0. Operads

$\mathbb{S}$ = groupoid of finite sets.

If $\mathcal{V}$ is enriching (= cocomplete, symmetric monoidal, and closed), then $\mathrm{Fun}(\mathbb{S},\mathcal{V})$ is monoidal:

$$(\mathcal{A}\odot\mathcal{B})(S) = \coprod_{T \text{ a partition of } S} \left(\bigotimes_{t\in T}\mathcal{A}(t)\right)\otimes B(S)$$

($\mathcal{V}$ is symmetric monoidal $\Rightarrow$ $\mathrm{Psh}(\mathcal{V})$ is enriching.)

$\mathcal{C}$ symmetric monoidal category $\Rightarrow$ enriched over $\mathrm{Fun}(\mathbb{S})$: $\mathcal{H}om(X,Y)(S) = \hom(X^{\otimes S}, Y)$

Defn: An *operad* $\mathcal{O}$ is a unital associative monoid object in $(\mathrm{Fun}(\mathbb{S}),\odot)$. An *$\mathcal{O}$-algebra* in $\mathcal{C}$ is $X\in\mathcal{C}$ and $\mathcal{O}\to\mathcal{H}om(X,X)$ morphism of operads. $\mathcal{V}$-enriched version is just as easy.

E.g.: • Unit object for $\odot$: $\mathbf{1}(S) = \begin{cases}\emptyset, & S\not\cong\{*\}\\ \{*\}, & S\cong\{*\}\end{cases}$.
- Terminal object in $\mathrm{Fun}(\mathbb{S})$: $\mathcal{C}om(S) = \{*\}$.
- $\mathcal{A}ssoc(S) = \{$total orderings of $S\}$.

1. Little 2-Disk Operads

$\mathrm{D} = \{(x,y)\in\mathbb{R}^2 \text{ s.t. } x^2+y^2\le 1\}$.

Defn: $\mathcal{E}_2, \tilde{\mathcal{E}}_2$ are operads in Spaces. $\mathcal{E}_2(S) = \{S$ many nonintersecting closed circular disks in $D\}$. Operad structure: shrink linearly, insert, erase seams.

Defn: $\tilde{\mathcal{E}}_2(S) = \mathcal{E}_2(S)\times(\mathrm{S}^1)^S$. Operad structure: rotate, shrink linearly, insert, erase seams.

$\mathrm{H}_\bullet X = \mathrm{H}_\bullet(X,\mathbb{Q})$. $\mathrm{H}_\bullet$ is monoidal: preserves operads.

$\mathrm{H}_0\mathcal{E}_2(S) = \mathbb{Q} \Rightarrow \mathrm{H}_0\mathcal{E}_2 = \mathbb{Q}\cdot\mathcal{C}om$ as linear operads.

$\mathrm{H}_1\mathcal{E}_2(2) = \mathbb{Q}$ with basis vector P. $P(a,b) = P(b,a)$.

$\mathrm{H}_1\mathcal{E}_2(3) = \mathbb{Q}^3$, $\mathrm{H}_2\mathcal{E}_2(3) = \mathbb{Q}^2 \Rightarrow$ relations.

Theorem [Getzler]: $\mathrm{H}_\bullet\mathcal{E}_2$ is generated as an operad in graded vector spaces by $\mathrm{H}_0\mathcal{E}_2 = \mathbb{Q}\cdot\mathcal{C}om$ and $P\in\mathrm{H}_1\mathcal{E}_2(2)$, with relations:

$P(a,b) = P(b,a)$
$P(a,bc) = P(a,b)c + P(a,c)b$
$P(P(a,b),c) + P(P(b,c),a) + P(P(c,a),b) = 0$

This is the *Gerstenhaber operad* (with corrected signs).

$\mathrm{H}_\bullet\tilde{\mathcal{E}}_2$ is generated by $\mathrm{H}_\bullet\mathcal{E}_2$ and $\Delta\in\mathrm{H}_1\tilde{\mathcal{E}}_2(1)$ with:

$\Delta(ab) = \Delta(a)\,b + a\,\Delta(b) + P(a,b)$

This is the *Batalin-Vilkovisky operad*.

E.g.: X is a manifold, $\mathscr{C}^\infty([1]\mathrm{T}^*X) = \Lambda^\bullet\Gamma(\mathrm{T}X)$ is Gerstenhaber with $P = \sum_i\left(\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial p_i} + \frac{\partial}{\partial p_i}\otimes\frac{\partial}{\partial x^i}\right)$ (indep of choice of coords x^i on X and Darboux fiber coords p_i).

Pick measure μ on X and x^i with $\mu = \mathrm{d}x^1\cdots\mathrm{d}x^n$. $\Delta = \sum_i\frac{\partial^2}{\partial x^i\partial p_i}$ depends only on μ and is a BV structure.

2. Drinfel'd associators and related

Defn: $\mathfrak{t}$ is an operad of Lie algebras ($\otimes = \oplus$) with:
- $\mathfrak{t}(S)$ generated by $\binom{S}{2}$ terms $t_{a,b} = t_{b,a}$, $a\neq b$;
- satisfying $[t_{a,b}, t_{c,d}] = 0$ if $\{a,b,c,d\}$ all distinct, and $[t_{a,b}, t_{b,c}] = [t_{b,c}, t_{c,a}]$.
- $t_{a,B} = \sum_{b\in B} t_{a,b}$ generates operad structure.

Defn: $\tilde{\mathfrak{t}}(S) = \mathfrak{t}(S)\oplus\mathbb{Q}^S$. $\{u_a\}$ basis for $\mathbb{Q}^S$. Operad structure: $u_A = \sum_{a\in A} u_a + \sum_{a,b\in A} t_{a,b}$.

U : Lie Algebras $\to$ Algebras is monoidals. $\mathrm{U}\mathfrak{t}$ and $\mathrm{U}\tilde{\mathfrak{t}}$ are operads of algebras ($\otimes = \otimes$). Adic-complete w.r.t. ideal generated by $t,u \Rightarrow$ operads $\hat{\mathrm{U}}\mathfrak{t}$ and $\hat{\mathrm{U}}\tilde{\mathfrak{t}}$.

Defn: A *Drinfel'd associator* is an element $\Phi = 1+\cdots\in\hat{\mathrm{U}}\mathfrak{t}$ satisfying:

$$\Phi_{a,b,c}\Phi_{a,\{b,c\},d}\Phi_{b,c,d} = \Phi_{\{a,b\},c,d}\Phi_{a,b,\{c,d\}} \qquad (\pentagon)$$

$$\exp\left(\pm\tfrac{1}{2}t_{a,b}\right)\Phi_{a,b,c}\exp\left(\pm\tfrac{1}{2}t_{b,c}\right) = \Phi_{b,a,c}\exp\left(\pm\tfrac{1}{2}t_{\{a,c\},b}\right)\Phi_{a,c,b} \qquad (\hexagon_\pm)$$

Theorem [Drinfel'd]: Drinfel'd associators exist. **Proof:** Knizhnik-Zamolodchikov equations $\Rightarrow$ associator over $\mathbb{C}$. Abstract argument $\Rightarrow$ then there must exist one over $\mathbb{Q}$.

Defn: $\mathcal{P}a\mathcal{B}$ is operad of groupoids with:
- Objects of $\mathcal{P}a\mathcal{B}(S)$ are parenthesized orderings of S.
- Morphisms are braids.
- Operadic structure is by cabling.

Defn: $\mathcal{P}\tilde{a}\mathcal{B}$ has morphisms = ribbon braids.

3. Formality of $\mathcal{E}_2$ and $\tilde{\mathcal{E}}_2$

Theorem [Kontsevich, Tamarkin, Ševera]: There is a zig-zag of quasiisomorphisms of dg operads connecting $\mathrm{C}_\bullet\mathcal{E}_2$ to $\mathrm{H}_\bullet\mathcal{E}_2$, and connecting $\mathrm{C}_\bullet\tilde{\mathcal{E}}_2$ to $\mathrm{H}_\bullet\tilde{\mathcal{E}}_2$.

Proof: $\mathrm{H}_\bullet\mathcal{E}_2 \xrightarrow{\sim} \Lambda^\bullet\mathfrak{t}$ = CE complex:
$\mathrm{H}_0 = \mathbb{Q}\cdot\mathcal{C}om = \Lambda^0$, $P(a,b)\mapsto t_{a,b}$, $\Delta(a)\mapsto u_a$.

$\Lambda^\bullet\mathfrak{t}\xrightarrow{\sim}\mathrm{C}_\bullet\mathrm{U}\mathfrak{t}\xrightarrow{\sim}\mathrm{C}_\bullet\hat{\mathrm{U}}\mathfrak{t}$ = complex computing $\mathrm{Tor}_{\hat{\mathrm{U}}\mathfrak{t}}(\mathbb{Q},\mathbb{Q})$, because the representation theories are (almost) the same.

$\mathrm{C}_\bullet\hat{\mathrm{U}}\mathfrak{t}$ = chains for $\mathbb{Q}$-linear category with one object. A choice of Drinfeld associator gives an equivalence of operads of categories $\mathbb{Q}\cdot\mathcal{P}a\mathcal{B}\to\hat{\mathrm{U}}\mathfrak{t}$, via "reassociate" $\mapsto\Phi$, "braid" $\mapsto\exp\left(\frac{1}{2}t\right)$, and "twist" $\mapsto u$.

$\Rightarrow \mathrm{C}_\bullet\mathcal{P}a\mathcal{B}\xrightarrow{\sim}\mathrm{C}_\bullet\hat{\mathrm{U}}\mathfrak{t}$, being an equivalence of categories.

$\mathcal{P}a\mathcal{B}\to$ pure braid group (one object groupoid) is equivalence of categories. $\mathcal{E}_2$ is a $K(G,1)$ for G = pure braid group. So there is a space X = (universal cover of (geometrical realization of nerve of $\mathcal{P}a\mathcal{B}$)$\times\mathcal{E}_2$)/(pure braid group), with $\mathrm{C}_\bullet\mathcal{P}a\mathcal{B}\xleftarrow{\sim}\mathrm{C}_\bullet X\xrightarrow{\sim}\mathrm{C}_\bullet\mathcal{E}_2$.

□