0. History and strategy

Feynman says:

$$U(t, q_0, q_1) = \int_{\substack{\text{paths } \varphi:[0,t]\to\mathcal{N} \\ \varphi(0)=q_0,\, \varphi(t)=q_1}} \exp\left(\frac{i}{\hbar}A(\varphi)\right) \mathrm{d}\varphi \qquad (*)$$

where:

- $U : \mathbb{R}\times\mathcal{N}\times\mathcal{N} \to \mathbb{C}$ is the integral kernel of the time evolution operator in quantum mechanics on $\mathcal{N}$;
- $A(\varphi) = \int_{\tau=0}^{t} L(\dot\varphi(\tau), \varphi(\tau))\mathrm{d}\tau$ is the *action* for a *Lagrangian function* $L : \mathrm{T}\mathcal{N} \to \mathbb{R}$;
- *Planck's constant* $\hbar \in \mathbb{R}$ is a small and positive;
- $\mathrm{d}\varphi = \prod_{0<\tau<t} \mathrm{dVol}(\varphi(t))$ does not exist (dVol is a chosen volume form on $\mathcal{N}$).

Strategy to make sense of $(*)$:

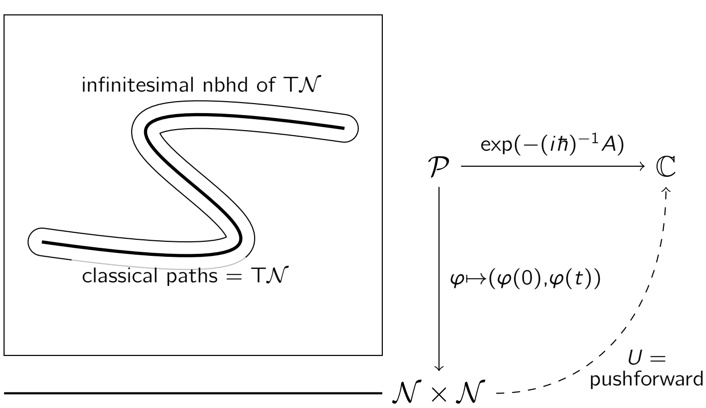
- Set $\hbar \ll 1$ infinitesimal;
- Define *formal oscillating integral* by copying $\hbar \to 0$ asymptotics of finite-dimensional integrals;
- Show that definition makes sense (hard!).

(Different possible strategy: use Wiener measure.)

Fact [folklore, c.f. Evans+Zworski]: $\mathcal{M}$ a f.d. manifold with volume form dVol; $f : \mathcal{M} \to \mathbb{R}$ smooth (with good growth near ∞). As $\hbar \to 0$, $\int_{\mathcal{M}} \exp\bigl(-(i\hbar)^{-1}f\bigr)\,\mathrm{dVol}$ is supported near critical points of f. Asymptotics known explicitly near each nondegenerate critical point of f.

Idea: $\mathcal{P} = \{\text{paths in } \mathcal{N} \text{ with duration } t\}$ is ∞-dim manifold. Fibered over $\mathcal{N}\times\mathcal{N}$. We want $\int$ over fibers. Fiberwise critical points = paths satisfying Euler-Lagrange equations = *classical paths* = copy of $\mathrm{T}\mathcal{N}$ if $\frac{\partial^2 L}{\partial v^2}$ is everywhere nondegenerate.

So *formal path integral* inputs $\mathcal{N}$, L, dVol, and a classical path γ satisfying nondegeneracy assumption, and outputs power-series = $\int$ over φ infinitesimally near γ with same boundary conditions.



Integral is supported on vertical cross sections of infinitesimal neighborhood of $\mathrm{T}\mathcal{N}$. So we should expect bad behavior near folds in $\mathrm{T}\mathcal{N} \to \mathcal{N}\times\mathcal{N}$.

1. Nondegeneracy conditions

Assume $\frac{\partial^2 L}{\partial v^2}$ is everywhere nondegenerate. Then "project, flow for duration t": $\mathrm{T}\mathcal{N} \to \mathcal{N}\times\mathcal{N}$ is well-defined; identity $\mathrm{T}\mathcal{N}$ = classical paths with duration t. (If $\mathcal{N}$ is noncompact, replace $\mathrm{T}\mathcal{N}$ with open nbhd thereof.) A classical path is *nonfocal* if this function is local diffeomorphism. *Focal* paths = "folds" in picture.

Fact [Morse, c.f. Milnor]: If $\frac{\partial^2 L}{\partial v^2}$ is everywhere positive-definite, then focal paths are few and far between.

Action A is smooth function on fibers of $\mathcal{P} \to \mathcal{N}\times\mathcal{N}$; at classical path γ, (fiberwise) Hessian $A^{(2)}_\gamma$ makes sense. $\mathrm{T}^{\text{fiber}}_\varphi\mathcal{P} = \{\text{sections of } \varphi^*\mathrm{T}\mathcal{N} \to [0,t] \text{ vanishing at } 0, t\}$; $A^{(2)}_\gamma$ is second-order differential operator on $\mathrm{T}^{\text{fiber}}_\gamma\mathcal{P}$.

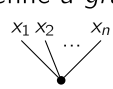
Theorem [folklore, c.f. TJF]: γ is classical path, $\gamma(0) = q_0$, $\gamma(t) = q_1$. TFAE:

- γ is nonfocal, so can make γ depend smoothly on q_a;
- $A^{(2)}_\gamma$ has no kernel on $\mathrm{T}^{\text{fiber}}_\gamma\mathcal{P}$;
- $A^{(2)}_\gamma$ has *Green's function* = nonsmooth section of $(\gamma^*\mathrm{T}\mathcal{N})^{\otimes 2} \to [0,t]^{\times 2}$ vanishing on boundary square, with $A^{(2)}_\gamma G(\varsigma,\tau) = \delta(\varsigma,\tau)$ ($A^{(2)}$ acts in ς slot);
- $S_\gamma(t, q_0, q_1) = A^{(0)}_\gamma$ satisfies $\det \frac{\partial^2 S}{\partial q_0 \partial q_1} \neq 0$.

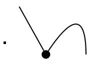
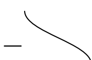
2. Feynman diagrams

$\mathcal{M}$ f.d. with dVol; $f : \mathcal{M} \to \mathbb{R}$ smooth with nondeg critical point; $\mathcal{O}$ compact nbhd of c.p. (no other c.p.s in $\mathcal{O}$); $x^i : \mathcal{O} \to \mathbb{R}^{\dim\mathcal{M}}$ local coords with $\mathrm{dVol} = \prod \mathrm{d}x^i$ (these exist by result of Moser) and $x(\text{c.p.}) = 0$. Set η = *Morse index* of f at c.p. = # negative eigenvalues of $f^{(2)}(0)$.

Define a *graphical calculus* / *Feynman rules*:



$= -f^{(n)}(0)\cdot(x_1\otimes\cdots\otimes x_n)$

 $= \bigl(f^{(2)}(0)\bigr)^{-1}$, i.e.  $= -$ 

A *Feynman diagram* is a combinatorial graph Γ (possibly empty, disconnected, etc.). $\mathrm{ev}(\Gamma)$ *evaluates* the diagram. $\chi(\Gamma) = |V| - |E|$. $|\mathrm{Aut}\,\Gamma|$ = number of symmetries.

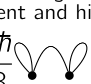
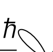
Example: ev() $= \bigl(-f^{(3)}(0)\bigr)^{\otimes 2} \circ \bigl((f^{(2)}(0))^{-1}\bigr)^{\otimes 3}$; $\chi = -1$; $|\mathrm{Aut}| = 8$.

Theorem [Feynman+Dyson, c.f. Evans+Zworski]:

$$\int_{\mathcal{O}} \exp\left(\frac{i}{\hbar}f\right)\mathrm{dVol} \overset{\hbar\to 0}{=} (2\pi i\hbar)^{\dim\mathcal{M}/2} \times$$

$$\times \exp\left(\frac{i}{\hbar}f(0)\right) \times (-i)^\eta \times \left|\det f^{(2)}(0)\right|^{-1/2} \times$$

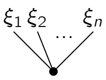
$$\times \sum_{\substack{\text{Feynman diagrams } \Gamma \\ \text{with only trivalent and higher vertices}}} \frac{(i\hbar)^{-\chi(\Gamma)}\,\mathrm{ev}(\Gamma)}{|\mathrm{Aut}\,\Gamma|}$$

$= (\dots)\times\Bigl(1 + \frac{i\hbar}{8}$  $+ \frac{i\hbar}{12}$  $+ \frac{i\hbar}{8}$  $+ \dots\Bigr)$.

3. Translation to infinite dimensions

$\mathcal{M} \rightsquigarrow$ fiber of $\mathcal{P} \to \mathcal{N} \times \mathcal{N}$. $f \rightsquigarrow A$. γ nondegenerate classical path; think of it as depending smoothly on boundary conditions $\gamma(0) = q_0$, $\gamma(t) = q_1$.

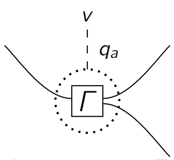
Pick local coords x^i on $\mathcal{N}$ with $\mathrm{dVol} = \prod \mathrm{d}x^i$. Then $A^{(n)} : \mathrm{T}^{\otimes n}\mathcal{P} \to \mathbb{R}$ makes sense, but depends on coords. For $\xi_1, \dots, \xi_n \in \mathrm{T}_\gamma\mathcal{P}$, an explicit Feynman rule:



$$= -A^{(n)}(\gamma)\cdot(\xi_1 \otimes \dots \otimes \xi_n) = $$
$$= -\int_0^t \prod_{k=1}^n \sum_{i_k=1}^{\dim \mathcal{N}} \left(\dot\xi_k^{i_k}(\tau)\frac{\partial}{\partial v^{i_k}} + \xi_k^{i_k}(\tau)\frac{\partial}{\partial q^{i_k}} \right) L \,\mathrm{d}\tau$$

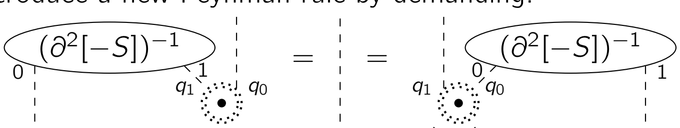
$\frac{\partial}{\partial v}, \frac{\partial}{\partial q}$ act only on L, then evaluated at $(\dot\gamma(\tau), \gamma(\tau))$.

If Γ is a Feynman diagram, $a = 0, 1$, and $v \in \mathrm{T}_{q_a}\mathcal{N}$,



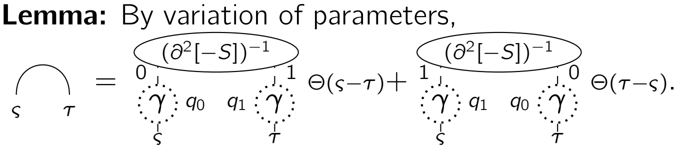
$$\stackrel{\text{def}}{=} \sum_{i=1}^{\dim \mathcal{N}} v^i \frac{\partial}{\partial q_a^i}\big[\mathrm{ev}(\Gamma)\big].$$

Introduce a new Feynman rule by demanding:



(Recall $\bullet = -A^{(0)} = -S_\gamma(q_0, q_1)$, so  is invertible if γ is nonfocal.)

Lemma: By variation of parameters,



The *Morse index* of a classical path is well-known, c.f. Milnor. It is finite iff $\frac{\partial^2 L}{\partial v^2}$ is positive-definite.

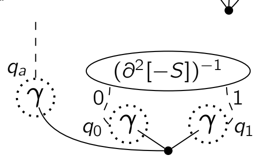
An ad hoc definition (chosen to make units work out):
$\dim(\text{fiber of } \mathcal{P} \to \mathcal{N}^{\times 2}) = -\dim \mathcal{N}$.

A more problematic ad hoc definition:

$$\left|\det A^{(2)}\right|^{-1} = \left|\det \frac{\partial^2\left[-S_\gamma(t, q_0, q_1)\right]}{\partial q_0\, \partial q_1}\right|$$

Justification:

- $\frac{\partial}{\partial q_a} \log \left|\det A^{(2)}\right|^{-1} \stackrel{\text{want}}{=} -\operatorname{Tr} \frac{\partial A^{(2)}}{\partial q_a}\left(A^{(2)}\right)^{-1} =$ 

- $\frac{\partial}{\partial q_a} \log \left|\det \frac{\partial^2\left[-S_\gamma(t,q_0,q_1)\right]}{\partial q_0\,\partial q_1}\right| \stackrel{\text{have}}{=}$ 

Lemma: If  converges, these agree.

This defines the *formal path integral* $U_\gamma(t, q_0, q_1)$. It depends on coordinates, might diverge, ...

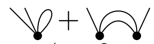
4. Divergences

Main Problem: Individual Feynman diagrams may diverge: $G_\gamma(\varsigma, \tau)$ has singularity like $|\varsigma - \tau|$; a vertex can differentiate w.r.t. ς or τ; so edge can include $\delta(\varsigma - \tau) = \frac{1}{2}\frac{\partial^2}{\partial\varsigma\partial\tau}|\varsigma - \tau|$; so each loop can contribute $\delta(0)$.

Example: $L(v, q) = \frac{v^2}{2q^2}$, $\mathrm{dVol} = \mathrm{d}q$. Then $U(t, q_0, q_1)$ diverges at one-loop order as $\frac{t^3}{24}(\delta(0))^2$.

Theorem [TJF]: If $L(v, q) = \frac{1}{2}a(q)\cdot v^{\otimes 2} + b(q)\cdot v + c(q)$ and $\mathrm{dVol} = \sqrt{\det a}$ is the volume form for the Riemannian metric a, then divergences cancel at each order.

Proof: L is quadratic in v, so divergent loops do not overlap. Consider sum (with symmetry factors) of degree-n "wagon wheels" = single loop with n external edges. Divergent part is
$\delta(0)\times\frac{1}{n!}\operatorname{Tr}\partial^n[\log a] = \delta(0)\times\frac{1}{n!}\partial^n[\log\det a] = 0$.

Example:  $=$
$= \delta(0)\times\operatorname{Tr}\big(-\partial^2 a\cdot a^{-1} + \partial a\cdot a^{-1}\cdot\partial a\cdot a^{-1}\big) + \text{finite}.$

5. Paths that leave a coordinate patch

Theorem [TJF]: If there are no diverges, $U_\gamma(t, q_0, q_1)$ does not depend on the choice of coordinates, only on that they be compatible with the volume form.

Theorem [TJF]: Let γ_{12} of duration $t_1 + t_2$ be nondegenerate and classical, and suppose that $\gamma_1 = \gamma_{12}|_{[0,t_1]}$ and $\gamma_2 = \gamma_{12}|_{[t_1,t_1+t_2]}$ are nondegenerate. Let U_{12}, U_1, U_2 be the corresponding formal path integrals. Then:
$U_{12}(t_1 + t_2, q_1, q_2) = \int U_1(t_1, q_1, q)\, U_2(t_2, q, q_2)\,\mathrm{d}q$
Integral is formal (Feynman-diagrammatic), supported in small neighborhood $q \approx \gamma_{12}(t)$.

Theorem [TJF]: The generically-infinite sum

$$U(t, q_0, q_1) \stackrel{\text{def}}{=} \sum_{\substack{\gamma \text{ nondegenerate s.t.}\\ \gamma(0)=q_0, \gamma(t)=q_1}} U_\gamma(t, q_0, q_1)$$

converges pointwise in the sense of distributions.

6. Schrödinger equation

Theorem [TJF]: Suppose $L(v, q) = \frac{1}{2}a(q)\cdot v^{\otimes 2} + b(q)\cdot v + c(q)$, where a is Riemannian metric (kinetic energy), b is 1-form (magnetic energy), c is function (potential energy). Pick local coordinates so that $\det a = 1$. Set:
$\hat H_q = \left(i\hbar\frac{\partial}{\partial q} + b(q)\right)\cdot\frac{a^{-1}(q)}{2}\cdot\left(i\hbar\frac{\partial}{\partial q} + b(q)\right) - c(q)$
Then $U_\gamma(t, q_0, q_1)$ satisfies
$i\hbar\frac{\partial}{\partial t}U_\gamma(t, q_0, q_1) = \hat H_{q_1}\big[U(t, q_0, q_1)\big]$.

Theorem [TJF]: Same conditions. As a pointwise limit of distributions:
$\lim_{t\to 0} U(t, q_0, q_1) = \delta(q_0 - q_1)$.