Lecture 3: Quotients and projectivity

This lecture has two main subjects: quotients and projectivity. In these notes—but not in the actual lectures—we already treated these topics in the context of groups and algebras: see **(1.48)** to the end of Lecture 1. Here we take up quotients by a symmetry of a field theory. Working in the "sandwich" picture, which separates out the abstract topological symmetry from the potentially nontopological field theory on which it acts, the quotient is effected by replacing the right regular boundary theory (Dirichlet) with an augmentation (Neumann). We also introduce *quotient defects*, which amounts to executing the quotient construction on a submanifold. In the literature these are often called "condensation defects", and the quotienting process is called "gauging". Quantum theory takes place in projective geometry, not linear geometry, and so in the second half of the lecture we take up projectivity in the context of field theory. It is expressed via invertible field theories. The projectivity of an n-dimensional field theory is an $(n+1)$-dimensional invertible field theory—its *anomaly (theory)*—and trivializations of the anomaly form a torsor over the group of n-dimensional invertible field theories. In other words, the group of n-dimensional invertible field theories acts on the space of all n-dimensional field theories, and theories in the same orbit share many of the same properties. A symmetry may only act projectively, in which case it is said to enjoy an *'t Hooft anomaly*, and that obstructs the existence of an augmentation, so too obstructs the existence of a quotient. We conclude with an example of quotients and twisted quotients for theories with a BA-symmetry for a finite abelian group A.

Preliminary remarks

I began the lecture with two remarks.

(3.1) *Invertibility.* Anytime we have a (higher categorical) monoid—a set with an associative composition law $*$ and unit 1—then we have a notion of invertibility: an element/object x is *invertible* if there exists y such that $x * y = 1$ (or $x * y \cong 1$ in a categorical context). This applies in two situations: (1) the composition law ("stacking") of field theories, which leads to the notion of an *invertible field theory*; and (2) the composition law ("fusion") of local defects, which leads to the notion of an *invertible defect*.

(3.2) *Symmetries of a boundary theory.* Not every left boundary theory F of a field theory α indicates that α is acting as *symmetries* on F. In these lectures we define *finite* symmetry in terms of a pair (σ, ρ); simply having a left module F for a theory α is not an action of symmetry. Furthermore, in this paragraph we do not require that α be topological.[27] On the other hand, if (σ, ρ) is finite symmetry data, then there is a notion of (σ, ρ) acting by symmetries on F. Namely,

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[27]Nor do we require that α be invertible; if it is, then we say F is *anomalous* with *anomaly theory* α.

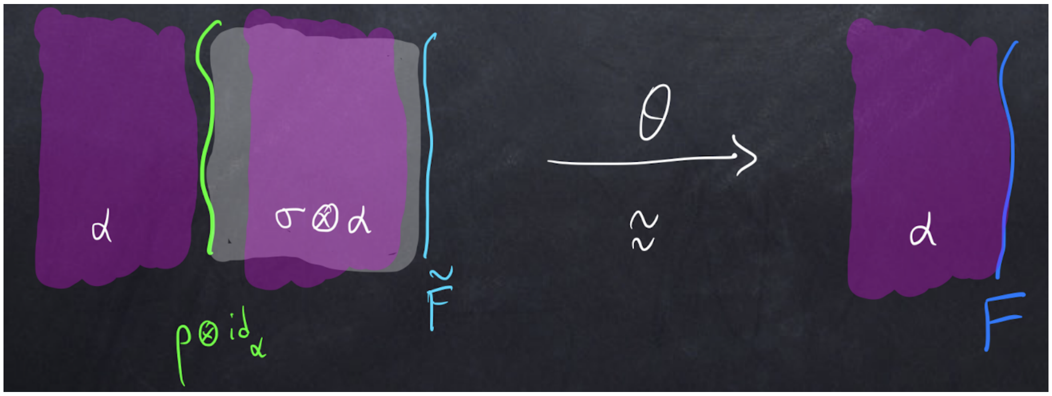

FIGURE 23. The action of (σ, ρ) on the boundary theory F of α

the *left module structure* data $(\widetilde{F}, \theta)$ is a left module $\widetilde{F}$ for $\sigma \otimes \alpha$ and an isomorphism as indicated in Figure 23.

Quotients by a symmetry in field theory

One should think that the quotients in this section are "derived" or "homotopical", though we do not deploy those modifiers. (See Exercise 1.56.)

(3.3) *Augmentations in higher categories.*

Definition 3.4. Let $\mathcal{C}'$ be a symmetric monoidal n-category, and set $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$. Suppose $A \in \mathcal{C}$ is an algebra object in $\mathcal{C}'$. Then an *augmentation* $\epsilon_A \colon A \to 1$ is an algebra homomorphism from A to the tensor unit $1 \in \mathcal{C}$.

Thus ϵ_A is a 1-morphism in $\mathcal{C}'$ equipped with data that exhibits the structure of an algebra homomorphism. Augmentations may not exist.

Remark 3.5. A general 1-morphism $A \to 1$ in $\mathcal{C}$ is an object of $\mathcal{C}'$ equipped with a right A-module structure. An augmentation is a right A-module structure on the tensor unit $1 \in \mathcal{C}'$.

(3.6) *Augmentations in field theory.*

Definition 3.7. Let $\mathcal{C}'$ be a symmetric monoidal n-category, and set $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$. Let $\mathcal{F}$ be a collection of $(n+1)$-dimensional fields, and suppose $\sigma \colon \mathrm{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is a topological field theory. A right boundary theory ϵ for σ is an *augmentation* of σ if $\epsilon(\mathrm{pt})$ is an augmentation of $\sigma(\mathrm{pt})$ in the sense of Definition 3.4.

An augmentation in this sense is often called a *Neumann boundary theory*.

(3.8) *The quotient theory.* We use notations in Definition 1.33 and Definition 1.37 in the following.

Definition 3.9. Suppose given finite symmetry data (σ, ρ) and a (σ, ρ)-module structure $(\widetilde{F}, \theta)$ on a quantum field theory F. Suppose ϵ is an augmentation of σ. Then the *quotient* of F by the symmetry σ is

$$F\big/_{\!\epsilon}\,\sigma = \epsilon \otimes_\sigma \widetilde{F}. \tag{3.10}$$

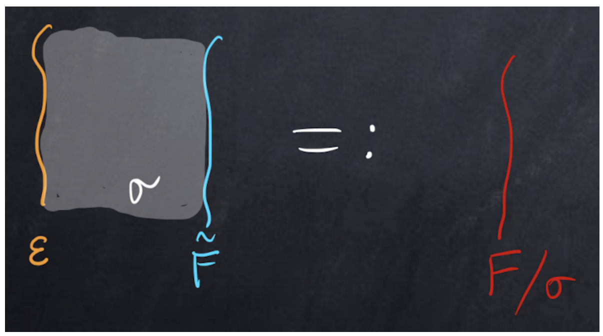

FIGURE 24. The quotient theory

We simply write 'F/σ' if the augmentation ϵ is understood from context. The right hand side of (3.10) is the sandwich in Figure 24.

(3.11) *Quotients in finite homotopy theories.* Recall the definition of a semiclassical boundary theory in **(2.51)**. We now tell what an augmentation is in this context.

Definition 3.12. Let $\mathfrak{X}$ be a π-finite space and suppose λ is a cocycle of degree m on $\mathfrak{X}$. A *semiclassical right augmentation* of $(\mathfrak{X}, \lambda)$ is a trivialization μ of $-\lambda$.

Observe that if $\lambda = 0$, then μ is a cocycle of degree m. Also, there is a canonical choice of μ in this instance: $\mu = 0$.

Remark 3.13. The cocycle λ encodes an *'t Hooft anomaly* in a finite homotopy type theory; it is the projectivity of the symmetry. If $\lambda = 0$, then a cocycle μ encodes a twist of the boundary theory, and it goes by various names: 'discrete torsion', 'θ-angles', etc., depending on the context.

Example 3.14. Let G be a finite group, and let $\sigma = \sigma_{BG}^{(n+1)}$ be the associated finite gauge theory. Use the canonical boundary theory $\mathrm{id}_{\mathfrak{X}}\colon \mathfrak{X} \to \mathfrak{X}$. In the semiclassical picture this corresponds to summing over all principal G-bundles with no additional fields on the ϵ-colored boundaries. This is the usual quotienting operation, oft called 'gauging'.

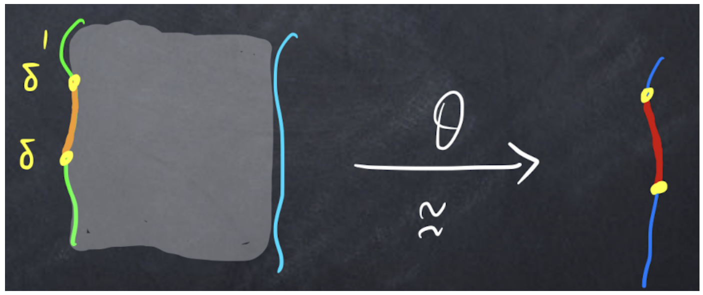

FIGURE 25. The domain walls δ and δ'

(3.15) *The Dirichlet-Neumann and Neumann-Dirichlet domain walls.*

Lemma 3.16. *Let σ be a topological field theory with codomain $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$, and suppose ρ is the right regular boundary theory of σ and ϵ is an augmentation of σ. Then the category of domain walls from σ to ϵ is the trivial theory, as is the category of domain walls from ϵ to σ.*

Roughly: Use the homomorphism $\epsilon(\mathrm{pt})\colon A \to 1$ to make 1 into a left A-module, where $A = \sigma(\mathrm{pt})$, and so construct a dual *left* boundary theory ϵ^L. Then the sandwich $\rho \otimes_\sigma \epsilon^L$ is the trivial theory: use the cobordism hypothesis to compute its value on a point as $A \otimes_A 1 \cong 1$. Let

$$\begin{aligned} \delta\colon \rho &\longrightarrow \epsilon \\ \delta'\colon \epsilon &\longrightarrow \rho \end{aligned} \tag{3.17}$$

be generating domain walls.

(3.18) *The composition.* Our task is to compute the composition

$$\delta' \circ \delta\colon \rho \longrightarrow \rho, \tag{3.19}$$

which is a self-domain wall of the boundary theory ρ. (The reverse composition $\delta \circ \delta'$ is similar.) As always, that computation is done by tracking the links as the points come together, and we obtain the pair of chaps extracted in Figure 26 and isolated in Figure 27. In that figure we have labeled the incoming boundary components in accordance with Lemma 3.16. For the outgoing boundary component we are assuming $\sigma(\mathrm{pt}) = A$ is an algebra in $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$; the label A in the figure is the underlying object of $\mathcal{C}'$. This evaluates to an object in $\mathrm{Hom}_{\mathcal{C}'}(1, A)$. We evaluate it in two cases.

Example 3.20 (Turaev-Viro symmetry)**.** Suppose $n = 2$ and the 3-dimensional theory σ is of Turaev-Viro type with $\sigma(\mathrm{pt}) = \mathcal{A}$ a fusion category. Assume ρ is given by the right regular module $\mathcal{A}_\mathcal{A}$ and ϵ is given by a fiber functor $\epsilon_\Phi\colon \mathcal{A} \to \mathrm{Vect}$. Then the codimension 1 quotient defect has local label the object $x_{\mathrm{reg}} \in \mathcal{A}$ defined as

$$x_{\mathrm{reg}} = \sum_x \epsilon_\Phi(x)^* \otimes x, \tag{3.21}$$

where the sum is over a representative set of simple objects x. See Proposition 8.9 in [23] for a very similar computation.

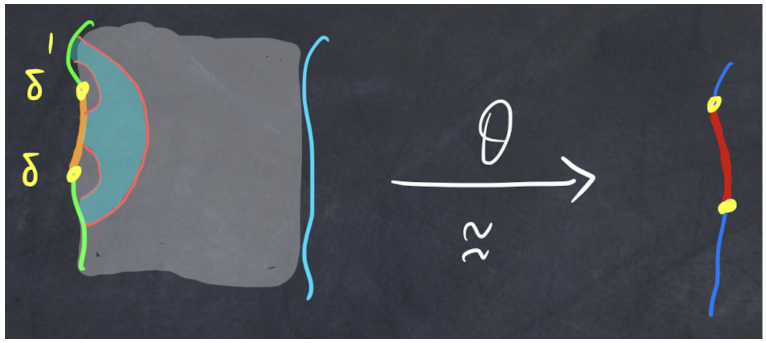

FIGURE 26. Computation of $\delta' \circ \delta$

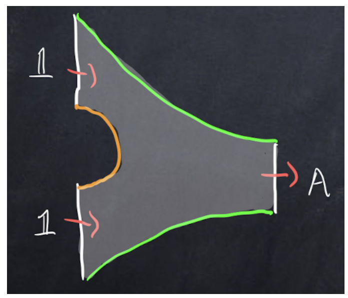

FIGURE 27. The pair of chaps: note the coloring of the boundary components

Example 3.22 (finite homotopy theories)**.** Let $\mathcal{X}$ be a π-finite space. Then the composition is the homotopy fiber product

$$\begin{tikzcd}
 & & \Omega\mathcal{X} \arrow[dl, dashed] \arrow[dr, dashed] & & \\
 & * \arrow[dl] \arrow[dr] & & * \arrow[dl] \arrow[dr] & \\
* \arrow[drr] & & \mathcal{X} \arrow[d] & & * \arrow[dll] \\
 & & \mathcal{X} & &
\end{tikzcd} \tag{3.23}$$

which is then the domain wall

$$\begin{tikzcd}
 & \Omega\mathcal{X} \arrow[dl] \arrow[dr] & \\
* \arrow[dr] & & * \arrow[dl] \\
 & \mathcal{X} &
\end{tikzcd} \tag{3.24}$$

Note that the points $*$ in the second row of (3.23) are obtained as fiber products from the parts of the diagram which lie below, and the fact that they are single points (contractible spaces) proves Lemma 3.16 in the finite homotopy theory case.

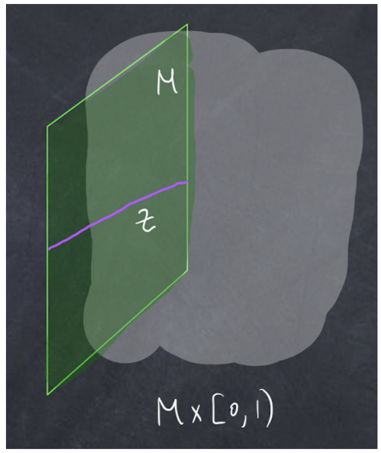

FIGURE 28. The defect $\delta' \circ \delta$ on Z

Example 3.25 (special case)**.** Now take $\mathcal{X} = BG$ for G a finite group. Then $\Omega\mathcal{X} \simeq G$, so the composition $\delta' \circ \delta$ sums over maps to G. Suppose M is a bordism on which we evaluate F, and suppose $Z \subset M$ is a cooriented codimension submanifold on which we place the defect $\delta' \circ \delta$. (As usual, we do not make background fields explicit.) Form the sandwich $[0,1) \times M$ with $\{0\} \times (M \backslash Z)$ colored with ρ and $\{0\} \times Z$ colored with $\delta' \circ \delta$. Then the theory sums over principal G-bundles together with a trivialization on the ρ-colored boundary $\{0\} \times (M \backslash Z)$. For each $g \in G$ there is a defect $\eta(g)$ (of "'t Hooft type") which constrains the jump in the trivializations across Z to be g. Then quantization using (3.24) shows

$$\delta' \circ \delta = \sum_{g \in G} \eta(g). \tag{3.26}$$

This equation appears in several recent physics papers. A similar equation holds for $\mathcal{X}$ in general, and in particular for $\mathcal{X} = B^q A$ for a finite abelian group A, only now automorphisms appear in the quantization.

Quotient defects: quotienting on a submanifold

The following discussion is inspired by the paper [28]. The basic idea is to execute the quotient construction on a submanifold, not necessarily to take the quotient of the entire theory.

Fix a positive integer n and finite n-dimensional symmetry data (σ, ρ). Suppose ϵ is an augmentation of σ, as in Definition 3.7. As explained in Definition 3.9, if $(\widetilde{F}, \theta)$ is a (σ, ρ)-module structure on an n-dimensional quantum field theory F, then dimensional reduction of σ depicted in Figure 24, which is the sandwich $\epsilon \otimes_\sigma \widetilde{F}$, is the quotient F/σ of F by the symmetry. This can be interpreted as placing the topological defect ϵ on the entire theory.

There is a generalization which places the defect on a submanifold. Suppose M is a bordism on which we evaluate F, and suppose $Z \subset M$ is a submanifold on which we place the defect. (As

[28] Konstantinos Roumpedakis, Sahand Seifnashri, and Shu-Heng Shao, *Higher Gauging and Non-invertible Condensation Defects*, arXiv:2204.02407.

usual, we do not make background fields explicit.) Form the sandwich $[0,1)\times M$ with $\{0\}\times M$ colored with ρ. Let $\nu\subset M$ be an open tubular neighborhood of $Z\subset M$ with projection $\pi\colon \nu\to Z$, and arrange that the closure $\bar\nu$ of ν is the total space of a disk bundle $\bar\nu\to Z$.

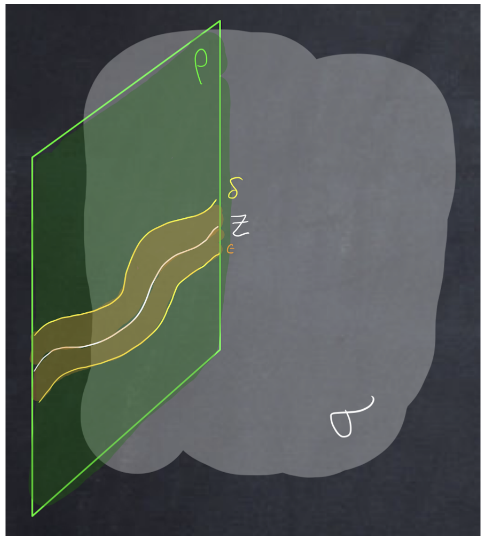

Figure 29. The quotient defect $\epsilon(Z)$

Definition 3.27. The *quotient defect* $\epsilon(Z)$ is the ρ-defect supported on $\{0\}\times\bar\nu$ with $\{0\}\times\nu$ colored with ϵ and $\{0\}\times\partial\bar\nu$ colored with δ.

This defect is depicted in Figure 29.

Next we compute the local label of the quotient defect $\epsilon(Z)$, as in Definition 2.65(1), and so express $\epsilon(Z)$ as a defect supported on Z. Consider a somewhat larger tubular neighborhood, now of $\{0\}\times Z\subset[0,1)\times M$. Let $\ell=\operatorname{codim}_M Z$. The tubular neighborhood for $\ell=1$ is depicted in Figure 30. It is a pair of chaps, two of whose incoming boundary components are δ-colored. Its value in the topological theory σ—with boundaries and defects ρ,ϵ,δ—is an object in $\operatorname{Hom}\bigl(1,\sigma(D^1,S^0_\delta)\bigr)$. (If $\mathcal{C}=\operatorname{Alg}(\mathcal{C}')$ is the codomain of σ, and $\sigma(\mathrm{pt})=A$ is an algebra object in $\mathcal{C}'$, then $\sigma(D^1,S^0_\delta)=A$ as an object of $\mathcal{C}'$.)

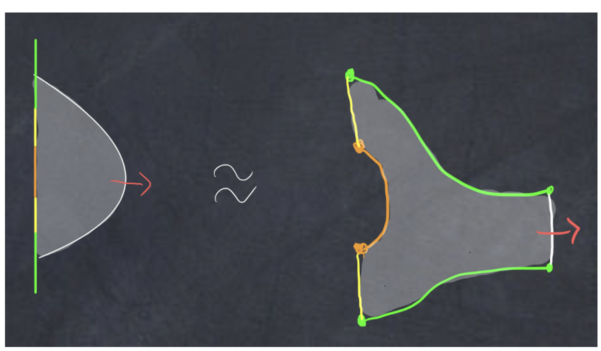

Figure 30. The local label of $\epsilon(Z)$ in codimension 1

Remark 3.28. The pair of chaps picture makes clear that the defect $\epsilon(Z)$ for $\ell = 1$ can be interpreted as follows, assuming $Z \subset M$ has trivialized normal bundle. Let Z_1, Z_2 be parallel normal translates of Z, color the region in between $\{0\} \times Z_1$ and $\{0\} \times Z_2$ with ϵ, color the remainder of $\{0\} \times M$ with ρ, and use the domain wall δ at $\{0\} \times Z_1$ and δ' at $\{0\} \times Z_2$. Then $\epsilon(Z)$ is the composition $\delta'(Z_2) * \delta(Z_1)$. If a quantum field theory F has a (σ, ρ)-module structure, then $\delta(Z_1)$ is a domain wall from F to F/σ and $\delta'(Z_2)$ is a domain wall from F/σ to F; the composition $\epsilon(Z)$ is a self domain wall of F, precisely the one computed in (**3.18**).

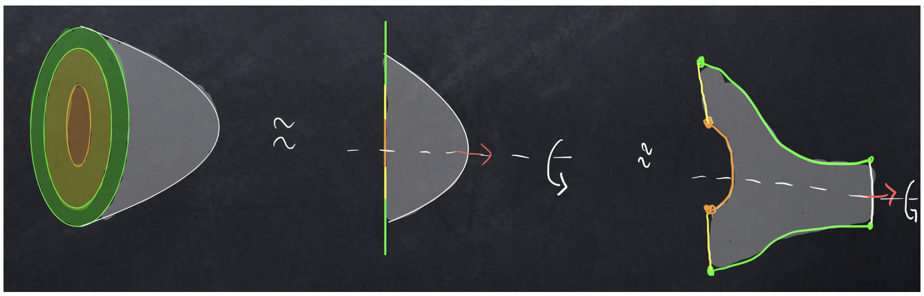

FIGURE 31. The local label of $\epsilon(Z)$ in codimension 2

The tubular neighborhood of $\{0\} \times Z \subset [0, 1) \times M$ for codimension $\ell = 2$ is the 3-dimensional bordism obtained from the pair of chaps by revolution in 3-space, as illustrated in Figure 31. For general $\ell > 1$, the bordism is the $(\ell + 1)$-disk $D^{\ell+1}$ with boundary S^ℓ partitioned as

$$\partial D^{\ell+1} = D^\ell_\epsilon \cup A^\ell_\rho \cup D^\ell \tag{3.29}$$

into disks D^ℓ and an annulus A^ℓ with the domain wall δ at the intersection of the ϵ and ρ-colored regions. (In Figure 31 that domain wall is thickened from a sphere $S^{\ell-1}$ to an annulus A^ℓ.)

Remark 3.30. These are the local defects. As always, the global defects are a section of a bundle (local system) of local defects over the submanifold $Z \subset M$.

Example 3.31 (finite homotopy theory)**.** Let $\sigma = \sigma^{(n+1)}_{\mathcal{X}}$ be the finite homotopy theory built from a π-finite space $\mathcal{X}$. Then we can use the semiclassical calculus for π-finite spaces to compute semiclassical spaces of defects. Suppose ρ is specified by a basepoint $* \to \mathcal{X}$ and ϵ is specified by the identity map $\mathcal{X} \xrightarrow{\text{id}} \mathcal{X}$. Then δ is specified by the homotopy fiber product

$$\begin{tikzcd} & * \arrow[dl, dashed] \arrow[dr, dashed] & \\ * \arrow[dr] & & \mathcal{X} \arrow[dl, "\text{id}"] \\ & \mathcal{X} & \end{tikzcd} \tag{3.32}$$

which is a point. This is the manifestation of the uniqueness of δ (Lemma 3.16), as already remarked after (3.24).

The semiclassical space of local ρ-defects of codimension ℓ is

$$\mathrm{Map}\big((D^\ell, S^{\ell-1}), (\mathcal{X}, *)\big) = \Omega^\ell \mathcal{X}. \tag{3.33}$$

Set

$$N^\ell = \big(D^{\ell+1}, D^\ell \cup A^\ell \cup D^\ell\big), \tag{3.34}$$

with boundary as in (3.29); see Figure 31. The semiclassical local label of the defect $\epsilon(Z)$ is

$$\mathrm{Map}(N^\ell, \mathcal{X}) \longrightarrow \Omega^\ell \mathcal{X}, \tag{3.35}$$

the map induced by restriction to $\Omega^\ell \mathcal{X}$.

Lemma 3.36. *There is a homotopy equivalence* $\mathrm{Map}(N^\ell, \mathcal{X}) \simeq \Omega^\ell \mathcal{X}$ *under which* (3.35) *is the identity map.*

Proof. Use the technique in Example 0.8 of [29]. First, deformation retract A^ℓ to $S^{\ell-1}$, and so define

$$\overline{N}^\ell = \big(D^{\ell+1}, D^\ell \cup S^{\ell-1} \cup D^\ell\big). \tag{3.37}$$

Choose a basepoint for $\overline{N}^\ell$ on $S^{\ell-1}$. Form the correspondence of pointed spaces

$$\begin{tikzcd} & \overline{N}^\ell \cup D^\ell \arrow[dl] \arrow[dr] & \\ \overline{N}^\ell / S^{\ell-1} & & D^\ell / S^{\ell-1} \end{tikzcd} \tag{3.38}$$

in which D^ℓ is attached to $S^{\ell-1} \subset \overline{N}^\ell$, the left map collapses this new D^ℓ, and the right map collapses $D^{\ell+1}$. Since $D^\ell, D^{\ell+1}$ are contractible, each of these arrows is a homotopy equivalence. Now take the pointed mapping spaces into $\mathcal{X}$. $\square$

The quantization of $\mathrm{id}\colon \Omega^\ell \mathcal{X} \to \Omega^\ell \mathcal{X}$ is typically a noninvertible object. For example, if the quantization is a vector space, then the vector space is[30] $\mathrm{Fun}(\pi_0 \Omega^\ell \mathcal{X}) = \mathrm{Fun}(\pi_\ell \mathcal{X})$; the local label is the constant function 1. If the quantization is a linear category, then it is the category $\mathrm{Vect}(\Omega^\ell \mathcal{X})$ of flat vector bundles over $\Omega^\ell \mathcal{X}$, i.e., vector bundles on the fundamental groupoid $\pi_{\leqslant 1} \Omega^\ell \mathcal{X}$; the local label is the trivial bundle with fiber $\mathbb{C}$.

Globally, we quantize

$$\mathrm{id}\colon \mathrm{Map}(Z^\nu, \mathcal{X}) \longrightarrow \mathrm{Map}(Z^\nu, \mathcal{X}), \tag{3.39}$$

[29] Allen Hatcher, *Algebraic topology.* available at https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.

[30] The homotopy group $\pi_\ell \mathcal{X} = \pi_\ell(\mathcal{X}, *)$ uses the basepoint $* \in \mathcal{X}$.

where Z^ν is the Thom space of the normal bundle. As an example, suppose $\ell = 1$ and assume that the normal bundle $\nu \to Z$ has been trivialized. (This amounts to a coorientation of the codimension 1 submanifold $Z \subset M$—a direction for the domain wall.) Then

$$\mathrm{Map}(Z^\nu, \mathfrak{X}) \simeq \mathrm{Map}(Z, \Omega\mathfrak{X}). \tag{3.40}$$

For example, if A is a finite abelian group and $\mathfrak{X} = B^2A$—so σ encodes a BA-symmetry—then $\mathrm{Map}(Z^\nu, B^2A) \simeq \mathrm{Map}(Z, BA)$ is the "space" of principal A-bundles $P \to Z$. One should, rather, treat it as a groupoid, the groupoid $\mathrm{Bun}_A(Z)$ of principal A-bundles over Z and isomorphisms between them. A point $* \to \mathrm{Bun}_A(Z)$ is a principal A-bundle $P \to Z$, and this map quantizes to a defect $\eta(P)$ on Z. The quantization of $\mathrm{id}\colon \mathrm{Bun}_A(Z) \to \mathrm{Bun}_A(Z)$ is a sum of the quantizations of $*/\mathrm{Aut}\,P \to \mathrm{Bun}_A(Z)$ over isomorphism classes of principal A-bundles $P \to Z$. Informally, we might write this as a sum of

$$\frac{1}{\mathrm{Aut}\,P}\,\eta(P) = \frac{1}{H^0(Z;A)}\,\eta(P). \tag{3.41}$$

This sort of expression appears in [31], for example; compare Example 3.25.

The ρ-defect $\eta(P)$ has a geometric semiclassical interpretation. Without the defect one is summing over A-gerbes on $[0,1) \times M$ which are trivialized on $\{0\} \times M$. The defect $\eta(P)$ on $\{0\} \times Z$ tells to only trivialize the A-gerbe on $\bigl(\{0\} \times M\bigr) \setminus \bigl(\{0\} \times Z\bigr)$ and to demand—relative to the coorientation of Z—that the trivialization jump by the A-bundle $P \to Z$.

Remark 3.42. If the π-finite space $\mathfrak{X}$ is equipped with a cocycle λ which represents a cohomology class $[\lambda] \in h^n(\mathfrak{X})$ for some cohomology theory h, then a codimension ℓ quotient defect has semiclassical label space $\Omega^\ell\mathfrak{X}$ with transgressed cocycle and its cohomology class $[\tau^\ell\lambda] \in h^{n-\ell}(\Omega^\ell\mathfrak{X})$. A nonzero cohomology class obstructs the quotient. However, as observed in [28] it is possible that $[\lambda] \neq 0$ but $[\tau^\ell\lambda] = 0$ for some ℓ, which means that the quotient by σ does not exist but quotient defects of sufficiently high codimension do exist.

Projectivity

We begin with some ruminations on projective symmetry in quantum theory, in part to make contact with Clay's lecture series. A review of **(1.60)** at this point is warranted.

(3.43) *Linear and projective geometry.* Let V be a *linear* space, say finite dimensional and complex. The automorphism group $\mathrm{Aut}\,V$ consists of invertible linear maps $T\colon V \to V$; after a choice of basis it is isomorphic to the group of invertible square complex matrices of size equal to $\dim V$. The *projective* space $\mathbb{P}V$ is the space of lines (1-dimensional subspaces) of V. A linear automorphism

[31] Yichul Choi, Clay Cordova, Po-Shen Hsin, Ho Tat Lam, and Shu-Heng Shao, *Non-Invertible Duality Defects in 3+1 Dimensions*, arXiv:2111.01139.

$T \in \operatorname{Aut} V$ induces an automorphism $\overline{T}$ of $\mathbb{P}V$; the linear map T takes lines to lines. A homothety (scalar multiplication) of V induces the identity map of $\mathbb{P}V$. So there is a group extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \operatorname{Aut} V \longrightarrow \operatorname{Aut} \mathbb{P}V \longrightarrow 1 \tag{3.44}$$

which is the definition of $\operatorname{Aut} \mathbb{P}V$. This extension is *central*; the kernel $\mathbb{C}^\times$ lies in the center of $\operatorname{Aut} V$ (and in fact equals the center). Each projective transformation in $\operatorname{Aut} \mathbb{P}V$ has a $\mathbb{C}^\times$-torsor of linear lifts.

The projective action of a group G on $\mathbb{P}V$ is a group homomorphism $G \to \operatorname{Aut} \mathbb{P}V$. One can use it to pull back the central extension (3.44) to a central extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow G^\tau \longrightarrow G \longrightarrow 1 \tag{3.45}$$

of G. The group G^τ acts linearly on V through a group homomorphism $G^\tau \to \operatorname{Aut} V$ and, as our starting point, the group G acts projectively on $\mathbb{P}V$. The central extension (3.45) is a measure of the projectivity of this projective action. If G is a discrete group, say a finite group, then the equivalence class of the central extension is an element of the cohomology group $H^2(G;\mathbb{C}^\times)$. If this class is zero, then there exist splittings of (3.45). Furthermore, the splittings form a torsor over the group of characters of G, i.e., over the cohomology group $H^1(G;\mathbb{C}^\times)$: given a splitting s any other splitting is the product of s with a character $\chi\colon G \to \mathbb{C}^\times$. In summary:

$$\begin{aligned} \text{existence: } & H^2(G;\mathbb{C}^\times) \\ \text{uniqueness: } & H^1(G;\mathbb{C}^\times) \end{aligned} \tag{3.46}$$

(3.47) *Quantum theory is projective, not linear.* The space of pure states of a quantum system is a projective space $\mathcal{P}$, at first with no topology. Instead there is a function

$$\mathcal{P} \times \mathcal{P} \longrightarrow [0,1] \tag{3.48}$$

which maps an ordered pair of states to the probability of transitioning from one state to the other.[32] Symmetries of the quantum system are automorphisms of $\mathcal{P}$ which preserve the function (3.48).

A fundamental theory due to Wigner (and von Neumann) states that for $\mathcal{P} = \mathbb{P}\mathcal{H}$ any symmetry of the quantum system lifts either to a unitary automorphism of $\mathcal{H}$ or an antiunitary automorphism of $\mathcal{H}$. (See [33] for geometric proofs.) In fact, up to a simple transformation (3.48) is the distance

[32] If we write $\mathcal{P} = \mathbb{P}\mathcal{H}$ for a Hilbert space $\mathcal{H}$, and $\psi_1, \psi_2 \in \mathcal{H}$ are nonzero vectors, then the value of (3.48) on the pair of lines generated is

$$\frac{|\langle \psi_1, \psi_2 \rangle|^2}{\|\psi_1\|^2\,\|\psi_2\|^2}. \tag{3.49}$$

[33] Daniel S. Freed, *On Wigner's theorem*, Proceedings of the Freedman Fest (Vyacheslav Krushkal Rob Kirby and Zhenghan Wang, eds.), Geometry & Topology Monographs, vol. 18, Mathematical Sciences Publishers, 2012, pp. 83–89. arXiv:1211.2133.

function for the Fubini-Study metric on $\mathbb{P}\mathcal{H}$, and so Wigner's theorem becomes a theorem about its isometries. In any case, it follows that the projectivity of a group G of symmetries of quantum theory is measured by a $\mathbb{Z}/2\mathbb{Z}$-graded central extension of G, where the $\mathbb{Z}/2\mathbb{Z}$-grading keeps track of the unitary vs. antiunitary dichotomy. (See [34] for a fuller discussion of symmetry in quantum mechanics.)

(3.50) *Projectivity in quantum field theory: anomalies.* Here the metaphor (2.2) comes in handy. A field theory $F\colon \mathrm{Bord}_n(\mathcal{F}) \to \mathcal{C}$ is a linear representation of $\mathrm{Bord}_n(\mathcal{F})$, and since quantum theory is projective we expect this representation to be projective, and furthermore its projectivity should be measured by some kind of "cocycle" on $\mathrm{Bord}_n(\mathcal{F})$. Indeed, the right kind of cocycle in this context is an *invertible field theory.* In this context the projectivity is called an *anomaly (theory)*;[35] it is an invertible field theory over $\mathcal{F}$ of dimension $n+1$. Observe that in the case of a finite group G acting on a quantum mechanical system ($n = 1$), this matches the cocycle of a group extension (3.45), which has degree 2.

Anomalies are not a sickness of a theory; to the contrary, they are useful tools for investigating its behavior. They are potentially a sickness when we want to "integrate out" some fields, i.e., turn some background fields into fluctuating fields. In that case we have a fiber bundle of fields, such as

$$\begin{array}{rl} \mathcal{F} & = \{\text{Riemannian metric, orientation, } H\text{-connection}\} \\ {\scriptstyle\pi}\big\downarrow & \\ \overline{\mathcal{F}} & = \{\text{Riemannian metric, orientation}\} \end{array} \tag{3.51}$$

Suppose we have a theory over $\mathcal{F}$ which has an anomaly α: an invertible $(n+1)$-dimensional theory over $\mathcal{F}$. Then to integrate over the fibers of π—to push forward under π—we need to provide *descent data* for α. In other words, we need to provide an $(n+1)$-dimensional field theory $\bar{\alpha}$ over $\overline{\mathcal{F}}$ and an isomorphism $\alpha \xrightarrow{\cong} \pi^*\bar{\alpha}$. This is the formal part—we must do the analysis required to integrate over an infinite dimensional space—but if so we obtain a pushforward theory over $\overline{\mathcal{F}}$ with anomaly $\bar{\alpha}$. This descent problem has existence and uniqueness aspects, analogous to (3.46), only now with invertible field theories.

Remark 3.52. Changing descent data by tensoring with an n-dimensional invertible theory is sometimes called "changing the scheme", and pushforwards which differ in this way share many physical properties.

(3.53) *'t Hooft anomalies in finite homotopy theories.* In the lectures I introduced "cocycles" on a π-finite space $\mathcal{X}$ at this point, but in these notes they appear in **(2.23)**, which I recommend you review at this point. In terms of the discussion above, the cocycle defines an invertible field theory

[34] Daniel S. Freed and Gregory W. Moore, *Twisted equivariant matter*, Ann. Henri Poincaré **14** (2013), no. 8, 1927–2023, `arXiv:1208.5055`. I apologize for the self-referential tendencies I learned from Gilderoy Lockhart. The paper on which these summer school notes are based will not be so disgustingly provincial.

[35] Note that the anomaly is a theory over the background fields $\mathcal{F}$, which may or may not have to do directly with symmetry.

in which the map to $\mathcal{X}$ remains a background field. Once we sum—the finite path integral—over these maps, with the cocycle as a weight, then we obtain a typically noninvertible theory σ. When equipped with a semiclassical regular boundary theory ρ, the quantization of data in (**2.51**), then (σ, ρ) represents a symmetry with an *'t Hooft* anomaly represented by the cocycle.

(3.54) *Twisted boundary theories.* At this point recall Definition 2.52. Note that if $\mathcal{X}$ is a π-finite space, and we take the zero cocycle, then a (right or left) semiclassical boundary theory is a map $\mathcal{Y} \to \mathcal{X}$ of π-finite spaces together with a *cocycle* μ on $\mathcal{Y}$. If we are working with an $m = (n+1)$-dimensional theory, then μ has degree m. The cocycle μ is used to weight/twist the quantization of the boundary.

Example: BA-symmetry in 4 dimensions and line defects

The following discussion is inspired by [36].

(3.55) *Symmetry data.* Let A be a finite abelian group, set $\mathcal{X} = B^2A$, and fix a basepoint $* \to B^2A$. This defines the semiclassical data of a BA-symmetry, what is often referred to as a "1-form A-symmetry". We set $\lambda = 0$: there is no 't Hooft anomaly. For definiteness set $n = 4$, so we use the 5-dimensional finite homotopy theory $\sigma = \sigma^{(5)}_{B^2A}$. The basepoint gives a regular right boundary theory, and we study the pair (σ, ρ) as abstract 4-dimensional symmetry data.

(3.56) *The left (σ, ρ)-module: 4-dimensional gauge theory.* Let H be a Lie group, and suppose $A \subset H$ is a subgroup its center. Set $\overline{H} = H/A$. From the exact sequence

$$A \longrightarrow H \longrightarrow \overline{H} \tag{3.57}$$

of Lie groups we obtain a sequence of fiberings

$$BA \longrightarrow BH \longrightarrow B\overline{H} \longrightarrow B^2A \tag{3.58}$$

In fact, we can promote (3.58) to a fibering of classifying spaces[37] of connections:

$$B_\nabla H \longrightarrow B_\nabla \overline{H} \longrightarrow B^2A \tag{3.59}$$

The fibering of $B_\nabla H$ over B^2A is the structure of the action of BA on it. (Given a principal H-bundle with connection and a principal A-bundle, use the *homomorphism* $A \times H \to H$ to construct a new principal H-bundle with connection.) If H is a finite group, and therefore $\overline{H}$ too is a finite group, then the map $B_\nabla \overline{H} = B\overline{H} \to B^2A$ would be a semiclassical left boundary theory. In general, of course, there is no finiteness nor is $\overline{H}$-gauge theory a topological field theory. Nonetheless, we set $\widetilde{F}$ to be 4-dimensional $\overline{H}$-gauge theory, and so obtain the (σ, ρ)-module exhibited in Figure 32.

[36] Ofer Aharony, Nathan Seiberg, and Yuji Tachikawa, *Reading between the lines of four-dimensional gauge theories*, JHEP **08** (2013), 115, `arXiv:1305.0318 [hep-th]`.

[37] These are simplicial sheaves on Man; see [11].

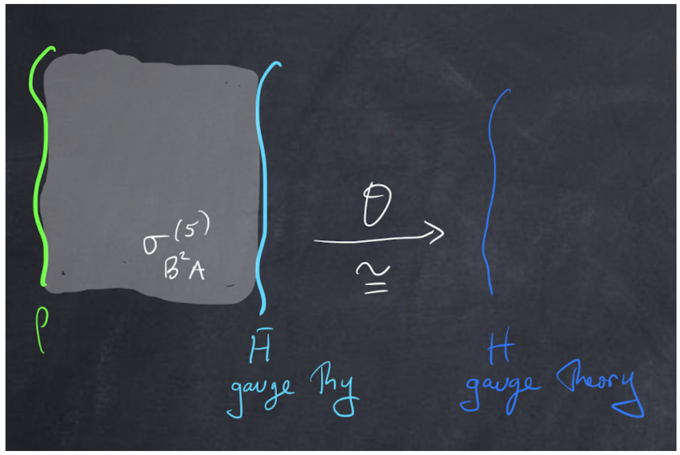

FIGURE 32. The BA-action on H-gauge theory

Remark 3.60. We have used the phrase "H-gauge theory" without specifying which one. All we need here that it is a 4-dimensional gauge theory with BA-symmetry. Other details do not enter this discussion.

(3.61) *Topological right* (σ,ρ)*-modules.* For any subgroup $A' \subset A$ there is a map of π-finite spaces

$$B^2A' \longrightarrow B^2A \tag{3.62}$$

We use this as semiclassical right boundary data, but we allow a twisting as in **(3.54)**, i.e., a degree 4 cocycle μ on B^2A'. In this case we use ordinary cohomology (so assume an orientation is among the background fields), and then the class of μ lives in the cohomology group $H^4(B^2A';\mathbb{C}^\times)$. A theorem of Eilenberg-MacLane computes

$$H^4(B^2A';\mathbb{C}^\times) \cong \{\text{quadratic functions } q\colon A' \longrightarrow \mathbb{C}^\times\} \tag{3.63}$$

Thus pairs (A',q) determine a right topological boundary theory $R_{A',q}$. The quotient, depicted in Figure 33, is a twisted form of H/A'-gauge theory.

Remark 3.64. Under the isomorphism (3.63), the quadratic form q gives rise to the *Pontrjagin square* cohomology operation

$$\mathcal{P}_q\colon H^2(X;A') \longrightarrow H^4(X;\mathbb{C}^\times) \tag{3.65}$$

on any space X.

(3.66) *Local line defects: interior label.* We study local line defects in the twisted H/A'-gauge theory using the sandwich picture in Figure 34. So we have the surface $[0,1]\times C$ built over a curve C (in some manifold M), with topological $R_{A',q}$-boundary at $\{0\}\times C$ and with nontopological

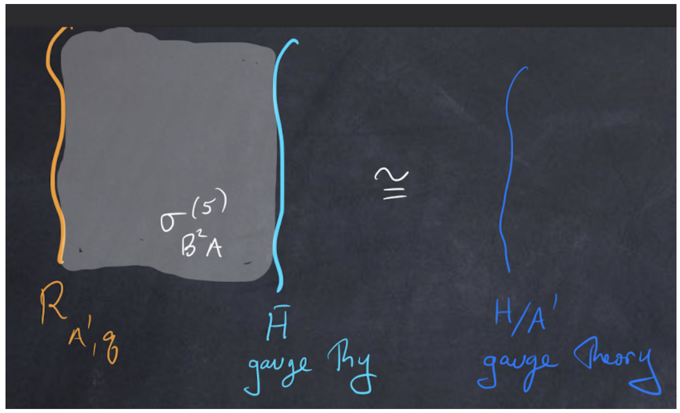

FIGURE 33. A quotient of H-gauge theory

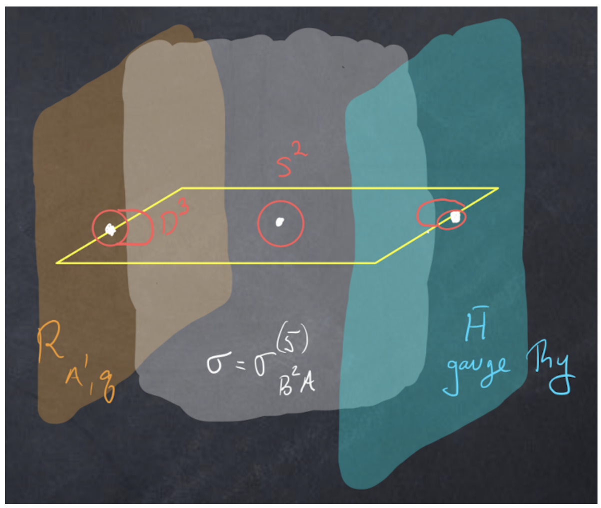

FIGURE 34. Line defects in twisted H/A'-gauge theory

$\widetilde{F}$-boundary at $\{1\} \times C$. See Figure 12 for an analogous figure in quantum mechanics for point defects. We find the relevant labels by working down in dimension of strata, starting at the top. The link of an interior point is S^2, and the quantization $\sigma(S^2)$ is, say, a linear 2-category—on step beyond our illustration in Example 2.27 (which I suggest you review). Observe that for the mapping space $\mathrm{Map}(S^2, B^2A)$ we have

$$\begin{aligned} \pi_0\big(\mathrm{Map}(S^2, B^2A)\big) &= H^2(S^2; A) \cong A \\ \pi_1\big(\mathrm{Map}(S^2, B^2A)\big) &= H^1(S^2; A) = 0 \\ \pi_2\big(\mathrm{Map}(S^2, B^2A)\big) &= H^0(S^2; A) \cong A \end{aligned} \tag{3.67}$$

The quantization consists of flat bundles (local systems) of linear categories over the 2-groupoid with these homotopy groups, depicted in Figure 35. In other words, for each

$$m \in H^2(S^2; A) \cong A \tag{3.68}$$

there is a linear category $\mathcal{L}_m$. Furthermore, π_2 based at m acts on $\mathcal{L}_m$ as automorphisms of the identity functor. Under suitable assumptions, then, we can decompose according to the characters

$$e \in H^0(S^2; A)^\vee \cong A^\vee \tag{3.69}$$

to obtain

$$\mathcal{L}_m = \bigoplus_e \mathcal{L}_{m,e} \cdot e \tag{3.70}$$

In summary, then, an object of $\sigma(S^2)$ is a collection of linear categories $\mathcal{L}_{m,e}$ labeled by $m \in A$ and $e \in A^\vee$.

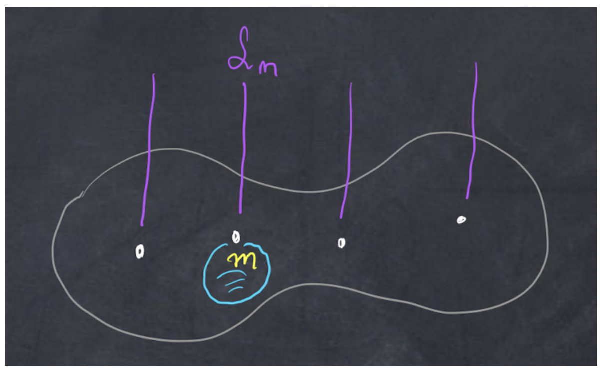

FIGURE 35. The 2-category $\mathrm{Cat}\big((B^2A)^{S^2}\big)$

(3.71) *Local line defects: collating the labels.* It remains to determine the labels on the boundary strata of $[0,1] \times C$, depicted with the links in Figure 34. A label on the $R_{A',q}$-colored boundary is an object $\mathcal{L}_0$ in this category—part of the topological field theory (σ, ρ)—and a label of the $\widetilde{F}$-colored boundary is another object $\mathcal{L}_1$. The image under θ of this configuration is the sum over m, e of $\mathrm{Hom}\big((\mathcal{L}_0)_{m,e}, (\mathcal{L}_1)_{m,e}\big)$, which is the category of defects in the twisted H/A'-gauge theory. What we will now compute is that $\mathcal{L}_0$ is supported at a subset of pairs (m,e) determined by the subgroup $A' \subset A$ and the quadratic function $q\colon A' \to \mathbb{C}^\times$. This is the information gained from the symmetry.

(3.72) *Higher Gauss law.* This paragraph can be made precise in the context of semiadditive categories.

We begin with the lower Gauss law. Suppose $\mathcal{G}$ is a finite 1-groupoid, and $L \to \mathcal{G}$ is a complex line bundle over $\mathcal{G}$. We want to compute the global sections (which is a limit; the colimit is equivalent). At a point $m \in \mathcal{G}$ the group $\pi_1(\mathcal{G}, m)$ acts by a character on the fiber L_m. The value of a global section at m must be a fixed point of this action, so it lies in the invariant subspace of L_m, that is, the section vanishes unless the character vanishes.

Now suppose $\mathcal{K} \to \mathcal{G}$ is a bundle of invertible complex linear categories over a 2-groupoid $\mathcal{G}$, the higher analog of a complex line bundle. (Equivalently, we could consider a $\mathbb{C}^\times$-gerbe.) Then at each $m \in \mathcal{G}$ there are two layers to consider, and we need fixed point data for both. If $f \in \pi_1(\mathcal{G}, m)$ and $x \in \mathcal{K}_m$, then the first piece of fixed point data is an isomorphism

$$\eta_f(x)\colon x \longrightarrow f(x) \tag{3.73}$$

Then if $a\colon f \to g$ is a 2-morphism in $\mathcal{G}$ at m, we need too an isomorphism $\lambda_a(x)$ in

$$\begin{tikzcd} & f(x) \arrow[dd, "\lambda_a(x)"] \\ x \arrow[ur, "\eta_f(x)"] \arrow[dr, "\eta_g(x)"'] & \\ & g(x) \end{tikzcd} \tag{3.74}$$

that makes the diagram commute. Now specialize to $f = g = \mathrm{id}_x$. Then (3.74) becomes

$$\begin{tikzcd} & x \arrow[dd, "\lambda_a(x)"] \\ x \arrow[ur, "\eta"] \arrow[dr, "\eta"'] & \\ & x \end{tikzcd} \tag{3.75}$$

for some automorphism η. This commutes only if $\lambda_a(x) = \mathrm{id}$. Hence, if $\pi_1\mathcal{G} = 0$ then the global sections are only nonzero on the components on which $\pi_2\mathcal{G}$ acts trivially.

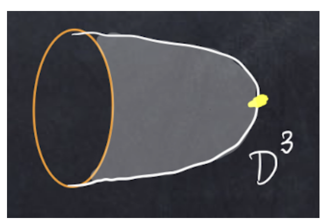

FIGURE 36. The link in Figure 34 at the $R_{A',q}$-colored boundary

(3.76) *Local line defects: the missing link.* First, associated to the quadratic function $q\colon A' \to \mathbb{C}^\times$ is a bihomomorphism

$$b\colon A' \times A' \longrightarrow \mathbb{C}^\times \tag{3.77}$$

It induces a Pontrjagin-Poincare duality

$$H^2(S^2; A') \times H^0(S^2; A') \longrightarrow \mathbb{C}^\times \tag{3.78}$$

and so an isomorphism

$$e'\colon H^2(S^2;A') \longrightarrow H^0(S^2;A')^\vee \tag{3.79}$$

Now let us quantize the link depicted in Figure 36, which is a 3-disk D^3 with boundary colored by $R_{A',q}$, and let us label the center point (m,e), which means the tensor unit category Vect sitting over m with π_2 acting via the character e; see (3.70). (Thus $\mathcal{L}_{m,e} = \mathrm{Vect}$ and the other linear categories are zero.) There is a semiclassical description of this defect (m,e) in terms of Definition 2.76: the space $\mathcal{Y} = B^2A$ is equipped with a 2-cocycle that represents the class in $H^2(B^2A;\mathbb{C}^\times) \cong A^\vee$ given by the character e; the map to $\mathrm{Map}(S^2,B^2A)$ is the identity onto the component indexed by m. At the $R_{A',q}$-colored boundary, in semiclassical terms the boundary theory is given as the map

$$\bigl(\mathrm{Map}(S^2,B^2A'),\tau^2(\mu_q)\bigr) \longrightarrow \mathrm{Map}(S^2,B^2A), \tag{3.80}$$

where $\tau^2\mu_q$ is the transgression of a cocycle which represents the class of the quadratic function in (3.63). Altogether we have a diagram

$$\begin{tikzcd}[column sep=small] \bigl(\mathrm{Map}(S^2,B^2A'),\tau^2(\mu_q)\bigr) \arrow[dr] & & \mathrm{Map}(D^3,B^2A) \arrow[dl] \arrow[dr] & & (B^2A,e) \arrow[dl, "m"] \\ & \mathrm{Map}(S^2,B^2A) & & \mathrm{Map}(S^2,B^2A) & \end{tikzcd} \tag{3.81}$$

Interpret in terms of the bordism obtained by cutting out a ball around the yellow defect; in the bottom row the first entry is the orange boundary and the second the link of the yellow defect. From this conclude that the quantization is supported on pairs (m,e) which satisfy

$$\begin{aligned} &m \in A' \\ &e = e'(m)^{-1} \end{aligned} \tag{3.82}$$

This selection rule is the one in [36].

Exercise 3.83. Check this last statement in some examples from [36].