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THE STUDY OF MATHEMATICS

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THE STUDY OF MATHEMATICS

IN regard to every form of human activity it is necessary that the question should be asked from time to time: What is its purpose and ideal? In what way does it contribute to the beauty of human existence? As respects those pursuits which contribute only remotely, by providing the mechanism of life, it is well to be reminded that not the mere fact of living is to be desired, but the art of living in the contemplation of great things. Still more in regard to those avocations which have no end outside themselves, which are to be justified, if at all, as actually adding to the sum of the world's permanent possessions, it is necessary to keep alive a knowledge of their aims, a clear prefiguring vision of the temple in which creative imagination is to be embodied.

The fulfilment of this need, in what concerns the studies forming the material upon which custom has decided to train the youthful mind, is indeed sadly remote—so remote as to make the mere statement of such a claim appear preposterous. Great men, fully alive to the beauty of the contemplations to whose service their lives are devoted, desiring that others may share in their joys, persuade mankind to impart to the successive generations the mechanical knowledge without which it is impossible to cross the threshold. Dry pedants possess themselves of the privilege of instilling this knowledge: they forget that it is to serve but as a key to open the doors of the temple; though they spend their lives on the steps leading up to those sacred doors, they turn their backs upon the temple so resolutely that its very existence is forgotten, and the eager youth who would press forward to be initiated to its domes and arches, is bidden to turn back and count the steps.

Mathematics, perhaps more even than the study of Greece and Rome, has suffered from this oblivion of its due place in civilisation. Although tradition has decreed that the great bulk of educated men shall know at least the elements of the subject, the reasons for which the tradition arose are forgotten, buried beneath a great rubbish-heap of pedantries and trivialities. To those who inquire as to the purpose of mathematics, the usual answer will be that it facilitates the making of machines, the travelling from place to place, and the victory over foreign nations, whether in war or commerce. If it be objected that these ends—all of which are of doubtful value—are not furthered by the merely elementary study imposed upon those who do not become expert mathematicians, the reply, it is true, will probably be that mathematics trains the reasoning faculties. Yet the very men who make this reply are, for the most part, unwilling to abandon the teaching of definite fallacies, known to be such, and instinctively rejected by the unsophisticated mind of every intelligent learner. And the reasoning faculty itself is generally conceived, by those who urge its cultivation, as merely a means for the avoidance of pitfalls, and a help in the discovery of rules for the guidance of practical life. All these are undeniably important achievements to the credit of mathematics; yet it is none of these that entitle mathematics to a place in every liberal education. Plato, we know, regarded the contemplation of mathematical truths as worthy of the deity; and Plato realised, more perhaps than any other single man, what those elements are in human life which merit a place in heaven. There is in mathematics, he says, "something which is *necessary* and cannot be set aside . . . and, if I mistake not, of divine necessity; for as to the human necessities of which the many talk in this connection, nothing can be more ridiculous than such an application of the words. CLE.: And what are these necessities of knowledge, stranger, which are divine and not human? ATH.: Those things without some use or knowledge of which, a man cannot become a god to the world, nor a spirit, nor yet a hero, nor able earnestly to think and care for man." (*Laws*, p. 818.)[1] Such was Plato's judgment of mathematics; but the mathematicians do not read Plato, while those who read him know no mathematics, and regard his opinion upon this question as merely a curious aberration.

Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry. What is best in mathematics deserves not merely to be learnt as a task, but to be assimilated as a part of daily thought, and brought again and again before the mind with ever renewed encouragement. Real life is, to most men, a long second-best, a perpetual compromise between the ideal and the possible; but the world of pure reason knows no compromise, no practical limitations, no barrier to the creative activity embodying in splendid edifices the passionate aspiration after the perfect from which all great work springs. Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home, and where one, at least, of our nobler

[1] This passage was pointed out to me by Mr. Gilbert Murray.

impulses can escape from the dreary exile of the actual world.

So little, however, have mathematicians aimed at beauty, that hardly anything in their work has had this conscious purpose. Much, owing to irrepressible instincts, which were better than avowed beliefs, has been moulded by an unconscious taste; but much also has been spoilt by false notions of what was fitting. The characteristic excellence of mathematics is only to be found where the reasoning is rigidly logical; the rules of logic are to mathematics what those of structure are to architecture. In the most beautiful work, a chain of argument is presented in which every link is important on its own account, in which an air of ease and lucidity reigns throughout, and the premisses achieve more than would have been thought possible, by means which appear natural and inevitable. Literature embodies what is general in particular circumstances whose universal significance shines through their individual dress; but mathematics endeavours to present whatever is most general in its purity, without any irrelevant trappings.

How should the teaching of mathematics be conducted so as to communicate to the learner as much as possible of this high ideal? Here experience must, in a great measure, be our guide; but some maxims may result from our consideration of the ultimate purpose to be achieved.

One of the chief ends served by mathematics, when rightly taught, is to awaken the learner's belief in reason, his confidence in the truth of what has been demonstrated, and in the value of demonstration. This purpose is not served by existing instruction; but it is easy to see ways in which it might be served. At present, in what concerns arithmetic, the boy or girl is given a set of rules, which present themselves as neither true nor false, but as merely the will of the teacher, the way in which, for some unfathomable reason, the teacher prefers to have the game played. To some degree, in a study of such definite practical utility, this is no doubt unavoidable; but as soon as possible, the reasons of rules should be set forth by whatever means most readily appeal to the childish mind. In geometry, instead of the tedious apparatus of fallacious proofs for obvious truisms which constitute the beginning of Euclid, the learner should be allowed at first to assume the truth of everything obvious, and should be instructed in the demonstrations of theorems which are at once startling and easily verifiable by actual drawing, such as those in which it is shown that three or more lines meet in a point. In this way, belief is generated; it is seen that reasoning may lead to startling conclusions, which nevertheless the facts will verify; and thus the instinctive distrust of whatever is abstract or rational is gradually overcome. Where theorems are difficult, they should be first taught as exercises in geometrical drawing, until the figure has become thoroughly familiar; it will then be an agreeable advance to be taught the logical connections of the various lines or circles that occur. It is desirable also that the figure illustrating a theorem should be drawn in all possible cases and shapes, that so the abstract relations with which geometry is concerned may of themselves emerge as the residue of similarity amid such great apparent diversity. In this way, the abstract demonstrations should form but a small part of the instruction, and should be given when, by familiarity with concrete illustrations, they have come to be felt as the natural embodiment of visible fact. In this early stage, proofs should not be given with pedantic fulness; definitely fallacious methods, such as that of superposition, should be rigidly excluded from the first, but where, without such methods, the proof would be very difficult, the result should be rendered acceptable by arguments and illustrations which are explicitly contrasted with demonstrations.

In the beginning of algebra, even the most intelligent child finds, as a rule, very great difficulty. The use of letters is a mystery, which seems to have no purpose except mystification. It is almost impossible, at first, not to think that every letter stands for some particular number, if only the teacher would reveal *what* number it stands for. The fact is, that in algebra the mind is first taught to consider general truths, truths which are not asserted to hold only of this or that particular thing, but of any one of a whole group of things. It is in the power of understanding and discovering such truths that the mastery of the intellect over the whole world of things actual and possible resides; and ability to deal with the general as such is one of the gifts that a mathematical education should bestow. But how little, as a rule, is the teacher of algebra able to explain the chasm which divides it from arithmetic, and how little is the learner assisted in his groping efforts at comprehension! Usually the method that has been adopted in arithmetic is continued: rules are set forth, with no adequate explanation of their grounds; the pupil learns to use the rules blindly, and presently, when he is able to obtain the answer that the teacher desires, he feels that he has mastered the difficulties of the subject. But of inner comprehension of the processes employed, he has probably acquired almost nothing.

When algebra has been learnt, all goes smoothly until we reach those studies in which the notion of infinity is employed—the infinitesimal calculus and the whole of higher mathematics. The solution of the difficulties which formerly surrounded the mathematical infinite is probably the greatest achievement of which our own age has to boast. Since the beginnings of Greek thought, these difficulties have been known; in every age, the finest intellects have vainly endeavoured to answer the apparently unanswerable questions that had been asked by Zeno the Eleatic; at last Georg Cantor has found the answer, and has conquered for the intellect a new and vast province which had been given over to Chaos and old Night. It was assumed as self-evident, until Cantor and Dedekind established the opposite, that if, from any collection of things, some were taken away, the number of things left must always be less than the original number of things. This assumption, as a matter of fact, holds only of finite collections; and the rejection of it, where the infinite is concerned, has been shown to remove all the difficulties that had hitherto baffled human reason in this matter. This stupendous fact ought to produce a revolution in the higher teaching of mathematics; it has itself added immeasurably to the educational value of the subject, and it has at last given the means of treating with logical precision many studies which, until lately, were wrapped in fallacy and obscurity. By those who were educated on the old lines, the new work is considered to be appallingly difficult, abstruse, and obscure; and it must be confessed that the discoverer, as is so often the case, has hardly himself emerged from the mists which the light of his intellect is dispelling. But inherently, the new doctrine of the infinite, to all candid and inquiring minds, has facilitated the mastery of higher mathematics; for, hitherto, it has been necessary to learn, by a long process of sophistication, to give assent to arguments which, on first acquaintance, were rightly judged to be confused and erroneous. So far from producing a fearless belief in reason, a bold rejection of whatever failed to fulfil the strictest requirements of logic, a mathematical training, during the past two centuries, encouraged the belief that many things, which a rigid inquiry would reject as fallacious, must yet be accepted because they work in what the mathematician calls "practice." By this means, a timid, compromising spirit, or else a sacerdotal belief in mysteries not intelligible to the profane, has been bred where reason alone should have ruled. All this it is now time to sweep away. Let those who wish to penetrate into the arcana of mathematics be taught at once the true theory in all its logical purity, and in the concatenation established by the very essence of the entities concerned.

If we are considering mathematics as an end in itself, and not as a technical training for engineers, it is very desirable to preserve the purity and strictness of its reasoning. Accordingly those who have attained a sufficient familiarity with its easier portions should be led backward from propositions to which they have assented as self-evident to more and more fundamental principles from which what had previously appeared as premisses can be deduced. They should be taught—what the theory of infinity very aptly illustrates—that many propositions seem self-evident to the untrained mind which, nevertheless, a nearer scrutiny shows to be false. By this means they will be led to a sceptical inquiry into first principles, an examination of the foundations upon which the whole edifice of reasoning is built, or, to take perhaps a more fitting metaphor, the great trunk from which the spreading branches spring. At this stage, it is well to study afresh the elementary portions of mathematics, asking no longer merely whether a given proposition is true, but also how it grows out of the central principles of logic. Questions of this nature can now be answered with a precision and certainty which were formerly quite impossible; and in the chains of reasoning that the answer requires, the unity of all mathematical studies at last unfolds itself.

In the great majority of mathematical text-books, there is a total lack of unity in method and of systematic development of a central theme. Propositions of very diverse kinds are proved by whatever means are thought most easily intelligible, and much space is devoted to mere curiosities which in no way contribute to the main argument. But in the greatest works, unity and inevitability are felt as in the unfolding of a drama; in the premisses a subject is proposed for consideration, and in every subsequent step some definite advance is made towards mastery of its nature. The love of system, of interconnection, which is perhaps the inmost essence of the intellectual impulse, can find free play in mathematics as nowhere else. The learner who feels this impulse must not be repelled by an array of meaningless examples, or distracted by amusing oddities, but must be encouraged to dwell upon central principles, to become familiar with the structure of the various subjects which are put before him, to travel easily over the steps of the more important deductions. In this way, a good tone of mind is cultivated, and selective attention is taught to dwell by preference upon what is weighty and essential.

When the separate studies into which mathematics is divided have each been viewed as a logical whole, as a natural growth from the propositions which constitute their principles, the learner will be able to understand the fundamental science which unifies and systematises the whole of deductive reasoning. This is Symbolic Logic—a study which, though it owes its inception to Aristotle, is yet, in its wider developments, a product, almost wholly, of the nineteenth century, and is indeed, in the present day, still growing with great rapidity. The true method of discovery, in Symbolic Logic, and probably also the best method for introducing the study to a learner acquainted with other parts of mathematics, is the analysis of actual examples of deductive reasoning, with a view to the discovery of the principles employed. These principles, for the most part, are so embedded in our ratiocinative instincts, that they are employed quite unconsciously, and can be dragged to light only by much patient effort. But when at last they have been found, they are seen to be few in number, and to be the sole source of everything in pure mathematics. The discovery that all mathematics follows inevitably from a small collection of fundamental laws, is one which immeasurably enhances the intellectual beauty of the whole; to those who have been oppressed by the fragmentary and incomplete nature of most existing chains of deduction, this discovery comes with all the overwhelming force of a revelation; like a palace emerging from the autumn mist as the traveller ascends an Italian hill-side, the stately storeys of the mathematical edifice appear in their due order and proportion, with a new perfection in every part.

Until Symbolic Logic had acquired its present development, the principles upon which mathematics depends were always supposed to be philosophical, and discoverable only by the uncertain, unprogressive methods hitherto employed by philosophers. So long as this was thought, mathematics seemed to be not autonomous, but dependent upon a study which had quite other methods than its own. Moreover, since the nature of the postulates from which arithmetic, analysis, and geometry are to be deduced, was wrapped in all the traditional obscurities of metaphysical discussion, the edifice built upon such dubious foundations began to be viewed as no better than a castle in the air. In this respect, the discovery that the true principles are as much a part of mathematics as any of their consequences, has very greatly increased the intellectual satisfaction to be obtained. This satisfaction ought not to be refused to learners capable of enjoying it, for it is of a kind to increase our respect for human powers and our knowledge of the beauties belonging to the abstract world.

Philosophers have commonly held that the laws of logic, which underlie mathematics, are laws of thought, laws regulating the operations of our minds. By this opinion, the true dignity of reason is very geatly lowered; it ceases to be an investigation into the very heart and immutable essence of all things actual and possible, becoming, instead, an inquiry into something more or less human and subject to our limitations. The contemplation of what is non-human, the discovery that our minds are capable of dealing with material not created by them, above all, the realisation that beauty belongs to the outer world as to the inner, are the chief means of overcoming the terrible sense of impotence, of weakness, of exile amid hostile powers, which is too apt to result from acknowledging the all-but omnipotence of alien forces. To reconcile us, by the exhibition of its awful beauty, to the reign of Fate—which is merely the literary personification of these forces—is the task of tragedy. But mathematics takes us still further from what is human, into the region of absolute necessity, to which not only the actual world, but every possible world, must conform; and even here it builds a habitation, or rather finds a habitation eternally standing, where our ideals are fully satisfied and our best hopes are not thwarted. It is only when we thoroughly understand the entire independence of ourselves, that belongs to this world which reason finds, that the profound importance of its beauty can be adequately realised.

Not only is mathematics independent of us and our thoughts, but in another sense we and the whole universe of existing things are independent of mathematics. The apprehension of this purely ideal character is indispensable, if we are to understand rightly the place of mathematics as one among the arts. It was formerly supposed that pure reason could decide, in some respects, as to the nature of the actual world: geometry, at least, was thought to deal with the space in which we live. But we now know that pure mathematics can never pronounce upon questions of actual existence: the world of reason, in a sense, controls the world of fact, but it is not at any point creative of fact, and in the application of its results to the world in time and space, its certainty and precision are lost among approximations and working hypotheses. The objects considered by mathematicians have, in the past, been mainly of a kind suggested by phenomena: but from such restrictions the abstract imagination should be wholly free. A reciprocal liberty must thus be accorded: reason cannot dictate to the world of facts, but the facts cannot restrict reason's privilege of dealing with whatever objects its love of beauty may cause to seem worthy of consideration. Here, as elsewhere, we build up our own ideals out of the fragments to be found in the world; and in the end it is hard to say whether the result is a creation or a discovery.

It is very desirable, in instruction, not merely to persuade the student of the accuracy of important theorems, but to persuade him in the way which itself has, of all possible ways, the most beauty. The true interest of a demonstration is not, as traditional modes of exposition suggest, concentrated wholly in the result; where this does occur, it must be viewed as a defect, to be remedied, if possible, by so generalising the steps of the proof that each becomes important in and for itself. An argument which serves only to prove a conclusion is like a story subordinated to some moral which it is meant to teach: for æsthetic perfection, no part of the whole should be merely a means. A certain practical spirit, a desire for rapid progress, for conquest of new realms, is responsible for the undue emphasis upon results which prevails in mathematical instruction. The better way is to propose some theme for consideration—in geometry, a figure having important properties, in analysis, a function of which the study is illuminating, and so on. Whenever proofs depend upon some only of the marks by which we define the object to be studied, these marks should be isolated and investigated on their own account. For it is a defect, in an argument, to employ more premisses than the conclusion demands; what mathematicians call elegance results from employing only the essential principles in virtue of which the thesis is true. It is a merit in Euclid that he advances as far as he is able to go without employing the axiom of parallels—not, as is often said, because this axiom is inherently objectionable, but because, in mathematics, every new axiom diminishes the generality of the resulting theorems, and the greatest possible generality is before all things to be sought.

Of the effects of mathematics outside its own sphere, more has been written than on the subject of its own proper ideal. The effect upon philosophy has, in the past, been most notable, but most varied; in the seventeenth century, idealism and rationalism, in the eighteenth, materialism and sensationalism, seemed equally its offspring. Of the effect which it is likely to have in the future, it would be very rash to say much; but in one respect a good result appears probable. Against that kind of scepticism which abandons the pursuit of ideals because the road is arduous and the goal not certainly attainable, mathematics, within its own sphere, is a complete answer. Too often it is said that there is no absolute truth, but only opinion and private judgment; that each of us is conditioned, in his view of the world, by his own peculiarities, his own taste and bias; that there is no external kingdom of truth to which, by patience, by austerity, by stern discipline, we may at last obtain admittance, but only truth for me, for you, for every separate person. By this habit of mind, one of the chief ends of human effort is denied, and the supreme virtue of candour, of fearless acknowledgment of what is, disappears from our moral vision. Of such scepticism, mathematics is a perpetual reproof; for its great edifice of truths stands unshakable and inexpugnable to all the weapons of doubting cynicism.

The effects of mathematics upon practical life, though they should not be regarded as the motive of our studies, may be used to answer a doubt to which the solitary student must always be liable. In a world so full of evil and suffering, retirement into the cloister of contemplation, to the enjoyment of delights which, however noble, must always be for the few only, cannot but appear as a somewhat selfish refusal to share the burden imposed upon others by accidents in which justice plays no part. Have any of us the right, we ask, to withdraw from present evils, to leave our fellow-men unaided, while we live a life which, though arduous and austere, is yet plainly good in its own nature? When these questions arise, the true answer is, no doubt, that some must keep alive the sacred fire, some must preserve, in every generation, the haunting vision which shadows forth the goal of so much striving. But when, as must sometimes occur, this answer seems too cold, when we are almost maddened by the spectacle of sorrows to which we bring no help, then we may reflect that indirectly the mathematician often does more for human happiness than any of his more practically active contemporaries. The history of science abundantly proves that a body of abstract propositions—even if, as in the case of conic sections, it remains two thousand years without effect upon daily life—may yet, at any moment, be used to cause a revolution in the habitual thoughts and occupations of every citizen. The use of steam and electricity—to take striking instances—is rendered possible only by mathematics. In the results of abstract thought, the world possesses a capital, of which the employment in enriching the common round has no hitherto discoverable limits. Nor does experience give any means of deciding what parts of mathematics will be found useful. Utility, therefore, can be only a consolation in moments of discouragement, not a guide in directing our studies.

For the health of the moral life, for ennobling the tone of an age or a nation, the austerer virtues have a strange power, exceeding the power of those not informed and purified by thought. Of these austerer virtues, the love of truth is the chief, and in mathematics, more than elsewhere, the love of truth may find encouragement for waning faith. Every great study is not only an end in itself, but also a means of creating and sustaining a lofty habit of mind; and this purpose should be kept always in view throughout the teaching and learning of mathematics.

Bertrand Russell.