary theory ρ via the right regular module Φ_Φ. The category of point ρ-defects is the linear category which underlies Φ, so a defect is labeled (locally) by an object of Φ. We can also have nontrivial stratified ρ-defects, such as illustrated in Figure 22. In the figure the x_i are objects of Φ and the label at the central point is a vector in $\mathrm{Hom}_{\Phi}(1, x_1 \otimes \cdots \otimes x_5)$.