

Poisson Lie linear algebra in the graphical language

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June 2, 2009

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1 Introduction

The two primary purposes of this article are to explain some basic notions from the theory of Lie bialgebras in a graphical language, and to teach the graphical language to readers who know some Lie bialgebra theory.¹

The graphical language, which we will not completely define here, was first introduced in 1971 by Roger Penrose in [5], and constitutes a merging of two computational conventions used in physics: Einstein's index summation convention, and Feynman's diagrams. The graphical language has been used to study Lie groups, c.f. [8] and [2]. The language has been expanded quite a lot, most notably to represent constructions in arbitrary tensor categories. For details, see for example [7] and references therein.

¹In fact, another goal of this article was to provide the author practice creating diagrams in the typesetting language *TikZ*. We refer the reader to [9] for a complete description. If the reader is interested in learning to draw such diagrams, she is welcome to inspect the source code for this document, available at <http://math.berkeley.edu/~theo/f/GraphicalLanguage.tex>.

For the purposes of this discussion, it suffices to adopt much closer to Penrose’s original meanings. We pick a finite-dimensional vector space, and denote it, its dual space, and the canonical pairing and unit maps by:



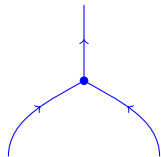
As we are working in the category of vector spaces, there’s nothing interesting to say about crossings (\times) and the like. We will not discuss infinite-dimensional spaces, but most of these constructions can be made to work in more generality.

It is a matter of taste which direction diagrams “go.” In this document we have elected to “let time go up”, so that morphisms are read from bottom-to-top. This is largely for ease of typesetting — Cartesian coordinates are more common than “matrix” coordinates — and because it more closely matches the physicists’ conventions from which the notation was derived.

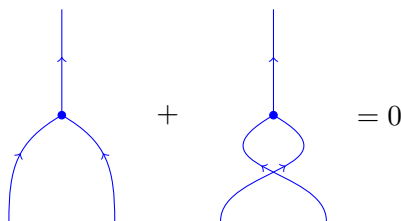
The author learned about the Drinfeld Double and the Chevalley Complex from a graduate seminar on Quantum Groups by Nicolai Reshetikhin. See [6] for class notes and references. In particular, [3] makes regular use of the graphical language, and treats not only the linear algebra of this article but the non-linear group theory and geometry of Poisson Lie and Quantum groups.

2 Lie algebras and coalgebras

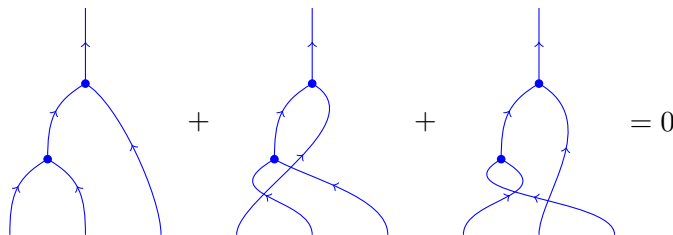
A *Lie algebra structure* on our vector space is a trivalent vertex




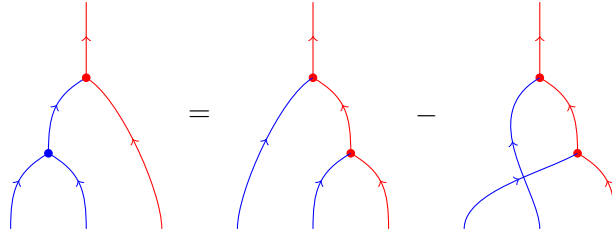
which should be *antisymmetric*:






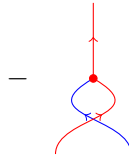
and also must satisfy the *Jacobi identity*:



Let \uparrow be another vector space. We define a *left action* of \downarrow on \uparrow to be a vertex  satisfying:

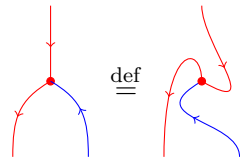


A *right action* is a vertex  with defining relation the reflection of the above. The Jacobi and antisymmetry identities assure that  is both a left and a right action of \downarrow on itself, called the *adjoint action*. Moreover, if  is a right action, then:

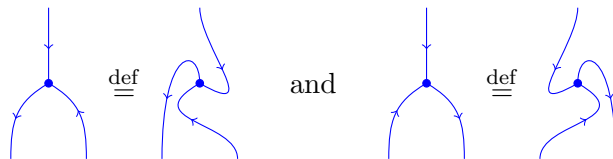


is a left action, and conversely. In particular, this relates the left and right adjoint actions.

If \downarrow acts on \uparrow from the left, and if \uparrow has a dual space \downarrow , then we can define a right action of \downarrow on \downarrow by:



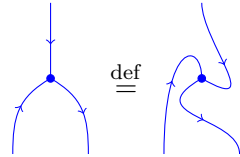
Similarly, we can turn a right action into a left action on the dual space (and, of course, switch between left and right actions at the cost of a minus sign). In this way, we can define the left and right *co-adjoint actions* of \downarrow on \downarrow :



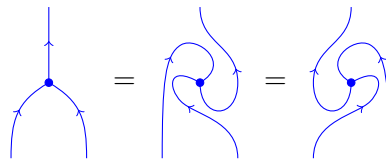
Just as with the adjoint action, the coadjoint action is antisymmetric in the following sense:



We remark that one can also easily check the following relationship between the two coadjoint actions:



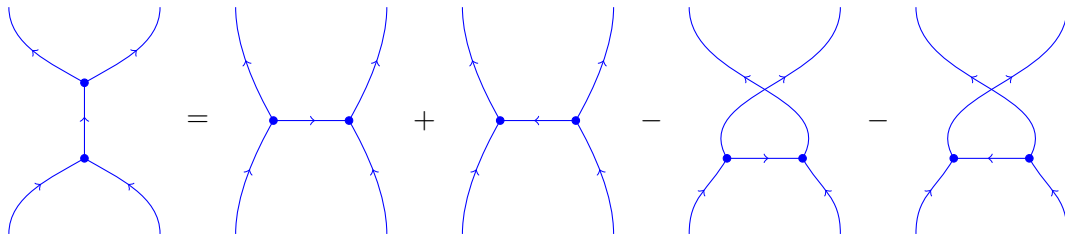
We now dualize all these notions. A *Lie coalgebra* structure on \mathfrak{g} is a map $\mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ satisfying the vertical reflections of the antisymmetry and Jacobi relations above. Equivalently, a Lie coalgebra structure on \mathfrak{g} is a Lie algebra structure on \mathfrak{g}^* . To define such an equivalence, we must choose a dual to the tensor product $\mathfrak{g} \otimes \mathfrak{g}$. Of course, the dual space is $\mathfrak{g}^* \otimes \mathfrak{g}^*$, but whereas [6] takes the pairing $\langle \cdot, \cdot \rangle$, we will take $\langle \cdot, \cdot \rangle$. Then we can relate the bracket on \mathfrak{g}^* to the cobracket on \mathfrak{g} by:



In this way, if \mathfrak{g} is equipped with both a Lie algebra and a Lie coalgebra structure, we can interpret any trivalent vertex with one or two incoming and two or one outgoing edges.

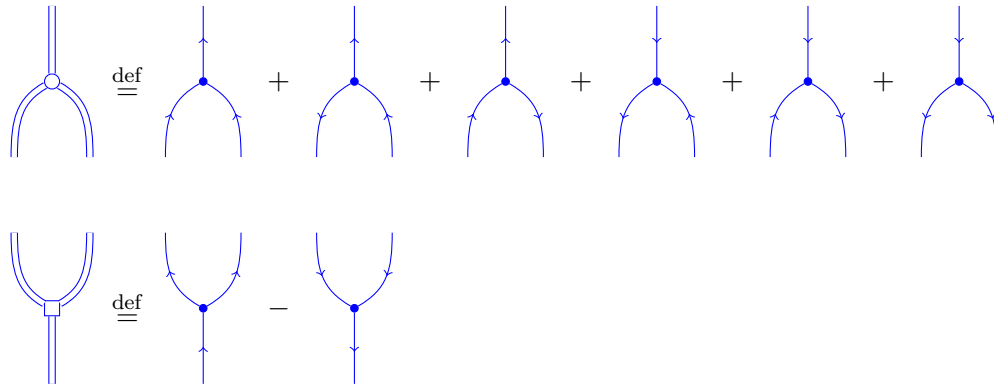
3 Lie bialgebras and the Drinfeld Double

Let \mathfrak{g} be equipped with both a Lie algebra structure and a Lie coalgebra structure. We say that this forms a *Lie bialgebra* if, in addition to the identities demanded in the previous section, the Lie algebra and coalgebra satisfy the following coherence relation:



One can easily see that this relation is symmetric, so that \mathfrak{g}^* is also a Lie bialgebra if \mathfrak{g} is.

If \mathfrak{g} is a Lie coalgebra, then we define the *Drinfeld Double* of \mathfrak{g} to be the vector space $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$, with the following bracket and cobracket:

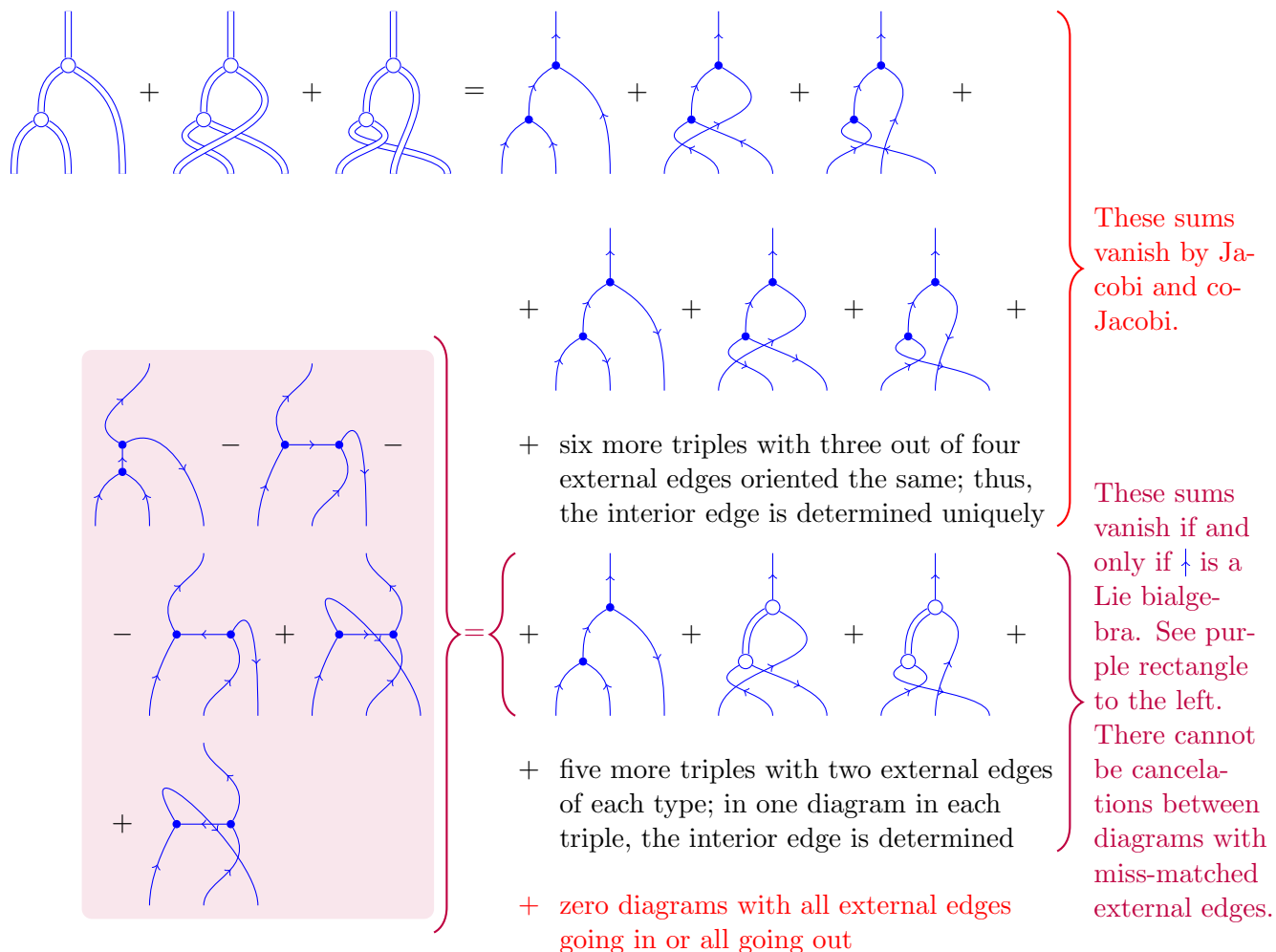


We should explain the meaning of the above sum. As a vector space, the double comes with a splitting into a direct sum. Thus, we can evaluate a tensor by projecting in all ways onto direct summands, evaluating on each projection, and summing. This is exactly what is expressed above.

It is immediate that the cobracket $\begin{array}{c} \cup \\ \sqcup \end{array}$ is a Lie coalgebra structure on \parallel : indeed, it makes \parallel into the direct sum of coalgebras (the coalgebra identities are homogeneous, and so any scalar times a Lie coalgebra structure gives another Lie coalgebra).

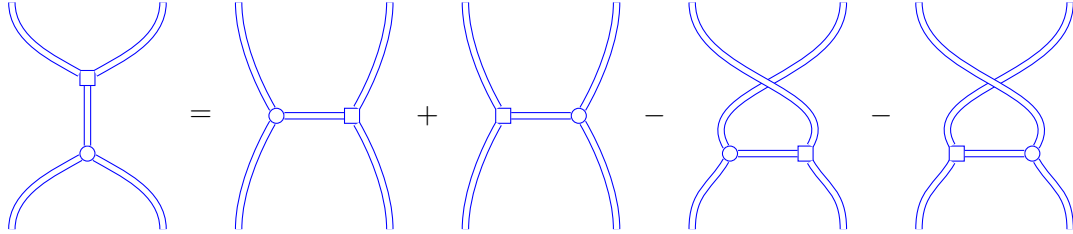
On the other hand, $\begin{array}{c} \cup \\ \cap \end{array}$ is only a Lie algebra structure if \downarrow is a Lie bialgebra. It is immediately

antisymmetric, but the Jacobi identity gives trouble:



We have defined two different (co)brackets on $\mathbb{1}$: $\mathbb{1} \circlearrowleft$, the sum of all canonical actions of \vdash and \dashv on each other and themselves, and $\mathbb{1} \square$, the difference of cobrackets on \vdash and \dashv . Repeating what we said above, the latter is always a Lie cobracket, and the former is a Lie cobracket if and only if \vdash is a Lie bialgebra. We have not marked a direction on $\mathbb{1}$ because it has a canonical pairing $\mathbb{1} \curvearrowright$. We have proposed a circle for the first bracket and a square for the second, because the former is rotation-invariant like $\mathbb{1} \circlearrowleft$ whereas the latter is not.

One now can ask if it is possible to turn $\mathbb{1}$ into a Lie bialgebra with some combination of these brackets. It is a direct calculation, in the manner of the check of Jacobi on the previous page, that neither bracket alone along with its rotation makes $\mathbb{1}$ into a Lie bialgebra. However, together the brackets do. It's standard to think of $\mathbb{1} \circlearrowleft$ as the “forward” bracket and $\mathbb{1} \square$ as the “backward” cobracket. The coherence relation reads:

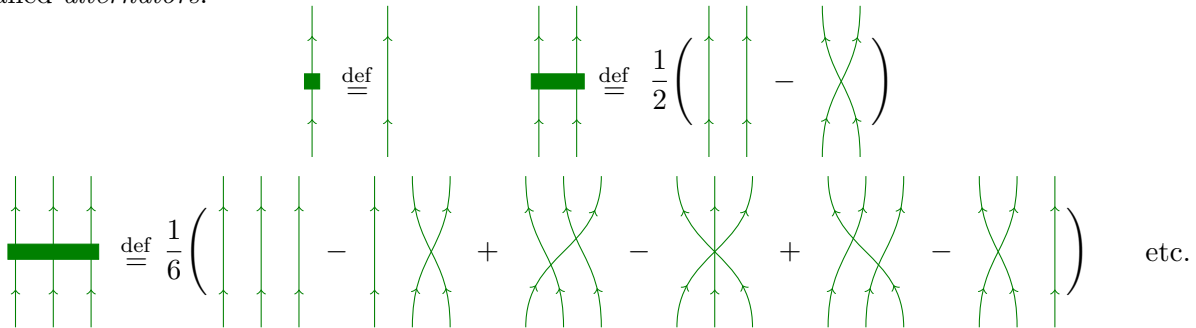


To check this, one rewrites the left hand side into a sum of six terms, and then converts each into two to four terms with the Jacobi and bialgebra coherence axioms. The minus sign in \square is vital: without it, the coherence relation does not hold.

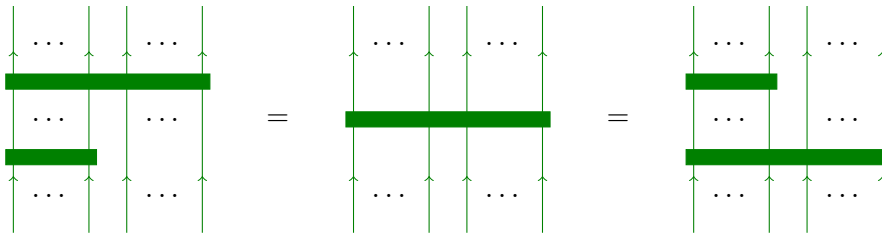
4 Lie Algebra Cohomology and the Chevalley Complex

The Chevalley cohomology of a Lie algebra is present throughout the literature. For example, in an introductory Lie Groups book (e.g. [4, section 4.4]) it controls the complete reducibility of modules of semisimple Lie groups and leads to Ado's Theorem. The reader may be interested in the perspective taken in [1], which generalizes the Chevalley cohomology to a cohomology theory of Poisson algebras.

Before we define the Chevalley Complex, we establish some notation. Henceforth, we work only in characteristic zero. Let \uparrow be a vector space. Then we define a sequence of projections, often called *alternators*:



In general, the n th projection is the signed average of the $n!$ permutations on n strands. It is easy to check that each alternator is a projection as claimed. Moreover, the composition of two alternators equals the longer one, in the following sense:

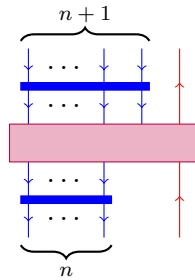


Conversely, we can decompose any alternator in terms of a signed average of alternators of one-shorter length:

$$\begin{array}{c} \overbrace{\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}}^{n+1} \\ \hline \vdots \\ \vdots \\ \underbrace{\vdots}_{n} \end{array} = \frac{1}{n+1} \sum_{j=0}^n (-1)^j \begin{array}{c} n \quad j \quad 0 \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ \hline \vdots \\ \vdots \\ \underbrace{\vdots}_{n} \end{array}$$

There are similar formulas on the other side. We will make a habit of identifying projections with their codomains; for example, we will write $\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}$ for the third wedge-power of \downarrow .

We know let \downarrow be a Lie algebra with bracket $\begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$, and let $\begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$ be an action of \downarrow on \downarrow . Then there are two natural maps, given by the two trivalent vertices, of the form:



We will let Q be a linear combination of these two maps, selected so that $Q^2 = 0$ (we leave to the reader the easy exercise of checking this):

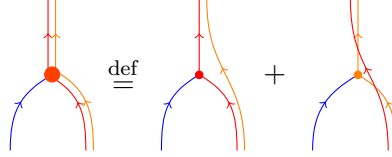
$$\begin{array}{c} \overbrace{\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}}^{n+1} \\ \hline \vdots \\ \vdots \\ \underbrace{\vdots}_{n} \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \stackrel{\text{def}}{=} \begin{array}{c} \overbrace{\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}}^{n+1} \\ \hline \vdots \\ \vdots \\ \underbrace{\vdots}_{n} \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} - \frac{n}{2} \begin{array}{c} \overbrace{\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}}^{n+1} \\ \hline \vdots \\ \vdots \\ \underbrace{\vdots}_{n} \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$$

Then Q is a differential on the Chevalley complex $(\wedge^\bullet \downarrow) \otimes \downarrow = \bigoplus \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$, and defines the BRST cohomology of \downarrow with coefficients in \downarrow . It is functorial in \downarrow in the following sense: if \downarrow is another module over \downarrow , an *intertwiner* is any map $\begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$ such that

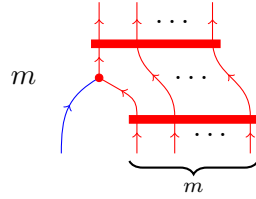
$$\begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$$

and the Chevalley complex construction commutes with intertwiners.

If \uparrow and \downarrow are two modules over \downarrow , then we can define an action of \downarrow on $\uparrow\downarrow$ by:

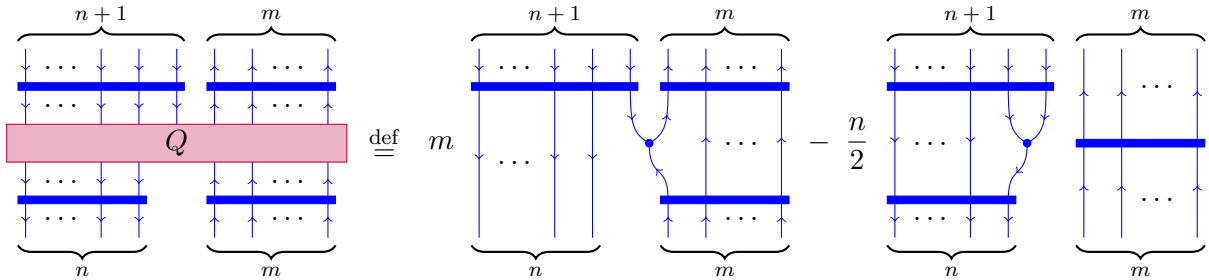


In this way, if \uparrow is a module over \downarrow , then we can define an action on $\uparrow\downarrow\uparrow$ by

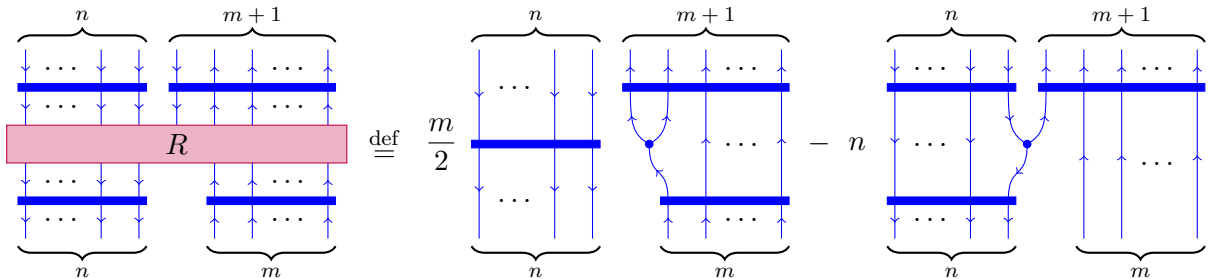


The multiplication by m is needed because the condition of being an action is not homogeneous in individual types of vertices.

In particular, the adjoint action defines \downarrow as a module over itself, and so the map Q above, with $\uparrow = \downarrow$, provides a family of maps



What if \downarrow is also a Lie coalgebra? Then we can switch all the arrows, getting another interesting map:



Then $RQ - QR$ vanishes for every m, n if and only if \downarrow is a Lie bialgebra. We sketch the argument, leaving the details to the interested reader as an exercise in manipulating the diagrams

of the graphical calculus: the terms in R and Q that do not mix \downarrow with \uparrow commute; upon expanding the mixed terms using the decomposition of an alternator into a sum of shorter-length alternators, every summand vanishes except those in which the two vertices are linked directly by an edge; when collected together, these terms sum to the difference of the two sides Lie bialgebra coherence relation, up to composition with alternators.

Thus Q and R provide the differentials for a bi-graded complex $\bigoplus_{n,m} \downarrow \cdots \downarrow \uparrow \cdots \uparrow$. We will compute the differential $T_k = \bigoplus_{m+n=k} (-1)^m Q_{n,m} \oplus (-1)^m R_{n,m}$ of the total complex², and observe that $T^2 = 0$ if \downarrow is a Lie bialgebra, thereby showing that $QR = RQ$.

Let \Downarrow and \Uparrow be the canonical injections, and \Downarrow and \Uparrow the canonical surjections, between \downarrow, \uparrow , and $\parallel = \downarrow \oplus \uparrow$. Moreover, we declare that $\downarrow = 0 = \uparrow$. We introduce the following notation, which we define by example:

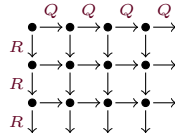
$$\text{sort}^{2,1} \stackrel{\text{def}}{=} \begin{array}{c} \downarrow \downarrow \uparrow \\ \parallel \parallel \parallel \end{array} \oplus - \begin{array}{c} \downarrow \downarrow \uparrow \\ \downarrow \uparrow \downarrow \end{array} \oplus \begin{array}{c} \downarrow \downarrow \uparrow \\ \downarrow \uparrow \downarrow \end{array}$$

The sign of a term is determined by the number of crossings, just as it is in the alternator projection. The reader should interpret the \oplus s in the above equation as living only on the upper half of the picture, so that $\text{sort}^{n,m}$ is a map $\parallel^{\otimes n+m} \rightarrow [\bigoplus \binom{n+m}{n}] \downarrow^{\otimes n} \otimes \uparrow^{\otimes m}$, by which we mean a direct sum of $\binom{n+m}{n}$ copies of the same vector space. We will mostly continue to suppress the diagonal and codiagonal (sum) maps $\Delta : \downarrow \rightarrow \downarrow \oplus \uparrow$ and $\nabla : \downarrow \oplus \uparrow \rightarrow \downarrow$ (so the composition $\Delta \circ \nabla$ is twice the identity operator)³. We remark only that using the codiagonal and the canonical injections, one can define the composition of sort maps: $\text{sort}^{n,k-n} \circ \text{sort}^{m,k-m}$ is zero if $n \neq m$. Since

$$\text{sort}^{n,m} = \begin{array}{c} \downarrow \cdots \downarrow \uparrow \cdots \uparrow \\ \parallel \cdots \parallel \end{array} = \begin{array}{c} \downarrow \cdots \downarrow \uparrow \cdots \uparrow \\ \downarrow \cdots \downarrow \uparrow \cdots \uparrow \end{array} \oplus 0 \oplus 0 \oplus \dots,$$

we see that $\nabla \circ \text{sort}^{n,m}$ is a projection onto $\downarrow^{\otimes n} \otimes \uparrow^{\otimes m}$.

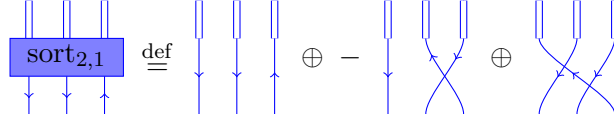
²Any bicomplex has many “total complexes”. In our convention, a bicomplex is a (possibly-infinite) grid of the form:



where each row and each column is a complex ($Q^2 = 0 = R^2$), and such that each square commutes ($QR = RQ$). Then the objects in the total complex are the direct sum of antidiagonals, but the differential depends on a choice of decoration of the edges of the bicomplex with $+$ s and $-$ s such that each square has an odd number of each sign. We have chosen the signs $T_k = \bigoplus_{m+n=k} (-1)^m Q_{n,m} \oplus (-1)^m R_{n,m}$ for later convenience.

³We remark that in the “time goes up” convention we have been adopting, it would make more visual sense to interchange the letters used for these two maps. However, we will continue with the traditional use of Δ for “diagonal”; it is at least the correct first letter.

In any case, we see that the dual map to $\text{sort}^{n,m}$ is the map $\text{sort}_{m,n}$ defined by example:

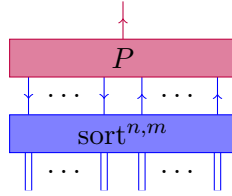


Here the \oplus are only on the bottom. We recall that we have defined the dualization functor to reverse the order of tensor products; we declare that it preserves the order of direct sums. In algebraic symbols: $(A \otimes B)^* = B^* \otimes A^*$; $(A \oplus B)^* = A^* \oplus B^*$. Then:

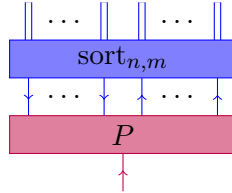
$$\left(\sum_{n+m=k} \text{sort}_{n,m} \right) \circ \left(\sum_{n+m=k} \text{sort}^{n,m} \right) = \text{identity on } \mathbb{1}^{\otimes k}$$

$$\text{sort}^{n,m} \circ \text{sort}_{n,m} = \text{identity on } \left[\bigoplus \binom{n+m}{n} \right] \downarrow^{\otimes n} \otimes \downarrow^{\otimes m}$$

As such, $\text{SORT} \stackrel{\text{def}}{=} \left(\bigoplus_{n+m=k} \text{sort}_{n,m} \right)$ and $\text{MIX} \stackrel{\text{def}}{=} \left(\bigoplus_{n+m=k} \text{sort}^{n,m} \right)$ are inverse isomorphisms, corresponding to the binomial expansion $(a+b)^k = \sum_{n=0}^k \binom{k}{n} a^n b^{k-n}$. If P is a map with domain $\downarrow^{\otimes n} \otimes \downarrow^{\otimes m}$, then we will write

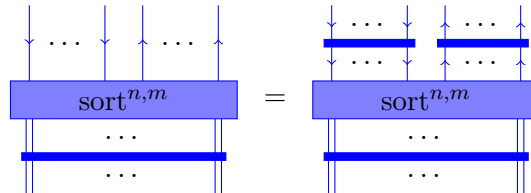


for the composition of $\text{sort}^{n,m}$ with the direct sum of $\binom{n+m}{n}$ copies of P . Similarly, if the codomain of P is $\downarrow^{\otimes n} \otimes \downarrow^{\otimes m}$, then

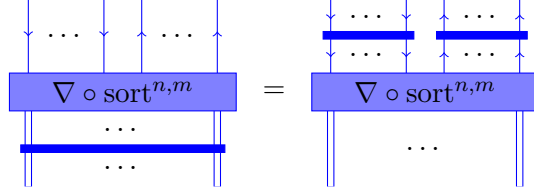


will denote the direct sum of P s composed with $\text{sort}_{n,m}$.

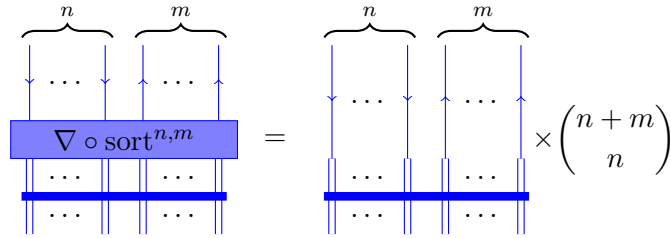
We invite the reader to check the following not-too-difficult fact:



Moreover, if we compose $\text{sort}^{n,m}$ with the codiagonal ∇ , then we can erase the lower alternator in the above equation:

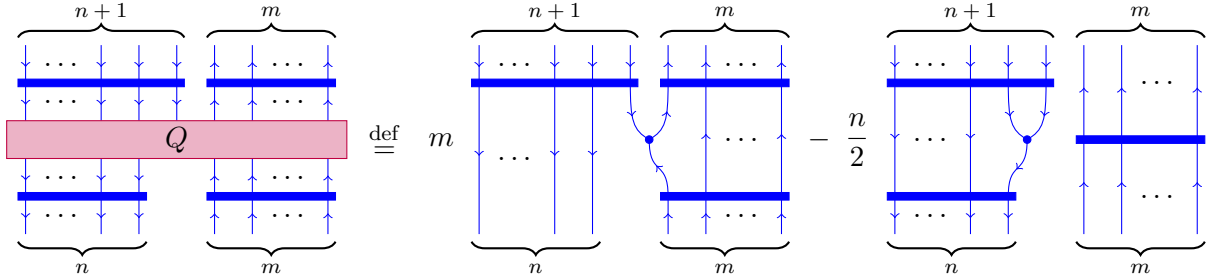


We define $\nabla \circ \text{SORT} \stackrel{\text{def}}{=} (\bigoplus_{n+m=k} \nabla \circ \text{sort}^{n,m})$, even though the different codiagonals in the direct sum have different domains and codomains. Then $\nabla \circ \text{SORT}$ is the canonical isomorphism of wedge powers $\mathbb{R}^k \rightarrow \bigoplus_{n+m=k} \mathbb{R}^n \wedge \mathbb{R}^m$. We will simply call this **SORT**, and also confuse **MIX** with $\text{MIX} \circ \Delta$; the mathematicians' careful distinction between $+$ and \bigoplus doesn't carry too well to the graphical language. However, we do need to be careful about normalization, to maintain the correct pairing. We have: We can write the components of this isomorphism even more simply:



whereas we do not add any coefficients to summands in **MIX**.

With this, we can now begin to study the total complex $T = (-1)^m Q \oplus (-1)^m R$. Recall:



We fix $m + n = k$, consider the composition $\text{MIX} \circ (-1)^m Q \circ \text{SORT}$, and sum over $0 \leq m \leq k$.

Then:

$$\begin{aligned}
 & \underbrace{\left(\begin{array}{c} k+1 \\ \dots \\ \text{MIX} \\ \sum (-1)^m Q \\ \text{SORT} \\ \dots \\ k \end{array} \right)} = \sum_{m=0}^k \binom{k}{m} (-1)^m m \underbrace{\left(\begin{array}{c} k-m \\ \dots \\ \dots \\ k-m \end{array} \right)}_{k-m} \underbrace{\left(\begin{array}{c} m-1 \\ \dots \\ \dots \\ m-1 \end{array} \right)}_{m-1} - \sum_{m=0}^k \binom{k}{m} (-1)^m \frac{k-m}{2} \underbrace{\left(\begin{array}{c} k-m-1 \\ \dots \\ \dots \\ k-m-1 \end{array} \right)}_{k-m-1} \underbrace{\left(\begin{array}{c} m \\ \dots \\ \dots \\ m \end{array} \right)}_m \\
 & = \sum_{m=1}^{k-1} \frac{k! (-1)^m (-1)^{m-1}}{(m-1)! (k-m)!} \underbrace{\left(\begin{array}{c} k-m \\ \dots \\ \dots \\ k-m \end{array} \right)}_{k-m} \underbrace{\left(\begin{array}{c} m-1 \\ \dots \\ \dots \\ m-1 \end{array} \right)}_{m-1} - \sum_{m=0}^{k-1} \frac{k! (-1)^m (-1)^m}{2 m! (k-m-1)!} \underbrace{\left(\begin{array}{c} k-m-1 \\ \dots \\ \dots \\ k-m-1 \end{array} \right)}_{k-m-1} \underbrace{\left(\begin{array}{c} m \\ \dots \\ \dots \\ m \end{array} \right)}_m \\
 & = -k \sum_{m=0}^{k-1} \binom{k-1}{m} \underbrace{\left(\begin{array}{c} k-m-1 \\ \dots \\ \dots \\ k-m-1 \end{array} \right)}_{k-m-1} \underbrace{\left(\begin{array}{c} m \\ \dots \\ \dots \\ m \end{array} \right)}_m - \frac{k}{2} \sum_{m=0}^{k-1} \binom{k-1}{m} \underbrace{\left(\begin{array}{c} k-m-1 \\ \dots \\ \dots \\ k-m-1 \end{array} \right)}_{k-m-1} \underbrace{\left(\begin{array}{c} m \\ \dots \\ \dots \\ m \end{array} \right)}_m \\
 & = -k \sum_{m=0}^{k-1} \underbrace{\left(\begin{array}{c} k-m-1 \\ \text{sort}_{k-m-1,m} \\ \dots \\ \text{sort}_{k-m-1,m} \\ k-m-1 \end{array} \right)}_{k-m-1} \underbrace{\left(\begin{array}{c} m \\ \dots \\ \dots \\ m \end{array} \right)}_m - \frac{k}{2} \sum_{m=0}^{k-1} \underbrace{\left(\begin{array}{c} k-m-1 \\ \text{sort}_{k-m-1} \\ \dots \\ \text{sort}_{k-m-1} \\ k-m-1 \end{array} \right)}_{k-m-1} \underbrace{\left(\begin{array}{c} m \\ \dots \\ \dots \\ m \end{array} \right)}_m \\
 & = -k \underbrace{\left(\begin{array}{c} k-m-1 \\ \dots \\ \dots \\ k-m-1 \end{array} \right)}_{k-m-1} \underbrace{\left(\begin{array}{c} m \\ \dots \\ \dots \\ m \end{array} \right)}_m - \frac{k}{2} \underbrace{\left(\begin{array}{c} k-m-1 \\ \dots \\ \dots \\ k-m-1 \end{array} \right)}_{k-m-1} \underbrace{\left(\begin{array}{c} m \\ \dots \\ \dots \\ m \end{array} \right)}_m
 \end{aligned}$$

Analogously to above, one can compute:

$$\begin{array}{c} \overbrace{\dots}^{k+1} \\ \text{MIX} \\ \Sigma(-1)^m R \\ \text{SORT} \\ \underbrace{\dots}_k \end{array} = -\frac{k}{2} \left(\begin{array}{c} \overbrace{\dots}^{k-m-1} \quad \overbrace{\dots}^m \\ \dots \\ \underbrace{\dots}_{k-m-1} \quad \underbrace{\dots}_m \end{array} - k \begin{array}{c} \overbrace{\dots}^{k-m-1} \quad \overbrace{\dots}^m \\ \dots \\ \underbrace{\dots}_{k-m-1} \quad \underbrace{\dots}_m \end{array} \right)$$

Then the total complex is:

$$\begin{array}{c} \overbrace{\dots}^{k+1} \\ \text{MIX} \\ T \\ \text{SORT} \\ \underbrace{\dots}_k \end{array} = \begin{array}{c} \overbrace{\dots}^{k+1} \\ \text{MIX} \\ \Sigma(-1)^m Q \\ \text{SORT} \\ \underbrace{\dots}_k \end{array} + \begin{array}{c} \overbrace{\dots}^{k+1} \\ \text{MIX} \\ \Sigma(-1)^m R \\ \text{SORT} \\ \underbrace{\dots}_k \end{array} = -\frac{k}{2} \begin{array}{c} \overbrace{\dots}^{k-1} \\ \dots \\ \underbrace{\dots}_{k-1} \end{array}$$

Here \square is a sum of vertices. We may as well include an alternator at the top (it will be absorbed by the larger alternator), whence:

$$\square = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$$

Thus T is exactly (the negative of) the Chevalley differential for \square with the trivial representation. In particular, $T^2 = 0$, and this proves that $QR = RQ$ as claimed. On the other hand, a direct proof that $QR = RQ$ implies that $T^2 = 0$, which is equivalent to \square being a Lie bracket. In particular, this provides an alternate proof of the Jacobi identity for \square .

5 Coboundary algebras, CYBE, and Triangular structures

Having developed, via the Chevalley complex, some cohomological language, we can now recast the above definitions, and describe a few more.

Recall that given a Lie algebra \mathfrak{g} and an action ρ over it, we defined the *Chevalley complex* of \mathfrak{g} with coefficients in V by the differential:

When the action ρ is trivial, the condition that $Q^2 = 0$ is equivalent to the Jacobi identity. Given the Jacobi identity, $Q^2 = 0$ if and only if ρ satisfies the conditions of a left action. In particular, \mathfrak{g} acts on itself via ad , and so we can consider (and did, above) the Chevalley complex with coefficients in \mathfrak{g} . The differential is:

Recall that given any vector space V , an *element* of V is a map from the ground field to V :



If \mathcal{C} is any complex, then an element α is a *cocycle* if $d\alpha = 0$. It is a *coboundary* if $\alpha = d\beta$ for some other element β . Of course, since \mathcal{C} is a complex, $d^2 = 0$, and so if an element is a coboundary, then it is a cocycle. The converse holds only if the complex is *exact*; most are not.

We unpack the condition of being a cocycle for the complex Q above for small n . When $n = 0$:

is a cocycle if and only if it is central.

i.e. its bracket with anything is 0. (For a general module, the 0-cocycles are the fixed points.)

When $n = 1$, we are interested in elements \downarrow such that:

$$\text{Diagram 1} - \frac{1}{2} \text{Diagram 2} = 0$$

This is equivalent to a map \rightarrow such that:

$$\text{Diagram 3} + \text{Diagram 4} = \text{Diagram 5}$$

Such a map is a *derivation* of the Lie algebra. (This notion also makes sense for any module; the element \downarrow is a map from \downarrow to \downarrow , and such a map is a *derivation* if it is a cocycle.)

Indeed, we can think of the bracket as an element \downarrow . Then it is easy to check that the Jacobi identity implies that the bracket is a 2-cocycle; however, the Jacobi identity is much stronger than this condition. There don't seem to be natural words to associate with higher cocycle conditions.⁴

Let us now vary the module \downarrow : we let $\downarrow = \underbrace{\downarrow \cdots \downarrow}_m = \downarrow^{\wedge m}$. When $m = n = 0$, the only elements are trivial, and the cocycle condition is trivially satisfied. When $m = 0$, the 1-cocycles are linear maps on \downarrow that are zero on the derived subalgebra; i.e. they are linear maps on the abelianization of \downarrow . We have explored already the case when $n = 0, 1$ and $m = 1$.

When $m = 2$, the 0-cocycles are strange. They are antisymmetric elements \downarrow such that for any element \downarrow , the element \downarrow is antisymmetric. We will now explore the cocycle condition when $m = 2$ and $n = 1$. We have an element \downarrow satisfying:

$$2 \text{Diagram 6} - \frac{1}{2} \text{Diagram 7} = 0$$

Consider the element \downarrow as a map \rightarrow . Then the cocycle condition reads:

$$\text{Diagram 8} = \text{Diagram 9} + \text{Diagram 10} - \text{Diagram 11} - \text{Diagram 12}$$

⁴We will see later that nondegenerate 2-cocycles in the Chevalley complex for the trivial representation (in the notation of the next paragraph: $m = 0, n = 2$) are in bijection with nondegenerate triangular structures.

In particular, if \mathfrak{g} is a Lie bialgebra, then the cobracket δ is a 1-cocycle. We remark that a 1-cocycle δ need not define a Lie bialgebra structure on \mathfrak{g} , because it may not satisfy the coJacobi identity. We write this identity in terms of elements of our Chevalley complex — the coJacobi identity asserts that the following element vanishes:

$$\text{Diagram 1} \stackrel{\text{def}}{=} \text{Diagram 2}$$

As we said above, that δ vanishes does not follow from the fact that δ is a 1-cocycle. But the cocycle condition does imply that δ is a 1-cocycle — checking this is a long but straightforward calculation which we do not repeat here, because it is not part of our primary story. Then it may occur that δ is a 1-coboundary. We say that the Lie algebra \mathfrak{g} along with a 1-cocycle δ is a *Lie quasibialgebra* if δ defined above is a 1-coboundary. We will not develop the theory of quasibialgebras here; see for example [3].

Let \mathfrak{g} be a Lie bialgebra. Then the cobracket δ is a 1-cocycle, as we have described. As in the previous paragraph, any time we find a cocycle, we should ask if it is a coboundary. We say that the Lie bialgebra \mathfrak{g} is *coboundary* if the cobracket is a 1-coboundary, i.e. if there is an element q such that $\delta = \delta_q + \delta_q$, and a choice of such an element is a *coboundary structure* on \mathfrak{g} , and a *coboundary Lie bialgebra* is a Lie bialgebra equipped with a choice of coboundary structure. We remark that a coboundary structure for a given Lie bialgebra is determined only up to a choice of 0-cocycle. A Lie bialgebra homomorphism of coboundary Lie bialgebras is *coboundary* if it preserves the coboundary structure.

Let \mathfrak{g} be a Lie algebra and let $q = -q$ be an antisymmetric element; then $\delta_q = \delta_q + \delta_q$ is a 1-coboundary and hence a 1-cocycle, where Q is the differential Q that we have used throughout these notes. Then it is not necessarily the case that δ_q is a Lie bialgebra structure on \mathfrak{g} : it may not satisfy the coJacobi identity. However, it is a theorem of Drinfeld that δ_q satisfies the coJacobi identity if and only if the element q is a 0-cocycle. The proof is straightforward and we leave it to the reader.⁵ We write **CYB** for the quadratic map $\mathfrak{g}^{\wedge 2} \rightarrow \mathfrak{g}^{\wedge 3}$ given by $q \mapsto \delta_q$. Then **CYB** extends to a map $\mathfrak{g}^{\otimes 2} \rightarrow \mathfrak{g}^{\otimes 3}$ by

$$\text{CYB} : \text{Diagram 3} \mapsto \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6}$$

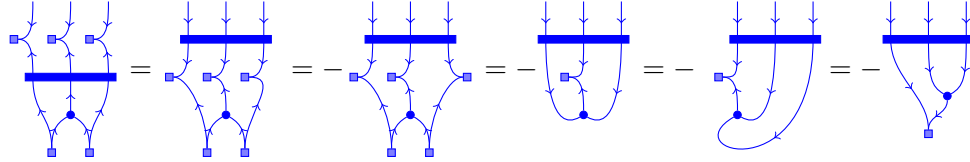
The map **CYB** is called the *classical Yang-Baxter map*, and the quadratic equation $\text{CYB}(q) = 0$ the *classical Yang-Baxter equation*. Certainly any antisymmetric solution to the classical Yang-Baxter equation gives rise to a Lie bialgebra. We say that a coboundary Lie bialgebra is *triangular*

⁵In [3], the proof is described as a “rather long direct computation using the Jacobi identity”. In fact, it is completely straightforward in the graphical language we have been using, but like the proof that δ is a cocycle, it is not deeply enlightening.

if the coboundary structure satisfies the classical Yang-Baxter equation; a *triangular structure* on a Lie bialgebra is a coboundary structure that is triangular, and a *triangular structure* on a Lie algebra is an antisymmetric solution to the classical Yang-Baxter equation. We say that any (possibly non-antisymmetric) solution to the classical Yang-Baxter equation is an *r-matrix*; thus a triangular structure is precisely an antisymmetric *r-matrix*. Following [3], we remark that the image of a coboundary structure under a Lie algebra homomorphism is not necessarily a coboundary structure, but that the image of an *r-matrix* necessarily is an *r-matrix*.

Recall that an arbitrary element $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array}$ is *nondegenerate* if the map $\begin{array}{c} \square \\ \downarrow \\ \downarrow \\ \downarrow \end{array}$ is invertible; by finite-dimensionality, the adjoint map $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array}$ is then also invertible. We say that a triangular Lie algebra is *nondegenerate* if the triangular structure is nondegenerate; we have seen that any triangular Lie algebra has a nondegenerate triangular subalgebra.

We will now describe nondegenerate triangular structures in terms of the cohomology we have developed. Let $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array}$ be nondegenerate and antisymmetric: that the map $\begin{array}{c} \square \\ \downarrow \\ \downarrow \\ \downarrow \end{array}$ is invertible and $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array} = -\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array}$. Define the element $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array}$ such that $\begin{array}{c} \square \\ \downarrow \\ \downarrow \\ \downarrow \end{array}$ is the inverse of $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array}$, i.e. $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \square \\ \downarrow \\ \downarrow \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array}$; then $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array}$ is also antisymmetric. Then:



Thus $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array}$ is a nondegenerate triangular structure on a Lie algebra \downarrow if and only if its inverse $\begin{array}{c} \square \\ \downarrow \\ \downarrow \\ \downarrow \end{array}$ is a 2-cocycle.⁶ Then the (possible degenerate) triangular structures on a Lie algebra are in bijection with the pairs consisting of a Lie subalgebra and a nondegenerate 2-cocycle (with coefficients in the trivial representation) on it.

6 Quasitriangularity

Let \downarrow be a Lie algebra. Recall that a possibly non-antisymmetric element $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array}$ is an *r-matrix* if it is a solution to the classical Yang-Baxter equation:

$$\text{CYB} \left(\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array} \right) \stackrel{\text{def}}{=} \begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array} + \begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array} + \begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array} = 0$$

As is clear from the diagrams, if $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array}$ is an *r-matrix*, then so is its reflection $\begin{array}{c} \square \\ \downarrow \\ \downarrow \\ \downarrow \end{array}$. The function **CYB** is homogeneous quadratic, so any scalar multiple of an *r-matrix* is an *r-matrix*, but the sum of two *r-matrices* is not necessarily an *r-matrix*. In particular, $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array} = \frac{1}{2} (\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array} - \begin{array}{c} \square \\ \downarrow \\ \downarrow \\ \downarrow \end{array})$ is not necessarily an *r-matrix*, and so does not make \downarrow into a triangular Lie bialgebra.

Recall that $\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array}$ makes \downarrow into a coboundary algebra if and only if $\text{CYB}(\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array})$ is invariant (a 0-cocycle). Since $\text{CYB}(\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array}) = 0 = \text{CYB}(\begin{array}{c} \square \\ \downarrow \\ \downarrow \\ \downarrow \end{array})$, it's clear that $\text{CYB}(\frac{1}{2}(\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array} - \begin{array}{c} \square \\ \downarrow \\ \downarrow \\ \downarrow \end{array})) = -\text{CYB}(\frac{1}{2}(\begin{array}{c} \downarrow \\ \downarrow \\ \square \\ \downarrow \\ \downarrow \end{array} + \begin{array}{c} \square \\ \downarrow \\ \downarrow \\ \downarrow \end{array}))$,

⁶This and the previous assertions are given geometric meaning and geometric proofs in [3].

and by the Jacobi identity this is definitely invariant provided that the symmetric element $\frac{1}{2}(\mathcal{Y} + \mathcal{Y}^{\text{sym}})$ is invariant. (There is nothing special about being symmetric: if \mathcal{Y} is invariant, then so is $\text{CYB}(\mathcal{Y})$.) This is a sufficiently natural condition that we give it a name: a *quasitriangular structure* on a Lie algebra \mathfrak{g} is an r -matrix \mathcal{Y} such that $\frac{1}{2}(\mathcal{Y} + \mathcal{Y}^{\text{sym}})$ is invariant under the action by \mathfrak{g} . In particular, if \mathcal{Y} is a quasitriangular structure, then $\mathcal{Y}^{\text{cob}} = \frac{1}{2}(\mathcal{Y} - \mathcal{Y}^{\text{sym}})$ makes \mathfrak{g} into a coboundary Lie bialgebra; a coboundary Lie bialgebra with a choice of quasitriangular r -matrix that gives rise to the cobracket is a *quasitriangular Lie bialgebra*. Although the image of a quasitriangular structure under a Lie algebra homomorphism is an r -matrix, it is not necessarily quasitriangular — its symmetrization need not be invariant — and does not necessarily define a coboundary structure.

Triangular Lie bialgebras are few and far between, but quasitriangular Lie bialgebras are of great importance. We will now describe a quasitriangular structure for the double of any Lie bialgebra \mathfrak{g} . Recall that as a vector space, the double is $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$, with bracket and cobracket given by:

$$\begin{aligned} \text{[Diagram: Lie bracket in double]} &\stackrel{\text{def}}{=} \text{[Diagram: Lie bracket in } \mathfrak{g} \text{]} + \text{[Diagram: Lie bracket in } \mathfrak{g}^* \text{]} + \text{[Diagram: Lie bracket in } \mathfrak{g} \text{]} + \text{[Diagram: Lie bracket in } \mathfrak{g}^* \text{]} + \text{[Diagram: Lie bracket in } \mathfrak{g} \text{]} + \text{[Diagram: Lie bracket in } \mathfrak{g}^* \text{]} \\ \text{[Diagram: Lie cobracket in double]} &\stackrel{\text{def}}{=} \text{[Diagram: Lie cobracket in } \mathfrak{g} \text{]} - \text{[Diagram: Lie cobracket in } \mathfrak{g}^* \text{]} \end{aligned}$$

Indeed, let $\mathcal{Y} \stackrel{\text{def}}{=} \mathcal{Y}$. Then its symmetrization $\frac{1}{2}(\mathcal{Y} + \mathcal{Y}^{\text{sym}}) = \frac{1}{2}\mathcal{Y}^{\text{sym}}$ is the inverse of the canonical invariant pairing $\langle \cdot, \cdot \rangle \stackrel{\text{def}}{=} \langle \cdot, \cdot \rangle + \langle \cdot, \cdot \rangle$, and so is invariant. We check the classical Yang-Baxter equation:

$$\begin{aligned} \text{CYB}(\mathcal{Y}) &= \text{[Diagram: CYB equation terms 1]} + \text{[Diagram: CYB equation terms 2]} + \text{[Diagram: CYB equation terms 3]} \\ &= \text{[Diagram: CYB equation terms 4]} + \text{[Diagram: CYB equation terms 5]} + \text{[Diagram: CYB equation terms 6]} \\ &= \text{[Diagram: CYB equation terms 7]} + \text{[Diagram: CYB equation terms 8]} + \text{[Diagram: CYB equation terms 9]} + \text{[Diagram: CYB equation terms 10]} \\ &= 0 + 0 = 0 \end{aligned}$$

by antisymmetry. Moreover:

$$\begin{aligned} \text{[Diagram: Double Lie bracket]} &= \text{[Diagram: Double Lie bracket terms 1]} + \text{[Diagram: Double Lie bracket terms 2]} \\ &= \text{[Diagram: Double Lie bracket terms 3]} + \text{[Diagram: Double Lie bracket terms 4]} + \text{[Diagram: Double Lie bracket terms 5]} + \text{[Diagram: Double Lie bracket terms 6]} + \text{[Diagram: Double Lie bracket terms 7]} + \text{[Diagram: Double Lie bracket terms 8]} \\ &= \text{[Diagram: Double Lie bracket terms 9]} + \text{[Diagram: Double Lie bracket terms 10]} + 0 + 0 \\ &= \text{[Diagram: Double Lie bracket terms 11]} - \text{[Diagram: Double Lie bracket terms 12]} \\ &= \text{[Diagram: Double Lie bracket terms 13]} \end{aligned}$$

Therefore \cup is a quasitriangular structure for \parallel . We will now prove that in some sense the double is a universal construction of quasitriangular Lie bialgebras.

Let \mathfrak{g} be a Lie algebra and let $\mathfrak{g}^{\mathfrak{g}}$ be an r -matrix, and consider the map $\mathfrak{g}^{\mathfrak{g}}$. It is immediately clear that the image $\mathfrak{g}^{\mathfrak{g}}$ of this map is a Lie subalgebra of \mathfrak{g} . Indeed, it follows directly from the classical Yang-Baxter equation that:

If furthermore $\mathfrak{g}^{\mathfrak{g}}$ is a quasitriangular structure on \mathfrak{g} , then the classical Yang-Baxter equation and the definition of the cobracket assure that:

This proves that $\mathfrak{g}^{\mathfrak{g}}$ is a Lie subbialgebra of \mathfrak{g} , and that $\mathfrak{g}^{\mathfrak{g}}$ is a Lie bialgebra homomorphism (in fact, it is an embedding, as can be seen by counting dimensions).

Let \parallel be the Drinfeld double of $\mathfrak{g}^{\mathfrak{g}}$. Then the embeddings $\mathfrak{g}^{\mathfrak{g}}$ and $\mathfrak{g}^{\mathfrak{g}}$ sum to a map \parallel . For it to be a Lie algebra homomorphism, we need only to check that

If we append $\mathfrak{g}^{\mathfrak{g}}$ to the lower-left string, we get precisely the classical Yang-Baxter equation. But since $\mathfrak{g}^{\mathfrak{g}}$ is onto, this suffices to verify the equation. We have shown that \parallel is a Lie algebra homomorphism. Moreover, it is clear that the quasitriangular structure \cup on \parallel maps to $\mathfrak{g}^{\mathfrak{g}}$ under this homomorphism. Thus we have shown that inside every quasitriangular Lie bialgebra there is a subalgebra whose double maps to the algebra whose double maps to the Lie bialgebra, such that this map is a bijection on the subalgebra and such that the quasitriangular structure is the image of the canonical one.

7 Standard structure

So far our entire story has applied to arbitrary Lie (bi)algebras. We will conclude with a description of the standard bialgebra structure on a simple Lie algebra; we will see that a simple Lie algebra is almost the double of its Borel subalgebra.

Let \mathfrak{g} be a semisimple Lie algebra. Then the Killing form \mathfrak{g} is a nondegenerate pairing. We choose a Cartan subalgebra and system of positive roots. Then the Killing form pairs the Cartan

with itself, so we will draw it without a direction: $|$. We now choose a system of positive roots, inducing upper- and lower-triangular subalgebras, which pair with each other under the Killing form. We write \uparrow for the upper-triangular subalgebra, and identify the lower-triangular subalgebra with its dual \downarrow . Then we have $\mathfrak{g} = \uparrow \oplus | \oplus \downarrow$, and $\uparrow \downarrow = \curvearrowright + \curvearrowleft + \curvearrowright$. Let us compare this to the Double construction: $\mathbb{||} = \uparrow \oplus \downarrow$ with pairing $\curvearrowright = \curvearrowright + \curvearrowleft$ and quasitriangular r -matrix \curvearrowleft . This suggests that for the semisimple Lie algebra \mathfrak{g} we consider the element $\mathbb{V} = \curvearrowleft + \frac{1}{2} \curvearrowright$. Then it is straightforward to check that this does indeed define an r -matrix. It is quasitriangular trivially: its symmetrization is the Casimir $\mathbb{V} \downarrow$. The r -matrix $\curvearrowleft + \frac{1}{2} \curvearrowright$ defines on a semisimple Lie algebra its *standard structure* as a quasitriangular Lie bialgebra. The standard structure depends on a choice of Cartan and system of positive roots; of course, any two such choices are related by an automorphism of \mathfrak{g} .

We conclude by describing the standard construction of the standard structure. Consider the Borel Lie algebra $\mathfrak{b} \stackrel{\text{def}}{=} \uparrow \oplus |$, a subalgebra of \mathfrak{g} . The Killing form on \mathfrak{b} pairs \uparrow invariantly with $\downarrow \stackrel{\text{def}}{=} \downarrow \oplus |$, and it's not too hard to see that the bracket on \downarrow pulls back to a bialgebra structure on \downarrow (namely, it definitely pulls back to a cobracket, and one can check the cocycle identity by hand). Thus we may construct the double $\mathbb{||}$ of \downarrow . We add arrows to the Cartan parts to tell them apart: we have $\mathbb{||} = \uparrow \oplus \uparrow \oplus \downarrow \oplus \downarrow$, and the nondegenerate invariant pairing is $\curvearrowright + \curvearrowleft + \curvearrowright + \curvearrowleft$. The reader is invited to check that the map $\mathbb{||} = \uparrow + \downarrow + \frac{1}{2} (\uparrow + \downarrow)$ that averages the two Cartan components is a Lie algebra homomorphism. Indeed, the kernel of this map is the image of the Cartan $|$ under the difference map $\uparrow - \downarrow$, and this image is central in $\mathbb{||}$. The projection $\mathbb{||}$ sends the quasitriangular structure $\curvearrowleft + \curvearrowright$ on $\mathbb{||}$ to the quasitriangular structure $\curvearrowleft + \frac{1}{2} \curvearrowright$ defined previously.⁷

Finally, a straightforward calculation shows that the cobracket when restricted to the Cartan $|$ is trivial — this is a strong reminder that a choice is involved in constructing the bialgebra structure, since any given element of a semisimple Lie algebra can be chosen to lie in a Cartan. The simplest semisimple Lie algebra is $\mathfrak{g} = \mathfrak{sl}_2$, for which the spaces \uparrow , $|$, and \downarrow are all one-dimensional. It is then a triviality to see that the cobracket defines on \mathfrak{g} the structure of the three-dimensional Heisenberg algebra. For a general semisimple Lie algebra one can compute the cobracket easily provided one has computed the coefficients of the bracket in terms of the root basis. Unfortunately, the graphical notation for linear algebra we have been promoting in this paper is not well suited for computations within a basis — this is entirely the point of the notation.

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⁷In [3] the authors present a slightly modified definition: rather than averaging the Cartans, they sum them, requiring extra 2s in their presentation.

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