RESEARCH STATEMENT: HIGHER ALGEBRA AND QUANTUM FIELD THEORY

THEO JOHNSON-FREYD

0. Overview

In my research, I apply higher algebra to the study of quantum field theory, and I apply quantum field theory to the study of higher algebra — two areas that have a rich history of interaction [Kap10, BL11]. As such, my results contribute to both mathematics and theoretical physics.[1]

Quantum field theory (QFT) arose in mid-twentieth-century high energy physics but has come in more recent years to provide an organizing principle to all areas of quantum systems, and I will use *quantum field theory* to refer both to the high-energy quantum systems traditionally studied under the "QFT" mantle as well as the low-energy systems studied under the name "condensed matter." Low-energy quantum field theory, and in particular Question 1 below, has become of great importance recently because of its applications to the development of novel quantum materials. A deep question, which will require both higher algebraic and functional analytic methods, is to give a precise mathematical definition of "quantum field theory" — many aspects of QFT have been mathematically axiomatized, but there are many aspects yet to make precise.

The name *higher algebra* is a twenty-first century invention: a twentieth-century higher algebraist might have said that they worked instead in category theory or algebraic topology or algebraic geometry or mathematical physics. The characterizing property of *higher algebra* is the presence of a hierarchy of algebraic structures. Examples of such hierarchies include the coefficients of the operator product expansion, the compositions in an n-category, derived functors and cohomology, and supersymmetric descendants.

One of the most important arenas[2] of interaction between higher algebra and quantum field theory arises when one asks to classify QFTs up to deformation, aka *phase*:

Question 1. *What is the homotopy type of the space[3] of quantum field theories (of a given spacetime dimension, number of supersymmetries, etc)?*

Asking about the *homotopy type* is a way of asking simultaneously about phases of QFTs with arbitrary symmetry group (what are called *symmetry enriched phases* in the condensed matter literature). Question 1 is a unifying question throughout much of contemporary theoretical and mathematical physics, and organizes and motivates all of my current research. It is fundamentally a higher algebraic question because the space of QFTs carries rich higher algebraic structures, and understanding these structures helps dramatically in the classification.

Address: Perimeter Institute for Theoretical Physics, Waterloo, Ontario.
Email: `theojf@pitp.ca`.

[1] As a mathematician, I have taken care in this Research Statement to distinguish my mathematically well-posed and rigorously proven Theorems from my results, marked Theorem*, that are proven conditional on providing a mathematical foundation for the physical objects under consideration.

[2] A second, overlapping arena where QFT and higher algebra interact comes from the fact that the information content of an individual QFT, for instance the product of operators, is inherently higher algebraic. My work on Feynman diagrams [JF10a, JF10b, JF10c, JF15, GJF18b] and on a Poisson generalization of the AKSZ formalism [JF14, JF16a, JF16c] fit into this arena.

[3] This space carries multiple topologies. The choice encodes which qualitative features one wishes to focus on when understanding "phases" of matter. For example, in some topologies "mass gaps" may close and open, whereas other topologies preserve gaps.

My goal in this Research Statement is to give an overview of my results from the last four years helping to answer Question 1, as well as the open questions and conjectures that I am pursuing. I organize my work into three threads:

First, I describe in Section 1 my answers to Question 1 in the case of minimally supersymmetric (1+1)-dimensional QFTs. Guiding this work is Conjecture 1, which predicts a tight relationship between such QFTs and elliptic cohomology, analogous to the relationship between quantum mechanics and K-theory. For physicists, it offers sensitive elliptic-cohomological invariants of quantum systems, which could surpass the K-theoretic methods currently at the cutting edge of quantum materials research. For mathematicians, it offers an analytic model of elliptic cohomology, which currently has only homotopy-algebraic constructions.

In Section 2 I describe my work on the 't Hooft anomalies arising in the "moonshine" of sporadic group actions on (1+1)-dimensional conformal field theories. 'T Hooft anomaly calculations are one of the main techniques available for predicting the low-energy behaviour of a given high-energy quantum system. On the mathematics side, 't Hooft anomalies are a generalization of characteristic classes, and play a vital role in my calculations of the cohomology of sporadic groups. My calculations help to understand where the sporadic groups come from in the classification of finite simple groups.

Finally, Section 3 focuses on my answers to Question 1 in the case of gapped topological phases of matter that are of so much recent interest [RSA16]. By considering the possible gapped topological phases that can be condensed from a given phase, I discovered new constructions in higher category theory. I end the Section by outlining my program to develop a higher-categorical version of Galois theory. This program requires the higher categorical constructions arising from the study of condensation, and leads, first, to a complete calculation of the space of invertible phases of matter, and, second, to a higher-categorical extension of the CPT and spin-statistics theorems. Thus quantum field theoretic questions about phases of matter lead to new pure mathematics, and higher algebraic questions lead to new theoretical physics.

1. TMF and phases of SQFTs

Much of my work focuses on the following specific case of Question 1. Let **SQFT** denote the space of unitary compact[4] minimally supersymmetric, aka $N{=}(0,1)$, (1+1)d QFTs.[5] In the sequel, I will write simply "SQFT" for "$N{=}(0,1)$, (1+1)d SQFT." Building on ideas of Witten and Segal [Wit87, Wit88, Seg88], Stolz and Teichner [ST04, ST11] conjectured a version of the following:

Conjecture 1. *The space* **SQFT** *provides a model for the generalized cohomology theory* TMF *of Topological Modular Forms.*[6]

It is worth emphasizing that the space **SQFT** does not have a robust mathematical definition, and providing one is an important step in verifying Conjecture 1. Nevertheless, Conjecture 1 suggests many mathematically well-posed conjectures. For instance, the subspace of **SQFT** consisting

[4] A unitary QFT is *compact* if its Wick-rotated partition function converges absolutely on arbitrary closed spacetime manifolds.

[5] The appropriate topology for **SQFT** should allow mass gaps to open and close (since a generic SQFT is very rarely gapped to begin with), and should be such that an SQFT is deformation-equivalent to its far-IR limit under renormalization group (RG) flow. The topology of **SQFT** can be understood through "zig-zags of RG flows" [GJFW19]: start with some SQFT, and flow it down to the far-IR; now flow back up to some other SQFT with the same far-IR limit; deform that SQFT by a supersymmetry-preserving marginal or relevant operator; flow down to the far-IR of the deformed theory, and repeat the cycle. For (1+1)d QFTs, and not in general, Zamolodchikov's "C-theorem" [Zam86] identifies RG flow as Morse flow for a function $C \in \mathcal{C}^\infty(\mathbf{SQFT})$ which is expected to be Morse–Bott. This "zig-zag" topology is then precisely the topology resulting from Morse theory.

[6] A universal version of elliptic cohomology, TMF was constructed by Hopkins and collaborators in [Hop95, AHS01, Hop02, DFHH14].

of antiholomorphic superconformal field theories (SCFTs) does have a precise mathematical definition in terms of vertex algebras.[7] There is a famous deformation invariant of SQFTs called the *elliptic genus* which is valued in modular forms (MF).[8] Conjecture 1 places constraints on the possible values of an elliptic genus of an SQFT, because the map TMF $\to$ MF is not surjective. In the case of an antiholomorphic SCFT of central charge $c = 12k$, the elliptic genus is equal to $m\Delta^k$, where $m \in \mathbb{Z}$ is the *index* of the SCFT and merely counts with signs the Ramond-sector ground states, and Δ is the modular discriminant. (The index vanishes automatically when $c \notin 12\mathbb{Z}$.) But $m\Delta^k$ is the elliptic genus of a TMF class if and only if m is divisible by $24/\gcd(k, 24)$.

Conjecture 2. *If V is an antiholomorphic SCFT of central charge $c = 12k$, then the index of V is divisible by $24/\gcd(k, 24)$. For each $k \in \mathbb{N}$, there exists an antiholomorphic SCFT of central charge $c = 12k$ with index $24/\gcd(k, 24)$.*

Conjecture 1 implies the first sentence of Conjecture 2 and suggests but does not imply the second sentence. Gaiotto and I showed, using the seemingly unrelated theory of error-correcting codes together with an exponentially slow computer search [GJF18a]:

Theorem 1. *The second sentence of Conjecture 2 is true for $k \leq 5$. The first sentence is true for $k \leq 2$, and supported by experimental data for $k \leq 5$.*

Question 2. *Is there a systematic construction of the "small-index" SCFTs predicted in Conjecture 2?*

Returning to the (as yet) mathematically ill-defined space **SQFT**, Conjecture 1 predicts the existence of "secondary" invariants that go far beyond the elliptic genus. Consider, for example, the $N{=}(0, 1)$ sigma model with target the round S^3 equipped with String structure of strength k.[9] Its elliptic genus vanishes for all k; nevertheless, Conjecture 1 leads to the prediction that this model can be deformed within **SQFT** to one with spontaneous supersymmetry breaking if and only if k is divisible by 24. In joint work with Gaiotto and Witten [GJFW19], I explained the "if" direction of this prediction:

Theorem* 2. *There is a deformation within* **SQFT** *connecting the direct sum (aka disjoint union) of $24/\gcd(k, 24)$ many copies of the S^3 sigma model with String structure k to an SQFT with spontaneous supersymmetry breaking.*

The "only if" direction requires constructing a "secondary" elliptic-genus-like invariant that can protect an SQFT from admitting such a deformation.[10] Together with Gaiotto, I found such an invariant in [GJF19b]:

Theorem* 3. *Consider the function on* **SQFT** *that outputs the class in $\mathbb{C}((q))/[\mathbb{Z}((q)) + \mathrm{MF}]$ of any generalized mock modular form with shadow the genus-one one-point function of the supersymmetry generator (computed in the far-IR limit of the SQFT). This function is a deformation invariant, called the* secondary invariant of the elliptic genus. *Its value on the S^3 sigma model with String structure k has order $24/\gcd(k, 24)$.*

---

[7] Namely, an *antiholomorphic SCFT* is a super vertex operator algebra which is unitary, is of strong CFT type, is equipped with a superconformal vector, and has trivial category of vertex modules.

[8] The elliptic genus of an SQFT is modular for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. More complicated modular objects arise when one considers the "flavoured elliptic genus" of SQFTs equipped with a global, aka flavour, symmetry. The usual elliptic genus is valued in (weight zero) modular functions which may have a multiplier under the action of $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. I work with a version corrected by powers of Dedekind's η function: this correction fixes the multiplier, and encodes the gravitational anomaly (aka central charge) as the weight of the modular form.

[9] Also called a quantum B-field, a String structure is the anomaly cancellation data needed to define the sigma model [MN85, Wit00]. For S^3, and not in general, the space of String structures is canonically isomorphic to $\mathbb{Z}$, generated by the String structure coming from recognizing S^3 as the group manifold $\mathrm{SU}(2)$.

[10] There is not a sharp definition of "secondary invariant." It means roughly an invariant of classes in the kernel of the "primary" invariant, which in the case at hand is the elliptic genus **SQFT** $\to$ MF.

Theorems 2 and 3 together pin the exact order in **SQFT** of the S^3 sigma models. The appearance of mock modularity in Theorem 3 connects TMF to the "umbral moonshine" of [Tro10, CDH14, CDD+15]. In spite of its strength, our secondary invariant, together with the "primary" elliptic genus, is not a complete invariant of **SQFT**, at least not if Conjecture 1 is correct.

Question 3. *Find "tertiary" invariants of SQFTs.*

Question 4. *Number theorists care about the subring* $\mathrm{mf} \subset \mathrm{MF}$ *of modular forms that are holomorphic at the cusp $q = 0$, and its ideal of "cusp forms" that vanish at $q = 0$. The former has a homotopical refinement called "*Tmf*," and the refinement of the latter is natural but has not been studied in the algebraic topology literature. What are the physical interpretations of these spaces?*

2. 'T HOOFT ANOMALIES AND MOONSHINE

Among the mysteries of TMF is that it is 576-fold periodic, analogous to the 8-fold Bott periodicity of real K-theory. Conjecture 1 raises the question of finding a "physical" description of this periodicity, which in turn leads to a search through the many proofs of Bott periodicity for one that might work for SQFTs. In [JF16b] I provided a new proof of Bott periodicity based on gauging (0+1)d free fermions by mildly-exceptional Lie groups.[11] The analogue for **SQFT** would involve the vertex algebra $\mathrm{Fer}(n)$ of n chiral free fermions.

Gauging a symmetry of a quantum system requires trivializing its 't Hooft anomaly,[12] and the 't Hooft anomalies of Lie group actions on vertex algebras are almost never trivializable. The result of gauging a finite group action on a vertex algebra is called its *orbifold*. My [JF16b] led me to ask:

Question 5. *Which (exceptional?) finite groups have privileged representations on chiral free fermions? What are their 't Hooft anomalies? What are their orbifolds?*

One answer to the first part of Question 5 comes from [Dun07, DMC15], which identifies Conway's largest sporadic group Co_1 as having an important action on $\mathrm{Fer}(24)$, namely the "Leech lattice" action of its Schur cover $\mathrm{Co}_0 = 2\mathrm{Co}_1$. Indeed, Co_1 is the full automorphism group of an antiholomorphic SCFT related to $\mathrm{Fer}(24)$, and in [JF20d] I identified a number of other interesting exceptional groups as the automorphism groups of SCFTs. For the Co_1 case, Treumann and I answered the second part of Question 5 in [JFT18]:

Theorem 4. *The 't Hooft anomaly of the Leech lattice action of* Co_0 *on* $\mathrm{Fer}(24)$ *has order* 24 *and generates* $\mathrm{H}^3(\mathrm{Co}_0; \mathrm{U}(1))$.

Schur covers of finite groups are the analogue of simply connected covers of Lie groups, and if G is a Schur cover of a finite simple group, then $\mathrm{H}^3(G; \mathrm{U}(1))$ is home to 't Hooft anomalies for actions of G on (1+1)d QFTs.[13] Theorem 4 provides a tantalizing hint about the 576-fold periodicity of TMF: since $576 = 24^2$, there is an nonanomalous action of Co_0 on $\mathrm{Fer}(576)$. The orbifold of this action is not trivial, but might play a role in constructing physically the periodicity of TMF.

---

[11] Mathematicians call the gauging procedure in (0+1)d "quantum Hamiltonian reduction" of an associative algebra.

[12] 'T Hooft anomalies generalize the obstruction to lifting a projective representation to an honest representation. They can be formalized in terms of the difference between "Schrödinger" and "Heisenberg" pictures [JF20b, JF20c], or in terms of "relative," aka "twisted," functorial field theory [ST11, FT12]; for the latter, Scheimbauer and I provided the complete definition in [JFS17]. In the (0+1)d case, the data trivializing the 't Hooft anomaly is known to mathematicians under the name "(co)moment map."

[13] The same statement holds for actions of arbitrary finite groups on bosonic (1+1)-dimensional QFTs. But for SQFTs, the 't Hooft anomaly lives in a generalized cohomology theory called *supercohomology* [GW14, WG17], which happens to be equal, in this degree, to ordinary cohomology if the group is a Schur cover of a simple group. I have developed the theory of supercohomology in my papers on vertex algebras, and Gaiotto and I connect supercohomology to gapped phases of matter in [GJF19c].

The work [JFT18] led me to be interested more generally in the cohomology of sporadic groups. In [JF19] I developed a "finite group T-duality" in order to calculate the anomaly of the action of the Monster group $\mathbb{M}$ on the Moonshine CFT $V^\natural$ (constructed in [FLM88]), and in [JFT19] Treumann and I calculated $\mathrm{H}^3(G; \mathrm{U}(1))$ for most sporadic groups. In particular:

Theorem 5. *The 't Hooft anomaly of the action of* $\mathbb{M}$ *on* $V^\natural$ *has order* 24 *in* $\mathrm{H}^3(\mathbb{M}; \mathrm{U}(1))$. *For most Schur covers* G *of sporadic finite simple groups,* $\mathrm{H}^3(G; \mathrm{U}(1))$ *is cyclic of order dividing* 24.

For comparison, for most Schur covers G of finite simple groups of Lie type defined over a field of order q, $\mathrm{H}^3(G; \mathrm{U}(1))$ is cyclic of order $q^2 - 1$ [Gro], and so Theorem 5 helps to understand the difference between sporadic and Lie-type groups.

Note the repeated occurrence, in all the Theorems 1–5, of the number 24. This number occurs throughout mathematics: as the critical dimension in string theory, as the denominator of a value of the Riemann ζ function, as the rank of the Leech lattice, as a stable homotopy group of spheres, as an algebraic K-group, as the number of Niemeier lattices, in the "cannonball problem" $1^2 + 2^2 + \cdots + 24^2 = 70^2$, ... Which of these 24s are "the same"? Specifically, [CN79] defined 24 as the largest number N such that if m is relatively prime to N, then $m^2 = 1 \pmod N$, and one can ask which 24s arise because of this property, and which don't. For example, Adams showed in [Ada65] that the 24s appearing in stable homotopy theory and K-theory (and hence in Theorems 1–3, if Conjecture 1 is to be believed) and in the Riemann ζ function are the 24 of [CN79], but it is unlikely that this 24 explains the number of Niemeier lattices, and the Leech and cannonball cases are unclear.

To show that the 24s in Theorems 4 and 5 are also the 24 of [CN79], I needed number theoretic control over 't Hooft anomalies, which I achieved by calculating the action of the absolute Galois group on 't Hooft anomalies of holomorphic CFTs.[14] Using that calculation, in [JF20a] I showed:

Theorem 6. *Suppose that a holomorphic CFT* V, *and a finite group* G *of symmetries thereof, are defined over* $\mathbb{Q}$. *Then the 't Hooft anomaly of the action of* G *on* V *has order dividing the* 24 *of* [CN79].

Since the corresponding Galois-theoretic control for (0+1)d 't Hooft anomalies can be used to explain the Galois-cohomological description of Brauer groups, Theorem 6 leads to:

Conjecture 3. *For each field* $\mathbb{F}$, *there is a "vertex Brauer group" of Morita-equivalence classes of holomorphic CFTs defined over* $\mathbb{F}$, *isomorphic to the Galois cohomology group* $\mathrm{H}^3(\mathbb{F}; \mu^{\otimes 2})$.

The physical interpretation of Conjecture 3 is that it predicts the home for 't Hooft anomalies of symmetries that act by nontrivial Galois automorphisms. Such symmetries are not completely foreign to physics: time reversal symmetry acts by complex conjugation.

3. Condensation, higher spin-statistics, and higher Galois theory

The other main focus of my work studies Question 1 for gapped quantum systems, including the lattice systems considered by condensed matter theorists. A quantum mechanical model is *gapped* if the lowest eigenvalue of the Hamiltonian appears with finite multiplicity, and there is a nonzero gap between the lowest and next-to-lowest eigenvalues; the topology on the space of gapped systems should be such that gaps do not close or open along deformations. In higher dimensions there is not yet a mathematical definition of "gapped" that captures locality, but many examples are known. In many cases, one expects the low-energy effective behaviour of a gapped system to be captured by a

[14] In order to have a well-defined Galois action, I used VOAs as my model of CFTs. In this model, Theorem 6 is conditional on a widely believed, and physically obvious, conjecture about rationality of VOA orbifolds [CM16]; without it there is not a VOA-theoretic definition of 't Hooft anomalies. For the special cases described in Theorems 4 and 5, one can establish rationality by a roundabout method using a different, operator algebraic model of conformal field theories called "conformal nets," but that model is not sufficiently algebraic to have a Galois action.

topological quantum field theory (TQFT), although the exact relationship should be a theorem, not an axiom. Although there are also "fracton" phases that violate this expectation [VHF15], in the sequel I will write simply "gapped" for what should more properly be called "gapped topological."

Gapped systems may be *stacked* (aka tensored) together, giving the space $\mathbf{GP}_d$ of $(d{+}1)$-dimensional gapped systems a multiplicative structure, and defining its subspace $\mathbf{GP}_d^\times$ of *invertible* (up to phase) gapped systems. Together with Gaiotto, in [GJF19c] I showed:

Theorem* 7. *The spaces $\mathbf{GP}_d^\times$ compile into an Ω-spectrum, i.e. they define a generalized cohomology theory. For any group G, the set of $(d{+}1)$-dimensional G-protected phases (i.e. phases enriched by G-symmetry, which are trivial in a non-G-symmetric way) is the reduced cohomology group $\widetilde{\mathrm{H}}^d(G; \mathbf{GP}_\bullet^\times)$.*

The physical import of Theorem 7 is that, whereas a priori the answer to Question 1 for invertible gapped phases might depend sensitively on the spatial dimension d, in fact the answer is essentially d-independent and may be studied using the mathematical techniques of stable homotopy.

To make further progress understanding the spaces $\mathbf{GP}_d$, one should construct a large class of gapped phases. The gapped defects between gapped phases enjoy a rich higher category theory, since they can be brought together (fused) without closing the gap.[15] In [GJF19a] we asked: given a gapped phase, what are all gapped phases one can reach from it by (physical) condensation? By studying the defects produced by gapped condensation, we discovered a construction in pure higher category theory, namely a higher-categorical version of the Karoubi envelope (aka idempotent completion) that appears throughout mathematics, that we call *categorical condensation*.

Theorem 8. *For each n-category $\mathcal{C}$, there is n-category* $\mathrm{Kar}(\mathcal{C})$*, described explicitly in terms of certain "condensation monad" diagrams in $\mathcal{C}$, called the* condensation completion *of $\mathcal{C}$. It satisfies a universal property and is complete not just for categorical condensation, but for arbitrary absolute limits.*

A standard construction in higher category theory converts a symmetric monoidal $(n-1)$-category $\mathcal{R}$ into a symmetric monoidal n-category $\mathrm{B}\mathcal{R}$ with one object.[16] Write $\Sigma\mathcal{R}$ for $\mathrm{Kar}(\mathrm{B}\mathcal{R})$. By iteration, one can define $\Sigma^d\mathrm{V{\scriptstyle EC}}_{\mathbb{C}}$, where $\mathrm{V{\scriptstyle EC}}_{\mathbb{C}}$ is the category of finite-dimensional Hilbert spaces.

Theorem* 9. *The $(d{+}1)$-category of gapped $(d{+}1)$-dimensional phases that can be condensed from the vacuum, in such a way that all the defects (of all dimensions) produced from the condensation procedure can themselves be condensed from the vacuum, is equivalent to $\Sigma^d\mathrm{V{\scriptstyle EC}}_{\mathbb{C}}$.*

Theorem 10. *There is an equivalence of $(d{+}1)$-categories between $\Sigma^d\mathrm{V{\scriptstyle EC}}_{\mathbb{C}}$ and the $(d{+}1)$-category of fully dualizable $\mathbb{C}$-linear d-categories. Applying the Cobordism Hypothesis of* [BD95, Lur09]*, this is equivalent to the $(d{+}1)$-category of fully extended functorial TQFTs (with target the space of all $\mathbb{C}$-linear d-categories).*

Theorem 11. *Every object of $\Sigma^{d+1}\mathbb{C}$ determines, uniquely up to phase, a $(d{+}1)$-dimensional commuting projector Hamiltonian system.*

Together, Theorems 9–11 establish an equivalence between large classes of gapped phases and TQFTs, and show that all of these phases can be represented by commuting projector Hamiltonian systems.

Moreover, the technology of categorical condensation allows for a complete axiomatization of the "higher fusion categories" of extended operators in a topological phase of matter. In [JF20c], following [Wen90, AKZ17, KWZ17], I called these higher algebras *topological orders*, to distinguish

[15] Specifically, the k-morphisms in this higher category are the gapped defects of codimension k.

[16] If $\mathcal{R}$ consists of all $(n-1)$-dimensional gapped phases, then $\mathrm{B}\mathcal{R}$ consists of the vacuum n-dimensional phase, together with all possible gapped defects from the vacuum to itself.

them from true phases of matter. Notably, invertible phases have trivial operator content: the map $\mathbf{GP}_d \to \{d\text{-dimensional topological orders}\}$ has $\mathbf{GP}_d^\times$ in its kernel. In [JF20c], I showed that this is exactly the kernel:

Theorem* 12. *The higher category of all topological orders, of arbitrary dimension, is the quotient of the higher category of gapped phases in which invertible gapped phases are declared trivial.*

I was also able to develop a number of technical results about higher fusion algebras and their centres. This provided the missing ingredient in order to complete the proof (and improve the statement) of the classification result announced in [LW19, LKW18]:

Theorem 13. *All* (3+1)*-dimensional topological orders arise as topological sigma models. The target of the sigma model is a groupoid $\mathcal{G}$, determined canonically by the topological order, equipped with a Dijkgraaf–Witten action living in the reduced twisted* $\mathrm{B}\mathbb{Z}_2$*-equivariant supercohomology of $\mathcal{G}$.*

Finally, this technology is what I need to pursue my project to develop a higher-categorical version of Galois theory. I started this program in [JF17] (building on my earlier work [CJF13, BCJF15] on categorified algebra, and continued in [JF20c]), where I recognized that the category $\mathrm{SV{\scriptstyle EC}}_{\mathbb{C}}$ of complex supervector spaces is a categorified version of a field (in the sense of abstract algebra), and that Deligne's "existence of fibre functors" [Del02] implies:

Theorem 14. *The extension* $\mathrm{V{\scriptstyle EC}}_{\mathbb{R}} \to \mathrm{SV{\scriptstyle EC}}_{\mathbb{C}}$ *is the "1-categorical algebraic closure." It is Galois with "1-categorical Galois group"* $\mathrm{Gal}(\mathrm{SV{\scriptstyle EC}}_{\mathbb{C}}/\mathrm{V{\scriptstyle EC}}_{\mathbb{R}}) = \mathbb{Z}_2 \times \mathrm{B}\mathbb{Z}_2$.

Moreover, I reinterpreted the CPT and spin-statistics theorems as saying:

Theorem 15. *There is a canonical equivalence between* $\mathrm{Gal}(\mathrm{SV{\scriptstyle EC}}_{\mathbb{C}}/\mathrm{V{\scriptstyle EC}}_{\mathbb{R}})$ *and the homotopy 1-type of the stable orthogonal group.*

Passing to ∞-categories, $\mathrm{V{\scriptstyle EC}}_{\mathbb{R}}$, $\mathrm{V{\scriptstyle EC}}_{\mathbb{C}}$, and $\mathrm{SV{\scriptstyle EC}}_{\mathbb{C}}$ become the "towers" $\Sigma^\bullet\mathrm{V{\scriptstyle EC}}_{\mathbb{R}}$, $\Sigma^\bullet\mathrm{V{\scriptstyle EC}}_{\mathbb{C}}$, and $\Sigma^\bullet\mathrm{SV{\scriptstyle EC}}_{\mathbb{C}}$.[17] Algebraic closures are not quite canonical — they are unique, but not up to unique isomorphism. This noncanonicity is why the following is hard to establish:

Conjecture 4. *The tower* $\Sigma^\bullet\mathrm{V{\scriptstyle EC}}_{\mathbb{R}}$ *has an algebraic closure* $\overline{\Sigma^\bullet\mathrm{V{\scriptstyle EC}}_{\mathbb{R}}}$.

Assuming Conjecture 4, the *higher spin-statistics theorem* would say:

Conjecture 5. *The ∞-categorical Galois group* $\mathrm{Gal}\big(\overline{\Sigma^\bullet\mathrm{V{\scriptstyle EC}}_{\mathbb{R}}}/\Sigma^\bullet\mathrm{V{\scriptstyle EC}}_{\mathbb{R}}\big)$ *is naturally equivalent to the stable orthogonal group.*

Conjecture 5 would have significant physical ramifications. For example, it would answer the main conjecture of [Kap14]:

Theorem* 16. *Subject to Conjectures 4 and 5, the Ω-spectra* $(\Sigma^\bullet\mathrm{V{\scriptstyle EC}}_{\mathbb{R}})^\times$, $(\Sigma^\bullet\mathrm{V{\scriptstyle EC}}_{\mathbb{C}})^\times$, *and* $(\Sigma^\bullet\mathrm{SV{\scriptstyle EC}}_{\mathbb{C}})^\times$ *of, respectively, invertible bosonic gapped phases with time reversal symmetry, invertible bosonic gapped phases without time reversal symmetry, and invertible fermionic gapped phases, are equivalent to the Pontryagin duals to unoriented, oriented, and spin cobordism, respectively.*

As Theorem 16 illustrates, questions in pure higher algebra will lead to deep results about the space of quantum field theories, and questions about the space of QFTs will lead to deep results in pure higher algebra.

[17] A *tower* is a categorical version of an Ω-spectrum. It consists of a sequence $(\mathcal{C}_0, \mathcal{C}_1, \dots)$, where for each n, $\mathcal{C}_n$ is a condensation-complete n-category equipped with a "base point" $1 \in \mathcal{C}_n$, together with equivalences $\mathcal{C}_{n-1} \simeq \Omega\mathcal{C}_n = \mathrm{End}_{\mathcal{C}_n}(1)$. This notion will appear in my upcoming work [JF21].

References

[Ada65] J. F. Adams. On the groups $J(X)$. II. *Topology*, 3:137–171, 1965. DOI:10.1016/0040-9383(65)90040-6. MR0198468.

[AHS01] M. Ando, M. J. Hopkins, and N. P. Strickland. Elliptic spectra, the Witten genus and the theorem of the cube. *Invent. Math.*, 146(3):595–687, 2001. DOI:10.1007/s002220100175. MR1869850.

[AKZ17] Yinghua Ai, Liang Kong, and Hao Zheng. Topological orders and factorization homology. *Adv. Theor. Math. Phys.*, 21(8):1845–1894, 2017. DOI:10.4310/ATMP.2017.v21.n8.a1. MR3783835. arXiv:1607.08422.

[BCJF15] Martin Brandenburg, Alexandru Chirvasitu, and Theo Johnson-Freyd. Reflexivity and dualizability in categorified linear algebra. *Theory Appl. Categ.*, 30:808–835, 2015. MR3361309. arXiv:1409.5934.

[BD95] J.C. Baez and J. Dolan. Higher dimensional algebra and topological quantum field theory. *J. Math. Phys.*, 36:6073–6105, 1995. MR1355899. arXiv:q-alg/9503002. DOI:10.1063/1.531236.

[BL11] John C. Baez and Aaron D. Lauda. A prehistory of n-categorical physics. In *Deep beauty*, pages 13–128. Cambridge Univ. Press, Cambridge, 2011. MR2752518.

[CDD+15] Miranda C. N. Cheng, Xi Dong, John F. R. Duncan, Sarah Harrison, Shamit Kachru, and Timm Wrase. Mock modular Mathieu moonshine modules. *Res. Math. Sci.*, 2:Art. 13, 89, 2015. DOI:10.1186/s40687-015-0034-9. MR3373711. arXiv:1406.5502.

[CDH14] Miranda C. N. Cheng, John F. R. Duncan, and Jeffrey A. Harvey. Umbral moonshine. *Commun. Number Theory Phys.*, 8(2):101–242, 2014. MR3271175. DOI:10.4310/CNTP.2014.v8.n2.a1. arXiv:1204.2779.

[CJF13] Alexandru Chirvasitu and Theo Johnson-Freyd. The fundamental pro-groupoid of an affine 2-scheme. *Appl. Categ. Structures*, 21(5):469–522, 2013. MR3097055. arXiv:1105.3104. DOI:10.1007/s10485-011-9275-y.

[CM16] Scott Carnahan and Masahiko Miyamoto. Regularity of fixed-point vertex operator subalgebras. 2016. arXiv:1603.05645.

[CN79] J. H. Conway and S. P. Norton. Monstrous moonshine. *Bull. London Math. Soc.*, 11(3):308–339, 1979. DOI:10.1112/blms/11.3.308. MR554399.

[Del02] P. Deligne. Catégories tensorielles. *Mosc. Math. J.*, 2(2):227–248, 2002. Dedicated to Yuri I. Manin on the occasion of his 65th birthday. MR1944506.

[DFHH14] Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill, editors. *Topological modular forms*, volume 201 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014. DOI:3223024.

[DMC15] John F. R. Duncan and Sander Mack-Crane. The moonshine module for Conway's group. *Forum Math. Sigma*, 3:e10, 52, 2015. DOI:10.1017/fms.2015.7. MR3376736. arXiv:1409.3829.

[Dun07] John F. Duncan. Super-moonshine for Conway's largest sporadic group. *Duke Math. J.*, 139(2):255–315, 2007. DOI:10.1215/S0012-7094-07-13922-X. MR2352133. arXiv:math/0502267.

[FLM88] Igor Frenkel, James Lepowsky, and Arne Meurman. *Vertex operator algebras and the Monster*, volume 134 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988. MR996026.

[FT12] Daniel S. Freed and Constantin Teleman. Relative quantum field theory. 2012. arXiv:1212.1692.

[GJF18a] Davide Gaiotto and Theo Johnson-Freyd. Holomorphic SCFTs with small index. 2018. arXiv:1811.00589.

[GJF18b] Owen Gwilliam and Theo Johnson-Freyd. How to derive Feynman diagrams for finite-dimensional integrals directly from the BV formalism. In *Topology and quantum theory in interaction*, volume 718 of *Contemp. Math.*, pages 175–185. Amer. Math. Soc., Providence, RI, 2018. arXiv:1202.1554. DOI:10.1090/conm/718/14479. MR3869644.

[GJF19a] Davide Gaiotto and Theo Johnson-Freyd. Condensations in higher categories. 2019. arXiv:1905.09566.

[GJF19b] Davide Gaiotto and Theo Johnson-Freyd. Mock modularity and a secondary elliptic genus. 2019. arXiv:1904.05788.

[GJF19c] Davide Gaiotto and Theo Johnson-Freyd. Symmetry protected topological phases and generalized cohomology. *J. High Energy Phys.*, (5):007, 34, 2019. DOI:10.1007/JHEP05(2019)007. arXiv:1712.07950. MR3978827.

[GJFW19] Davide Gaiotto, Theo Johnson-Freyd, and Edward Witten. A note on some minimally supersymmetric models in two dimensions. 2019. arXiv:1902.10249.

[Gro] Jesper Grodal. Low dimensional homology of finite groups of Lie type, away from the characteristic. Unpublished.

[GW14] Zheng-Cheng Gu and Xiao-Gang Wen. Symmetry-protected topological orders for interacting fermions: Fermionic topological nonlinear σ models and a special group supercohomology theory. *Phys. Rev. B*, 90(115141), 2014. DOI:10.1103/PhysRevB.90.115141. arXiv:1201.2648.

[Hop95] Michael J. Hopkins. Topological modular forms, the Witten genus, and the theorem of the cube. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 554–565. Birkhäuser, Basel, 1995. DOI:10.1007/978-3-0348-9078-6_49. MR1403956.

[Hop02] M. J. Hopkins. Algebraic topology and modular forms. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, pages 291–317. Higher Ed. Press, Beijing, 2002. MR1989190. arXiv:math/0212397.

[JF10a] Theo Johnson-Freyd. Feynman-diagrammatic description of the asymptotics of the time evolution operator in quantum mechanics. *Lett. Math. Phys.*, 94(2):123–149, 2010. DOI:10.1007/s11005-010-0424-2. MR2733558. arXiv:1003.1156.

[JF10b] Theo Johnson-Freyd. The formal path integral and quantum mechanics. *J. Math. Phys.*, 51(12):122103, 31, 2010. DOI:10.1063/1.3503472. MR2779164. arXiv:1004.4305.

[JF10c] Theo Johnson-Freyd. On the coordinate (in)dependence of the formal path integral. 2010. arXiv:1003.5730.

[JF14] Theo Johnson-Freyd. Poisson AKSZ theories and their quantizations. In *Proceedings of the conference String-Math 2013*, volume 88 of *Proceedings of Symposia in Pure Mathematics*, pages 291–306, Providence, RI, 2014. Amer. Math. Soc. DOI:10.1090/pspum/088. arXiv:1307.5812.

[JF15] Theo Johnson-Freyd. Homological perturbation theory for nonperturbative integrals. *Lett. Math. Phys.*, 2015. DOI:10.1007/s11005-015-0791-9. arXiv:1206.5319.

[JF16a] Theo Johnson-Freyd. Exact triangles, Koszul duality, and coisotopic boundary conditions. 2016. arXiv:1608.08598.

[JF16b] Theo Johnson-Freyd. The quaternions and Bott periodicity are quantum Hamiltonian reductions. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 12:Paper No. 116, 6, 2016. DOI:10.3842/SIGMA.2016.116. MR3581593. arXiv:1603.06603.

[JF16c] Theo Johnson-Freyd. Tree- versus graph-level quasilocal Poincaré duality on S^1. *J. Homotopy Relat. Struct.*, 11(2):333–374, 2016. DOI:10.1007/s40062-015-0110-2. MR3511825. arXiv:1412.4664.

[JF17] Theo Johnson-Freyd. Spin, statistics, orientations, unitarity. *Algebr. Geom. Topol.*, 17(2):917–956, 2017. DOI:10.2140/agt.2017.17.917. MR3623677. arXiv:1507.06297.

[JF19] Theo Johnson-Freyd. The moonshine anomaly. *Comm. Math. Phys.*, 365(3):943–970, 2019. arXiv:1707.08388. DOI:10.1007/s00220-019-03300-2. MR3916985.

[JF20a] Theo Johnson-Freyd. Galois action on VOA gauge anomalies. *Progr. Math. (to appear)*, 2020. arXiv:1811.06495.

[JF20b] Theo Johnson-Freyd. Heisenberg-picture quantum field theory. *Progr. Math. (to appear)*, 2020. arXiv:1508.05908.

[JF20c] Theo Johnson-Freyd. On the classification of topological orders. 2020. arXiv:2003.06663.

[JF20d] Theo Johnson-Freyd. Supersymmetry and the Suzuki chain. *Tunis. J. Math.*, 2020. arXiv:1908.11012.

[JF21] Theo Johnson-Freyd. Higher-categorical galois extensions. In preparation, 2021.

[JFS17] Theo Johnson-Freyd and Claudia Scheimbauer. (Op)lax natural transformations, twisted field theories, and "even higher" Morita categories. *Adv. Math.*, 307:147–223, 2017. DOI:10.1016/j.aim.2016.11.014. arXiv:1502.06526. MR3590516.

[JFT18] Theo Johnson-Freyd and David Treumann. $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) = \mathbf{Z}/24$. *Int. Math. Res. Not. IMRN*, 2018. arXiv:1707.07587.

[JFT19] Theo Johnson-Freyd and David Treumann. Third homology of some sporadic finite groups. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 15:059, 38 pages, 2019. arXiv:1810.00463. DOI:10.3842/SIGMA.2019.059. MR3990846.

[Kap10] Anton Kapustin. Topological field theory, higher categories, and their applications. In *Proceedings of the International Congress of Mathematicians. Volume III*, pages 2021–2043. Hindustan Book Agency, New Delhi, 2010. MR2827874. arXiv:1004.2307.

[Kap14] Anton Kapustin. Symmetry protected topological phases, anomalies, and cobordisms: Beyond group cohomology. 2014. arXiv:1403.1467.

[KWZ17] Liang Kong, Xiao-Gang Wen, and Hao Zheng. Boundary-bulk relation in topological orders. *Nuclear Phys. B*, 922:62–76, 9 2017. DOI:10.1016/j.nuclphysb.2017.06.023. arXiv:1702.00673.

[LKW18] Tian Lan, Liang Kong, and Xiao-Gang Wen. Classification of $(3+1)$D bosonic topological orders: The case when pointlike excitations are all bosons. *Phys. Rev. X*, 8:021074, Jun 2018. DOI:10.1103/PhysRevX.8.021074. arXiv:1704.04221.

[Lur09] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009. MR2555928. arXiv:0905.0465.

[LW19] Tian Lan and Xiao-Gang Wen. Classification of $3+1$D bosonic topological orders (ii): The case when some pointlike excitations are fermions. *Phys. Rev. X*, 9:021005, Apr 2019. DOI:10.1103/PhysRevX.9.021005. arXiv:1801.08530.

[MN85] Gregory Moore and Philip Nelson. The ætiology of sigma model anomalies. *Comm. Math. Phys.*, 100(1):83–132, 1985. MR796163.

[RSA16] The Royal Swedish Academy. Topological phase transitions and topological phases of matter. In *Scientific Background on the Nobel Prize in Physics 2016*, 2016. https://www.nobelprize.org/uploads/2018/06/advanced-physicsprize2016-1.pdf.

[Seg88] Graeme Segal. Elliptic cohomology (after Landweber-Stong, Ochanine, Witten, and others). *Astérisque*, (161-162):Exp. No. 695, 4, 187–201 (1989), 1988. Séminaire Bourbaki, Vol. 1987/88.

[ST04] Stephan Stolz and Peter Teichner. What is an elliptic object? In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 247–343. Cambridge Univ. Press, Cambridge, 2004. MR2079378. DOI:10.1017/CBO9780511526398.013.

[ST11] Stephan Stolz and Peter Teichner. Supersymmetric field theories and generalized cohomology. In *Mathematical foundations of quantum field theory and perturbative string theory*, volume 83 of *Proc. Sympos. Pure Math.*, pages 279–340. Amer. Math. Soc., Providence, RI, 2011. MR2742432. arXiv:1108.0189.

[Tro10] Jan Troost. The non-compact elliptic genus: mock or modular. *J. High Energy Phys.*, (6):104, 18, 2010. DOI:10.1007/JHEP06(2010)104. arXiv:1004.3649.

[VHF15] Sagar Vijay, Jeongwan Haah, and Liang Fu. A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations. *Phys. Rev.*, B92(23):235136, 2015. DOI:10.1103/PhysRevB.92.235136. arXiv:1505.02576.

[Wen90] Xiao-Gang Wen. Topological orders in rigid states. *Internat. J. Modern Phys. B*, 4(2):239–271, 1990. DOI:10.1142/S0217979290000139.

[WG17] Qing-Rui Wang and Zheng-Cheng Gu. Towards a complete classification of fermionic symmetry protected topological phases in 3d and a general group supercohomology theory. 2017. arXiv:1703.10937.

[Wit87] Edward Witten. Elliptic genera and quantum field theory. *Comm. Math. Phys.*, 109(4):525–536, 1987. MR885560.

[Wit88] Edward Witten. The index of the Dirac operator in loop space. In *Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986)*, volume 1326 of *Lecture Notes in Math.*, pages 161–181. Springer, Berlin, 1988. DOI:10.1007/BFb0078045. MR970288.

[Wit00] Edward Witten. World-sheet corrections via D-instantons. *J. High Energy Phys.*, (2):Paper 30, 18, 2000. DOI:10.1088/1126-6708/2000/02/030. arXiv:hep-th/9907041.

[Zam86] A. B. Zamolodchikov. “Irreversibility” of the flux of the renormalization group in a 2D field theory. *Pis′ma Zh. Èksper. Teoret. Fiz.*, 43(12):565–567, 1986. MR865077.