Algebraically closed higher categories

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These slides available at
http://categorified.net/Haifa.pdf

Preamble

My goal for this talk is to construct the algebraic closure of $n\mathrm{VEC}$.

Actually, although I will call it the "algebraic closure," what I will really construct is the separable closure. This should be good enough for Galois theory.

Recall: a field extension $\mathbb{K} \to \mathbb{L}$ is ind-finite if $\mathbb{L}$ is the union of its finite-degree subfields; the algebraic closure of $\mathbb{K}$ is the ind-finite extension which contains all finite-degree extensions; the separable closure merely contains all finite-degree separable extensions.

I will work over a fixed field $\mathbb{K}$ of characteristic zero. I expect a similar story applies in positive characteristic, but with more finicky details.

Symmetric multifusion n-categories

Suppose $\mathcal{A}$ is a symmetric monoidal $\mathbb{K}$-linear n-category, with unit object $1 \in \mathcal{A}$. **Convenient notation:** $\Omega\mathcal{A} := \mathrm{End}_{\mathcal{A}}(1)$.

Suppose further that $\mathcal{A}$ is:

- Karoubi complete: all higher idempotents in $\mathcal{A}$ split.
- locally semisimple: all hom-1-categories are semisimple.
- rigid: all objects and morphisms in $\mathcal{A}$ had duals and adjoints.

Then $\mathcal{A}$ has a notion of simple object, and every object decomposes into a direct sum of simples.

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***n*-categorical Schur's lemma:** If X, Y are simple, and $f : X \to Y$ is nonzero, then Y is the image of an idempotent at X (with $f : X \twoheadrightarrow Y$ the projection) and X is the image of an idempotent at Y (with $f : X \hookrightarrow Y$ the inclusion). The relation "$X \sim Y$ if $\hom(X, Y) \neq 0$" is an equivalence relation on $\{$simple objects$\}$.

Definition (Douglas–Reutter)**:** The (Schur) components of $\mathcal{A}$ are $\pi_0\mathcal{A} := \{\text{simple objects}\}/\sim$. For $k<n$, $\pi_k\mathcal{A} := \pi_0\Omega^k\mathcal{A}$.

Definition: $\mathcal{A}$ is multifusion if $|\pi_k\mathcal{A}| < \infty$ for all $k<n$.

Symmetric multifusion *n*-categories

Symmetric multifusion *n*-categories are the "finite-dimensional separable commutative algebras" in the *n*-categorical world. Because I will use them a lot, I will call them simply *n*-gebras.

Convenient notation: $\Sigma : \{(n-1)\text{-gebras}\} \to \{n\text{-gebras}\}$ is the left adjoint to $\Omega : \mathcal{A} \mapsto \mathrm{End}_{\mathcal{A}}(1)$. If $\pi_0\mathcal{A} = \mathrm{pt}$, then $\Sigma\Omega\mathcal{A} = \mathcal{A}$.

$\Sigma\mathcal{A} = \{\text{finite semisimple } \mathcal{A}\text{-modules}\}$. If you drop finiteness and semisimplicity, then Σ certainly exists: it is the Karoubi completion of the 1-object delooping. I think we know how to show that it preserves finiteness and local semisimplicity (in char=0), building on Décoppet, Douglas–Schommer-Pries–Snyder, ...

Example: $\Sigma\mathbb{K} = \mathrm{V\small{EC}}$. $\Sigma\mathrm{V\small{EC}} = \{\text{finite semisimple } \mathbb{K}\text{-algebras, finite bimodules}\}$. $n\mathrm{V\small{EC}} := \Sigma^n\mathbb{K}$.

Nullstellensätze

An (ind-)n-gebra $\mathcal{V}^n$ is algebraically closed if, for any $\mathcal{V}^n$-linear (ind-)n-gebra $\mathcal{A}^n$ (i.e. an equipped with a map $\mathcal{V}^n \to \mathcal{A}^n$):

Weak: If $\mathcal{A}^n \neq 0$, then there exists a $\mathcal{V}^n$-linear map $\mathcal{A}^n \to \mathcal{V}^n$.

Strong: The n-groupoid $\hom_{\mathcal{V}^n}(\mathcal{A}^n, \mathcal{V}^n)$ of morphisms of $\mathcal{V}^n$-linear n-gebras recovers $\mathcal{A}^n$: the canonical map

$$\mathcal{A}^n \to \textsc{Functors}(\hom_{\mathcal{V}^n}(\mathcal{A}^n, \mathcal{V}^n), \mathcal{V}^n)$$

is an equivalence.

Defining property in the homotopy category of n-gebras:

$$\pi_0 \hom_{\mathcal{V}^n}(\mathcal{A}^n, \mathcal{V}^n) = \hom_{\overline{\mathbb{K}}}(\Omega^n \mathcal{A}^n, \overline{\mathbb{K}}).$$

It is not immediately obvious that these conditions are equivalent, nor that they are satisfied by any $\mathcal{V}$, but it will follow from the construction.

Nullstellensätze

When $n = 0$, that the weak and strong Nullstellensätze both characterize $\overline{\mathbb{K}}$ is essentially a theorem of Hilbert.

When $n = 1$, $\Sigma\overline{\mathbb{K}} = \mathrm{V{\small EC}}_{\overline{\mathbb{K}}}$ is not algebraically closed.
Theorem (Deligne): $\mathrm{s}\mathrm{V{\small EC}}_{\overline{\mathbb{K}}}$ is algebraically closed:

- The weak Nullstellensatz is existence of super fibre functors.
- The strong Nullstellensatz is super Tannakian duality.
- The defining property in the homotopy category is (existence of super fibre functors and) if $1 \in \mathcal{A}$ is simple (i.e. if $\mathcal{A}$ is fusion) then all fibre functors $\mathcal{A} \to \mathrm{s}\mathrm{V{\small EC}}$ are isomorphic.

Deligne didn't think of it in this language.

Immediate properties of algebraic closedness

Suppose $\mathcal{V}^n$ is the algebraically closed n-gebra (if it exists).

Lemma: $\Omega\mathcal{V}^n$ is the algebraically closed $(n-1)$-gebra $\mathcal{V}^{n-1}$. In other words, $\mathcal{V}^\bullet$ is a categorical loop spectrum.

Proof: Given an $(n-1)$-gebra $\mathcal{A}$,

$$\pi_0 \hom(\mathcal{A}, \Omega\mathcal{V}^n) = \pi_0 \hom(\Sigma\mathcal{A}, \mathcal{V}^n) = \hom(\Omega^n\Sigma\mathcal{A}, \overline{\mathbb{K}})$$
$$= \hom(\Omega^{n-1}\mathcal{A}, \overline{\mathbb{K}}) = \pi_0 \hom(\mathcal{A}, \mathcal{V}^{n-1}).$$

Lemma: $\mathcal{V}^\times = \mathrm{I}\overline{\mathbb{K}}^\times$ is the $\overline{\mathbb{K}}^\times$-dual to the sphere spectrum. This answers a request of Freed and Hopkins.

Proof: Given an (π-finite) infinite loop space A, build the "group n-gebra" $\Sigma^n\mathbb{K}[A]$.

$$\pi_0 \hom(A, (\mathcal{V}^n)^\times) = \pi_0 \hom(\Sigma^n\mathbb{K}[A], \mathcal{V}^n) = \hom(\Omega^n(\Sigma^n\mathbb{K}[A]), \overline{\mathbb{K}})$$
$$= \hom(\overline{\mathbb{K}}[\pi_n A], \overline{\mathbb{K}}) = \hom(\pi_n A, \overline{\mathbb{K}}^\times) = \pi_0 \hom(A, \Sigma^n \mathrm{I}\overline{\mathbb{K}}^\times)$$

Constructing the algebraic closure

Suppose by induction that we have constructed the algebraically closed $(n-1)$-gebra $\mathcal{V}^{n-1}$, and we want to extend $\Sigma\mathcal{V}^{n-1}$ to $\mathcal{V}^n$.

Lemma (extending on a result of JF–Yu): For any $\mathcal{A}$ with $\Omega\mathcal{A} = \mathcal{V}^{n-1}$, $\pi_0\mathcal{A}$ is a group.

Higher ENO extension theory: The data of an n-gebra $\mathcal{A}$ with prescribed identity component $\Sigma\Omega\mathcal{A}$ and with $A = \pi_0\mathcal{A}$ a group consists of: the abelian group A, and a map of infinite loop spaces

$$A \to (\Sigma^2\Omega\mathcal{A})^\times.$$

I.e. to build $\mathcal{V}^n$ is to find the universal abelian group U, to be $\pi_0\mathcal{V}^n$, such that, for any abelian group A,

$$\hom(A, U) = \pi_0 \hom(A, (\Sigma^2\mathcal{V}^{n-1})^\times).$$

Such a U exists iff $A \mapsto$ RHS takes cokernels to kernels.

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Theorem: This U exists. **Corollary:** $\mathcal{V}^n$ exists.

Hint at proof: The canonical map $(\Sigma^2\mathcal{V}^{n-1})^\times \to \mathrm{I}\overline{\mathbb{K}}^\times[n+1]$ is an iso on $\pi_{\geq 2}$ and an inclusion on π_1. Now apply some LESs to prove the needed exactness of the RHS.

Remark: You can also show that $\mathcal{V}^n$ exists directly, by showing that the presheaf described by universal property takes finite colimits to finite limits, and hence must be ind-representable.

Constructing the algebraic closure

Set $\mathsf{Inv}(-) := \pi_0((-)^\times) = \{$invertible objects up to isomorphism$\}$. For example, $\mathsf{Inv}(\mathcal{V}^n) = \pi_{-n}\mathrm{I}\overline{\mathbb{K}}^\times = \widehat{\pi_n\mathbb{S}}$, where $\pi_n\mathbb{S}$ is the nth stable homotopy group of spheres, and $\widehat{A} := \hom(A, \overline{\mathbb{K}}^\times)$.

Unpacking the proof, our universal $U = \pi_0\mathcal{V}^n$ fits into a LES

$$0 \to \mathsf{Inv}(\Sigma\mathcal{V}^{n-1}) \to \widehat{\pi_n\mathbb{S}} \to U \to \mathsf{Inv}(\Sigma^2\mathcal{V}^{n-1}) \to \widehat{\pi_{n+1}\mathbb{S}} \to \dots$$

Example: When $n = 3$, $\mathsf{Inv}(\Sigma\mathcal{V}^{3-1})$ is trivial and $\mathsf{Inv}(\Sigma^2\mathcal{V}^{3-1})$ is the quantum Witt group qWitt of slightly-degenerate braided fusion categories studied by Davydov, Nikshych, and Ostrik. Thus $\pi_0\mathcal{V}^3 = (\mathbb{Z}/24).\mathrm{qWitt}$. This recovers (and was inspired by) the construction due to Freed, Scheimbauer, and Teleman.

More precisely, for the separable closure, use just the torsion subgroup of qWitt.

Classifying invertibles

What are $\mathrm{Inv}(\Sigma\mathcal{V}^{n-1})$ and $\mathrm{Inv}(\Sigma^2\mathcal{V}^{n-1})$? Identify $\mathcal{V}^{n+1}$ = n+1D framed TFTs (valued in $\mathcal{V}^{n+1}$). Then:

- $\mathrm{Inv}(\mathcal{V}^{n+1})$ = (iso classes of) invertible n+1D TFTs α. Think of these as potential anomaly theories.
- $\mathrm{Inv}(\Sigma\mathcal{V}^{n})$ = (iso classes of) invertible n+1D TFTs α equipped with a nonzero topological boundary condition $\beta : 1 \to \alpha$, modulo "any two boundary conditions are equivalent."
- $\mathrm{Inv}(\Sigma^2\mathcal{V}^{n-1})$ = (iso classes of) invertible n+1D TFTs α equipped with a nonzero topological boundary condition β, modulo nonzero topological interfaces between boundary conditions.

Note: In particular, the map $\mathrm{Inv}(\Sigma^2\mathcal{V}^{n-1}) \twoheadrightarrow \mathrm{Inv}(\Sigma\mathcal{V}^{n})$ is surjective.

Classifying invertibles

Together, the pair (α, β), where $\alpha \in (\mathcal{V}^{n+1})^{\times}$ is invertible and $\beta : 1 \to \alpha$ is nonzero, is an anomalous nD TFT valued in $\mathcal{V}^{\bullet}$.

For $n \neq 3$, the method of Lan, Kong, and Wen provides a complete classification of these. Their method is "dual to surgery theory": you condense all operators of dim $< \frac{n-1}{2}$; this kills the operators of dim $> \frac{n-1}{2}$ by what I want to call quantum Poincaré duality; if n is odd, there is some residual nondegenerate S-matrix pairing information on the operators of dim $= \frac{n-1}{2}$.

LKW don't compare with surgery theory. They focus on the 4D case, but the method works in general, and requires only that $\mathcal{V}$ be algebraically closed. The method also "works" in 3D, except the "pairing information" is described by an arbitrary MTC, and classifying these is probably hopeless.

Suppose A is a finite abelian group equipped with a nondegenerate symmetric bilinear form $\langle , \rangle : A \times A \to \overline{\mathbb{K}}^{\times}$. A subgroup $I \subset A$ is isotropic if $\langle , \rangle|_I = 1$. The isotropic reduction is $A /\!\!/ I = I^{\perp}/I$.

Definition: The Witt group is the quotient monoid (under $\times$) of pairs $(A, \langle , \rangle)$ modulo the equiv relation generated by $A \sim (A /\!\!/ I)$.

$$\mathrm{Witt} = \mathbb{Z}/2 \oplus \bigoplus_{p=1 \pmod 4} (\mathbb{Z}/2)^2 \oplus \bigoplus_{p=3 \pmod 4} \mathbb{Z}/4.$$

Theorem: Applying the LKW method gives:

$$\mathrm{Inv}(\Sigma^2 \mathcal{V}^{n-1}) = \begin{cases} 0, & n = 2k, \\ \mathbb{Z}/2, & n = 4k+1, \\ \mathrm{Witt}, & n = 4k-1, n \neq 3, \\ \mathrm{qWitt}, & n = 3. \end{cases}$$

Classifying invertibles

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$\mathrm{Inv}(\Sigma\mathcal{V}^n)$ is the image of $\mathrm{Inv}(\Sigma^2\mathcal{V}^{n-1})$ inside $\mathrm{Inv}(\mathcal{V}^{n+1}) = \widehat{\pi_{n+1}\mathbb{S}}$.

The generator of the $\mathbb{Z}/2$ is the $4k+2$D Arf–Kervaire invariant.
Cor: $\mathrm{Inv}(\Sigma\mathcal{V}^{4k+1}) = 0$ except when Arf–Kervaire $\neq 0 \in \widehat{\pi_{4k+2}\mathbb{S}}$.

Every class in Witt can be lifted to an oriented TFT, and so vanishes in $\widehat{\pi_{4k}\mathbb{S}}$. **Cor:** $\mathrm{Inv}(\Sigma\mathcal{V}^{4k-1}) = \{0\} \subset \widehat{\pi_{4k}\mathbb{S}}$.

And $\mathrm{Inv}(\Sigma\mathcal{V}^{2k}) = \{0\}$. In other words, the only nontrivial anomalies realized by TFTs are the Arf–Kervaire invariants.

The absolute Galois group

Each extension $\Sigma\mathcal{V}^{n-1} \subset \mathcal{V}^n$ is Galois. All together, the higher-categorical absolute Galois group is $\mathrm{G{\scriptstyle AL}} := \mathrm{Aut}(\mathcal{V}^\bullet/\Sigma^\bullet\mathbb{K})$. By construction, $\pi_n\mathrm{G{\scriptstyle AL}} = \widehat{\pi_0\mathcal{V}^n}$. Our calculations provide a LES:

$$\begin{array}{ccccccc}
 & & & \dots & \to & 0 \\
\to & \pi_{4k+2}\mathrm{G{\scriptstyle AL}} & \to & \pi_{4k+2}\mathbb{S} & \to & \mathbb{Z}/2 \\
\to & \pi_{4k+1}\mathrm{G{\scriptstyle AL}} & \to & \pi_{4k+1}\mathbb{S} & \to & 0 \\
\to & \pi_{4k}\mathrm{G{\scriptstyle AL}} & \to & \pi_{4k}\mathbb{S} & \overset{0}{\to} & \widehat{\mathrm{Witt}} \\
\to & \pi_{4k-1}\mathrm{G{\scriptstyle AL}} & \to & \pi_{4k-1}\mathbb{S} & \to & 0 \\
\to & \dots & & & &
\end{array}$$

Remark: The right-hand column are almost the L-groups $L_n(\mathbb{Z})$!

In fact, they are (products of) L-groups:

$$\widehat{\mathrm{Witt}} = \prod_p L_{4k}(\mathbb{Z}/p), \qquad \mathbb{Z}/2 = \prod_p L_{4k+2}(\mathbb{Z}/p).$$

The absolute Galois group

If the right-hand column were $L_n(\mathbb{Z})$, then $\mathrm{G{\scriptsize AL}}$ would be the group PL of piecewise-linear automorphisms of $\mathbb{R}^\infty$. At least, this is true at the level of $\pi_\bullet$. Said another way:

Summary of results: $\mathrm{G{\scriptsize AL}}$ is what you get by taking PL and replacing:

- The $8\mathbb{Z} = \pi_{4k}(\mathbb{S}^\times/\mathrm{PL})$, $k > 1$, with $\widehat{\mathrm{Witt}}$.
- The $24\mathbb{Z} = \pi_4(\mathbb{S}^\times/\mathrm{PL})$ with $\widehat{\mathrm{qWitt}}$.
- The $\mathbb{Z}/2 = \pi_0\mathrm{PL}$ with $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$.

When $\mathbb{K} = \mathbb{R}$, the last one is no replacement! This should have something to do with higher unitarity and reflection positivity, but that's a talk for another day...