TMF and SQFT: questions and conjectures

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Background: $\mathrm{SQFT}^\bullet$

The spectrum $\mathrm{TMF}^\bullet$ of Topological Modular Forms, as defined by Lurie, Hopkins–Miller, Goerss, . . . , is an object of derived number theory: the global sections of the derived structure sheaf of the derived stack of derived elliptic curves. To work with TMF involves tracking Galois actions, descent data, spectral sequences, ...

Proposal [Segal, Stolz–Teichner]: $\mathrm{TMF}^\bullet$ has an analytic model given by:

$$\mathrm{TMF}^\bullet \simeq \mathrm{SQFT}^\bullet := \{\text{compact 2D } \mathcal{N}{=}(0,1) \text{ SQFTs of degree } \bullet\}$$

Motivation: $\mathrm{K}^\bullet$ can be defined derived-algebraically, but can also be modelled analytically in terms of super Hilbert spaces. There are many closely related models, one of which is:

$$\mathrm{K}^\bullet \simeq \{\text{compact } \mathcal{N}{=}1 \text{ SQM models of degree } \bullet\}$$

Background: $\mathrm{SQFT}^\bullet$

$\mathrm{SQFT}^\bullet = \{$compact 2D $\mathcal{N}{=}(0,1)$ SQFTs of degree $\bullet\}$

SQFT $=$ supersymmetric quantum field theory $=$ (unitary, Poincaré-invariant, etc.) QFT equipped with some susy. The minimal nonzero susy in 2D is $\mathcal{N} = (0,1)$: there is one susy, which transforms as a right-handed chiral spinor under $\mathrm{Spin}(2)$.

The topology on $\{$(S)QFTs$\}$ should be one in which two SQFTs are close if they have the same low-energy ("effective") behaviour. **Example [Henriques]:** For (S)QM models, can use the topology of strong convergence of the resolvent (of $\hat{H}$).

Any (Spin, say) nD QFT can have a gravitational anomaly valued in $\mathrm{I}\Omega^{n+2}_{\mathrm{Spin}}$. When $n{=}2$, the iso class of the anomaly is the degree $2(c_L - c_R) \in \mathrm{I}\Omega^4_{\mathrm{Spin}}(\mathrm{pt}) \cong \mathbb{Z}$. More precisely, we must give an isomorphism between the anomaly of our SQFT and a reference anomomaly in order to resolve some sign ambiguities.

Background: $\mathrm{SQFT}^\bullet$

$\mathrm{SQFT}^\bullet$ = {compact 2D $\mathcal{N}$=(0, 1) SQFTs of degree •}

Definition [Segal]: A (unitary) QFT is compact if its Wick-rotated partition function converges absolutely on all closed spacetimes. For QM: $\exp(-\tau\hat{H})$ should be trace-class for all $\tau > 0$.

Examples: Sigma models with compact target. Massive boson (aka harmonic oscillator). **Nonexample:** Massless boson.

Lemma: SQFT^{4n} is not contractible. **Proof:** The (appropriately normalized) partition function on tori with nonbounding spin structure provides a nontrivial locally-constant map $\mathrm{SQFT}^{4n} \to \{$weakly holomorphic modular forms of weight $-2n\}$.

Conjecture [Seiberg]: For any fixed anomaly, dimension, and susy, {(possibly) noncompact QFTs} is contractible.

RG flow

Spaces of (S)QFTs come with a canonical one-parameter renormalization group (RG) flow which rescales the metric on spacetime. RG fixed points = (super)conformal field theories.

Zamolodchikov: In 2D, RG flow is Morse flow for $C = \frac{1}{2}(c_L + c_R) \geq 0$. **Expect:** C is Morse–Bott. (C is not Morse.)

Question: Does (downward) RG flow converge in $\{$compact 2D (S)QFTs$\}$? In other words, if a 2D QFT is compact, is its deep IR limit again compact?

If so, then $\mathrm{SQFT}^\bullet$ can be studied Morse-theoretically in terms of "zig-zags along RG flow lines" between compact SCFTs.
Question: Develop a model of $\mathrm{TMF}^\bullet$ built out of SCFTs.

Question: $\mathrm{SQFT}^\bullet$ isn't really a space — it is a stack. Develop Morse theory on stacks.

Spectrum structure

Basepoint := $0 \in \mathrm{SQFT}^\bullet$

The basepoint for $\mathrm{SQFT}^\bullet$ is the Zero TQFT: for nonempty inputs, partition function $\equiv 0$, Hilbert space $\equiv \mathbb{C}^0$, etc.

An SQFT has spontaneous susy breaking if the operator 1 is a superdescendant (=is susy-exact). **Expect:** Spont. susy breaking iff deep IR limit is Zero. (This should be an example of susy localization. It is also a condition on the topology on $\mathrm{SQFT}^\bullet$.)

Example: $\mathrm{Fer}(1) :=$ a single chiral massless fermion λ. Choose the $\mathcal{N}{=}(0,1)$ susy generated by the supercurrent $:\!\lambda\!:$. At length scale L, this flows to $\mathrm{Fer}(1)$ with susy $L^{1/2}{:}\lambda{:}$. Renormalize λ. In the $L \to \infty$ limit, λ renormalizes to 0, and $\mathrm{Fer}(1) \to$ Zero.

Question: In some models, "Zero" is not a valid QFT. Is {SQFTs with spontaneous susy breaking} in any case contractible?

Spectrum structure
$\Omega\mathrm{SQFT}^\bullet \simeq \mathrm{SQFT}^{\bullet-1}$

Let $\mathbb{R} \to \mathrm{SQFT}^n$, $x \mapsto \mathcal{F}_x$ be a smooth family of SQFTs. Can dynamicalize the parameter x by promoting it to a scalar multiplet: a boson $\hat{x}$ and its susy partner ψ, an antichiral fermion. "$\int_x \mathcal{F}_x$"

Example: $\mathrm{Fer}(1)_x :=$ ($\mathrm{Fer}(1)$ with susy $x{:}\lambda{:}$). Then $\int_x \mathrm{Fer}(1)_x =$ (chiral fermion λ, full boson $\hat{x}$, antichiral fermion ψ, with $\mathcal{L} = \|\hat{x}\|^2 + \lambda\partial\lambda + \psi\partial\psi$). Theory is massive: the deep IR is the One tqft with partition function $\equiv 1$, $\mathcal{H} \equiv \mathbb{C}^1$, etc.

$$\otimes\mathrm{Fer}(1)_x : \mathrm{SQFT}^{\bullet-1} \leftrightarrows \Omega\mathrm{SQFT}^\bullet : \int_x$$

Question: Suppose $\mathcal{F}_x$ is compact for every x and that $\mathcal{F}_x \to$ Zero as $x \to \pm\infty$. How quickly must this converge for $\int_x \mathcal{F}_x$ to be compact?

The Witten genus $\mathrm{Wit} : \mathrm{SQFT}^\bullet \to \{\text{modular forms}\}$ is

$$\mathrm{Tr}_{\text{R-sector}}\left((-1)^F q^{L_0 - c_L/24}\right) \times \eta(\tau)^{2(c_R - c_L)},$$

up to a convention-dependent $\sqrt[8]{1}$. Recall (Dedekind):
$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$, $\Delta(\tau) = \eta(\tau)^{24}$.

The Witten genus $\mathrm{Wit} : \mathrm{TMF}^\bullet \to \{\text{modular forms}\}$ is the edge map for the elliptic s.s. $\mathrm{H}^\bullet(\mathcal{M}_{\text{ell}}, \pi_\bullet \mathcal{O}^{\text{der}}) \Rightarrow \pi_\bullet \Gamma(\mathcal{M}_{\text{ell}}; \mathcal{O}^{\text{der}})$. Its image was calculated by Hopkins.

Example: If the $\mathcal{N}{=}(0,1)$ susy in $\mathcal{F}$ enhances to $(1,1)$, then automatically $\mathrm{Wit}(\mathcal{F}) \in \mathbb{Z}[\Delta]$. Coefficient counts Ramond-sector ground states. **Question:** Can every class in $\mathrm{Wit}(\mathrm{TMF}) \cap \mathbb{Z}[\Delta]$ be represented by an $\mathcal{N}{=}(1,1)$ SQFT?

Calculation [Hopkins]: $k\Delta^m \in \mathrm{Wit}(\mathrm{TMF})$ iff $km \in 24\mathbb{Z}$. So minimum nonzero k is $24/\gcd(24, m)$.

Example: Duncan's "supermoonshine" holomorphic $\mathcal{N}{=}1$ SCFT $V^{f\natural}$ realizes $24\Delta^{-1}$. Antiholomorphic $\bar{V}^{f\natural}$ realizes 24Δ.

Thus $(\bar{V}^{f\natural})^m$ realizes $24^m\Delta^m$. Let $k(m) \in \mathbb{Z}$ so that permutation orbifold $(\bar{V}^{f\natural})^m /\!\!/ S_m$ realizes $k(m)\Delta^m$.
Calculation [Gaiotto]: $\gcd(k(m), 24^m) = 24/\gcd(24, m)$.
Cor: The minimum expected values can be realized by systems with massively ($\sim 24^m$) many vacua and a massive cancellation.

Question: Realize $\frac{24}{\gcd(24,m)}\Delta^m$ by an antiholomorphic SCFT with only one vacuum.

Let L be an even unimodular lattice. Then the theta function $\Theta_L = \sum_{\ell \in L} q^\ell$ is in the image of $\mathrm{Wit} : \mathrm{TMF}^\bullet \to \{\text{modular forms}\}$. For example, $L = \mathrm{E}_8$ lattice $\rightsquigarrow \Theta_L =$ weight-4 Eistenstein series.

Question: Realize Θ_L.

Nonsolution: Purely holomorphic lattice VOA V_L has $\mathcal{N}{=}(0,1)$ susy because its right-moving sector is trivial. But this realizes $\Theta_L/\eta^{2\mathrm{rank}(L)}$. To get Θ_L, we need $c_R - c_L = \mathrm{rank}(L)/2$.

The $\mathcal{N}{=}(0,1)$ sigma model with target the torus hom$(L, \mathrm{U}(1))$ has correct $c_R - c_L$, but it has fermion zero-modes which make the Witten genus vanish. Maybe some clever orbifold procedure will kill the zero-modes?

Equivariant SQFT$^{\bullet}$

A G-equivariant QFT is sometimes said to have G-flavour symmetry (to distinguish from gauge ~~symmetry~~redundancy).

Expect: For any anomaly $\omega \in \mathrm{I}\Omega^4_{\mathrm{Spin}}(\mathrm{B}G) \ominus \mathrm{I}\Omega^4_{\mathrm{Spin}}(\mathrm{pt})$,

$$\begin{aligned}\mathrm{SQFT}^{\omega}_{G} &:= \{\text{SQFTs with } G\text{-flavour symmetry and anomaly } \omega\} \\ &\simeq \mathrm{TMF}^{\omega}_{G} := \omega\text{-twisted } G\text{-equivariant } \mathrm{TMF}.\end{aligned}$$

Remark: A standard way to study $\mathcal{Q} \in \mathrm{SQFT}^{\omega}_{G}$ is to study the "$\mathcal{Q}$+Neumann" boundary condition for 3D G-gauge theory. (Anomaly inflow mechanism: $\omega =$ gauge theory Lagrangian.)
Can recover $\mathcal{Q}$ with its G-action by forming a sandwich:

QFTs can also have noninvertible aka categorical symmetries. For 2D SQFTs, the most general type of finite categorical symmetry is described by a super modular tensor category $\mathcal{C}$: a $\mathcal{C}$-equivariant SQFT is a supersymmetric boundary condition for the 3D (Reshetikhin–Turaev type) Spin TQFT determined by $\mathcal{C}$.

Conjecture: There is a meaningful notion of $\mathcal{C}$-equivariant TMF $\mathrm{TMF}_{\mathcal{C}}^\bullet$ for any super MTC $\mathcal{C}$. The assignment $\mathcal{C} \mapsto \mathrm{TMF}_{\mathcal{C}}$ is functorial for super Witt equivalences (=topological interfaces of 3D TQFTs). **E.g.:** $\mathrm{TMF}_G^{\omega+\bullet}$ depends only on $\mathcal{C} = \mathcal{Z}(\mathrm{sVec}^\omega[G])$.

Theorem [Henriques–Morrison]: $\mathrm{TMF}_{\mathcal{C}}^\bullet \otimes \mathbb{Q}$ does meaningfully exist (at least for $\mathcal{C}$ bosonic). Hard part: the Galois action.

A 2D SQFT with G-flavour symmetry can be placed on any worldsheet Σ equipped with a G-bundle. The strength of the G-bundle is called the fugacity of the flavour symmetry.

Definition: An SQFT is G-flavoured-compact if its Wick-rotated partition function converges absolutely whenever the fugacity is nonzero, but it is allowed to diverge at fugacity 0.

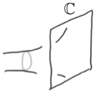
Example: Take the sigma model with target $\mathbb{C} = \mathbb{R}^2$ and $G = \mathrm{U}(1) = \mathrm{SO}(2)$ acting in the canonical way. At nonzero fugacity, if the string tries to wander away from $0 \in \mathbb{C}$, it will get stretched, which costs energy. So the theory is flavoured-compact.

Equivariant $\mathrm{SQFT}^\bullet$

Flavoured-compactness

Question: Define and analyze flavoured-compact bordism spectra and flavoured-compact TMF. Interpret:

- level-N TMF as $\mathbb{Z}/N$-flavoured-compact TMF.
- Jacobi forms with a pole at $z = 0$ as $\mathrm{U}(1)$-flavoured-compact modular forms.
- trumpet geometries as flavoured-compact-nullbordisms. E.g.:



$$S^1 \longrightarrow \emptyset$$

- formulas relating meromorphic Jacobi and mock-modular forms (e.g. polar decomposition) in terms of adding/removing trumpets.

$\mathrm{Tmf}^\bullet$ etc.

Like for ordinary modular forms, it makes sense to ask for topological modular forms whose growth at the cusp $\tau \to i\infty$ is no worse than q^D. $\mathrm{Tmf}^\bullet :=$ holomorphic at the cusp ($D = 0$); $\mathrm{Tcf}^\bullet :=$ topological cusp forms $=$ vanish at the cusp ($D = 1$).

This roughly translates to the request that, in the Ramond sector, the spectrum of L_0 be bounded below by $D + \frac{c_L}{8} - \frac{c_R}{12}$.
Question: What is the physical meaning of this request?

Unlike for ordinary modular forms, there are holomorphic topological modular forms of negative weight.
Example: $\pi_{-21}\mathrm{Tmf} \cong \mathbb{Z}$ generated by $\Delta^{-1}\nu$. $\pi_{-21}\mathrm{TMF} \cong 0$.
Theorem [Stojanoska]: $\mathrm{I}_{\mathbb{Z}}\mathrm{Tmf}^\bullet \cong \Sigma^{21}\mathrm{Tmf}^\bullet$.

Question: Physically describe these negative-weight Tmf classes.

For $\mathrm{TMF}^\bullet_G$, space of cusps $\approx$ adjoint quotient $\frac{G}{G} = \mathrm{L}(\mathrm{B}G)$.

Moonshine is about G-equivariant modular objects which grow as q^D near the cusp $e \in \frac{G}{G}$, and as q^{D+1} near all other cusps.

E.g.: $\mathbb{Z}/N$-modular $\approx$ modular for $\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \middle| c \equiv 0\,(N) \right\}$.
If $f(\tau)$ is $\Gamma_0(N)$-modular of weight$=0$ and grows as q^{-1} near e and as q^0 near the other cusps, then f is a hauptmodul: an iso (upper half plane)$/\Gamma_0(N) \xrightarrow{\sim} \mathbb{P}^1$.

Question: Define and study a version of $\mathrm{Tmf}^\bullet_G$ with this type of mixed cuspidal behaviour.

Question: Monstrous moonshine involves hauptmoduln for subgps of $\mathrm{SL}_2(\mathbb{R})$ not contained in $\mathrm{SL}_2(\mathbb{Z})$. Do these make sense in TMF?