Perturbative Methods in Path Integration

by

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Abstract

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This dissertation addresses a number of related questions concerning perturbative "path" integrals. Perturbative methods are one of the few successful ways physicists have worked with (or even defined) these infinite-dimensional integrals, and it is important as mathematicians to check that they are correct. Chapter 0 provides a detailed introduction.

We take a classical approach to path integrals in Chapter 1. Following standard arguments, we posit a Feynman-diagrammatic description of the asymptotics of the time-evolution operator for the quantum mechanics of a charged particle moving nonrelativistically through a curved manifold under the influence of an external electromagnetic field. We check that our sum of Feynman diagrams has all desired properties: it is coordinate-independent and well-defined without ultraviolet divergences, it satisfies the correct composition law, and it satisfies Schrödinger's equation thought of as a boundary-value problem in PDE.

Path integrals in quantum mechanics and elsewhere in quantum field theory are almost always of the shape $\int f\, e^s$ for some functions f (the "observable") and s (the "action"). In Chapter 2 we step back to analyze integrals of this type more generally. Integration by parts provides algebraic relations between the values of $\int(-)\, e^s$ for different inputs, which can be packaged into a Batalin–Vilkovisky-type chain complex. Using some simple homological perturbation theory, we study the version of this complex that arises when f and s are taken to be polynomial functions, and power series are banished. We find that in such cases, the entire scheme-theoretic critical locus (complex points included) of s plays an important role, and that one can uniformly (but noncanonically) integrate out in a purely algebraic way the contributions to the integral from all "higher modes," reducing $\int f\, e^s$ to an integral over the critical locus. This may help explain the presence of analytic continuation in questions like the Volume Conjecture.

We end with Chapter 3, in which the role of integration is somewhat obscured, but perturbation theory is prominent. The Batalin–Vilkovisky homological approach to integration illustrates that there are generalizations of the notion of "integral" analogous to the generalization from cotangent bundles to Poisson manifolds. The AKSZ construction of topological quantum field theories fits into this approach; in what is usually called "AKSZ theory,"

everything is still required to be symplectic. Using factorization algebras as a framework for (topological) quantum field theory, we construct a one-dimensional Poisson AKSZ field theory for any formal Poisson manifold M. Quantizations of our field theory correspond to formal star-products on M. By using a "universal" formal Poisson manifold and abandoning configuration-space integrals in favor of other homological-perturbation techniques, we construct a universal formal star-product all of whose coefficients are manifestly rational numbers.

Chapters 1 and 2 are based on the papers [JF10a, JF10b, JF10c, GJF12, JF12]; the material in Chapter 3 has not previously appeared. Other than small modifications to formatting, etc., there is one major improvement: whereas [JF10a] verifies Schrödinger's equation only on flat space, in [JF10b] the formal path integral is constructed also for curved target manifolds, and here we verify Schrödinger's equation in that generality.

Dedication

There are two men who have been particularly important in my life.

My father, J.Q. Johnson, became ill with cancer midway through my graduate school career. He battled the disease for two and a half years, with many good times and a few bad, and passed away in the summer before my last year. My father always supported and encouraged me in mathematics, and introduced me to Möbius strips when I had just learned to count past two. This dissertation is for him.

My husband, Brian Gillis, and I met our first semester in graduate school, and married the summer before our last year. He is the love of my life and my favorite companion. This dissertation is for him.

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This dissertation would never have happened without the guidance, suggestions, and always-excellent discussions provided by my advisor, Nicolai Reshetikhin. I don't think I will forget his comment when I gave him my first attempt at what became Chapter 1: "This is good student work." Later versions received much more positive reviews.

This dissertation has also been influenced by conversations with many people, and I'm certain to forget somebody. Of particular importance is Owen Gwilliam, who introduced me to factorization algebras. Our conversations (as well as our coauthored note [GJF12] and Owen's thesis [Gwi12]) motivated much of Chapters 2 and 3.

Dan Berwick Evans, Alex Chirvasitu, and Harold Williams were constant discussion partners at UC Berkeley — I have probably discussed with them every aspect of this dissertation, among many other topics. I would like to thank Jørgen Andersen, Kiril Datchev, Arturo Prat-Walden, George Thompson, and Ivan Ventura for discussions related to Chapter 1, and Chris Schommer-Pries specifically for the proof of Lemma 1.3.9 [SP09]. The anonymous referees for [JF10a, JF10b] also provided very useful feedback that is now in that chapter. Chapter 2 was particularly influenced by my conversations with Morgan Brown, Tom Goodwillie, Boris Hanin, Eric Peterson, Josh Shadlen, Shamil Shakirov, Yuan Shen, Zack Sylvan, and PhilSang Yoo, and an anonymous referee's comments on [JF12] were also quite valuable. I discussed the material in Chapter 3 with Damien Calaque, Alberto Cattaneo, Kevin Costello, Dmitry Pavlov, Dmitri Tamarkin, Hiro Tanaka, and Arkady Vaintrob, and my particular gratitude goes to Martin Callies and Callum Sleigh for patiently listening as I caught errors in various computations.

I described parts of this dissertation on the online discussion forum http://mathoverflow.net/, where I am a frequent visitor. I was a physical visitor during stages of this project at Aarhus Universitet, Northwestern University, and Universität Zürich, and I would like to thank all three institutions for their hospitality and generosity. This work is supported by the NSF grant DMS-0901431.

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Chapter 0

Introduction

This dissertation addresses a series of questions related to path integration in quantum field theory. Our focus is on methods that can broadly be defined as "perturbative": Feynman diagrams, homological perturbation theory, deformation quantization. The dissertation combines all but one of the research papers its author produced while a Ph.D. student at UC Berkeley. (The exception [CJF11], joint with Alex Chirvasitu, is not unrelated to the category theory in Chapter 3, but is far from our main storyline.) Compared to an article, a Ph.D. dissertation provides a chance to tell a longer story, with more background and motivation. To help the reader appreciate the entire narative, this introduction gives an overview of the various topics discussed. Each section of this introduction ends with a list of new results in the corresponding chapter.

An attempt has also been made to make the individual chapters readable separately from each other. Chapter 1 is based on the papers [JF10a, JF10b, JF10c], and proves that the traditional method of Feynman diagrams gives all the correct answers when applied to the quantum mechanics of a charged particle on a curved manifold. Chapter 2, based on [GJF12, JF12], steps back to discuss more generally integrals of the type appearing in quantum field theory, and applies some homological perturbation theory to study *nonperturbative* integrals in which all functions are polynomials (rather than the usual power series). Finally, Chapter 3 consists of material that is new to this dissertations; it and uses the same homological algebra as in Chapter 2 to construct a non-path-integral one-dimensional quantum field theory that provides a universal (and manifestly rational) $\star$-quantization of formal Poisson manifolds.

0.1 Feynman diagrams and an overview of Chapter 1

In [Fey85], Feynman introduces a historical remark as "a physicist's history of physics": although morally correct, the details were simplified and probably inaccurate. As mathematicians, we are even less connected to the real world than our physicist friends, and what follows is "a mathematician's history of physics." For a sociologist's history of the physics we are about to discuss, [Kai05] is highly recommended.

0.1.1 Preamble: lightning review of classical mechanics To set the stage, we quickly review some classical mechanics. More details are in Section 1.3 and many textbooks.

A *classical Lagrangian system* consists of a manifold $\mathcal{N}$, thought of as the possible *configurations* of a particle, and a function $L : \mathbb{R} \times \mathrm{T}\mathcal{N} \to \mathbb{R}$ called the *Lagrangian*, which may be required to satisfy some convexity or other nondegeneracy properties. Given a (piecewise) smooth *path* $\gamma : [t_0, t_1] \to \mathcal{N}$, lifted in the canonical way to $(\dot\gamma, \gamma) : [t_0, t_1] \to \mathrm{T}\mathcal{N}$, its *action* is $\mathcal{S}(\gamma) = \int_{\tau=t_0}^{t_1} L\big(\tau, \dot\gamma(\tau), \gamma(\tau)\big)\mathrm{d}\tau$. We define a mechanics on $\mathcal{N}$ by saying that a path γ is *classically allowed* if it is a critical point of $\mathcal{S}$ among all paths with the same values at t_0 and t_1. Provided L satisfies sufficient nondegeneracy conditions, such mechanics is equivalent to a nondegenerate second-order ordinary differential equation on $\mathcal{N}$ called the *Euler–Lagrange equation*.

0.1.2 The path integral in quantum mechanics Feynman, in his thesis [Fey48] (also available in [FBD05]), proposed a new formulation of quantum mechanics. We begin with the previously-developed story.

The *Hamiltonian* $H : \mathbb{R} \times \mathrm{T}^*\mathcal{N} \to \mathbb{R}$ corresponding to a Lagrangian L is the fiberwise Legendre transform of L. Under the (time-varying) isomorphism $\mathrm{T}^*\mathcal{N} \cong \mathrm{T}\mathcal{N}$ defined by the fiberwise second derivatives of L, the Euler–Lagrange equation is equivalent to the Hamiltonian flow "$\frac{\mathrm{d}}{\mathrm{d}t} = \{H, -\}$" on $\mathrm{T}^*\mathcal{N}$, where $\{,\}$ is the canonical Poisson structure on $\mathrm{T}^*\mathcal{N}$. The usual approach to quantization replaces functions on $\mathrm{T}^*\mathcal{N}$ with differential operators on $\mathcal{N}$, the Poisson structure with the commutator, and H with a *quantization* $\hat H$, called the *Schrödinger operator* and generating a flow "$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\sqrt{-1}}{\hbar}[\hat H, -]$" on the space of functions on $\mathcal{N}$. It is usually requested that the space of functions be completed to the Hilbert space $\mathrm{L}^2(\mathcal{N}, \mathrm{dVol})$, which requires the introduction of a volume form dVol on $\mathcal{N}$. We denote by $U(t_0, t_1) : \mathrm{L}^2(\mathcal{N}, \mathrm{dVol}) \to \mathrm{L}^2(\mathcal{N}, \mathrm{dVol})$ the operator describing the evolution of a state from time t_0 to time t_1.

At a physicist's level of rigor, $\mathrm{L}^2(\mathcal{N}, \mathrm{dVol})$ has a basis consisting of the Dirac delta functions δ_q where q ranges over $\mathcal{N}$, and the operator $U(t_0, t_1)$ can be given in terms of the amplitudes $U(t_0, q_0, t_1, q_1)$ to transition from state δ_{q_0} at time t_0 to state δ_{q_1} at time t_1. Since the δ states are a complete basis, the transition amplitudes satisfy a semigroup law of the form

$$U(t_0, q_0, t_1, q_1) = \int_{q \in \mathcal{N}} U(t_0, q_0, t, q)\, U(t, q, t_1, q_1)\ \mathrm{dVol}(q)$$

for any fixed t. Consider subdividing the interval $[t_0, t_1]$ into a very fine partition $t_0 = \tau_0 <$

$\tau_1 < \cdots < \tau_n = t_1$, with step size $\tau_i - \tau_{i-1} = \Delta\tau$. Then we have

$$U(t_0, q_0, t_1, q_1) = \int\limits_{\substack{\gamma:\{\tau_0=t_0,\ldots,\tau_n=t_1\}\to\mathcal{N} \\ \gamma(t_0)=q_0,\,\gamma(t_1)=q_1}} \prod_{\tau\in\{\tau_0,\tau_1,\ldots,\tau_{n-1}\}} U\big(\tau,\gamma(\tau),\tau+\Delta\tau,\gamma(\tau+\Delta\tau)\big) \times \prod_{\tau\in\{\tau_1,\ldots,\tau_{n-1}\}} \mathrm{dVol}\big(\gamma(\tau)\big)$$

If we now take the step size $\Delta\tau$ to be very small, two things happen. First, we can think of the domain of integration as being over all functions $\gamma : [t_0, t_1] \to \mathcal{N}$ with boundary conditions $\gamma(t_a) = q_a$ for $a = 0, 1$, and the measure as being $\mathrm{d}\gamma = \prod_{\tau\in(t_0,t_1)} \mathrm{dVol}(\gamma(\tau))$. Second, for very small $\Delta\tau$, the evolution $U(\tau, q, \tau+\Delta\tau, q')$ is almost a delta function $\delta(q'-q)$, forcing γ to be continuous, and a more detailed analysis gives a correction of the form $\exp\big(\frac{\sqrt{-1}}{\hbar}L(\tau, v, q)\big)$, where L is the Lagrangian for the corresponding classical system and $v = (q'-q)/\Delta\tau$.

Thus Feynman posits the following integral expression for the time evolution operator:

$$U(t_0, q_0, t_1, q_1) = \int\limits_{\substack{\gamma:[t_0,t_1]\to\mathcal{N} \\ \gamma(t_0)=q_0,\,\gamma(t_1)=q_1}} \exp\left(\frac{\sqrt{-1}}{\hbar}\mathcal{S}(\gamma)\right) \prod_{\tau\in(t_0,t_1)} \mathrm{dVol}\big(\gamma(\tau)\big)$$

He then argues the converse direction: that the above integral expression for U satisfies all requirements to be the time evolution operator. His arguments do not claim to go beyond a physicist's level of rigor, and apply only when $\mathcal{N} = \mathbb{R}^d$ and the Lagrangian L is a quadratic function on $\mathrm{T}\mathcal{N} = \mathbb{R}^{2d}$ plus an infinitesimal perturbation. In such a situation the infinite-dimensional integral can be defined algebraically as a Gaussian distribution. In other cases, defining the integral requires some work [Dri04].

0.1.3 The path integral in quantum field theory and Feynman diagrams Shortly after the published version of his thesis appeared, Feynman attended a conference in the Poconos. He had the misfortune of following a virtuosic performance by Schwinger. His goal was to explain his "path integral" formulation of quantum field theory, but instead he drew all sorts of diagrams. Oppenheimer, who organized the conference, could easily reject the whole thing: it was well-known at the time that quantum mechanics forbids a particle from having both position and momentum, which is what it seemed Feynman's diagrams required. It took Dyson [Dys48, Dys49] to explain what was going on.

Classical Lagrangian field theory is a generalization of classical mechanics in which the interval $[t_0, t_1]$ is replaced by a higher-dimensional manifold. Let $\pi : P \to X$ be a smooth bundle, where X is a finite-dimensional manifold with boundary ∂X, and denote by $\Gamma(P \to X)$ the space of sections of $P \to X$ (of some chosen regularity — physicists tend to be very vague about what regularity is required). For each *Dirichlet boundary condition* $\phi \in \Gamma(\partial P \to \partial X)$, let $\Gamma_\phi(P \to X) = \{\gamma \in \Gamma(P \to X) \text{ s.t. } \gamma|_{\partial X} = \phi\}$. Then $\Gamma(P \to X)$ and

$\Gamma_\phi(P \to X)$ are infinite-dimensional smooth manifolds. The mechanics is controlled by an *action* $\mathcal{S} : \Gamma(P \to X) \to \mathbb{R}$ by declaring that the classically-allowed field configurations with boundary conditions ϕ are precisely the critical points of $\mathcal{S}$ restricted to $\Gamma_\phi(P \to X)$. *Quantum Lagrangian field theory* requires in addition the data of a section dVol of the *fiberwise density bundle* $\mathscr{D}_{\text{fiber}} \to P$, which is the bundle that restricts to the density bundles $\mathscr{D}_{\pi^{-1}(x)} \to \pi^{-1}(x)$ on the fibers of $\pi : P \to X$. The *quantum mechanical amplitude* of the boundary condition ϕ is given by the following infinite-dimensional "path" integral:

$$U(\phi) = \int_{\gamma \in \Gamma_\phi(P \to X)} \exp\left(\frac{\sqrt{-1}}{\hbar}\,\mathcal{S}(\gamma)\right) \prod_{x \in X \smallsetminus \partial X} \mathrm{dVol}\big(\gamma(x)\big)$$

For action functions of the form "kinetic minus potential" there do exist analytic definitions of the quantum-mechanical path integral, but in higher dimensions the analysis seems intractable. Instead, the standard approach is to approximate the integral by a Gaussian. In Chapter 1, as is often done, we will posit that these infinite-dimensional path integrals enjoy the same $\hbar \to 0$ asymptotics as do finite-dimensional oscillating integrals. Such asymptotics are studied by the "WKB" or "stationary phase" method, which we review in Section 1.2: the contributions from non-critical points of the action $\mathcal{S}$ cancel out, and so the integral is concentrated near classically allowed paths; provided $\mathcal{S}$ is a Morse function, the asymptotics are computed by keeping the quadratic part of $\mathcal{S}$ as a Gaussian term, expanding the rest in Taylor series, and integrating via Wick's formula. An even easier situation appears in quantum electrodynamics: $P \to X$ is a vector bundle, and $\mathcal{S}$ is quadratic on the vector space $\Gamma(P \to X)$ plus a small "interaction" term. Supposing that the word "small" is controlled by a parameter λ and that we are interested in the $\lambda \to 0$ asymptotics of U, we can again define the integral in terms of a Gaussian.

In either case, the combinatorics involved in the Gaussian integrals and the various Taylor series are well-described by certain labeled graphs, called *Feynman diagrams*. To evaluate expectations values of integrals against the measure $e^{\frac{\sqrt{-1}}{\hbar}\mathcal{S}} \prod \mathrm{dVol}$, one computes a sum over all Feynman diagrams, where each diagram is weighted by the value of a finite-dimensional integral in which each variable of integration ranges over X. For Dyson, that's all Feynman diagrams are: a description of an integral expression which he shows to be equivalent to Schwinger's formulation of quantum electrodyamics. But to Feynman the diagrams are also pictures of particle interactions. The reason is that when $P \to X$ is a vector bundle and $\mathcal{S} = \text{quadratic} + \lambda \cdot \text{interaction}$, then the weight of a graph Γ is also computed by a *quantum-mechanical* path integral over all maps $\Gamma \to X$. This interpretation of Feynman diagrams does not apply well in the stationary-phase approximation, so we will not use it in this dissertation.

0.1.4 What Chapter 1 proves Feynman diagrams are used successfully in quantum field theory, but there are many questions left about them and whether Feynman-diagrammatic methods are well-founded. (An analogous situation occurred in the centuries between Newton's introduction of Calculus and Cauchy's placement of its infinitesimal and infinite quan-

tities on sure footing.) Thus our goal is to test these methods, and especially the stationary-phase approximation, against analytically well-defined quantum mechanics.

We begin at the same level of generality as in 0.1.1 and 0.1.2, but eventually restrict our attention to *nonrelativistic motion of a charged particle in a curved manifold through an external electromagnetic field*, by which we mean that the Lagrangian L consists of three parts: a term which is quadratic in velocity, mathematically a Riemannian metric $a \in \Gamma((\mathrm{T}^*)^{\otimes 2}\mathcal{N})$ and physically a "mass" term; a term which is linear in velocity, mathematically a one-form $b \in \Gamma(\mathrm{T}^*\mathcal{N})$ and physically a "magnetic potential"; and a term which is constant in velocity, mathematically a function $c \in \mathscr{C}^\infty(\mathcal{N})$ and physically an "electric potential" (all three terms may also depend on the external "time" parameter). We choose $\mathrm{dVol} = \sqrt{\det a}$ to be the Riemannian volume form associated to the metric a. Together, Theorems 1.5.1, 1.6.1, and 1.7.4 show that Definition 1.3.22 (in which we posit a Feynman-diagrammatic sum for the $\hbar \to 0$ asymptotics of the quantum-mechanical path integral) provides a well-defined formal path integral which is coordinate-independent, free of ultraviolet divergences, and satisfies a composition law. Then in Theorems 1.8.2 and 1.9.4 we show that our *formal path integral* $U(t_0, q_0, t_1, q_1)$ satisfies Schrödinger's equation (with a scalar curvature term) and converges as a distribution to $\delta(q_1 - q_0)$ as $t_1 \to t_0$.

Most of our arguments are diagrammatic, and so the reader is encouraged to flip through Chapter 1 if only to admire the pretty pictures.

0.2 Batalin–Vilkovisky complex and an overview of Chapter 2

The stationary phase approximation has long been known to give an asymptotic description of finite-dimensional oscillating integrals of the form $\int_{\mathcal{M}} f \exp\left(\frac{\sqrt{-1}}{\hbar} s\right) \mathrm{dVol}$ near nondegenerate critical points of s. The case of degenerate critical points, which occur in all physically-relevant quantum field theories, is more subtle. One situation in which degenerate critical points arise is when a compact Lie group G acts on $\mathcal{M}$ preserving f, s, and dVol. Then the critical points of s form G-orbits, corresponding to critical points in the quotient stack $\mathcal{M}/G$. If these critical points are nondegenerate in $\mathcal{M}/G$, then the Faddeev–Popov procedure introduces new fermionic variables to $\mathcal{M}$ so that all together the integral has a nondegenerate critical point [FP67]. Choices are required to implement the Faddeev–Popov procedure, and in order to translate the procedure to infinite dimensions one would like an *algebraic* proof that the choices are irrelevant. Such a proof was supplied by Becchi, Rouet, and Stora [BRS74, BRS75, BRS76] and independently by Tyutin [Tyu75]: the Faddeev–Popov procedure can be understood as a choice of chain complex computing the homology of the G-action on $\mathcal{M}$, and the different choices involved in the procedure differ by an exact term.

This was one of the first introductions of homological algebra into physics, but what we are concerned with in Chapter 2 is the further work by Batalin and Vilkovisky [BV83, BV84,

BV85]. In their search for a quantum theory of supergravity, they faced degenerate critical points with no apparent group action. Their solution also involved additional fermionic variables as well as homological algebra. We will ignore their superspace integrals, and focus on the homological algebra.

0.2.1 A version of the de Rham complex The basic idea of the Batalin–Vilkovisky construction is quite simple, as explained by Witten in [Wit90]. For convenience, suppose that $\mathcal{M}$ is compact, connected, and oriented. Any nowhere-vanishing probability measure dVol determines an isomorphism between the space $\mathscr{C}^\infty(\mathcal{M})$ of functions and the space $\Omega^{\dim \mathcal{M}}(\mathcal{M})$ of top-degree differential forms. More generally, denote by $\mathrm{MV}_k = \Gamma(\mathrm{T}^{\wedge k}\mathcal{M})$ the space of k-linear antisymmetric multivector fields on $\mathcal{M}$. Then by contracting with dVol, we get an isomorphism $\mathrm{MV}_k \cong \Omega^{\dim \mathcal{M}-k}(M)$. We can move the de Rham differential d_{dR} through this isomorphism to get a differential Δ_{dVol} on the graded vector space $\mathrm{MV}_\bullet$; Δ_{dVol} is also called the operator of *divergence* with respect to dVol. Note that since $\mathcal{M}$ is assumed to be connected, $\mathrm{H}^{\dim \mathcal{M}}(\Omega^{\dim \mathcal{M}}, \mathrm{d}_{\mathrm{dR}}) = \mathrm{H}_0(\mathrm{MV}_\bullet, \Delta_{\mathrm{dVol}})$ is one-dimensional. Thus the unique isomorphism $\mathrm{H}_0(\mathrm{MV}_\bullet, \Delta_{\mathrm{dVol}}) \cong \mathbb{R}$ identifying the class of $1 \in \mathscr{C}^\infty(\mathcal{M}) = \mathrm{MV}_0$ with $1 \in \mathbb{R}$ must identify the class of $f \in \mathscr{C}^\infty(\mathcal{M})$ with $\int f$ dVol.

We call the differential "Δ" for the following reason. Just like the de Rham complex $\Omega^\bullet(\mathcal{M}) = \Gamma((\mathrm{T}^*)^{\wedge\bullet}\mathcal{M})$, the BV complex $\mathrm{MV}_\bullet$ is a graded-commutative algebra with "wedge" multiplication. But unlike $\Omega^\bullet(\mathcal{M})$, the BV differential Δ_{dVol} is not a derivation, so $(\mathrm{MV}_\bullet, \Delta_{\mathrm{dVol}})$ is not a dga. Rather, Δ_{dVol} is a second-order differential operator for the algebra structure on $\mathrm{MV}_\bullet$. Since $\mathrm{MV}_\bullet$ is the algebra of multivector fields on $\mathcal{M}$, it does come with a canonical structure independent of a choice of volume form, namely the *Schouten–Nijenhuis bracket* $\{,\}$. In fact, $\{,\}$ is precisely the *symbol* of the second-order operator Δ_{dVol}: for any dVol, the bracket measures the failure of Δ_{dVol} to be a derivation.

If, in addition to being compact, connected, and oriented, $\mathcal{M}$ has trivial first homology, then there is a bijection between probability measures dVol on $\mathcal{M}$ and differentials on $\mathrm{MV}_\bullet$ which are second-order differential operators with symbol $\{,\}$ [Kos85]. Now replacing $\mathcal{M}$ by an infinite-dimensional manifold, although we no longer have $\Omega^{\dim \mathcal{M}}(M)$, we can still define $\mathrm{MV}_\bullet$ and the Schouten–Nijenhuis bracket, and such *BV probability measures* are a homological-algebraic way to define integration.

Finally, consider changing dVol to $\exp\left(\frac{\sqrt{-1}}{\hbar}s\right)\mathrm{dVol}$ for some function s. Then Δ_{dVol} changes to $\Delta_{\exp(\frac{\sqrt{-1}}{\hbar}s)\,\mathrm{dVol}} = \Delta_{\mathrm{dVol}} + \frac{\sqrt{-1}}{\hbar}\{s,-\}$. To encode integration, we need only the homology for this differential, and so if we are interested in the $\hbar \to 0$ asymptotics we can multiply through by $\frac{\hbar}{\sqrt{-1}}$ and study the differential $\mathcal{Q} = \{s,-\} - \sqrt{-1}\,\hbar\,\Delta_{\mathrm{dVol}}$. This gives an algebraic way to see the stationary-phase approximation: when $\hbar = 0$, the differential $\{s,-\}$ on $\mathrm{MV}_\bullet$ is precisely the differential for the Koszul resolution of the intersection of the zero section with the graph of $\mathrm{d}s$ in $\mathrm{T}^*\mathcal{M}$.

0.2.2 What Chapter 2 proves The BV complex is traditionally studied when $\hbar$ is a formal variable, and in Extended Example 2.2.5 we rederive the method of Feynman diagrams

from this perspective. But our primary interest in Chapter 2 is to understand *nonperturbative* integrals of the form $\int f\, e^s$ dVol when f and s are both polynomials in n variables (and the integral is taken over some contour in $\mathbb{C}^n$ along which e^s enjoys exponential decay). Such integrals are controlled by an polynomial version of the BV complex which is not a shifted version of the usual de Rham complex, but rather of a twisted de Rham complex. We set $\mathrm{MV}_\bullet$ to be the algebra of *polynomial* multivector fields. Let $x_1, \ldots, x_n$ denote the standard coordinates on $\mathbb{C}^n$ (so that $\mathrm{dVol} = \mathrm{d}x_1 \cdots \mathrm{d}x_n$), and let $\xi_1, \ldots, \xi_n$ denote the "odd" generators of $\mathrm{MV}_\bullet$ corresponding to the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \in \Gamma(\mathrm{T}\mathbb{C}^n)$. The "BV differential" that we are interested in is:

$$\partial_{\mathrm{BV}} = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i \partial \xi_i} + \frac{\partial s}{\partial x_i} \frac{\partial}{\partial \xi_i} \right)$$

For most of Chapter 2 we assume that the polynomial s has maximal total degree d and that the *top part* $s^{(d)}$ is *nonsingular* in the sense that the corresponding hypersurface $\{s^{(d)} = 0\} \subseteq \mathbb{CP}^{n-1}$ is smooth. Under this assumption, both the space of valid contours of integration and the space of functions on the scheme-theoretic critical locus of s are $(d-1)^n$-dimensional. Denote by $\mathcal{O}(\{\mathrm{d}s = 0\}) = \mathbb{C}[x_1, \ldots, x_n]/(\frac{\partial s}{\partial x_1}, \cdots, \frac{\partial s}{\partial x_n})$ the latter space of functions. In Theorem 2.3.4 we prove that any choice of splitting $\mathcal{O}(\{\mathrm{d}s = 0\}) \to \mathcal{O}(\mathbb{C}^n) = \mathbb{C}[x_1, \ldots, x_n]$ of the restriction map — thought of as a choice of how to extend each function on the critical locus to a function on the ambient space — determines an isomorphism $\mathcal{O}(\{\mathrm{d}s = 0\}) \cong \mathrm{H}_0(\mathrm{MV}_\bullet, \partial_{\mathrm{BV}})$, and that all other homology groups of the BV complex vanish. The composition $\mathcal{O}(\mathbb{C}^n) = \mathrm{MV}_0 \to \mathrm{H}_0(\mathrm{MV}_\bullet, \partial_{\mathrm{BV}}) \cong \mathcal{O}(\{\mathrm{d}s = 0\})$ is not the usual restriction, and rather can be thought of as an algebrogeometric version of "integrating out the higher modes." In Theorem 2.3.8 we provide a uniform coordinate-dependent way to choose the splitting $\mathcal{O}(\{\mathrm{d}s = 0\}) \to \mathcal{O}(\mathbb{C}^n)$ which works for very general actions s. The primary tool for both theorems is the Homological Perturbation Lemma, which we recall in 2.3.10.

We end Chapter 2 with some speculation towards a generalization of these ideas to nonperturbative field theory, focussing our attention on Chern–Simons Theory and the Volume Conjecture.

0.3 Poisson AKSZ theory and an overview of Chapter 3

In Chapter 3, we construct a manifestly rational universal $\star$-quantization of formal Poisson manifolds. Our construction is combinatorial and homological-algebraic, and except for the motivational Remark 3.1.4 it is not written in the language of quantum field theory. But the meaning of the construction is very much quantum-field-theoretic, and is a natural extension of the ideas in the previous chapters. So here we will explain the construction from that point of view.

0.3.1 Costello–Gwilliam quantum field theory The BV complex suggests the following generalization of the notion of "oscillating integral." A *Pois$_0$ algebra* is a dg commutative algebra $(A_\bullet, \partial)$ equipped with a compatible degree-(-1) bracket P, which is a derivation in each variable and satisfies a Jacobi identity (equivalently, a Gerstenhaber algebra in which the bracket P and any other differential ∂ have the same, rather than opposite, homological degree). A *quantization* of a Pois_0 algebra $(A_\bullet, \partial, P)$ is a Maurer–Cartan element Δ on $A_\bullet$ which is a second-order differential operator with symbol P. In many cases, $A_\bullet$ is defined over formal power series in a variable $\hbar$, and $P = O(\hbar)$, so one can expect that Δ will provide a small deformation of the chain complex $A_\bullet$, and thus the deformed complex $(A_\bullet, \partial + \Delta)$ can be studied using the Homological Perturbation Lemma. A *Pois$_0$ manifold* is a $\mathbb{Z}$-graded manifold whose algebra of functions is equipped with a Pois_0 structure. For details on $\mathbb{Z}$-graded manifolds, see [Meh06].

Based on the work of Beilinson and Drinfel′d on conformal field theory [BD04], Costello and Gwilliam [CG11, Gwi12] have proposed the following picture of classical and quantum field theory. A *classical field theory* on a spacetime X is nothing more than a sheaf $\mathcal{E}$ on X, usually valued in infinite-dimensional dg manifolds. Namely, one imagines assigning to each open $U \subseteq X$ its space of classically-allowed field configurations, but to homological algebraists it is more natural to assign the *derived* space of classically-allowed field configurations. For a Lagrangian field theory based on a bundle $P \to X$, let $\mathcal{M}_U = \Gamma(P|_U \to U)$ denote the infinite-dimensional manifold of all fields over U, and $\mathrm{MV}_\bullet(U) = \Gamma(\mathrm{T}^{\wedge\bullet}\mathcal{M}_U)$ its algebra of multivector fields, made into a Pois_0 algebra with the Schouten–Nijenhuis bracket $\{,\}$. If the action function in question is denoted $s \in \mathscr{C}^\infty(\mathcal{M})$, then the classical field theory assigns to U the dg manifold $\mathcal{E}(U) = \mathrm{spec}\big(\mathrm{MV}_\bullet(U), \{s, -\}\big)$.

As in the example of Lagrangian classical field theory, each space $\mathcal{E}(U)$ is naturally a Pois_0 manifold. However, $\mathcal{E}(-)$ is not a sheaf valued in Pois_0 manifolds, for reasons related to the following fact from usual Poisson (or *Pois$_1$*) geometry: if $\mathcal{P}$ is a Poisson manifold with nontrivial Poisson structure, then the diagonal map $\mathcal{P} \to \mathcal{P} \times \mathcal{P}$ is not a Poisson map. Rather, $\mathcal{E}(-)$ is a *factorization coalgebra* valued in Pois_0 manifolds, or equivalently $U \mapsto \big(\mathrm{MV}_\bullet(U), \{s, -\}, \{,\}\big)$ is a *factorization algebra* valued in Pois_0 algebras. Factorization coalgebras are a certain weakening of the notion of "sheaf": they have restriction and locality axioms, but can take values in categories (like PoisMan or Vect) where the most natural monoidal structure is not the cartesian product. A *quantization of $\mathcal{E}$* is then a quantization $\Delta(U)$ of each Pois_0 algebra $\big(\mathrm{MV}_\bullet(U), \{s, -\}, \{,\}\big)$ such that the deformed complexes $\big(\mathrm{MV}_\bullet(U), \Delta(U)\big)$ fit together into a factorization algebra valued in chain complexes. More generally, a *quantum field theory* in the Costello–Gwilliam picture is precisely a factorization algebra.

0.3.2 Poisson AKSZ theory In [AKSZ97], Alexandrov, Kontsevich, Schwarz, and Zaboronsky develop a general method to construct Pois_0 manifolds. Although they don't say so, their construction breaks into two pieces, which we call the *first AKSZ trick* and the *second AKSZ trick*. We will explain both, although only the first trick is used in Chapter 3.

By definition, a *Pois_n manifold* is a dg manifold equipped with a degree-$(n-1)$ Poisson (or Gerstenhaber) bracket. Suppose that $\mathcal{M}$ is a Pois_n manifold and Y is a dg manifold equipped with a degree-m volume form. Then, except for the very real problem of ultraviolet divergences, the mapping space $\underline{\mathrm{Maps}}(Y, \mathcal{M})$ is a Pois_{n-m} manifold. For any manifold U, its *affinized de Rham space* is the dg manifold $U_{\mathrm{dR}} = \mathrm{spec}(\Omega^\bullet(U), \mathrm{d}_{\mathrm{dR}})$. If U is m-dimensional and oriented, then U_{dR} comes equipped with a distinguished degree-m volume form. The *first AKSZ trick* inputs a Pois_n-manifold $\mathcal{M}$ and outputs a classical field theory on any oriented n-dimensional manifold X by assigning to each open $U \subseteq X$ the Pois_0 manifold $\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, \mathcal{M})$. Such field theories are interesting to quantize: as we will explain in Remark 3.1.4, quantizations of this field theory correspond to E_n quantizations of $\mathscr{C}^\infty(\mathcal{M})$.

Just as the functions on the cotangent bundle $\mathrm{T}^*\mathcal{M}$ that are polynomial in the fibers are precisely the symmetric polyvector fields on $\mathcal{M}$, the algebra $\mathrm{MV}_\bullet = \Gamma(\mathrm{T}^{\wedge\bullet}\mathcal{M})$ of antisymmetric multivector fields is the algebra of functions on the *shifted cotangent bundle* $\mathrm{T}^*[-1]\,\mathcal{M}$. The original BV construction from 0.2.1 related quantizations of $\mathrm{T}^*[-1]\,\mathcal{M}$ with integration on $\mathcal{M}$. If $\mathcal{M}$ is Pois_n, the shifted cotangent bundle $\mathrm{T}^*[n]\,\mathcal{M}$ can be given a Pois_{n+1} structure in which the bracket is the canonical one and the differential encodes the Pois_n structure on $\mathcal{M}$. Suppose now that U is $(n+1)$-dimensional. Then, still ignoring the problem of ultraviolet divergences, there is an isomorphism of Pois_0 manifolds $\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, \mathrm{T}^*[n]\,\mathcal{M}) \cong \mathrm{T}^*[-1]\,\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, \mathcal{M})$. Thus to a Pois_n manifold we can also assign an $(n+1)$-dimensional quantum field theory, again with interesting quantizations (especially in the presence of boundary conditions). This is the *second AKSZ trick*.

0.3.3 What Chapter 3 proves In the last chapter of this dissertation, we implement the first AKSZ trick when $n = 1$. Although the above ideas are written in terms of smooth manifolds over $\mathbb{R}$, we work combinatorially and over $\mathbb{Q}$. As our target manifold, we take $\mathcal{M}$ to be the *universal formal Poisson manifold*, which is defined in Section 3.2. In Sections 3.3 and 3.4, we construct a factorization algebra on $\mathbb{R}$ that deserves to be thought of as the field theory $U \mapsto \underline{\mathrm{Maps}}(U_{\mathrm{dR}}, \mathcal{M})$. Our factorization algebra has the flavor of a "lattice approximation" of a smooth field theory: since functions on U_{dR} are precisely de Rham forms on U, linear functions on $\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, \mathcal{M})$ are a version of $\mathrm{Chains}_\bullet(U)$, and we choose a very combinatorial model of $\underline{\mathrm{Chains}}_\bullet$ which is well-suited to our later purposes.

Then in Sections 3.5 and 3.6 we quantize our theory. When implementing the AKSZ tricks honestly, ultraviolet divergences are unavoidable, and so our quantization is not quite as strict as what we outlined above. For example, only modulo $\hbar$ is our Maurer–Cartan element a second-order operator. Nevertheless, it is sufficient for our purposes: using the same Homological Perturbation Lemma as in Chapter 2, in Section 3.7 we turn our quantum field theory into a $\star$-quantization of $\mathcal{M}$.

All together, Theorems 3.6.3 and 3.7.2 and Propositions 3.7.4, 3.7.8, 3.7.9, and 3.7.11 prove that there exist universal polynomials with rational coefficients in the Taylor expansion of a Poisson structure on any formal power series algebra, which package together into a deformation quantization of the corresponding Poisson algebra. At each order in the

deformation parameter $\hbar$, the new *star product* is a finite-order bidifferential operator. This universal quantization is combinatorially defined and amenable to direct computation. For fixed n, to compute up to order $\hbar^n$ requires making finitely many choices, and the number of choices and the number of ensuing computations does grow rapidly with n.

Understanding the relation between our universal $\star$-quantization and those of Kontsevich [Kon03] and Tamarkin [TT00], as well as understanding the behavior of our $\star$-quantization under changes of coordinates, will have to wait until future work.

Chapter 1

Classical methods: Feynman diagrams for quantum mechanics

The primary goal of this chapter is to clarify the definition and construction of the formal path integral as it applies to quantum mechanics on possibly-curved spaces. One motivation comes from quantum field theories in which the fields take values in a fiber bundle with curved fibers. We will prove that for the quantum mechanics of a nonrelativistic charged particle moving in a Riemannian manifold under the influence of an external electromagnetic field, the formal path integral is well-defined, satisfies a Fubini-style composition law, and satisfies Schrödinger's equation with the correct initial value. This chapter is based closely on [JF10a, JF10b, JF10c].

It's worth highlighting two improvements over those papers. First, the precise statement of the asymptotics of oscillating integrals is almost, but not quite, correct in [JF10a, JF10b], where sharp cut-offs are implied, but in fact smooth cut-offs are required [Zwo12]. This has been corrected; see Fact 1.2.12. Second and more importantly, in [JF10a] Schrödinger's equation was verified only in the case of quantum mechanics on $\mathbb{R}^n$. As asserted in [JF10b], that proof translates more-or-less directly to the curved case, but more care is needed, and in particular a curvature term is required. Here we give the general proof.

The outline for the chapter is as follows. Section 1.1 is introductory; it explains the basic ideas behind the formal path integral, and includes a more detailed overview of the chapter. Section 1.2 reviews the Feynman-diagrammatic description of the asymptotics of oscillating integrals. In Section 1.3 we recall some of the basics of Lagrangian mechanics, developing enough to give a coordinate-full definition of the formal path integral supported near a chosen critical point (Definition 1.3.22). We continue to develop our diagrammatic language in Section 1.4, where we record some useful facts. Then in Section 1.5 we come to our first main theorem, Theorem 1.5.1, on the coordinate independence of the formal path integral for "short" paths. Provided there are no ultraviolet divergences, in Section 1.6 we prove a composition law for the formal path integral (Theorem 1.6.1), which explains how to define the formal path integral on arbitrary manifolds. Completing the proof that the formal path integral is well-defined, in Section 1.7 we discuss ultraviolet divergences,

and show in Theorem 1.7.4 that all ultraviolet divergences cancel for the quantization of nonrelativistic motion of an electron moving in a curved space under the influence of an external electromagnetic field. This is precisely the situation where there is a well-behaved Schrödinger operator, and in Section 1.8 we prove that our formal path integral satisfies Schrödinger's equation (Theorem 1.8.2). We end the chapter with Section 1.9, in which we address the sum over all critical points of our formal path integral, and prove in Theorem 1.9.4 that it converges and has the correct initial value.

1.1 Idea of the path integral

Feynman introduced the path integral in 1948 in his thesis [Fey48] (available in [FBD05]) as a new formalism for quantum mechanics. In 1949, based on his path integral and his powerful physical intuition, Feynman introduced his famous diagrams as a tool for studying quantum electrodynamics [Fey49a]. In the subsequent years, path integrals and Feynman diagrams became universal in the study of quantum field theories; for a detailed history, see [Kai05]. These applications are usually "formal," in the sense that they return formal power series in the physical variables: analytic definitions of path integrals remain elusive in most cases. Among physically important quantum field theories, only quantum mechanics (a one-dimensional quantum field theory) exists analytically (see e.g. [Tak08]), but the diagrammatic methods have not been rigorously checked against the analytic theory. Diagrammatic path integrals do exist in the work of Kleinert and Chervyakov [KC02] and in the work of DeWitt-Morette [DM76]. DeWitt-Morette's diagrammatic expansion is equivalent to ours, whereas Kleinert and Chervyakov's methods apply directly only to perturbations of free motion (the same case as Feynman considered). In all cases that we could find, the results in this chapter are essentially taken for granted, rather than checked directly.

In the remainder of this introductory section, we describe the basic idea of the path integral, and then overview our version of the formal path integral and the results of the chapter.

1.1.1 Classical and quantum Lagrangian mechanics Let us very briefly review some basics of Lagrangian mechanics. More details are in Section 1.3 and in many textbooks. A *configuration space* is a finite-dimensional smooth manifold $\mathcal{N}$, thought of as the collection of possible "locations" of a particle. A moving particle also has a velocity, and together the position and velocity are a point in the tangent bundle $\mathrm{T}\mathcal{N}$, the *phase space* for the particle. The classically-allowed motion of the particle is controlled by a *Lagrangian* $L : \mathbb{R} \times \mathrm{T}\mathcal{N} \to \mathbb{R}$. The first variable allows for the possibility that the forces acting on the particle vary with "time." In Sections 1.3–1.4 we will work quite generally, but for our main theorems in Sections 1.6–1.9 we will be primarily interested in the case when L is inhomogeneous quadratic in the fiber directions. Let $q^1, \dots, q^{\dim \mathcal{N}}$ denote local coordinates

on $\mathcal{N}$, and $v^1, \dots, v^{\dim \mathcal{N}}$ the corresponding fiber coordinates on $\mathrm{T}\mathcal{N}$. Then L is of the form:

$$L(t, v, q) = \left(\sum_{i,j=1}^{\dim \mathcal{N}} \frac{1}{2} a_{ij}(t, q)\, v^i v^j \right) + \left(\sum_{i=1}^{\dim \mathcal{N}} b_i(t, q)\, v^i \right) + c(t, q)$$

We will suppose furthermore that for each $(t, q) \in \mathbb{R} \times \mathcal{N}$, the matrix (a_{ij}) is positive-definite; i.e. $a(t, -)$ is a Riemannian metric on $\mathcal{N}$. Then a can be interpreted as describing the "mass" of the particle. Moreover, $b(t, -)$ must transform as a one-form on $\mathcal{N}$, and describes a "magnetic potential," while $c(t, -)$ is a function that describes an "electric potential." So we will think of such Lagrangians as describing "nonrelativistic motion of an electron through curved space and through an external electromagnetic field."

A *path* is a function $\varphi : [t_0, t_1] \to \mathcal{N}$; physicists tend towards imprecision in the allowed regularity, and it will be convenient for us to use paths that are continuous and piecewise-smooth, with singularities like the absolute value function. Then φ lifts canonically to $(\dot{\varphi}, \varphi) : [t_0, t_1] \to \mathrm{T}\mathcal{N}$, which has singularities like Heaviside's step function (see Definition 1.3.1). Given a path φ and a Lagrangian L, the *action* is

$$\mathcal{S}(\varphi) = \int_{\tau=t_0}^{t_1} L\big(\tau, \dot{\varphi}(\tau), \varphi(\tau)\big)\, \mathrm{d}\tau.$$

It should be understood as a smooth function on the infinite-dimensional manifold of paths.

The *boundary conditions* of a path $\varphi : [t_0, t_1] \to \mathcal{N}$ are the data $\big(t_0, q_0 = \varphi(t_0), t_1, q_1 = \varphi(t_1)\big)$. A path γ is *classically allowed*, which we will abbreviate to *classical*, if it is a critical point of $\mathcal{S}$ among all paths with the same boundary conditions. Under mild conditions on L, this amounts to requiring that γ satisfy a nondegenerate second-order ordinary differential equation. Thus classical paths are smooth if L is.

To sum up, classical Lagrangian mechanics assigns to each boundary condition $(t_0, q_0, t_1, q_1) \in \mathbb{R} \times \mathcal{N} \times \mathbb{R} \times \mathcal{N}$ the *set* of classical paths γ with said boundary conditions, and each such path comes equipped with its action.

Feynman's proposal for *quantum* Lagrangian mechanics is the following. Choose a nonzero real number $\hbar$, called *Planck's constant*. (For most of this chapter, $\hbar$ will denote a formal parameter.) In addition to the choice of a configuration space $\mathcal{N}$ and a Lagrangian L, choose on $\mathcal{N}$ a *volume form* dVol, which like everything else may depend on the external "time" parameter. (When L is quadratic, there is a distinguished choice for the volume form, namely $\mathrm{dVol} = \sqrt{\det a}$ = the Riemannian volume form, and we will use this choice in Sections 1.6–1.9.) Then *Feynman's path integral* assigns to the boundary condition (t_0, q_0, t_1, q_1) the *complex number* $U(t_0, q_0, t_1, q_1)$ given by the following infinite-dimensional integral:

$$U(t_0, q_0, t_1, q_1) = \int_{\substack{\text{paths } \varphi:[t_0,t_1]\to\mathcal{N} \text{ with} \\ \varphi(t_0)=q_0 \text{ and } \varphi(t_1)=q_1}} \exp\left(\frac{\sqrt{-1}}{\hbar}\mathcal{S}(\varphi)\right) \prod_{\tau \in (t_0,t_1)} \mathrm{dVol}\big(\tau, \varphi(\tau)\big)$$

Making analytic sense of this integral can be accomplished in certain cases using Wiener measures [AHK77, KD84, JL00], but in general the integral is ill-defined.

Feynman asserts that if such an integral can be defined, then U will be a fundamental solution to the Schrödinger equation corresponding to the Lagrangian L: i.e. U will be the kernel of the "time evolution" operator, a unitary operator on the Hilbert space $\mathrm{L}^2(\mathcal{N}, \mathrm{dVol})$. His justifications in [Fey48, FH65] hold only to a "physical" level of rigor, and break down when the Lagrangian is not inhomogeneous quadratic along fibers with flat quadratic part, and even in this special case the arguments break down when the Lagrangian grows too quickly in the position coordinates. The problem is well-illustrated by a particle moving in one dimension under a quartic potential: for any $t > 0$ there are infinitely many classical trajectories of duration t connecting a chosen pair of points, most of which involve the particle flying very far away very quickly. These "high energy" classical solutions invalidate Feynman's estimates.

In cases where the path integral can be defined, it goes a long way towards explaining both the basic "probabilistic" nature of quantum mechanics — all paths occur, but their contributions to the physical world are weighted — and the emergence of "classical" phenomena in certain limits, as oscillating integrals are concentrated near the critical points of the phase function.

1.1.2 How the formal path integral works The basic idea of the *formal path integral* is to make $\hbar$ into a formal variable at the very beginning. This is in some ways a disappointing step: in general, the order in which limits are taken matters, and to truly describe our semiclassical real world, we should take the limit as $\hbar \to 0$ at the very end. But it allows us to define the path integral by simply declaring that infinite-dimensional oscillating integrals enjoy the same asymptotic description as do finite-dimensional ones.

Let s be a smooth real-valued function on a finite-dimensional manifold $\mathcal{M}$ (equipped with a volume form dVol), and suppose that s has a unique critical point, which is nondegenerate in the Morse-theoretic sense. Then the asymptotic expansion as $\hbar \to 0$ of the oscillating integral $\int_{\mathcal{M}} \exp\left(\frac{\sqrt{-1}}{\hbar}\, s\right)$ dVol is well-understood (see e.g. [Zwo12, Pol05, Res10a]). In particular, the coefficients depend only on the ∞-jet of s at the critical point. After using [Mos65] to choose a coordinate systems that trivializes dVol, the coefficients are straightforward to compute: each coefficient is described succinctly by a finite sum of "Feynman diagrams," which were generalized by Penrose [Pen71] to describe tensor contractions in arbitrary vector spaces. (A famous aesthetic split developed in the theoretical physics community over the interpretation of Feynman diagrams [Kai05]. Feynman thought of his diagrams as pictures of fundamental interactions of basic particles. On the other hand, Dyson [Dys48, Dys49], who deserves most of the credit for codifying and popularizing the use of Feynman diagrams in quantum electrodynamics, believed that the diagrams were devoid of physical meaning, representing only a useful nemonic for the complicated integrals in Schwinger's field theory. We are firmly in Dyson's camp: the diagrams provide a powerful notation, which we use throughout this chapter, but do not represent particle interactions.) We review this material in Section 1.2.

Based on this, by definition the *formal path integral* does the following:

1. As input, it takes a configuration space $\mathcal{N}$, a Lagrangian L (satisfying some mild nondegeneracy conditions), and a volume form dVol on $\mathcal{N}$.

2. Using [Mos65], it covers $\mathcal{N}$ with local (time-varying) coordinates that trivialize dVol.

3. It chooses a classical trajectory $\gamma : [t_0, t_1] \to \mathcal{N}$, which is nondegenerate in the Morse-theoretic sense and "short" enough to be contained within a coordinate patch. Nondegeneracy of γ is equivalent (Definition 1.3.11 and Proposition 1.3.14) to γ being a member of a family of classical paths depending smoothly on their boundary conditions, so everything constructed out of γ is actually a function defined on an open neighborhood of $(t_0, \gamma(t_0), t_1, \gamma(t_1)) \in \mathbb{R} \times \mathcal{N} \times \mathbb{R} \times \mathcal{N}$.

4. It cuts and pastes the Feynman-diagrammatic description of the asymptotics of finite-dimensional integrals to the infinite-dimensional setting.

5. As output, the formal path integral produces a function U_γ of the boundary conditions valued in formal expressions of the form:

$$U_\gamma = \pm\big(\hbar\sqrt{-1}\big)^{-\dim\mathcal{N}/2} \exp\left(\sum_{n=-1}^{\infty} r_n \big(\hbar\sqrt{-1}\big)^n\right)$$

   The r_ns are real coefficients, each given by a finite sum of finite-dimensional integrals, with one major proviso that we will describe in Problem 1.1.3.

The full details are in Definition 1.3.22.

In a sense, writing out the full details is the first main contribution of this Chapter. It should not be — it ought to be given as an exercise in an advanced quantum mechanics textbook. The closest we can find is Exercise 10.11 from [HKK+03], in which only Lagrangians on $\mathbb{R}^d$ of the form "kinetic energy minus potential energy" are considered: "Formulate Feynman diagram perturbation theory for quantum mechanics by following steps similar to those for the zero-dimensional QFT." The authors do not suggest any of our main results, and we almost certainly they did not try to prove these theorems themselves.

1.1.3 Problem: Ultraviolet divergences The major proviso in Step 5. is that the integrals contributing to each r_n are of products of distributions. Because there is only one "time" parameter in quantum mechanics, these distributions are never worse than Dirac's delta function δ (see Definition 1.3.1), but they can be present (see Example 1.7.2), and in fact each integral evaluates to a polynomial in the "infinite" number $\delta(0)$. These infinities correspond to the *ultraviolet divergences* in quantum field theory, but usually are not considered in the physics literature on quantum mechanics, where ultraviolet divergence are normally thought of as a feature of higher-dimensional quantum field theories. If, upon summing these polynomials in $\delta(0)$ to compute the coefficient r_n, only the constant term survives, then we will say that the path integral is *free of ultraviolet divergences*. One of the

main results of this chapter is Theorem 1.7.4, which says that in the case of nonrelativistic motion of an electron — i.e. when the Lagrangian is quadratic in velocity — and when the volume form chosen is the Riemannian one, then the formal path integral is divergence-free.

An important case in which divergences have been considered is when the corresponding classical mechanics involves singular potentials. Certain examples have been studied thoroughly within the framework of renormalization [MT94, DK82].

1.1.4 Problem: Coordinate independence for "short" paths So far, so good, but what about the choice of coordinates in Step 2.? In Theorem 1.5.1 we prove that this choice is irrelevant if the classical path γ is sufficiently short.

Under a somewhat different setup, the problem of whether the path integral is invariant under changes of coordinates has also been addressed by Kleinert and Chervyakov [KC99, KC00a, KC00b, KC01, KC02, KC03]. Under changes of coordinates, the dependence of the path integral shows up in the form of divergent quantities: Theorem 1.5.1 says that these divergences cancel if the coordinate-change preserves the volume form, whereas Example 1.7.2 shows that they need not cancel for general coordinate changes. Rather than considering only volume-compatible coordinate changes, Kleinert and Chervyakov follow ideas from higher-dimensional field theories and adopt a "dimensional renormalization" scheme from the beginning to handle these divergences. They consider a more restricted case of examples than we do: they consider only situations of the form "kinetic energy minus potential energy." Moreover, their methods are equivalent to forcing the potential energy to be infinitesimal — indeed, any approach that begins by setting $\hbar = 1$ must then find some other parameter to use in the perturbation series, and the standard choice is the strength of the potential energy function.

1.1.5 Problem: from "short" paths to "long" paths That U_γ is well-defined for "long" γ follows from a cut-and-glue argument. Theorem 1.6.1 says that U_γ satisfies the following composition law. Let $\gamma : [t_0, t_1] \to \mathcal{N}$ be a nondegenerate classical path, and choose $t \in (t_0, t_1)$ so that the restrictions $\gamma_0 = \gamma|_{[t_0,t]}$ and $\gamma_1 = \gamma|_{[t,t_1]}$ are nondegenerate (they are automatically classical). Define a *formal* finite-dimensional integral by declaring the answer to be the Feynman-diagrammatic asymptotics of honest finite-dimensional integrals (see Definition 1.2.14). Then:

$$U_\gamma(t_0, q_0, t_1, q_1) = \int_{\approx\gamma(t)}^{\text{formal}} U_{\gamma_0}(t_0, q_0, t, q)\, U_{\gamma_1}(t, q, t_1, q_1)\ \mathrm{dVol}(q)$$

By a standard argument (e.g. [Mil63]), there are only finitely many $t \in (t_0, t_1)$ for which the restrictions γ_0, γ_1 fail to be nondegenerate, provided the Lagrangian L is convex along the fibers of $\mathbb{R} \times \mathrm{T}\mathcal{N} \to \mathbb{R} \times \mathcal{N}$. Thus, to define the path integral for a general nondegenerate classical path, we can cut it into small pieces, each contained within some coordinate patch, compute each path integral, and integrate the answer. For Theorem 1.6.1, we must assume that the path integral is free of ultraviolet divergences, or at least that the integral represented

by the Feynman diagram  converges; otherwise, our *ad hoc* choice for the determinant term in Definition 1.3.22 should be corrected to a divergent one.

1.1.6 Schrödinger's equation and initial value problem With everything well-defined, at least in the most important case of a nonrelativistic electron, we turn our attention to checking a formal-power-series version of Schrödinger's equation, which asserts that a *wave function* $\psi \in \mathrm{L}^2(\mathcal{N})$ evolves via $\frac{\partial}{\partial t}\psi = \frac{1}{\hbar\sqrt{-1}}\hat{H}\psi$. The *Schrödinger operator* $\hat{H}$ is a version of the Laplace–Beltrami operator, corrected to include the magnetic and potential terms, and, importantly, with a "curvature" correction if the metric is not flat (see Definition 1.8.1). Evolution from time t_0 to t_1 via Schrödinger's equation is controlled by an operator $U(t_0, t_1)$ on $\mathrm{L}^2(\mathcal{N})$, whose integral kernel is the distribution $U(t_0, q_0, t_1, q_1)$ determined by the following initial value problem:

$$\hbar\sqrt{-1}\,\frac{\partial}{\partial t_1}U(t_0, q_0, t_1, q_1) = \hat{H}_{q_1}\big[U(t_0, q_0, t_1, q_1)\big] \qquad U(t_0, q_0, t_0, q_1) = \delta(q_1 - q_0)$$

The operator $\hat{H}_{q_1}$ acts on the q_1 variable, leaving the other variables fixed.

Consider the following (generically infinite) sum of formal path integrals:

$$U(t_0, q_0, t_1, q_1) = \sum_{\substack{\gamma:[t_0,t_1]\to\mathcal{N} \text{ classical and nonfocal} \\ \gamma(t_0)=q_0,\,\gamma(t_1)=q_1}} U_\gamma(t_0, q_0, t_1, q_1)$$

If U_γ represents an integral over all paths that are "near" γ, then U represents an integral over *all* paths. We claim that this U satisfies the above initial value problem.

We prove this claim in two steps. First, in Theorem 1.8.2, we show that each U_γ satisfies Schrödinger's equation. By construction, $U_\gamma = \exp\big(\frac{\sqrt{-1}}{\hbar}J_\gamma\big)\,O(1)$, where J_γ is the Hamilton principal function for γ (Definition 1.3.12), and $O(1)$ is a power series in $\hbar\sqrt{-1}$. So J_γ satisfies the Hamilton–Jacobi equation, which is the $\hbar \to 0$ limit of Schrödinger's equation. Remaining to check is a transport equation. Second, in Theorem 1.9.4 we verify that U has the correct limit as $t_1 \to t_0$, where the limit is taken in the topology of pointwise convergence of distributions (Definition 1.2.1). Together, these results show that the Feynman-diagrammatic path integral does compute the asymptotics of the above initial value problem, as it is known that these asymptotics are given by the Hamilton function and the transport equation [Tak08].

Comparisons between the path-integral and Schrödinger equations have been made, primarily using the Wiener-measure definition of the integral. For the more algebraic, Feynman-diagrammatic definition we will use, the comparison is straightforward in the *harmonic oscillator* case when $\mathcal{N} = \mathbb{R}^d$, the metric a is constant, the one-form b is linear in position, and the function c is quadratic; then the path integral is purely Gaussian, and agrees with Schrödinger's picture [Tak08]. Feynman's original arguments apply best to formal perturbations of free motion on $\mathbb{R}^n$; formal perturbations of the harmonic oscillator are not much harder. Duru and Kleinert [DK82] apply the path integral approach to calculate the energy

spectrum of a particle moving in the Coulomb potential, and compare their approach to the Schrödinger methods. Since they set $\hbar = 1$, their perturbation parameter is the "charge" of the potential, so their case is an infinitesimal perturbation of the flat-space case. Note that we disallow singular potentials but consider arbitrary smooth ones, so our results do not apply to the Coulomb potential; the singularity does make the case considered by Duru and Kleinert more interesting than a purely-smooth perturbation of free motion.

1.1.7 Some questions we will not resolve We leave open the following three questions, each of which deserves its own paper:

1. When there are ultraviolet divergences in the formal path integral, what do they measure?

2. When the Lagrangian is not inhomogeneous quadratic, is there a choice of measure in which the formal path integral converges? Does this require a "measure" on path-space that is more general than "$\mathrm{d}\varphi = \prod_{t_0<\tau<t_1} \mathrm{dVol}\big(\tau, \varphi(\tau)\big)$," say by depending explicitly on the velocities of paths?

3. When the Lagrangian is not inhomogeneous quadratic, what version of Schrödinger's equation does the diagrammatic formal path integral satisfy? What does this say about "canonical" quantization?

1.1.8 Definition (repeated index convention and the Kronecker δ) For the remainder of this chapter, we will use *Einstein's repeated index convention*: given vectors $(a^i) = (a^1, \dots, a^d)$ and $(b_i) = (b_1, \dots, b_d)$, the expression $a^i b_i$ means $\sum_{i=1}^d a^i b_i$. According to [Pai82], Einstein once called the suppression of the summation sign his greatest discovery in mathematics.

Kronecker's delta matrix is:

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

1.2 Review: Oscillating integrals and Feynman diagrams

This section is a review of standard material. We derive the $\hbar \to 0$ asymptotics of oscillating integrals of the form $\int \exp\big(\frac{\sqrt{-1}}{\hbar} s\big)\, \mathrm{dVol}$, and recall their description in terms of Feynman diagrams.

1.2.1 Definition (volume forms, distributions, and big-O notation) Let $\mathcal{M}$ be a finite-dimensional (real) smooth manifold. The line bundle $\mathscr{D}_{\mathcal{M}} = |(\mathrm{T}^*)^{\wedge \mathrm{top}}\mathcal{M}|$ of *densities* is given in local coordinates by the transition amplitudes $\mu(\tilde{x}) = \mu(x)\big|\det \frac{\partial \tilde{x}}{\partial x}\big|$ for any change of

local coordinates $\tilde{x}(x)$. The line bundle $\mathscr{D}_{\mathcal{M}}$ is trivializable but not canonically trivialized. By construction there is a canonical *integration* map $\int : G_{\text{cpt}}(\mathscr{D}_{\mathcal{M}}) \to \mathbb{R}$ of compactly-supported smooth sections of $\mathscr{D}_{\mathcal{M}}$. We choose a nowhere-vanishing *volume form* $\text{dVol} \in G(\mathscr{D}_{\mathcal{M}})$.

Write $\mathscr{C}^\infty(\mathcal{M})$ for the algebra of smooth functions on $\mathcal{M}$, and $\mathscr{C}^\infty_{\text{cpt}}(\mathcal{M})$ for the subspace of compactly-supported smooth functions. A *real-valued distribution* on $\mathcal{M}$ is a continuous linear functions $\mathscr{C}^\infty_{\text{cpt}}(\mathcal{M}) \to \mathbb{R}$. For example, $f \mapsto \int f \,\text{dVol}$ is a distribution. Distributions valued in other topological vector spaces are defined similarly. A sequence $\mu_1, \mu_2, \ldots$ of distributions on $\mathcal{M}$ *converges pointwise* to a distribution μ_∞ if for every $f \in \mathscr{C}^\infty_{\text{cpt}}(\mathcal{M})$ we have $\lim_{i\to\infty} \mu_i(f) = \mu_\infty(f)$; in this case we write $\lim_{i\to\infty} \mu_i = \mu_\infty$.

We will work with a parameter $\hbar$, which at first will range over positive real numbers and later will denote a formal variable. If X, Y, Z are expressions in $\hbar$, we write $X = Y + O(Z)$ to mean "$(X-Y)/Z$ is bounded as $\hbar \to 0$." We write $O(\hbar^\infty)$ to mean "$O(\hbar^n)$ for all n."

1.2.2 Fact (basic theorem of oscillating integrals) Fix $s \in \mathscr{C}^\infty(\mathcal{M})$, and consider the family of complex-valued volume forms $\exp\left(\frac{\sqrt{-1}}{\hbar}s\right)\text{dVol}$. The basic fact is that as $\hbar \to 0$, the distributions $\exp\left(\frac{\sqrt{-1}}{\hbar}s\right)\text{dVol}$ are asymptotic (in the topology of pointwise convergence) to a distribution concentrated near the *critical locus* $\{\mathrm{d}s = 0\} = \{m \in \mathcal{M} \text{ s.t. } \mathrm{d}s_m = 0 \in \mathrm{T}^*_m\mathcal{M}\} \subseteq \mathcal{M}$. More precisely, let $\psi \in \mathscr{C}^\infty(\mathcal{M})$ be any function which takes constant value 1 on a neighborhood of $\{\mathrm{d}s = 0\}$. Then for any $g \in \mathscr{C}^\infty_{\text{cpt}}(\mathcal{M})$,

$$\int_{\mathcal{M}} g \exp\left(\frac{\sqrt{-1}}{\hbar}s\right) \text{dVol} = \int_{\mathcal{M}} g\,\psi \exp\left(\frac{\sqrt{-1}}{\hbar}s\right) \text{dVol} + O(\hbar^\infty).$$

For a proof, see [Zwo12].

1.2.3 Remark (conditional convergence and switching limits) If $\mathcal{M}$ is not compact, then $1 \notin \mathscr{C}^\infty_{\text{cpt}}(\mathcal{M})$. Nevertheless, in this chapter, we are primarily interested in the number $\int_{\mathcal{M}} 1 \exp\left(\frac{\sqrt{-1}}{\hbar}s\right)\text{dVol}$, and in particular in its $\hbar \to 0$ asymptotics. This integral does not converge absolutely, but under mild assumptions on the growth rate of s it does converge conditionally: 1 is a limit of compactly-supported functions (in many ways). Our strategy will be to study the asymptotics of $\exp\left(\frac{\sqrt{-1}}{\hbar}s\right)\text{dVol}$ in the pointwise-convergence topology on distributions. This is particularly justified when the critical locus $\{\mathrm{d}s = 0\}$ is compact, by Fact 1.2.2. Note that there is a minor cheat involved: we take the $\hbar \to 0$ limit first, and only afterward approximate 1 by compactly-supported functions.

1.2.4 Definition (jets, nondegenerate critical points, determinant, Morse index) If $\mathcal{M}$ has no more structure than that of a smooth manifold, then for $n \geq 2$ the nth derivative $s^{(n)}$ is not a well-defined coordinate-independent object. Rather, under changes of coordinates $s^{(n)}$ transforms in an affine way, with a shift depending linearly on $s^{(1)}, \ldots, s^{(n-1)}$. (Thus, the *nth jet of* s, consisting of the data $\{s^{(0)}, \ldots, s^{(n)}\}$, does exist without coordinates.) In particular, the first derivative $s^{(1)} = \mathrm{d}s$ is always defined. If $c \in \{\mathrm{d}s = 0\}$, then the *Hessian*

$s^{(2)}(c)$ exists as a symmetric element of $(\mathrm{T}^*_c\mathcal{M})^{\otimes 2}$. The critical point c is *nondegenerate* if the Hessian, though of as a map $s^{(2)}(c) : \mathrm{T}_c\mathcal{M} \to \mathrm{T}^*_c\mathcal{M}$, has trivial kernel. The function s is *Morse* if $\{\mathrm{d}s = 0\}$ consists of finitely many nondegenerate critical points.

The volume form dVol determines a *determinant* $\det_{\mathrm{dVol}}\big(s^{(2)}(c)\big) = \big(s^{(2)}(c)\big)^{\wedge \dim \mathcal{M}} \cdot \big((\mathrm{dVol})^{-1}\big)^{\otimes 2} \in \mathbb{R}$, which is nonzero iff c is nondegenerate. The *Morse index* $\eta(c)$ is the dimension of any maximal subspace of $\mathrm{T}_c\mathcal{M}$ on which the inner product $s^{(2)}(c) : (\mathrm{T}_c\mathcal{M})^{\otimes 2} \to \mathbb{R}$ is negative-definite. For details and well-definedness, see Definition 1.3.16.

1.2.5 Fact (Gaussian approximation) Suppose that $c \in \{\mathrm{d}s = 0\}$ is nondegenerate, and choose a cut-off function $\psi \in \mathscr{C}^\infty_{\mathrm{cpt}}(\mathcal{M})$ to take constant-value 1 on a neighborhood of c and 0 on a neighborhood of $\{\mathrm{d}s = 0\} \smallsetminus \{c\}$. Then:

$$\int \psi\, e^{\frac{\sqrt{-1}}{\hbar} s}\, \mathrm{dVol} = \big(2\pi\hbar\sqrt{-1}\big)^{\dim \mathcal{M}/2}\, e^{\frac{\sqrt{-1}}{\hbar} s^{(0)}}\, \big(\sqrt{-1}\big)^{-\eta}\, \big|\det s^{(2)}\big|^{-1/2}\, \big(1 + O(\hbar)\big)$$

We have suppressed the point c from the notation, writing $s^{(0)} = s(c)$ and $s^{(2)} = s^{(2)}(c)$. Note that by definition $s^{(1)}(c) = \mathrm{d}s(c) = 0$.

To venture further into the asymptotics requires choosing coordinates: higher asymptotics depend on higher jets of s, but we lack good notation for manipulating jets, and the individual derivatives $s^{(n)}$ are not tensors (see Definition 1.2.4). We henceforth fix a nondegenerate critical point $c \in \{\mathrm{d}s = 0\}$, and will generally drop the cut-off function ψ from the notation.

1.2.6 Fact (existence of coordinates trivializing a volume form) By a theorem of Moser's [Mos65], there exist local coordinates $x^1, \dots, x^{\dim \mathcal{M}}$ near c such that $\mathrm{dVol} = \mathrm{d}x^1\mathrm{d}x^2 \dots \mathrm{d}x^{\dim \mathcal{M}}$. Such a coordinate system is called *compatible* with dVol. We choose dVol-compatible coordinates and impose $x^i(c) = 0$.

1.2.7 Remark (expansion of $\int \exp\big(\frac{\sqrt{-1}}{\hbar} s\big)\, \mathrm{dVol}$ near c) We now outline the usual derivation of the higher $\hbar \to 0$ asymptotics of $\int \psi \exp\big(\frac{\sqrt{-1}}{\hbar} s\big)\, \mathrm{dVol}$, where ψ, as in Fact 1.2.5, is a smooth cut-off concentrating the integral near a particular nondegenerate critical point c. Our derivation will be "at a physicist's level of rigor": we will be sloppy with notation, leaving off ψ, c, and big-O corrections. We derive similar formulas in Chapter 2.

With a choice of coordinates, the Taylor coefficients $s^{(n)}$ become symmetric n-linear tensors:

$$s(x) = \sum_{n=0}^{\infty} \frac{1}{n!} s^{(n)}_{i_1 \cdots i_n} x^{i_1} \cdots x^{i_n} = \sum_{n=0}^{\infty} \frac{1}{n!} s^{(n)} \cdot x^{\otimes n}.$$

We pull $\exp\big(\frac{\sqrt{-1}}{\hbar} s^{(0)}\big)$ out of the integral, separate the remaining exponent into a quadratic part and a cubic-and-higher part, and expand the latter in Taylor series:

$$\int e^{\frac{\sqrt{-1}}{\hbar} s}\, \mathrm{dVol} = e^{\frac{\sqrt{-1}}{\hbar} s^{(0)}} \int \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\sqrt{-1}}{\hbar} \sum_{n=3}^{\infty} \frac{1}{n!} s^{(n)}_{i_1 \cdots i_n} x^{i_1} \cdots x^{i_n} \right)^m e^{\frac{\sqrt{-1}}{\hbar} \frac{1}{2} s^{(2)}_{ij} x^i x^j}\, \mathrm{dVol}$$

The freshman-calculus fact that $\int_{-\infty}^{\infty} x^{2k} \exp(-x^2/2)\,\mathrm{d}x = \sqrt{2\pi}\,\frac{(2k)!}{2^k k!}$ has a multi-dimensional analog, called Wick's Theorem by the physicists and due to Isserlis in 1918 [Iss18]. Let a_{ij} be a symmetric bilinear form on $\mathbb{R}^{\dim \mathcal{M}}$ with positive-definite real part, and $b_{i_1 \cdots i_n}$ any symmetric n-linear tensor. Then:

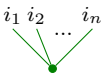
$$\int_{\mathbb{R}^{\mathcal{N}}} b_{i_1 \cdots i_n} x^{i_1} \cdots x^{i_n} \exp\left(-\frac{1}{2} a_{ij} x^i x^j\right) \mathrm{dVol} =$$

$$= \begin{cases} 0, & n \text{ odd} \\ \sqrt{\det(2\pi a^{-1})}\,\dfrac{(2k)!}{2^k k!}\, b_{i_1 \cdots i_n} (a^{-1})^{i_1 i_2} \cdots (a^{-1})^{i_{n-1} i_n}, & n = 2k \end{cases}$$

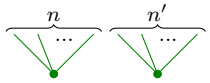
When b is not symmetric, it must be symmetrized before contracting with the (a^{-1})s. (The contraction with the xs automatically symmetrizes the left-hand side.) This involves an averaging of precisely $\frac{(2k)!}{2^k k!}$ terms: we must pick two indices from $\{i_1, \ldots, i_n\}$ to contract with the first a^{-1}, then pick two more to contract with the second, and so on. Thus the right-hand side in the non-symmetric case is actually a *sum over pairings* of the n inputs into the tensor b.

If b is partially symmetric, some of these pairings are equivalent. There is a general philosophy that holds that "divided power" polynomials like $\frac{1}{n!}x^n$ are more natural than non-divided power polynomials — the denominator "counts" the number of permutations of the xs. Dividing both sides of Isserlis's result by $(2k)!$ gives an example of this philosophy; the theory of Taylor expansions is another. For "partially symmetric" tensors, we should divide by a count of their symmetries.

Pairings can naturally be represented visually as collections of arcs  . The natural representation of a term like $\frac{1}{n!} s^{(n)} \cdot x^{\otimes n}$ is a vertex with n incoming edges: then the $n!$ counts its total number of symmetries:

$$s^{(n)}_{i_1 \cdots i_n} \leftrightarrow$$ 

If $n \neq n'$, then the product $\left(\frac{1}{n!} s^{(n)} \cdot x^{\otimes n}\right)\left(\frac{1}{n'!} s^{(n')} \cdot x^{\otimes n'}\right)$ is naturally represented as a pair of vertices, one with valence n and the other with valence n'. If $n = n'$, such a diagram has an extra factor of 2 in its symmetry group, and naturally represents $\frac{1}{2}\left(\frac{1}{n!} s^{(n)} \cdot x^{\otimes n}\right)^2$.

$$s^{(n)} s^{(n')} \leftrightarrow$$ 

Note that the numbers of symmetries of such diagrams are precisely the corresponding denominators appearing in $\exp\left(\sum \frac{1}{n!} s^{(n)} \cdot x^{\otimes n}\right)$.

There are some signs and factors of $\frac{\sqrt{-1}}{\hbar}$ to discuss. Since the inner product a_{ij} appears most naturally with a factor of -1, we switch from $\frac{\sqrt{-1}}{\hbar}$ to $-(\hbar\sqrt{-1})^{-1}$. Then $-s^{(n)}$ is more natural than $s^{(n)}$. As for the factors of $\hbar\sqrt{-1}$, one appears with each "edge" $(s^{(2)})^{-1}$, and each "vertex" $-s^{(n)}$ appears with a factor of $(\hbar\sqrt{-1})^{-1}$.

After stating more precisely the rules for diagrams, we state the conclusion of the above derivation in Fact 1.2.12.

1.2.8 Definition (Feynman diagrams) A Feynman diagram is a combinatorial graph Γ (it may be empty, disconnected, etc.). More precisely, a *partition* of a finite set H is a finite collection B of pairwise-disjoint subsets (*blocks*) of H such that $\bigcup_{b\in B} b = H$, and a *Feynman diagram* is a finite collection H of "half-edges" along with a finite partition V of H into blocks (the *vertices*) and a finite partition E of H into blocks of size 2 (the *edges*). The *valence* of a vertex is its number of half-edges. An *isomorphism* of Feynman diagrams $\phi : (H, E, V) \to (H', E', V')$ is a bijection $H \to H'$ that induces bijections $E \to E'$ and $V \to V'$. For a Feynman diagram $\Gamma = (H, E, V)$, we will write $|\mathrm{Aut}\,\Gamma|$ for the number of isomorphisms $\Gamma \to \Gamma$. The *Euler characteristic* of a Feynman diagram is $\chi(H, E, V) = |V| - |E|$.

In later sections, we will use more elaborate Feynman diagrams, with multiple types of vertices and edges. We will trust the reader to write down the necessary combinatorial definitions. For now, we will draw all vertices on the bottom of diagrams:

1.2.9 Remark (half-edges over adjacency matrices) A more standard definition of a combinatorial graph is a finite set V of "vertices" and an "adjacency matrix" $V \times V \to Nn\mathcal{N}$. We prefer Definition 1.2.8 because we want its corresponding notion of "number of automorphisms." In particular, if Γ is the diagram with one (bivalent) vertex and one edge connecting this vertex to itself, then we want $|\mathrm{Aut}\,\Gamma| = 2$.

1.2.10 Definition (evaluation of diagrams and Feynman rules) A Feynman diagram is a picture of a contraction of tensors. We *evaluate* Feynman diagrams as follows.

The vertices and edges are evaluated via the following *Feynman rules*, which depend on a function $s \in \mathscr{C}^\infty(\mathcal{M})$ and a nondegenerate critical point $c \in \{\mathrm{d}s = 0\}$:

$$\mathrm{ev}\left(\begin{array}{c} i_1\ i_2\ \cdots\ i_n \\ \diagdown\ \diagup\ \diagdown \\ \bullet \end{array}\right) = -s^{(n)}_{i_1\cdots i_n} \qquad \mathrm{ev}\left(\begin{array}{c} \frown \\ i\quad j \end{array}\right) = \left(\left(s^{(2)}\right)^{-1}\right)^{ij}$$

Let Γ be an (unlabeled) Feynman diagram. Consider all ways to attach a label $i = 1, \ldots, \dim \mathcal{M}$ to each half edge of Γ. For each, multiply together the assigned values of the vertices and edges. Then sum over all labelings. This defines the value of Γ:

$$\mathrm{ev}(\Gamma) = \sum_{\vec{\imath}\colon \{\text{half-edges of } \Gamma\} \to \{1,\ldots,\dim \mathcal{M}\}} \left(\prod_{\text{vertices and edges}} \mathrm{ev}(\text{vertex or edge}) \right).$$

1.2.11 Remark (Penrose's graphical language) Our diagrams have their natural home among the language introduced in Penrose's thesis [Pen71] (and made precise in [JS91], and implicit in Feynman's and Dyson's work on path integrals [Fey48, Fey49b, Fey49a, Dys48, Dys49]). These more general diagrams need not have every half edge paired to another half

edge; rather, some can connect to the top of the page, and others to the bottom. Such an "open" diagram with n half-edges heading to the top and n' half-edges heading to the bottom evaluates to a linear map $(\mathrm{T}_c\mathcal{M})^{\otimes n} \to (\mathrm{T}_c\mathcal{M})^{\otimes n'}$. For example, the second Feynman rule from Definition 1.2.10 implies:

$$\mathrm{ev}\Big($$  $$\Big) = -\,\mathrm{ev}\Big($$  $$\Big)$$

1.2.12 Fact (asymptotics of oscillating integrals) Choose a finite-dimensional smooth manifold $\mathcal{M}$ with volume form dVol, a smooth function $s \in \mathscr{C}^\infty(\mathcal{M})$, and a nondegenerate critical point $c \in \{\mathrm{d}s = 0\}$. As in Fact 1.2.5, choose a smooth cut-off function ψ that takes constant value 1 in a neighborhood of c, and takes constant value 0 in a neighborhood of the rest of the critical locus $\{\mathrm{d}s = 0\} \smallsetminus \{c\}$. As in Fact 1.2.6, choose coordinates x^i near c such that $\mathrm{dVol} = \mathrm{d}x^1 \cdots \mathrm{d}x^{\dim \mathcal{M}}$. Letting η denote the Morse index of c (Fact 1.2.5) and adopting Definitions 1.2.8 and 1.2.10, the complete asymptotics of the oscillating integral $\int_{\mathcal{M}} \psi \exp\big(\frac{\sqrt{-1}}{\hbar} s\big)\,\mathrm{dVol}$ are given by:

$$\int \psi\, e^{-(\hbar\sqrt{-1})^{-1}s}\,\mathrm{dVol} = O(\hbar^\infty) + \\ \big(2\pi\hbar\sqrt{-1}\big)^{\dim\mathcal{M}/2}\, e^{-(\hbar\sqrt{-1})^{-1}s^{(0)}}\, \big(\sqrt{-1}\big)^{-\eta}\, \big|\det s^{(2)}\big|^{-1/2} \sum_{\Gamma} \frac{(\hbar\sqrt{-1})^{-\chi(\Gamma)}\,\mathrm{ev}(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

The sum is over those Feynman diagrams all of whose vertices have valence three or more. By convention, the sum does include the empty diagram, which evaluates to $\mathrm{ev}(\emptyset) = 1$.

The diagrammatic formula is directly equivalent to the following description of the asymptotics, which is proved in [Zwo12]:

$$\int \psi\, e^{-(\hbar\sqrt{-1})^{-1}s}\,\mathrm{dVol} = O(\hbar^\infty) + (2\pi\hbar)^{\dim\mathcal{M}/2}\, e^{\mathrm{sign}(s^{(2)})\pi\sqrt{-1}/4}\, \big|\det s^{(2)}\big|^{-1/2} \times \\ \exp\left(-\frac{\hbar\sqrt{-1}}{2} \sum_{j,k=1}^{\dim\mathcal{M}} \big((s^{(2)})^{-1}\big)^{jk}\, \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}\bigg|_{x=0}\right) \exp\left(-(\hbar\sqrt{-1})^{-1}\left(s(x) - s^{(2)}\cdot\frac{x^{\otimes 2}}{2}\right)\right)$$

As before, our coordinates x^i are chosen with $x^i(c) = 0$. By definition, $\mathrm{sign}\big(s^{(2)}\big) = \dim_+ - \dim_-$, where $\dim_\pm$ is the dimension of the largest subspace of $\mathrm{T}_c\mathcal{M}$ on which $\pm s^{(2)}$ is positive-definite.

1.2.13 Remark (number of diagrams) Since every vertex is required to have valence at least three, the Euler characteristic of any nonempty Feynman diagram is negative, and a diagram with Euler characteristic $-n$ has at most $6n$ half-edges. Thus there are finitely many Feynman diagrams with a given Euler characteristic, and the sum is well-defined as a formal power series in $\hbar\sqrt{-1}$. However, there are factorially many Feynman diagrams for a given Euler characteristic, and so one expects the sum to have zero radius of convergence.

For example, a version of the *Airy function* is $A(\hbar) = \int_{-\infty}^{\infty} \exp\left(\frac{\sqrt{-1}}{\hbar}(x^2/2 - x^3/3)\right) \mathrm{d}x$. Its asymptotics as $\hbar \to 0$ are $A(\hbar) = \left(2\pi\hbar\sqrt{-1}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(6n)!}{2^{3n}\,(3n)!\,6^{2n}\,(2n)!} \left(\hbar\sqrt{-1}\right)^n + O(\hbar^\infty)$, and $\frac{(6n)!}{2^{3n}\,(3n)!\,6^{2n}\,(2n)!} \sim n!\left(\frac{1}{288}\right)^n$, so the power series has zero radius of convergence.

1.2.14 Definition (formal integral) We now treat $\hbar\sqrt{-1}$ as a formal variable. Let V be a (possibly infinite-dimensional) real vector space and $s \in \mathscr{C}^\infty(V)$ a smooth function, with a nondegenerate critical point at the origin. (We won't need more "smoothness" of s beyond that it has a well-defined Taylor expansion with nth derivative $s^{(n)} : V^{\otimes n} \to \mathbb{R}$.)

Suppose that one can make sense of the following pieces (all straightforward when V is finite-dimensional):

1. $\dim V$;
2. $\eta =$ Morse index of s at the origin;
3. $\det s^{(2)}$;
4. $(s^{(2)})^{-1}$;
5. the tensor contractions involved in evaluating Feynman diagrams.

Then the *formal integral near* 0 is given by the formal expression in $\hbar\sqrt{-1}$ appearing on the right-hand side of Fact 1.2.12:

$$\int_{\approx 0}^{\text{formal}} e^{-(\hbar\sqrt{-1})^{-1}s}\,\mathrm{dVol} =$$
$$\left(2\pi\hbar\sqrt{-1}\right)^{\dim V/2} e^{-(\hbar\sqrt{-1})^{-1}s^{(0)}} \left(\sqrt{-1}\right)^{-\eta} \left|\det s^{(2)}\right|^{-1/2} \sum_{\substack{\text{Feynman diagrams } \Gamma \\ \text{with all vertices of valence} \geq 3}} \frac{(\hbar\sqrt{-1})^{-\chi(\Gamma)}\,\mathrm{ev}(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

Suppose further that $f \in \mathscr{C}^\infty(V)$ is another smooth function. The *formal integral* $\int_{\approx 0}^{\text{formal}} f \exp\left(\frac{\sqrt{-1}}{\hbar}s\right) \mathrm{dVol}$ is given by the same expression, with the following modification. A Feynman diagram is allowed two types of vertices, a basic kind and a marked kind, and we include a Feynman rule evaluating the marked vertices to Taylor coefficients of f:

$$\mathrm{ev}\left(\begin{array}{c}\diagup\,\cdots\,\diagdown\\ \bullet\end{array}\right) = -s^{(n)} : V^{\otimes n} \to \mathbb{R} \qquad \mathrm{ev}\left(\begin{array}{c}\diagup\,\cdots\,\diagdown\\ \star\end{array}\right) = f^{(n)} : V^{\otimes n} \to \mathbb{R}$$

In the sum of diagrams, all basic vertices are to have valence at least three, and each diagram is to have precisely one marked vertex, which is allowed arbitrary valence. The exponent replacing $\chi(\Gamma)$ for a diagram with a marked vertex is the number of *unmarked* vertices minus the number of edges, i.e. one less than the topological Euler characteristic. A calculation similar to the one in Remark 1.2.7 justifies this second definition.

1.2.15 Remark (compatibility in Definition 1.2.14) It's clear that to define $\int_{\approx 0}^{\text{formal}} f\, e^{-(\hbar\sqrt{-1})^{-1}s}\,\text{dVol}$, it suffices to replace f and s by formal power series centered at the origin. Moreover, f and s can be $\mathbb{R}[\![\hbar\sqrt{-1}]\!]$-valued.

Suppose that $f = \exp(g)$ for some $\mathbb{R}[\![\hbar\sqrt{-1}]\!]$-valued formal power series g centered at $0 \in V$. Then Definition 1.2.14 gives two prescriptions for evaluating $\int_{\approx 0}^{\text{formal}} f\, e^{-(\hbar\sqrt{-1})^{-1}s}\,\text{dVol}$: we can include a marked vertex for f, or we can use the function $\tilde{s} = s - (\hbar\sqrt{-1})g$ in $\int_{\approx 0}^{\text{formal}} e^{-(\hbar\sqrt{-1})^{-1}\tilde{s}}\,\text{dVol}$. (If $g^{(1)} \neq 0$, the origin will not be a critical point of $\tilde{s}$. Rather, there will be a critical point located at $O(\hbar)$ distance from the origin, and by manipulating formal power series it is straightforward to write down the Taylor expansion of $\tilde{s}$ at that critical point.) The reader is invited to check that the two definitions in fact give the same answer.

1.2.16 Convention (sum of diagrams) In the remainder of this chapter, we will regularly refer to "sums of diagrams." We adopt the following conventions. First, implicit in the notion of *sum* is the division by the number of automorphisms of the summand — this is made precise in Baez's theory of "groupoid integrals" [BD01]. Second, in sums over diagrams, each diagram Γ is always weighted by $\left(\hbar\sqrt{-1}\right)^{-\chi(\Gamma)}$. Third, we will write equations involving diagrams without including the operator ev:

1.2.17 Remark (more on jets and coordinates) Our goal will be to apply Definition 1.2.14 to the case when $\mathcal{M}$ is an infinite-dimensional manifold, $s \in \mathscr{C}^\infty(\mathcal{M})$, and V is replaced by the tangent space $\mathrm{T}_c\mathcal{M}$ for some nondegenerate critical point c of s. But as we said in Definition 1.2.4, jets do not transform as tensors. Thus, even once the dimension, Morse index, and so on have been defined, it is not clear that Definition 1.2.14 without a choice of "coordinates" identifying a neighborhood of $c \in \mathcal{M}$ with a neighborhood of $0 \in \mathrm{T}_c\mathcal{M}$. We will prove a result in this direction in Theorem 1.5.1; more general results are possible [JF10c].

1.2.18 Remark ($\log \int^{\text{formal}}$) We hinted at the following fact in Remark 1.2.7. Suppose that X is a sum of *connected* diagrams. Then $\exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$ is the sum over all possible disjoint unions of diagrams from X.

Recall that $\bullet = -s^{(0)}$ with $\chi(\bullet) = 1$, and univalent vertices evaluate to 0 since we are at a critical point of s. Provided we can make sense of $\operatorname{tr}\log s^{(2)}$, Definition 1.2.14 is equivalent to:

$$\log \int_{\approx 0}^{\text{formal}} e^{-(\hbar\sqrt{-1})^{-1}s}\,\text{dVol} = \frac{\pi\sqrt{-1}}{2}\eta + \frac{\dim \mathcal{M}}{2}\log\left(2\pi\hbar\sqrt{-1}\right) - \frac{1}{2}\operatorname{tr}\log s^{(2)} + \sum_{\substack{\text{connected Feynman diagrams } \Gamma \\ \text{with no bivalent vertices}}} \frac{\left(\hbar\sqrt{-1}\right)^{-\chi(\Gamma)}\operatorname{ev}(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

Moreover, using $-\log(1+X) = \sum_{n\geq 1} \frac{1}{n}(-X)^n$ and $(1+X)^{-1} = \sum_{n\geq 0}(-X)^n$, there is a way to expand the edges and incorporate the logarithmic term into a sum of diagrams that allow bivalent vertices, but we will leave the details to the reader.

1.3 Lagrangian mechanics and the coordinate-full definition of the formal path integral

In this section we explain how to implement the formal path integral in the case of quantum mechanics on $\mathbb{R}^d$ with arbitrary Lagrangian L. To define the vertices of our Feynman diagrams we need the notion of *functional derivative*, which we recall in Definition 1.3.3. The edges of the Feynman diagrams are Green's functions for a second-order differential equation, which we solve (provided the chosen classical path is nondegenerate) by variation of parameters in Proposition 1.3.14. We then define the Morse index of a classical path; it is necessarily finite if the matrix $\frac{\partial^2 L}{\partial v^i \partial v^j}$ is everywhere positive-definite (Corollary 1.3.21). Finally, we make a few *ad hoc* decisions and summarize everything in Definition 1.3.22.

1.3.1 Definition (Heaviside's step function and Dirac's delta function) *Heaviside's step function* is:

$$\Theta(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} & t = 0 \\ 1 & t > 0 \end{cases}$$

Its derivative is the *Dirac delta distribution*, which satisfies

$$\int_{t=-\infty}^{\infty} f(t)\,\delta(t) = f(0)$$

for all functions f.

In a few places, we will need to multiply distributions. Much care is required when doing so [dBPSvN95]. For example, as distributions Θ and Θ^2 agree, but we will not identify them, as otherwise $\delta = \Theta' = (\Theta^3)' = 3\Theta^2\Theta' = 3\Theta\Theta' = \frac{3}{2}(\Theta^2)' = \frac{3}{2}\Theta' = \frac{3}{2}\delta$, forcing $\delta = 0$. Instead, we will treat Θ as a function and δ as a distribution, and enforce the above equation even when f is not smooth. Note then that Θ and Θ^2 have the same derivative, so we have no choice but to allow functions like $\Theta^2 - \Theta$ that differentiate everywhere to 0 but are not constant.

1.3.2 Definition (path, Lagrangian function, action) We work on the configuration space $\mathbb{R}^d$ with tangent bundle $\mathrm{T}\mathbb{R}^d = \mathbb{R}^{2d}$; the standard coordinates on $\mathrm{T}\mathbb{R}^d$ are (v^i, q^i) for $i = 1, \ldots, d$. A *(piecewise-smooth parameterized) path in* $\mathbb{R}^d$ is a continuous map $\gamma : [t_0, t_1] \to \mathbb{R}^d$ such that there exists a finite division $t_0 = \tau_0 < \tau_1 < \cdots < \tau_n = t_1$ with

$\gamma|_{[\tau_j,\tau_{j+1}]}$ smooth for each $j = 0, \dots, n-1$. We think of the space of paths as an infinite-dimensional manifold. For fixed $t_0 < t_1$, the space of paths $\gamma : [t_0, t_1] \to \mathbb{R}^d$ is an infinite-dimensional vector space, and for fixed $q_0, q_1 \in \mathbb{R}^d$, the space of paths γ with $\gamma(t_a) = q_a$, $a = 0, 1$ is an affine subspace thereof; this affine subspace is modeled on the vector space $\{\gamma : [t_0, t_1] \to \mathbb{R}^d \text{ s.t. } \gamma(t_0) = 0 = \gamma(t_1)\}$ of *piecewise-smooth loops based at* 0. Thus we can identify this loop space with the tangent space at any path to the subspace of paths with the same boundary conditions. We abbreviate "piecewise-smooth loop based at 0" by *based loop*.

A *Lagrangian function on* $\mathbb{R}^d$ is any smooth function $L : \mathbb{R} \times \mathrm{T}\mathbb{R}^d \to \mathbb{R}$. For a chosen Lagrangian function L, the corresponding *action* $\mathcal{S}$ assigns to each path $\gamma : [t_0, t_1] \to \mathbb{R}^d$ the number $\mathcal{S}(\gamma) = \int_{t_0}^{t_1} L(\tau, \dot\gamma(\tau), \gamma(\tau))\,\mathrm{d}\tau$.

1.3.3 Definition (derivatives of the action) Let $L : \mathbb{R} \times \mathrm{T}\mathbb{R}^d \to \mathbb{R}$ be a Lagrangian on $\mathbb{R}^d$ and $\mathcal{S} : \gamma \mapsto \int_{t_0}^{t_1} L\big(\tau, \dot\gamma(\tau), \gamma(\tau)\big)\,\mathrm{d}\tau$ the corresponding action. If $\xi : [t_0, t_1] \to \mathbb{R}^d$ is another path, then by the chain rule:

$$\mathcal{S}(\gamma + \xi) = \mathcal{S}(\gamma) + \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial v^i}\bigg|_{\gamma(\tau)} \dot\xi^i(\tau) + \frac{\partial L}{\partial q^i}\bigg|_{\gamma(\tau)} \xi^i(\tau) \right) \mathrm{d}\tau + o(\xi)$$

where $o(\xi)$ is a quantity that vanishes faster than linearly under rescaling $\xi^i \mapsto \epsilon\xi^i$. Thus we define the *functional derivative* $\delta\mathcal{S}/\delta\gamma = \mathcal{S}^{(1)}(\gamma)$ to be the following linear operator:

$$\mathcal{S}^{(1)}(\gamma) \cdot \xi = \mathcal{S}^{(1)}_i(\gamma) \cdot \xi^i = \int_{t_0}^{t_1} \left(\dot\xi^i(\tau)\, \frac{\partial L}{\partial v^i}\bigg|_{\gamma(\tau)} + \xi^i(\tau)\, \frac{\partial L}{\partial q^i}\bigg|_{\gamma(\tau)} \right) \mathrm{d}\tau$$

Differentiating repeatedly gives:

$$\mathcal{S}^{(n)}_{i_1 \dots i_n}(\gamma) \cdot \xi^{i_1}_1 \dots \xi^{i_n}_n = \int_{t_0}^{t_1} \prod_{k=1}^{n} \left(\dot\xi^{i_k}_k(\tau) \frac{\partial}{\partial v^{i_k}} + \xi^{i_k}_k(\tau) \frac{\partial}{\partial q^{i_k}} \right) L\bigg|_{\gamma(\tau)} \mathrm{d}\tau$$

These will correspond to the vertices in Definition 1.2.14.

It should be understood that the partial derivatives act only on L and commute with the ξs. For example:

$$\mathcal{S}^{(2)}_{ij} \cdot \xi^i \zeta^j = \int_{t_0}^{t_1} \left(\frac{\partial^2 L}{\partial v^i \partial v^j}\bigg|_\gamma \dot\xi^i \dot\zeta^j + \frac{\partial^2 L}{\partial q^i \partial v^j}\bigg|_\gamma \xi^i \dot\zeta^j + \frac{\partial^2 L}{\partial v^i \partial q^j}\bigg|_\gamma \dot\xi^i \zeta^j + \frac{\partial^2 L}{\partial q^i \partial q^j}\bigg|_\gamma \xi^i \zeta^j \right) \mathrm{d}\tau$$

1.3.4 Definition (nondegenerate classical path) A path γ is *classical* if $\mathcal{S}^{(1)}(\gamma) \cdot \xi = 0$ for all based loops ξ; as is well-known, such paths are precisely the classically-allowed trajectories for the mechanical system with Lagrangian L. By integrating by parts, a path γ is classical if and only if it satisfies the *Euler–Lagrange equation*:

$$\frac{\partial L}{\partial q^i}\bigg|_{(\tau, \dot\gamma(\tau), \gamma(\tau))} = \frac{\mathrm{d}}{\mathrm{d}\tau} \left[\frac{\partial L}{\partial v^i}\bigg|_{(\tau, \dot\gamma(\tau), \gamma(\tau))} \right]$$

We will always assume that the Euler–Lagrange equation is a nondegenerate second-order differential equation; equivalently, we assume that the symmetric matrix $\frac{\partial^2 L}{\partial v^i \partial v^j}(\tau, v, q)$ is invertible for every $(\tau, v, q) \in \mathbb{R} \times \mathrm{T}\mathbb{R}^d = \mathbb{R}^{2d+1}$. For Newtonian systems, the Euler–Lagrange equation reduces to Newton's law $F = ma$.

Our convention is that near a corner of a piecewise-smooth path γ, the velocity $\dot\gamma$ is discontinuous like Heaviside's step function Θ, and the acceleration $\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\gamma(\tau)$ has a discontinuity like Dirac's delta function $\delta(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\Theta(\tau)$. For a path to be classical, we impose the Euler–Lagrange equation even at these points of discontinuity, understanding the equation in the sense of distributions. Provided $\frac{\partial^2 L}{\partial v^i \partial v^j}(v, q)$ is invertible, all classical paths are smooth, by a classical "bootstrapping" argument: the acceleration enters only once in the Euler–Lagrange equation, as $\frac{\partial^2 L}{\partial v^i \partial v^j}\frac{\mathrm{d}^2\gamma^j}{\mathrm{d}\tau^2}$, and so can have a discontinuity no worse than the step function, but then $\dot\gamma$ is continuous, hence so is $\frac{\mathrm{d}^2\gamma^j}{\mathrm{d}\tau^2}$, etc.

In Definition 1.2.14, we insisted that each critical point be nondegenerate. Let us say that a classical path γ is *nondegenerate* if, when restricted to based loops, the operator $\mathcal{S}^{(2)}(\gamma)$ from Definition 1.3.3 has no zero modes. We will now describe this operator in more detail and compute its inverse.

1.3.5 Definition ($\mathcal{D}_\gamma$) Let $\gamma : [t_0, t_1] \to \mathbb{R}^d$ be classical and let $\xi, \zeta : [0, t] \to \mathbb{R}^d$ be based loops. By integrating the formula for $\mathcal{S}^{(2)}$ from Definition 1.3.3 by parts, we have $\mathcal{S}^{(2)} \cdot \xi\zeta = \int_{t_0}^{t_1} \mathcal{D}_\gamma[\xi(\tau)]_j\, \zeta^i(\tau)\, \mathrm{d}\tau$, where $\mathcal{D}_\gamma$ is the second-order linear differential operator given by:

$$\begin{aligned}\mathcal{D}_\gamma[\xi]_j(\tau) = &- \left.\frac{\partial^2 L}{\partial v^i \partial v^j}\right|_{\gamma(\tau)} \ddot\xi^i(\tau) \\ &+ \left(-\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left.\frac{\partial^2 L}{\partial v^i \partial v^j}\right|_{\gamma(\tau)}\right] - \left.\frac{\partial^2 L}{\partial q^i \partial v^j}\right|_{\gamma(\tau)} + \left.\frac{\partial^2 L}{\partial v^i \partial q^j}\right|_{\gamma(\tau)}\right) \dot\xi^i(\tau) \\ &+ \left(-\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left.\frac{\partial^2 L}{\partial q^i \partial v^j}\right|_{\gamma(\tau)}\right] + \left.\frac{\partial^2 L}{\partial q^i \partial q^j}\right|_{\gamma(\tau)}\right) \xi^i(\tau)\end{aligned}$$

As we mentioned in Defintion 1.3.4, the second derivative of a piecewise-smooth function can have a discontinuity like $\delta(\tau)$, and the integral expression for $\mathcal{S}^{(2)}$ should be understood accordingly. We will show that when γ is nondegenerate and $\frac{\partial^2 L}{\partial v^2}$ is invertible, then $\mathcal{D}_\gamma$ has an inverse.

1.3.6 Definition (Green's function) Let $\mathcal{D}$ be a second-order linear differential operator on the space of paths $[t_0, t_1] \to \mathbb{R}^d$. A *Green's function for $\mathcal{D}$* is a matrix-valued function of two variables $g : [t_0, t_1]^2 \to \mathrm{Mat}(\mathbb{R}^d) = \mathbb{R}^{d^2}$ such that $\mathcal{D}[g(\varsigma, -)]^j_k(\tau) = \delta^j_k \delta(\varsigma, \tau)$ (the product of Dirac's delta function with Kronecker's delta matrix), and $g(\varsigma, t_0) = 0 = g(\varsigma, t_1)$, so that $g(\varsigma, -)$ is a based loop for each $\varsigma \in [t_0, t_1]$.

Let L be a Lagrangian on $\mathbb{R}^d$ and $\gamma : [t_0, t_1] \to \mathbb{R}^d$ a classical path. The *Green's function for γ*, if it exists, is the Green's function G_γ for the operator $\mathcal{D}_\gamma$ in Definition 1.3.5.

Let us justify the word "the" in the previous sentence. Since $\mathcal{S}^{(2)} \cdot \xi\zeta = \mathcal{S}^{(2)} \cdot \zeta\xi$, we see that if $G^{ij}_\gamma(\varsigma, \tau)$ is a Green's function for $\mathcal{D}_\gamma$, then so is $G^{ji}_\gamma(\tau, \varsigma)$. Now suppose that G, G' are two Green's functions for $\mathcal{D}_\gamma$, and consider $\mathcal{S}^{(2)} \cdot G(\varsigma, -)G'(-, \varsigma')$. By integrating by parts, this equals both $G(\varsigma, \varsigma')$ and $G'(\varsigma, \varsigma')$. Moreover, by uniqueness, we see that γ cannot have a Green's function if it is not nondegenerate. We will prove the converse in Proposition 1.3.14.

The best way to solve an inhomogeneous linear differential equation, if the solutions to the corresponding homogeneous equation are known, is to use the method of "variation of parameters," which works for matrix-valued functions just as well as it does for scalars:

1.3.7 Lemma (variation of parameters) Let $\mathcal{D}$ be a second-order linear differential operator on the space of paths $[t_0, t_1] \to \mathbb{R}^d$ of the form $\mathcal{D}[\varphi]^j = \ddot{\varphi}^j + B^j_i \dot{\varphi}^i + C^j_i \varphi^i$, where B and C are smooth matrix-valued functions on $[t_0, t_1]$. Suppose that there are functions $\phi^a : [t_0, t_1] \to \mathrm{Mat}(\mathbb{R}^d)$, $a = 0, 1$, satisfying $\mathcal{D}[\phi^a] = 0$ with boundary values $\phi^{a,i}_k(t_b) = \delta^i_k \delta^a_b$ (here the indices i, j, k range from $1, \dots, d$, but $a, b \in \{0, 1\}$). Then the $2d \times 2d$ matrix $M(\tau) = \begin{pmatrix} \phi^0(\tau) & \phi^1(\tau) \\ \dot{\phi}^0(\tau) & \dot{\phi}^1(\tau) \end{pmatrix}$ is invertible for each $\tau \in [t_0, t_1]$. Let $\psi_0, \psi_1 : [t_0, t_1] \to \mathrm{Mat}(\mathbb{R}^d)$ comprise the right $d \times 2d$ half of M^{-1}. The function $g : [t_0, t_1]^2 \to \mathrm{Mat}(\mathbb{R}^n)$ given by

$$g^i_k(\varsigma, \tau) = \Theta(\tau - \varsigma)\, \phi^{0,i}_j(\tau)\, \psi^j_{0,k}(\varsigma) - \Theta(\varsigma - \tau)\, \phi^{1,i}_j(\tau)\, \psi^j_{1,k}(\varsigma)$$

is a Green's function for $\mathcal{D}$. Here and throughout, Θ denotes Heaviside's step function (Definition 1.3.1).

1.3.8 Proof of Lemma 1.3.7 We first prove that $M(\tau)$ is invertible for each τ. A solution $\varphi(\tau)$ to $\mathcal{D}[\varphi] = 0$ with $\varphi(t_0) = 0$ is determined by $\dot{\varphi}(t_0)$, and thus $\mathcal{D}$ determines a (constant) matrix D satisfying $D^j_i \dot{\varphi}^i(t_0) = \varphi^j(t_1)$. In particular, $D^j_i \dot{\phi}^{1,i}_k(t_0) = \phi^{1,j}_k(t_1) = \delta^j_k$, and so $\dot{\phi}^{1,i}_k(t_0)$ has full rank. Thus $M(t_0) = \begin{pmatrix} \delta & 0 \\ \dot{\phi}^0(t_0) & \dot{\phi}^1(t_0) \end{pmatrix}$ is invertible. By Liouville's formula, $\det M(\tau) = \exp\bigl(-\int_{t_0}^\tau \operatorname{tr} B\bigr) \det M(t_0)$, and in particular it is never 0.

The boundary conditions for g are immediate — the ψ_a satisfy $\psi_0(t_0) = 0 = \psi_1(t_1)$, because by definition $\phi^{0,i}_j(\tau)\psi^j_{0,k}(\tau) + \phi^{1,i}_j(\tau)\psi^j_{1,k}(\tau) = 0$ — and g is continuous near the diagonal $\varsigma = \tau$ for the same reason. Finally, one must check the derivatives of g, but the only terms in $\mathcal{D}[g]$ that survive are $\delta(\tau - \varsigma)\dot{\phi}^{0,i}_j(\tau)\psi^j_{0,k}(\varsigma) + \delta(\tau - \varsigma)\dot{\phi}^{1,i}_j(\tau)\psi^j_{1,k}(\varsigma) = \delta(\tau - \varsigma)\delta^i_k$, because $\dot{\phi}^{0,i}_j(\tau)\psi^j_{0,k}(\tau) + \dot{\phi}^{1,i}_j(\tau)\psi^j_{1,k}(\tau) = \delta^i_k$.

1.3.9 Lemma (nondegeneracy implies a local diffeomorphism) Let $\pi : \mathcal{E} \to \mathcal{B}$ be a smooth bundle, where $\mathcal{B}$ is a finite-dimensional smooth manifold and $\mathcal{E}$ is a possibly-infinite-dimensional smooth manifold. Let $f : \mathcal{E} \to \mathbb{R}$ be a smooth map. For each $b \in \mathcal{B}$, consider the restriction $f|_{\pi^{-1}(b)}$ of f to the fiber $\pi^{-1}(b) \subseteq \mathcal{E}$. Define $\mathcal{C} \subseteq \mathcal{E}$ to be the set of $c \in \mathcal{E}$ so that $\bigl(f|_{\pi^{-1}(\pi(c))}\bigr)^{(1)}(c) = 0$ — here $\pi^{-1}(\pi(c))$ is the fiber containing p, and $\bigl(f|_{\pi^{-1}(\pi(c))}\bigr)^{(1)}$ is

the first derivative of f along the fiber, so that $\big(f|_{\pi^{-1}(\pi(c))}\big)^{(1)}(c) \in \mathrm{T}^*_c\big(\pi^{-1}(\pi(c))\big)$. Assume that $\mathcal{C}$ is a manifold of the same dimension as $\mathcal{B}$.

Let $c \in \mathcal{C}$ be nondegenerate in the sense that the second derivative $\big(f|_{\pi^{-1}(\pi(c))}\big)^{(2)}(c)$, thought of as a map $\mathrm{T}_c\big(\pi^{-1}(\pi(c))\big) \to \mathrm{T}^*_c\big(\pi^{-1}(\pi(c))\big)$, has zero kernel. Then $\pi|_{\mathcal{C}} : \mathcal{C} \to \mathcal{B}$ is a local diffeomorphism near $c \in \mathcal{C}$; i.e. there are open neighborhoods $c \in U \subseteq \mathcal{C}$ and $\pi(c) \in \mathcal{O} \subseteq \mathcal{B}$ with $\pi|_U : U \xrightarrow{\sim} \mathcal{O}$.

1.3.10 Proof of Lemma 1.3.9 We reproduce the proof from [SP09]:

Since the statement is local, to save space we restrict $\mathcal{B}$ to an open neighborhood of $\pi(c)$ and choose a trivialization $\mathcal{E} = \mathcal{F} \times \mathcal{B}$, so that we can identify all fibers $\pi^{-1}(b)$ with $\mathcal{F}$. Then for $e \in \mathcal{E}$, the sequence $\mathcal{F} \to \mathcal{E} \to \mathcal{B}$ gives a short-exact sequence $\ker \pi = \mathrm{T}_e\mathcal{F} \to \mathrm{T}_e\mathcal{E} \to \mathrm{T}_{\pi(e)}\mathcal{B}$. The function $f : \mathcal{E} \to \mathbb{R}$ defines a map $f^{(1)} = \mathrm{d}f : \mathrm{T}\mathcal{E} \to \mathrm{T}\mathbb{R} = \mathbb{R} \times \mathbb{R}$, and $\mathcal{C} = \{c \in \mathcal{E} \text{ s.t. the restriction of } \mathrm{d}f \text{ to } \mathrm{T}_c\mathcal{F} \text{ is } 0\}$.

Pick $c \in \mathcal{C}$. Then $f^{(2)}$ determines a linear map $h : \mathrm{T}_c\mathcal{E} \to \mathrm{T}^*_c\mathcal{F}$, which deserves to be called the *Hessian*; it transforms as a tensor because $f^{(1)}$ vanishes on $\mathrm{T}_c\mathcal{F}$ (compare Definition 1.2.4). The key fact is that $\mathrm{T}_c\mathcal{C} \subseteq \ker h$, easily checked by considering the derivative of $f^{(1)}$ along paths in $\mathcal{C}$. But c is nondegenerate if and only if $\mathrm{T}_c\mathcal{F} \cap \ker h = 0$. Thus if c is nondegenerate, then $\mathrm{d}\pi : \mathrm{T}_c\mathcal{C} \to \mathrm{T}_{\pi(c)}\mathcal{B}$ is an injection. On the other hand, by assumption the dimensions of $\mathrm{T}_c\mathcal{C}$ and $\mathrm{T}_{\pi(c)}\mathcal{B}$ agree, so $\mathrm{d}\pi$ is full-rank and π is a local diffeomorphism.

1.3.11 Definition (nonfocality) Suppose that L is a Lagrangian on $\mathbb{R}^d$ such that the matrix $\frac{\partial^2 L}{\partial v^i \partial v^j}(\tau, v, q)$ is invertible for every $(\tau, v, q) \in \mathbb{R} \times \mathrm{T}\mathbb{R}^d$. Then a classical path $\gamma : [t_0, t_1] \to \mathbb{R}^d$ is determined by its initial conditions $\big(\dot\gamma(t_0), \gamma(t_0)\big) \in \mathrm{T}\mathbb{R}^d$. Let $\mathrm{Flow} : \mathbb{R} \times \mathbb{R} \times \mathrm{T}\mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ be the smooth function satisfying

$$\mathrm{Flow}\big(t_0, t_1, \dot\gamma(t_0), \gamma(t_0)\big) = \big(t_0, \gamma(t_0), t_1, \gamma(t_1)\big)$$

for classical paths γ — we write this as having domain $\mathbb{R}^2 \times \mathrm{T}\mathbb{R}^d$, but of course really the domain is some open neighborhood in $\mathbb{R}^2 \times \mathrm{T}\mathbb{R}^d$ containing $\{(t_0, t_1, v, q) \text{ s.t. } t_0 = t_1\}$.

A classical path $\gamma : [t_0, t_1] \to \mathbb{R}^d$ is *nonfocal* if Flow is a local diffeomorphism near $\big(t_0, t_1, \dot\gamma(t_0), \gamma(t_0)\big) \in \mathbb{R}^2 \times \mathrm{T}\mathbb{R}^d$. In fact, it suffices that for fixed t_0, t_1 the function $\mathrm{Flow}_{[t_0,t_1]} : \mathrm{T}\mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ be a local diffeomorphism near $\big(\dot\gamma(t_0), \gamma(t_0)\big)$, as this is clearly an open condition in t_0, t_1.

By identifying classical paths with their initial conditions (and domains), we see that a classical path γ is nonfocal if and only if it extends to a family of classical paths smoothly parameterized by "Dirichlet" boundary conditions. More precisely, if $\gamma : [t_0, t_1] \to \mathbb{R}^d$ is classical and nonfocal, then there is an open neighborhood $\mathcal{O}$ of $\big(t_0, \gamma(t_0), t_1, \gamma(t_1)\big) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ and a smooth function

$$\hat\gamma : \big\{(t_0', q_0, t_1', q_1, \tau) \in \mathbb{R}^{2d+3} \text{ s.t. } (t_0', q_0, t_1', q_1) \in \mathcal{O} \text{ and } \tau \in [t_0', t_1']\big\} \to \mathbb{R}^d$$

with the following properties: (i) for each $(t_0', q_0, t_1', q_1) \in \mathcal{O}$, the path $\hat\gamma(t_0', q_0, t_1', q_1; -)$ is classical; (ii) for $a = 0, 1$, we have $\hat\gamma(t_0', q_0, t_1', q_1; t_a') = q_a$; (iii) $\hat\gamma(t_0, \gamma(t_0), t_1, \gamma(t_1); -) = \gamma$.

Henceforth we will drop the ˆs and ′s, and we will feel free to confuse nonfocal classical paths with their extensions to families.

1.3.12 Definition (Hamilton function) Let γ be a nonfocal classical path and $\mathcal{O}$ the corresponding neighborhood in $\mathbb{R}^{2d+2}$. Then the corresponding *Hamilton principal function* $J_\gamma : \mathcal{O} \to \mathbb{R}$ is:

$$J_\gamma(t_0, q_0, t_1, q_1) = \mathcal{S}\big(\gamma(t_0, q_0, t_1, q_1; -)\big) = \int_{t_0}^{t_1} L\big(\tau, \dot\gamma(t_0, q_0, t_1, q_1; \tau), \gamma(t_0, q_0, t_1, q_1; \tau)\big)\,\mathrm{d}\tau$$

Here and throughout by $\dot\gamma(t_0, q_0, t_1, q_1; \tau)$ we mean $\frac{\partial\gamma}{\partial\tau}(t_0, q_0, t_1, q_1; \tau)$.

1.3.13 Fact (derivatives of J_γ) The following equations are well-known, and can be checked by differentiating under the integral and applying the Euler–Lagrange equations (Definition 1.3.4):

$$\frac{\partial J_\gamma}{\partial q_0} = -\frac{\partial L}{\partial v}\bigg|_{(\tau,v,q)=(t_0,\dot\gamma(t_0),\gamma(t_0))} \qquad \frac{\partial J_\gamma}{\partial q_1} = \frac{\partial L}{\partial v}\bigg|_{(\tau,v,q)=(t_1,\dot\gamma(t_1),\gamma(t_1))}$$

1.3.14 Proposition (the Green's function for γ) Let L be a Lagrangian on $\mathbb{R}^d$ with $\frac{\partial^2 L}{\partial v^2}$ everywhere invertible, and let γ be a classical path. Then the following are equivalent:

1. γ is nondegenerate;
2. γ is nonfocal;
3. a Green's function G_γ exists for γ.

If it exists, G_γ is given explicitly by:

$$G^{ij}(\varsigma,\tau) = \Theta(\tau-\varsigma)\,\frac{\partial\gamma^i}{\partial q_1^k}(\varsigma)\left(\left(\frac{\partial^2(-J_\gamma)}{\partial q_1\partial q_0}\right)^{-1}\right)^{kl}\frac{\partial\gamma^j}{\partial q_0^l}(\tau) + \\ + \Theta(\varsigma-\tau)\,\frac{\partial\gamma^i}{\partial q_0^k}(\varsigma)\left(\left(\frac{\partial^2(-J_\gamma)}{\partial q_0\partial q_1}\right)^{-1}\right)^{kl}\frac{\partial\gamma^j}{\partial q_1^l}(\tau)$$

In particular, the inverse matrices exist. Our index convention is $\left(\left(\frac{\partial^2(-J)}{\partial q_1\partial q_0}\right)^{-1}\right)^{kl}\frac{\partial^2(-J)}{\partial q_0^l\partial q_1^m} = \delta^k_m$.

1.3.15 Proof of Proposition 1.3.14 We argued already (in Definition 1.3.6) that 3 implies 1. To show that 1 implies 2, we use Lemma 1.3.9: we let $\mathcal{E}$ be the space of all paths in $\mathbb{R}^d$ (with arbitrary domain), $\mathcal{B} = \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$ with the natural projections, and $f = \mathcal{S}$. Then $\mathcal{C}$ is the set of classical paths, and it is a naturally a $(2d+2)$-dimensional manifold (in fact, an open subset of $\mathbb{R}^2\times\mathrm{T}\mathbb{R}^d$) by the remarks in Definition 1.3.11. Finally, to show that

2 implies 3, we observe that if γ is nonfocal, then the paths $\phi_k^{a,i} = \frac{\partial \gamma^i}{\partial q_a^k}$ satisfy $\mathcal{D}_\gamma[\phi^a] = 0$ and $\phi_k^{a,i}(t_b) = \delta_k^i \delta_b^a$. To apply Lemma 1.3.7, we use the fact that if $A : [t_0, t_1] \to \mathrm{Mat}(\mathbb{R}^d)$ is a smooth function such that $A(\tau)$ is invertible for every τ, then a Green's function for $A\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + AB\frac{\mathrm{d}}{\mathrm{d}\tau} + AC$ is given by $G(\varsigma, \tau) = g(\varsigma, \tau)(A(\varsigma))^{-1}$, where g is the Green's function from Lemma 1.3.7.

Therefore, taking advantage of the Einstein index notation to permute some factors and adopting the notation of Lemma 1.3.7, the Green's function for γ is given by:

$$G^{ij}(\varsigma,\tau) = -\left(\left(\frac{\partial^2 L}{\partial v^2}\Big|_{\gamma(\varsigma)}\right)^{-1}\right)^{ik} \psi_{0,k}^l(\varsigma)\ \frac{\partial \gamma^j}{\partial q_0^l}(\tau)\ \Theta(\tau-\varsigma) + \\ + \left(\left(\frac{\partial^2 L}{\partial v^2}\Big|_{\gamma(\varsigma)}\right)^{-1}\right)^{ik} \psi_{1,k}^l(\varsigma)\ \frac{\partial \gamma^j}{\partial q_1^l}(\tau)\ \Theta(\varsigma-\tau)$$

But $G^{ij}(\varsigma,\tau) = G^{ji}(\tau,\varsigma)$ by the symmetry of $\mathcal{S}^{(2)}$, and so $\eta_a^{jl} = \left(\left(\frac{\partial^2 L}{\partial v^2}\big|_\gamma\right)^{-1}\right)^{jk} \psi_{a,k}^l$ is a solution to $\mathcal{D}_{ij}[\eta_a^{jl}] = 0$ and therefore a linear combination of the $\phi^b = \frac{\partial \gamma}{\partial q_b}$. By checking the boundary conditions, we see that:

$$\left(\left(\frac{\partial^2 L}{\partial v^2}\Big|_{\gamma(\varsigma)}\right)^{-1}\right)^{ik} \psi_{0,k}^l(\varsigma) = \frac{\partial \gamma^i}{\partial q_1^j}(\varsigma)\left(\left(\frac{\partial^2 L}{\partial v^2}\Big|_{\gamma(t_1)}\right)^{-1}\right)^{jk} \psi_{0,k}^l(t_1) \\ = \frac{\partial \gamma^i}{\partial q_1^j}(\varsigma)\left(\left(\frac{\partial^2 L}{\partial v^2}\Big|_{\gamma(t_1)}\right)^{-1}\right)^{jk}\left(\left(\frac{\partial \dot\gamma}{\partial q_0}(t_1)\right)^{-1}\right)_k^l$$

$$\left(\left(\frac{\partial^2 L}{\partial v^2}\Big|_{\gamma(\varsigma)}\right)^{-1}\right)^{ik} \psi_{1,k}^l(\varsigma) = \frac{\partial \gamma^i}{\partial q_0^j}(\varsigma)\left(\left(\frac{\partial^2 L}{\partial v^2}\Big|_{\gamma(t_0)}\right)^{-1}\right)^{jk}\left(\left(\frac{\partial \dot\gamma}{\partial q_1}(t_0)\right)^{-1}\right)_k^l$$

Finally, we should study $\left(\left(\frac{\partial^2 L}{\partial v^2}\big|_{\gamma(t_1)}\right)^{-1}\right)^{jk}\left(\left(\frac{\partial \dot\gamma}{\partial q_0}(t_1)\right)^{-1}\right)_k^l$ and $\left(\left(\frac{\partial^2 L}{\partial v^2}\big|_{\gamma(t_0)}\right)^{-1}\right)^{jk}\left(\left(\frac{\partial \dot\gamma}{\partial q_1}(t_0)\right)^{-1}\right)_k^l$. The former is the inverse matrix to $\frac{\partial^2 L}{\partial v^j \partial v^k}\big|_{\gamma(t_1)} \frac{\partial \dot\gamma^k}{\partial q_0^l}(t_1)$. But:

$$\frac{\partial}{\partial q_0^l}\left[\frac{\partial L}{\partial v^j}\Big|_{\gamma(t_1)}\right] = \frac{\partial^2 L}{\partial v^j \partial v^k}\Big|_{\gamma(t_1)} \frac{\partial \dot\gamma^k(t_1)}{\partial q_0^l} + \frac{\partial^2 L}{\partial v^j \partial q^k}\Big|_{\gamma(t_1)} \frac{\partial \gamma^k(t_1)}{\partial q_0^l} = \frac{\partial^2 L}{\partial v^j \partial v^k}\Big|_{\gamma(t_1)} \frac{\partial \dot\gamma^k}{\partial q_0^l}(t_1)$$

since $\frac{\partial \gamma(t_1)}{\partial q_0} = \frac{\partial q_1}{\partial q_0} = 0$. Then Fact 1.3.13 completes the proof.

1.3.16 Definition (Morse index) Let γ be a classical path; then $\mathcal{S}^{(2)}(\gamma)$ is a symmetric bilinear pairing on the space of loops based at 0. The *Morse index* $\eta(\gamma)$ is the dimension of any maximal subspace of the space of based loops for which $\mathcal{S}^{(2)}(\gamma)$ is negative definite. Following the classical argument (see e.g. [Mil63]), in Corollary 1.3.21 we will prove this is a well-defined finite number provided the Lagrangian L has positive second derivatives in all velocity directions. Well-definedness follows from an easy lemma:

1.3.17 Lemma (well-definedness of the Morse index) Let V be any vector space and $a : V \otimes V \to \mathbb{R}$ a symmetric pairing. Suppose that a is negative-definite on a finite-dimensional subspace $V_- \subseteq V$, and that V_- cannot be extended to any larger subspace on which a is negative-definite. Then we claim that any subspace of V on which a is negative-definite has dimension at most $\dim V_-$.

1.3.18 Proof of Lemma 1.3.17 Indeed, it suffices to consider the kernel $(V_-)^\perp$ of the map $V \to (V_-)^*$ given by $v \mapsto a(v, -)$. If $W \subseteq V$ has dimension $> \dim V_- = \dim(V_-)^*$, then it intersects nontrivially with $(V_-)^\perp$ as it cannot inject into $(V_-)^*$, but if $v \in W \cap (V_-)^\perp$ has $a(v, v) < 0$, then a is negative-definite on $V_- \oplus v\mathbb{R}$. Thus the Morse index is well-defined.

1.3.19 Proposition ($\mathcal{S}^{(2)}$ is positive-definite on strings of very short based loops) Pick a Lagrangian L on $\mathbb{R}^d$ and let $\gamma : [t_0, t_1] \to \mathbb{R}^d$ be classical. Suppose that the symmetric matrix $\frac{\partial^2 L}{\partial v^i \partial v^j}\big(\tau, \dot\gamma(\tau), \gamma(\tau)\big)$ is positive-definite for each $\tau \in [t_0, t_1]$. Then for sufficiently fine subdivisions $t_0 = \tau_0 \le \tau_1 \le \cdots \le \tau_n = t_1$ of the interval $[t_0, t_1]$, the pairing $\mathcal{S}^{(2)}(\gamma)$ is positive-definite on the space of paths ξ with $\xi(\tau_k) = 0$ for $k = 0, \ldots, n$.

1.3.20 Proof of Proposition 1.3.19 We will find $\epsilon > 0$ so that the statement holds whenever $\tau_k - \tau_{k-1} < \epsilon$. Let V be the space of based loops with domain $[t_0, t_1]$. The space $V_{\vec\tau}$ of paths $\xi \in V$ that vanish at each τ_k splits as a direct sum $V_{\vec\tau} = \bigoplus_{k=1}^n V_k$, where V_k is the space of based loops with domain $[\tau_{k-1}, \tau_k]$, and the direct summands are mutually orthogonal with respect to the pairing $\mathcal{S}^{(2)}(\gamma)$. Thus it suffices to show that $\mathcal{S}^{(2)}(\gamma) \cdot \xi\xi > 0$ whenever $\xi \in V$ has support a subinterval of $[t_0, t_1]$ of length less than ϵ. Upon integrating $\mathcal{S}^{(2)}(\gamma) \cdot \xi\xi$ by parts, the $\dot\xi\xi$ integrands cancel out, so for $C_{ij}(\tau) = \frac{\partial^2 L}{\partial q^i \partial q^j}(\gamma(\tau)) - \frac{\mathrm{d}}{\mathrm{d}\tau}\big[\frac{\partial^2 L}{\partial q^i \partial v^j}(\gamma(\tau))\big]$, there is some $t' \in [0, t]$ such that:

$$\mathcal{S}^{(2)}(\gamma) \cdot \xi\xi = \int_{t'}^{t'+\epsilon} \left(\left.\frac{\partial^2 L}{\partial v^i \partial v^j}\right|_{\gamma(\tau)} \dot\xi^i(\tau)\, \dot\xi^j(\tau) + C_{ij}(\tau)\, \xi^i(\tau)\, \xi^j(\tau) \right) d\tau$$

Let $\lambda_1 > 0$ be the minimal eigenvalue of $\frac{\partial^2 L}{\partial v^i \partial v^j}\big(\tau, \dot\gamma(\tau), \gamma(\tau)\big)$ as τ ranges over $[t_0, t_1]$, and let $\lambda_2 > 0$ be the maximum eigenvalue of the $-\frac{1}{2}\big(C_{ij}(\tau) + C_{ji}(\tau)\big)$ for $\tau \in [t_0, t_1]$ (if C is always positive-semidefinite, then the conclusion of Proposition 1.3.19 is immediate). Then:

$$\mathcal{S}^{(2)}(\gamma) \cdot \xi\xi \ge \int_{t'}^{t'+\epsilon} \Big(\lambda_1 \big|\dot\xi(\tau)\big|^2 - \lambda_2 \big|\xi(\tau)\big|^2 \Big)\, d\tau \ge \lambda_1 \int_{t'}^{t'+\epsilon} \big|\dot\xi(\tau)\big|^2 d\tau - \epsilon\lambda_2 \sup |\xi|^2$$

But by the Cauchy-Schwarz inequality, $\int_{t'}^{t'+\epsilon} |\dot\xi(\tau)|^2 d\tau \ge \frac{1}{\epsilon}\big(\int_{t'}^{t'+\epsilon} |\dot\xi(\tau)| d\tau\big)^2 \ge \frac{1}{\epsilon}\big(2 \sup |\xi|\big)^2$. Thus:

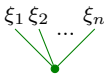
$$\mathcal{S}^{(2)}(\gamma) \cdot \xi\xi \ge \left(\frac{\lambda_1}{\epsilon} - \epsilon\lambda_2 \right) \sup_{\tau \in [t', t'+\epsilon]} \big|\xi(\tau)\big|^2$$

Taking $\epsilon < \sqrt{\lambda_1 \lambda_2}$ completes the proof of Proposition 1.3.19.

1.3.21 Corollary (classical paths for convex Lagrangians have finite Morse index) Thus within the space V of based loops with domain $[t_0, t_1]$ we have found a large subspace $V_{\vec{\tau}}$ on which $\mathcal{S}^{(2)}(\gamma)$ is positive-definite. The restriction of γ to each interval $[\tau_{k-1}, \tau_k]$ gives a nondegenerate classical path γ_k. Let $W \subseteq V$ be the space of based loops ξ that are solutions to $\mathcal{D}_\gamma[\xi] = 0$ except at the times τ_k, $k = 1, \dots, n-1$, where $\mathcal{D}_\gamma$ is the second-order differential operator from Definition 1.3.5. By Proposition 1.3.14, such a path ξ depends only on its values at the times τ_k, so that $W \cong (\mathbb{R}^d)^{\otimes(n-1)}$. The vector spaces W and $V_{\vec{\tau}}$ are mutually orthogonal with respect to $\mathcal{S}^{(2)}(\gamma)$, and $W \oplus V_{\vec{\tau}} = V$. Since $\mathcal{S}^{(2)}(\gamma)$ is positive-definite on V, any subspace of $V \oplus W$ on which it is negative-definite cannot have dimension greater than $\dim W = d^{n-1} < \infty$. In particular:

Let L be a Lagrangian function on $\mathbb{R}^d$ so that the matrix $\frac{\partial^2 L}{\partial v^i \partial v^j}(\tau, v, q)$ is positive-definite for each $(\tau, v, q) \in \mathbb{R} \times \mathrm{T}\mathbb{R}^d$. Then every classical path has finite Morse index.

1.3.22 Definition (coordinate-full formal path integral for quantum mechanics) Now look back over the ingredients needed to define the formal integral in Definition 1.2.14. Fix a Lagrangian function L on $\mathbb{R}^d$ and corresponding action function $\mathcal{S}$, as in Definition 1.3.2. Choose a nonfocal classical path $\gamma : [t_0, t_1] \to \mathbb{R}^d$, and consider the vector space V of based loops $[t_0, t_1] \to \mathbb{R}^d$, which is the tangent space to the space of paths with the same boundary conditions as γ. On V we consider the function $s(\xi) = \mathcal{S}(\gamma + \xi)$; since γ is classical, s has a critical point at $\xi = 0$, and since γ is nonfocal, this critical point is nondegenerate (Proposition 1.3.14). We can thus declare the Feynman rules for the formal integral. From Definition 1.3.3,



$$= -\mathcal{S}^{(n)} \cdot \xi_1 \cdots \xi_n = -\int_{\tau=0}^{t} \prod_{k=1}^{n} \left(\dot{\xi}_k^{i_k}(\tau) \frac{\partial}{\partial v^{i_k}} + \xi_k^{i_k}(\tau) \frac{\partial}{\partial q^{i_k}} \right) L \Big|_{(\dot{\gamma}(\tau), \gamma(\tau))} \mathrm{d}t,$$

and from Proposition 1.3.14,

$$= G^{ij}(\varsigma, \tau) = \Theta(\tau - \varsigma) \frac{\partial \gamma^i}{\partial q_1^k}(\varsigma) \left(\left(\frac{\partial^2(-J_\gamma)}{\partial q_1 \partial q_0} \right)^{-1} \right)^{kl} \frac{\partial \gamma^j}{\partial q_0^l}(\tau)$$
$$+ \Theta(\varsigma - \tau) \frac{\partial \gamma^i}{\partial q_0^k}(\varsigma) \left(\left(\frac{\partial^2(-J_\gamma)}{\partial q_0 \partial q_1} \right)^{-1} \right)^{kl} \frac{\partial \gamma^j}{\partial q_1^l}(\tau).$$

The Morse index $\eta(\gamma)$ is defined provided the matrix $\frac{\partial^2 L}{\partial v^i \partial v^j}$ is everywhere positive-definite (Definition 1.3.16 and Corollary 1.3.21). We declare that $\dim V = -\dim \mathbb{R}^d = -d$, using the following justification: there is a natural isomorphism $V \oplus \mathbb{R}^d \oplus V \xrightarrow{\sim} V$ taking (ξ_0, q, ξ_1) to the based loop

$$\tau \mapsto \begin{cases} \xi_0(2\tau - t_0) + 2\,\frac{\tau - t_0}{t_1 - t_0}\, q, & \tau \le \frac{t_0 + t_1}{2} \\ \xi_1(2\tau - t_1) + 2\,\frac{t_1 - \tau}{t_1 - t_0}\, q, & \tau \ge \frac{t_0 + t_1}{2} \end{cases},$$

and so $\dim V$ should solve $2 \dim V + d = \dim V$.

The last required piece is the determinant, which we declare entirely *ad hoc*: $|\det \mathcal{S}^{(2)}|^{-1} = \left|\det \frac{\partial^2[-J_\gamma]}{\partial q_0 \partial q_1}\right|$. We will provide some justification for this choice in Remark 1.4.10. All together, we arrive at the following definition:

Let L be a Lagrangian on the configuration space $\mathbb{R}^d$ such that the matrix $\frac{\partial^2 L}{\partial v^2}(\tau, v, q)$ is positive definite for every $(\tau, v, q) \in \mathbb{R} \times \mathrm{T}\mathbb{R}^d$, and let $\gamma : [t_0, t_1] \to \mathbb{R}^d$ be a nonfocal classical path, extended to a smooth family of classical paths parameterized by the Dirichlet boundary conditions $\gamma(t_0) = q_0$ and $\gamma(t_1) = q_1$. Then the *formal path integral supported near* γ is:

$$U_\gamma(t_0, q_0, t_1, q_1) = \left(2\pi\hbar\sqrt{-1}\right)^{-d/2}\left(\sqrt{-1}\right)^{-\eta(\gamma)} \times \exp\left(\frac{\sqrt{-1}}{\hbar} J_\gamma(t_0, q_0, t_1, q_1)\right) \sqrt{\left|\det \frac{\partial^2[-J_\gamma]}{\partial q_0 \partial q_1}\right|} \sum_\Gamma \frac{\left(\hbar\sqrt{-1}\right)^{-\chi(\Gamma)} \mathrm{ev}(\Gamma)}{|\mathrm{Aut}\,\Gamma|}$$

The sum ranges over Feynman diagrams in which all vertices have valence at least three, and Feynman diagrams are evaluated as above.

1.3.23 Remark (ultraviolet divergences and *ad hoc* choices) The Feynman diagrams in Definition 1.3.22 in general do not evaluate to finite numbers. We will discuss these *ultraviolet divergences* in more detail in Section 1.7. Because of the one-dimensionality of quantum mechanics, the ultraviolet divergences are not terrible: every diagram evaluates to a real polynomial in an "infinite" parameter $\delta(0)$, as we prove in Lemma 1.7.3. We will say that the formal path integral is *divergence free* if in the sum of diagrams all terms that are linear-or-higher in $\delta(0)$ cancel (see Definition 1.7.1).

The choices made in Definition 1.3.22 are all reasonably justified except for the choice to use $\left|\det \frac{\partial^2[-J_\gamma]}{\partial q_0 \partial q_1}\right|$ as the meaning of "$|\det \mathcal{S}^{(2)}|^{-1}$." The only justification for this choice is that $\left|\det \frac{\partial^2[-J_\gamma]}{\partial q_0 \partial q_1}\right|$ solves a certain differential equation necessary to have a Fubini-type theorem guaranteeing the composition law for quantum mechanics (Theorem 1.6.1). In fact, the composition law holds only when the formal path integral is divergence free. When the ultraviolet divergences do not cancel, a different choice for "$|\det \mathcal{S}^{(2)}|^{-1}$" is required, but we do not know what it is.

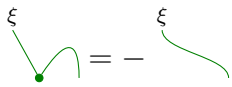
We will prove in Theorem 1.7.4 that in the most important case — when the Lagrangian is of the form $L(\tau, v, q) = \frac{1}{2} a_{ij}(\tau, q)\, v^i v^j + b_i(\tau, q)\, v^i + c(\tau, q)$ and $\det a$ is identically 1 — the formal path integral is free of ultraviolet divergences. We know of no other divergence-free examples.

1.4 Some useful Feynman rules: derivatives of U_γ

Pick a classical nonfocal path $\gamma : [t_0, t_1] \to \mathbb{R}^d$, extended to a family that depends smoothly on its boundary conditions $\gamma(t_0) = q_0$ and $\gamma(t_1) = q_1$ via Definition 1.3.11. As in Definition 1.3.22, we will use solid vertical lines in our Feynman diagrams as referring to the vector

space of all paths in $\mathbb{R}^d$ with domain $[t_0, t_1]$. (This is a small generalization of the notation: in Definition 1.2.14, solid lines in Feynman diagrams referred specifically to the vector space over which we perform the formal integral, i.e. the space of based loops; but all vertices make sense when contracted with paths that are not based loops, and we will use such paths.) In this section we introduce a few more Feynman rules and record some useful facts that we will use in later sections.

1.4.1 Remark (Green's function and Euler–Lagrange equations) Definition 1.3.6 says that  is a based loop in each output strand, and satisfies



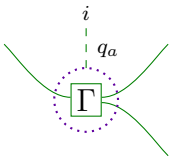
for all based loops ξ. Note that the above equation is not satisfied if ξ is not a based loop.

Similarly,

if ξ is a based loop, but in general not otherwise.

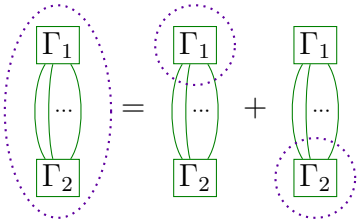
We will say more about the values of the left-hand sides of these equations when ξ is not a based loop in Lemma 1.6.13.

1.4.2 Definition (derivatives of Feynman diagrams) We will use dashed lines ⋮ to denote the vector space $\mathbb{R}^d$; equivalently, a dashed line carries an index $i = 1, \dots, d$ but no time variable. Let Γ be a Feynman diagram, possibly not closed. Then its value depends on the classical path γ, and in particular on the boundary conditions q_0, q_1. We represent differentiation with a dotted circle:



$$= \frac{\partial}{\partial q_a^i}\big[\mathrm{ev}(\Gamma)\big]\,, \qquad a = 0, 1$$

If Γ consists of two components Γ_1, Γ_2, possibly connected to each other, then the product rule can be written graphically as:



Suppose then that Γ_1 is a subdiagram of Γ whose images $\Gamma_1, \dots, \Gamma_n$ under the group of automorphisms of Γ do not intersect, so that $\Gamma = \bar{\Gamma} \cup \Gamma_1 \cup \cdots \cup \Gamma_n$ (we do allow automorphism of Γ to induce nontrivial automorphisms of Γ_1). Then:

$$\Gamma = \bar{\Gamma}\,\Gamma_1\,\Gamma_2 \cdots \Gamma_n = \bar{\Gamma}\,\Gamma_1\,\Gamma_2 \cdots \Gamma_n + \bar{\Gamma}\,\Gamma_1\,\Gamma_2 \cdots \Gamma_n + \cdots + \bar{\Gamma}\,\Gamma_1\,\Gamma_2 \cdots \Gamma_n =$$
$$= \bar{\Gamma}\,\Gamma_1\,\Gamma_2 \cdots \Gamma_n + n\,\bar{\Gamma}\,\Gamma_1\,\Gamma_2 \cdots \Gamma_n$$

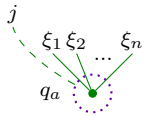
It is an elementary counting lemma that $\mathrm{Aut}\left(\bar{\Gamma}\,\Gamma_1\,\Gamma_2 \cdots \Gamma_n\right) = \frac{1}{n}\,\mathrm{Aut}\left(\bar{\Gamma}\,\Gamma_1\,\Gamma_2 \cdots \Gamma_n\right)$. From this observation, we derive the following fundamental result:

1.4.3 Lemma (Product Rule) For $a = 0, 1$, we have:

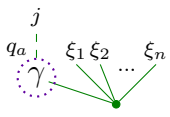
$$\frac{\partial}{\partial q_a} \sum_{\Gamma} \frac{(\hbar\sqrt{-1})^{\chi(\Gamma)}\,\mathrm{ev}(\Gamma)}{|\,\mathrm{Aut}\,\Gamma|} = \sum_{\Gamma \text{ with one } \cdot \text{ or } \frown} \frac{(\hbar\sqrt{-1})^{\chi(\Gamma)}\,\mathrm{ev}(\Gamma)}{|\,\mathrm{Aut}\,\Gamma|}$$

The sum on the right-hand side ranges over diagrams with precisely one differentiated basic subgraph — either a single differentiated vertex (of valence three or more) or a single differentiated edge.

1.4.4 Lemma (derivative of a vertex) Assume that $\xi_1, \dots, \xi_n$ do not depend on q_a. Then:



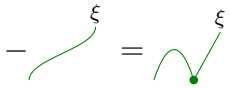
$$= \frac{\partial}{\partial q_a^j}\left[-\int_{t_0}^{t_1} \prod_{k=1}^{n} \left(\dot{\xi}_k^{i_k}(\tau)\frac{\partial}{\partial v^{i_k}} + \xi_k^{i_k}(\tau)\frac{\partial}{\partial q^{i_k}}\right) L\Big|_{\gamma(\tau)}\, d\tau\right]$$
$$= -\int_{t_0}^{t_1} \frac{\partial}{\partial q_a^j} \prod_{k=1}^{n} \left(\dot{\xi}_k^{i_k}(\tau)\frac{\partial}{\partial v^{i_k}} + \xi_k^{i_k}(\tau)\frac{\partial}{\partial q^{i_k}}\right) L\Big|_{\gamma(\tau)}\, d\tau$$
$$= -\int_{t_0}^{t_1} \left(\frac{\partial \dot{\gamma}^i}{\partial q_a^j}\frac{\partial}{\partial v^i} + \frac{\partial \gamma^i}{\partial q_a^j}\frac{\partial}{\partial q^i}\right) \prod_{k=1}^{n} \left(\dot{\xi}_k^{i_k}(\tau)\frac{\partial}{\partial v^{i_k}} + \xi_k^{i_k}(\tau)\frac{\partial}{\partial q^{i_k}}\right) L\Big|_{\gamma(\tau)}\, d\tau$$
$$=$$



since the only q_a dependence is in the classical path γ.

Note that the final line is non-zero even when $n = 0$, as $\frac{\partial \gamma}{\partial q_a}$ is not a based loop; compare with Remark 1.4.1.

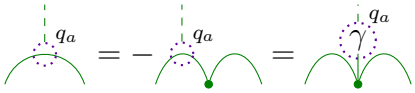
1.4.5 Lemma (derivative of an edge) We can now use Remark 1.4.1 and Lemmas 1.4.3 and 1.4.4 to compute the derivatives of the Green's function. Let ξ be a based loop. We have:



, provided ξ is a based loop.

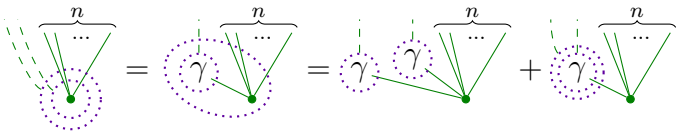
Differentiating gives:

Contracting with an edge, which is a based loop in each end, gives:

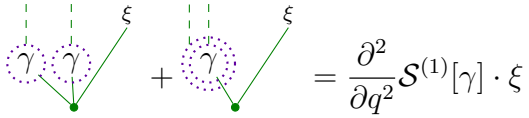


The first equality requires that $\frac{\partial G}{\partial q_a}$ vanishes at both endpoints, which follows from differentiating $G(\varsigma, t_0) = 0 = G(\varsigma, t_1)$ with respect to q_a.

1.4.6 Lemma (higher derivatives of γ) When evaluating formal integrals, we need not only the first derivative of the integrand but its full Taylor expansion. To expand higher derivatives of diagrams, we use the product rule again:

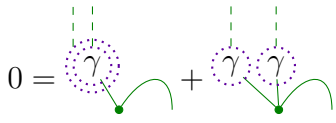


In particular:



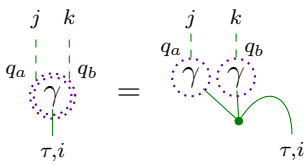
$$= \frac{\partial^2}{\partial q^2} \mathcal{S}^{(1)}[\gamma] \cdot \xi$$

This vanishes if ξ is a based loop, by the Euler–Lagrange equations. Thus:



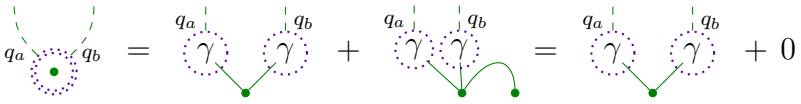
And $\frac{\partial^2 \gamma}{\partial q_a \partial q_b}$ vanishes at both endpoints. Therefore:

$$\frac{\partial^2 \gamma^i}{\partial q_a^j \partial q_b^k}(\tau) =$$



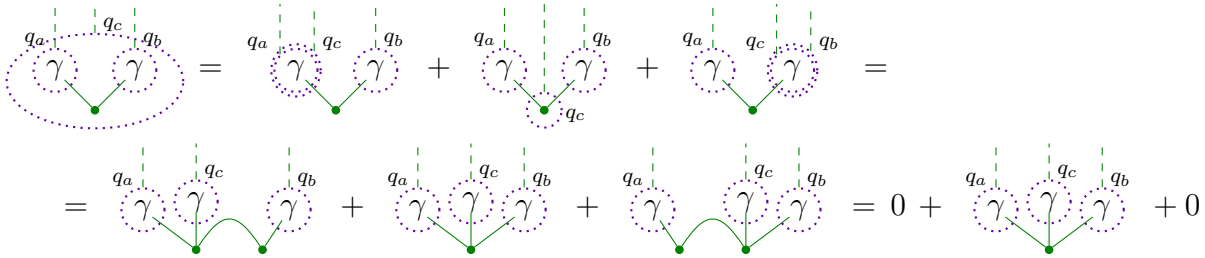
Thus in general to take the second derivative of a vertex one either adds two $\frac{\partial \gamma}{\partial q_a}$s or an edge connecting to a trivalent vertex with two $\frac{\partial \gamma}{\partial q_a}$s.

1.4.7 Example ($\partial^2 J_\gamma/\partial q^2$) The second derivative of a zero-valent vertex is:

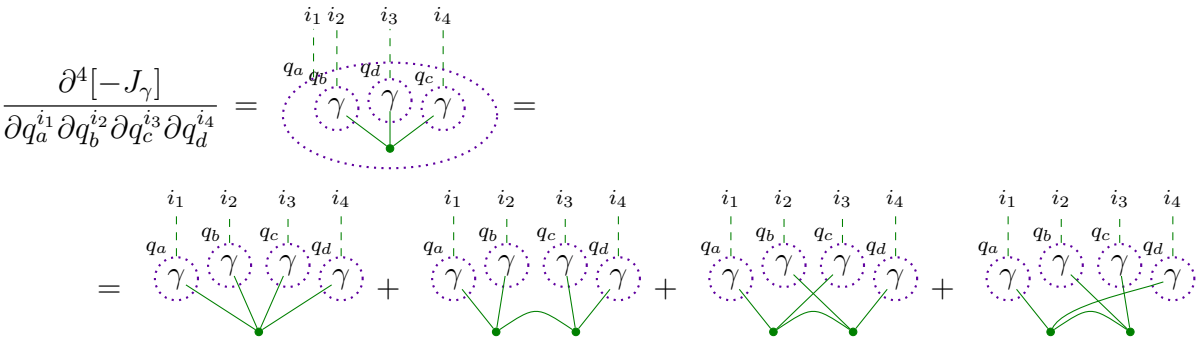


The second summand vanishes because G is a based loop in each variable.

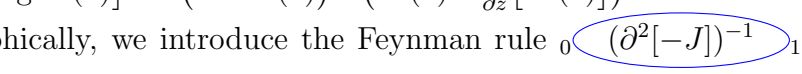
1.4.8 Remark ($\partial^n J/\partial q^n$ is a sum of trees) Taking another derivative:



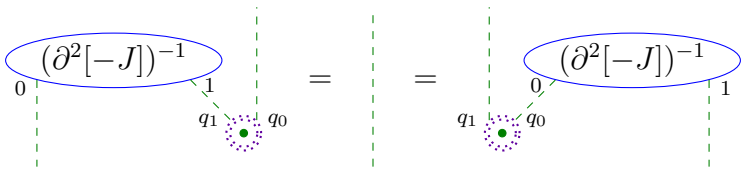
The outer terms vanish because $\mathcal{S}^{(2)} \cdot \frac{\partial \gamma}{\partial q_a} \xi = 0$ when ξ is a based loop. Yet another derivative gives:



In general, the nth derivative of $-J_\gamma$ will be a sum of trees.

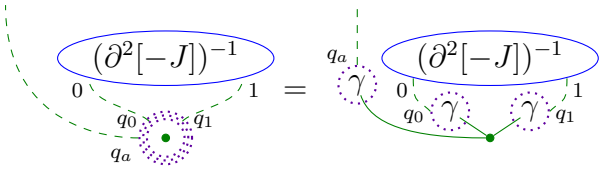
1.4.9 Lemma (derivative of the determinant) The final component of Definition 1.3.22 that is not locally constant in q_0, q_1 is the determinant $\det \frac{\partial^2[-S]}{\partial q_0 \partial q_1}$. Recall the derivative of a determinant of a matrix-valued function: $\frac{\partial}{\partial z}\big[\det M(z)\big] = \big(\det M(z)\big) \frac{\partial}{\partial z}\big[\log \det M(z)\big] = \big(\det M(z)\big) \frac{\partial}{\partial z}\big[\operatorname{tr} \log M(z)\big] = \big(\det M(z)\big) \operatorname{tr}\big(M(z)^{-1} \frac{\partial}{\partial z}\big[M(z)\big]\big)$. To denote the derivatives of $\det \frac{\partial^2[-J]}{\partial q_0 \partial q_1}$ graphically, we introduce the Feynman rule 

for the inverse matrix $\big(\frac{\partial^2[-J]}{\partial q_0 \partial q_1}\big)^{-1}$:

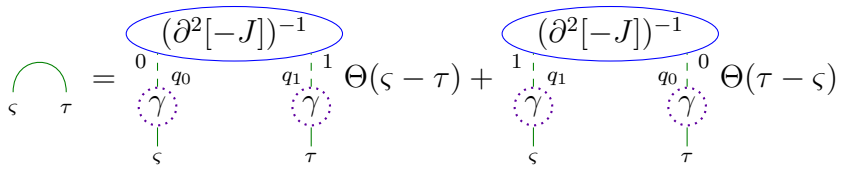


Then Example 1.4.8 gives:

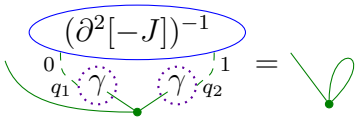
$$\frac{\partial}{\partial q_a}\left[\log\left|\det\frac{\partial^2[-J]}{\partial q_0\partial q_1}\right|\right] =$$



1.4.10 Remark (justification for the determinant) An important comparison is in order. In the graphical notation, Proposition 1.3.14 reads:



Recall that implicit in vertices are integrals, which might not converge: the Green's function is not smooth at $\varsigma = \tau$. Suppose, however, that the diagram  does converge. Then we claim that:



Indeed, the difference comes only from derivatives of the Heaviside step functions in  , and of these only $\delta(\varsigma-\tau)$ can contribute, but if it contributes at all to  then it contributes a divergent term proportional to $\delta(0)$. This proves the claimed equality when the formal path integral has no ultraviolet divergences.

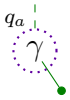
Thus our *ad hoc* determinant has the same derivatives as would be had by the undefined determinant $\left|\det \mathcal{S}^{(2)}\right|^{-1}$, which is what Definition 1.2.14 says should be in Definition 1.3.22. When there are ultraviolet divergences, we believe that some sort of "divergent" derivative should replace our *ad hoc* definition.

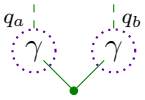
1.4.11 Proposition (summary of derivatives) In the formal path integral, the determinant appears with exponent $\frac{1}{2}$. This fraction now develops new meaning: $\partial\left[\sqrt{\left|\det\frac{\partial^2[-J]}{\partial q_0\partial q_1}\right|}\right] = \sqrt{\left|\det\frac{\partial^2[-J]}{\partial q_0\partial q_1}\right|} \times \frac{1}{2}$  , and on the right-hand-side the $\frac{1}{2}$ can be understood as the symmetry factor of the diagram. All together, Lemmas 1.4.3, 1.4.4, 1.4.5, and 1.4.6, and Remark 1.4.10 imply:

Let L be a Lagrangian on $\mathbb{R}^d$ and γ a classical nonfocal path depending smoothly on its boundary conditions such that the formal path integral $U_\gamma(q_0, q_1)$ has no ultraviolet

divergences. Then:

$$-J_\gamma = \bullet \qquad \frac{\partial[-J_\gamma]}{\partial q_a} = \qquad \frac{\partial^2[-J_\gamma]}{\partial q_a \partial q_b} =$$





$$\frac{\partial^n[-J_\gamma]}{(\partial q)^n} = \sum_{\substack{\text{trees } \Gamma \text{ with trivalent} \\ \text{and higher vertices} \\ \text{and } n \text{ leaves}}} \operatorname{ev}(\Gamma), \quad n \geq 3$$

$$\frac{\partial^n}{(\partial q)^n}\left[\sqrt{\left|\det \frac{\partial^2[-J_\gamma]}{\partial q_0 \partial q_1}\right|} \sum_{\substack{\text{diagrams } \Gamma \text{ with} \\ \text{no leaves and trivalent} \\ \text{and higher vertices}}} \frac{(\hbar\sqrt{-1})^{-\chi(\Gamma)} \operatorname{ev}(\Gamma)}{|\operatorname{Aut}\Gamma|}\right] =$$

$$= \sqrt{\left|\det \frac{\partial^2[-J_\gamma]}{\partial q_0 \partial q_1}\right|} \sum_{\substack{\text{diagrams } \Gamma \text{ with} \\ n \text{ leaves and trivalent} \\ \text{and higher vertices} \\ \text{and no trees}}} \frac{(\hbar\sqrt{-1})^{-\chi(\Gamma)} \operatorname{ev}(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

In all cases, "leaves" of a diagram are attached to $\frac{\partial\gamma}{\partial q}$s, and are totally ordered. Note that a tree with totally ordered leaves has no automorphisms. In each case, you could just as well have divided the left-hand side by $n!$, working with $\frac{1}{n!}\frac{\partial^n}{\partial q^n}$, in which case we would not order the leaves on the right-hand sides and would have to divide by $|\operatorname{Aut}\Gamma|$ in the equation for $\frac{\partial^n[-J_\gamma]}{(\partial q)^n}$.

1.4.12 Remark (interpretation of Proposition 1.4.11) There is a cute way of rewriting Proposition 1.4.11. By recalling the formal integral of Definition 1.2.14, and making the same *ad hoc* choices as in Definition 1.3.22, the results of Proposition 1.4.11 can be packaged together as saying that we can "differentiate under the formal path integral":

$$\frac{\partial^n}{(\partial q_a)^n}\big[U_\gamma(q_0, q_1)\big] = \int_{\approx\gamma}^{\text{formal}} \frac{\partial^n}{(\partial q_a)^n}\left[\exp\big(-(\hbar\sqrt{-1})^{-1}\mathcal{S}(\varphi)\big)\right] \mathrm{d}\varphi$$

By $\frac{\partial}{\partial q_0}[\mathcal{B}(\varphi)]$, say, we mean the following. The paths φ range among paths with boundary values $\varphi(t_a) = q_a$, $a = 0, 1$. Arbitrarily pick a collection of path $\xi_j : [t_0, t_1] \to \mathbb{R}^d$ with $\xi(t_1) = 0$ and $\xi_j^i(t_0) = \delta_j^i$. Then $\frac{\partial}{\partial q_0^j}[\mathcal{B}(\varphi)]$ is the functional derivative $\lim_{\epsilon\to 0} \epsilon^{-1}\big(\mathcal{B}(\varphi + \epsilon\xi_j) - \mathcal{B}(\varphi)\big) = \mathcal{B}^{(1)}(\varphi) \cdot \xi_j$. By the Euler–Lagrange equations, the choice of ξ does not effect the value of the formal integral. If the integral made sense analytically, the choice of ξ would be "integrated out."

1.5 Independence of the choice of volume-compatible coordinates

Let $\mathcal{N}$ be a smooth manifold with volume form dVol and Lagrangian function $L : \mathbb{R} \times \mathrm{T}\mathcal{N} \to \mathbb{R}$. Choose a nonfocal classical path γ, and suppose that γ can be contained within a coordinate patch of $\mathcal{N}$. Using Fact 1.2.6, suppose that the coordinates are compatible with dVol. Since the formal path integral from Definition 1.3.22 only depends on the value of the Lagrangian in a neighborhood of the chosen classical path, we have enough data to define the formal path integral supported near γ for the quantum mechanics on $\mathcal{N}$ with Lagrangian L.

But, as we mentioned in Definition 1.2.4, jets do not transform as tensors: *a priori*, we have no guarantee that the value of the formal path integral does not depend on the choice of volume-compatible coordinates. In this section, we prove that in fact the formal path integral is independent of the choice of coordinates. This will move us a step closer to having a definition of the formal path integral on an arbitrary manifold. More precisely, we will prove:

1.5.1 Theorem (coordinate independence for the formal path integral) Let $\mathcal{O} \subseteq \mathbb{R}^d$ be an open neighborhood which is *star-shaped*: for each $q \in \mathcal{O}$ and each $s \in [0, 1]$, we have $sq \in \mathcal{O}$. Let $f : \mathcal{O} \to \mathbb{R}^d$ a *locally volume-preserving* smooth function: when restricted to small enough neighborhoods, it is a volume-preserving diffeomorphism onto its image. Pick a Lagrangian $L : \mathbb{R} \times \mathrm{T}\mathbb{R}^d \to \mathbb{R}$ and a path $\tilde{\gamma} : [t_0, t_1] \to \mathcal{O}$ such that $\gamma = f \circ \tilde{\gamma}$ is a classical nondegenerate path for L, and let U_γ be the formal path integral defined in Definition 1.3.22 for (γ, L). Let $\tilde{L} = L \circ (\mathrm{id}, \mathrm{d}f, f) : \mathbb{R} \times \mathrm{T}\mathcal{O} \to \mathbb{R}$ and write $\tilde{U}_{\tilde{\gamma}}$ for the formal path integral for $(\tilde{\gamma}, \tilde{L})$. Then $\tilde{U}_{\tilde{\gamma}}(t_0, q_0, t_1, q_1) = U_\gamma\big(t_0, f(q_0), t_1, f(q_1)\big)$ for $q_0, q_1 \in \mathcal{O}$.

1.5.2 Remark (generalizations) The proof we will give works with almost no changes when f is allowed to depend on an external time parameter; we have not included it only out of a desire to keep notation simpler. More generally, with minimal changes this proof can be adapted to the applications of the formal path integral in the Lagrangian formulation of quantum field theory [JF10c].

Since we have no guarantee that the path integral is free of ultraviolet divergences (Remark 1.3.23), in the conclusion of Theorem 1.5.1 we mean an equality of polynomials in $\delta(0)$.

1.5.3 Remark (simplification) Upon inspection of Definition 1.3.22, the following is clear: the dimension, Morse index, and classical action terms are invariant under arbitrary changes of coordinates; the determinant term is invariant under volume-preserving changes of coordinates; and the individual Feynman diagrams are invariant under affine changes of coordinates. Therefore to prove Theorem 1.5.1 we need only to consider the sum of diagrams, and by composing f with various affine maps, we can suppose that $f(0) = 0$ and $\frac{\partial f^i}{\partial q^j}(0) = \delta^i_j$.

1.5.4 Lemma (infinitesimal transformations suffice) Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a star-shaped open neighborhood, and suppose that $f : \mathcal{O} \to \mathbb{R}^d$ is orientation- and locally-volume-preserving. Suppose furthermore that $f(0) = 0$ and $\frac{\partial f^i}{\partial q^j}(0) = \delta^i_j$. Then there exists a smooth function $F : [0,1] \times \mathcal{O} \to \mathbb{R}^d$ such for each $s \in [0,1]$, $F(s,-)$ is orientation- and locally-volume preserving with $F(s,0) = 0$ and $\frac{\partial F^i}{\partial q^j}(s,0) = \delta^i_j$, and such that $F(0,q) = q$ and $F(1,q) = f(q)$.

1.5.5 Proof of Lemma 1.5.4 Let $f'^i_{\ j} = \frac{\partial f^i}{\partial q^j}$. Then $f' : \mathcal{O} \to \mathrm{Mat}(d)$ satisfies the following conditions:

$$\text{(i)} \quad f'^i_{\ j}(0) = \delta^i_j; \qquad \text{(ii)} \quad \det f' = 1; \qquad \text{(iii)} \quad \frac{\partial f'^i_{\ j}}{\partial q^k} = \frac{\partial f'^i_{\ k}}{\partial q^j}.$$

By the fundamental theorem of calculus, since $\mathcal{O}$ is connected, the function $f : \mathcal{O} \to \mathbb{R}^d$ is completely determined by f' and $f(0) = 0$. Conversely, since $\mathcal{O}$ is simply-connected, any f' satisfying the last of the above three conditions determines some function $f : \mathcal{O} \to \mathbb{R}^d$ with $f(0) = 0$ and $f' = \frac{\partial f}{\partial q}$; by the middle condition, f is locally-volume-preserving.

For f as in the lemma, let $F' : [0,1] \times \mathcal{O} \to \mathbb{R}^d$ be given by $F'(s,q) = \frac{\partial f}{\partial q}(sq)$; this is well-defined since $\mathcal{O}$ is star-shaped. Then for each $s \in [0,1]$, $f' = F'(s,-)$ satisfies conditions (i–iii) above; the third follows by the chain rule. Therefore, for each $s \in [0,1]$, there is a unique function $F(s,-) : \mathcal{O} \to \mathbb{R}^d$ with $\frac{\partial F}{\partial q} = F'$ and $F(s,0) = 0$, and F is smooth in s. When $s = 1$, $F(1,q) = f(q)$, and when $s = 0$, we have $F(0,q) = q$, as $\frac{\partial F^i}{\partial q^j}(0,q) = F'^i_{\ j}(0,q) = \delta^i_j$. Therefore F is the desired homotopy.

1.5.6 Fact (Faà di Bruno's formula) Let W and V be vector spaces, $\mathcal{S} : V \to \mathbb{R}$ any smooth function, and $\mathcal{F} : W \to V$ any smooth function satisfying $\mathcal{F}(0) = 0$. Write $\mathcal{S}^{(n)} : V^{\otimes n} \to \mathbb{R}$ (resp. $\mathcal{F}^{(n)} : W^{\otimes n} \to V$) for the nth Taylor coefficient of $\mathcal{S}$ (resp. $\mathcal{F}$) at the origin. Then:

$$\big(\mathcal{S} \circ \mathcal{F}\big)^{(n)} \cdot \big(\xi_1 \otimes \xi_n\big) = \sum_{\text{partitions } B \text{ of } \{1,\dots,n\}} \big(\mathcal{S}^{(|B|)} \circ \mathcal{F}\big) \cdot \bigotimes_{b \in B} \Big(\big(\mathcal{F}\big)^{(|s|)} \cdot \bigotimes_{j \in s} \xi_j \Big)$$

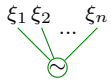
The partition determines how to contract (abbreviated "$\cdot$") the tensors. For a proof, see [Har06].

1.5.7 Proof of Theorem 1.5.1 By Remark 1.5.3 and Lemma 1.5.4, we can suppose that our volume-preserving function is homotopic to the identity: there is a function $F : [0,1] \times \mathcal{O} \to \mathbb{R}^d$ so that for each $s \in [0,1]$, $F(s,-)$ is locally volume-preserving. Then $E = \frac{\partial F}{\partial s}$ makes sense as a vector field on $F(s,\mathcal{O})$, and in particular determines a family of locally-volume-preserving functions $F(s_1,s_2,-)$ with $F(0,s,-) = F(s,-)$ and $F(s_1,s_2,-) \circ F(s_0,s_1,-) = F(s_0,s_2,-)$. Let $L^s = L \circ \big(\mathrm{d}F(0,s,-), F(0,s,-)\big)$ be a Lagrangian on $F(0,s,\mathcal{O})$, and let

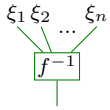
$\gamma^s = F(s,1,-)\circ\tilde{\gamma}$. Let U^s be the formal path integral for L^s and its classical path γ^s. Then $U^0 = U_\gamma$ and $U^1 = \tilde{U}_{\tilde{\gamma}}$, and so to prove Theorem 1.5.1, it suffices to show that $\frac{\partial}{\partial s}\big[U^s\big] = 0$.

And for this, it suffices to consider Theorem 1.5.1 when f is an "infinitesimal change of coordinates". I.e.: $f(q) = q + \epsilon E(q)$, where E is a fixed vector field on $\mathcal{O}$ and $\epsilon^2 = 0$. We will also make the following abuse of notation: we denote the map $\gamma \mapsto f\circ\gamma$ on the space of paths in $\mathcal{O}$ by the same letter as we use for the function $f : \mathcal{O} \to \mathcal{O}$. With all these assumptions and notation, the Feynman diagrams in the path integral $\tilde{U}$ are based on the action $\tilde{\mathcal{S}} = \mathcal{S}\circ f^{-1}$, which is the action determined by the Lagrangian $\tilde{L}$. As we can ignore terms of order ϵ^2, we have $f^{-1}(q) = q - \epsilon E(q)$.

We introduce the following Feynman rules:

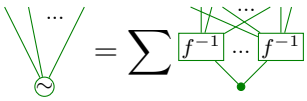


$$= -\tilde{\mathcal{S}}^{(n)}(\tilde{\gamma})\cdot(\xi_1\otimes\cdots\otimes\xi_n), \qquad = \big(\tilde{\mathcal{S}}^{(2)}\big)^{-1},$$



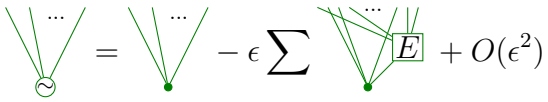
$$= \Big\{x \mapsto \big(f^{-1}\big)^{(n)}(\gamma(x))\cdot\big(\xi_1(x)\otimes\cdots\otimes\xi_n(x)\big)\Big\} \in \mathrm{T}_\gamma\big(\Gamma(Q\to X)\big)$$

Then Fact 1.5.6 with $\mathcal{F} = f^{-1}$ reads:

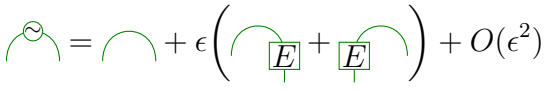


(1.1)

The sum ranges over isomorphism classes of diagrams with ordered leaves but unordered $\boxed{f^{-1}}$ vertices. The • vertex can be of arbitrary valence (non-zero, if the left-hand-side has non-zero valence), and each $\boxed{f^{-1}}$ vertex has one output strand and at least one input strand. The ∾ vertex on the left-hand side and the $\boxed{f^{-1}}$ vertices on the right hand side are evaluated at $\tilde{\gamma}$, and the • vertex on the right hand side is evaluated at $\gamma = f^{-1}\circ\tilde{\gamma}$. Given that $f^{-1}(q) = q - \epsilon E(q)$, we have:

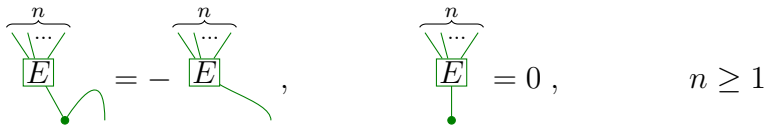


Keeping with our conventions, the $\boxed{E}$ vertices are the obvious derivatives. Moreover:



Finally, we consider the sum of diagrams in $\tilde{U}$, and show in three steps that the extra diagrams — those with $\boxed{E}$s — cancel to first order in ϵ. The first step in the cancellation is essentially immediate: the extra diagrams in the expansions of  and  appear with opposite signs, and the symmetry factors $|\mathrm{Aut}\,\Gamma|$ work out, so we can cancel the diagrams from  with those from  in which the $\boxed{E}$ vertex has precisely one input string. The

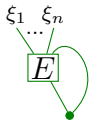
second cancelation is almost as quick. The path γ is classical, and the Green's function is a based loop in each variable, so we can apply Remark 1.4.1:



After the first cancellation, diagrams with either side of the first part of these equalities appear exactly when the $\boxed{E}$ vertex has at least two input strings, and diagrams with a component like the second equality appear whenever the $\boxed{E}$ vertex has at least three incoming strings.

Only in the final cancellation does the fact that f is volume-preserving play a role. After the cancelations in the previous paragraph, the remaining diagrams with $\boxed{E}$s in them have components of the form:

But $f(q) = q + \epsilon E(q)$ is volume-preserving up to $O(\epsilon^2)$ if and only if $\frac{\partial E^i}{\partial q^i} = 0$. As above, we can apply Remark 1.4.1 since $n \geq 1$. Thus:



$$= -\int_0^t \delta(0)\, \frac{\partial^n}{\partial q^{j_1} \dots \partial q^{j_n}} \left[\frac{\partial E^i}{\partial q^i}\right]\Big|_{q=\gamma(\tau)} \mathrm{d}\tau = -\int_0^t \delta(0) \cdot 0 \, \mathrm{d}\tau = 0$$

This completes the proof of Theorem 1.5.1.

1.6 Fubini's theorem for formal path integrals: the semigroup law

In this section, we prove the following "composition law" for formal path integrals:

1.6.1 Theorem (semigroup law for formal path integrals) Fix a Lagrangian L on the configuration space $\mathbb{R}^d$ with $\frac{\partial^2 L}{\partial v^2}$ everywhere positive-definite. Let $\gamma : [t_0, t_1] \to \mathbb{R}^d$ be classical and nonfocal, and pick $t \in [t_0, t_1]$ such that both restrictions $\gamma_0 = \gamma|_{[t_0,t]}$ and $\gamma_1 = \gamma|_{[t,t_1]}$ are nonfocal. Then $\gamma(t)$ is a nondegenerate critical point for $S_{\gamma_0}(t_0, q_0, t, -) + S_{\gamma_1}(t, -, t_1, q_1)$. Furthermore, suppose that the formal path integrals for L have no ultraviolet divergences. Then:

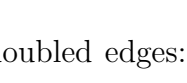
$$\int_{\approx\gamma(t)}^{\text{formal}} U_{\gamma_0}(t_0, q_0, t, q)\, U_{\gamma_1}(t, q, t_1, q_1)\, \mathrm{d}q = U_\gamma(t_0, q_0, t_1, q_1)$$

The integral is to be understood in the sense of Definition 1.2.14. The volume form $\mathrm{d}q$ is the standard volume form on $\mathbb{R}^d$.

1.6.2 Remark (path integrals on curved manifolds) Theorem 1.6.1 provides justification for the *ad hoc* choices in Definition 1.3.22 and so is interesting in its own right. But it also indicates how to define formal path integrals in the absence of global coordinate systems. Let $\mathcal{N}$ be some classical configuration space with Lagrangian L and volume form dVol, and let γ be a classical nonfocal path in $\mathcal{N}$. By Proposition 1.3.19, sufficiently small pieces of γ are nondegenerate, and by Fact 1.2.6 each sufficiently small piece can be included in a coordinate chart such that dVol is the pullback along the chart of the canonical volume form on $\mathbb{R}^{\dim \mathcal{N}}$. (In [Mos65] this is proved even if dVol is allowed to depend on the external time parameter τ, provided the coordinates also are allowed to depend on τ.) Then we can calculate the formal path integrals for each piece, and by Theorem 1.5.1 their values do not depend on the chosen charts. To define the path integral for γ, we integrate the contributions from each piece. By interleaving different ways to cut γ into short pieces, we see via Theorem 1.6.1 that the total path integral for γ does not depend on the choice of cuts.

1.6.3 Remark (overview of proof) To prove Theorem 1.6.1, we must compare two formal expressions in $\hbar\sqrt{-1}$. On the right-hand side, we have (recalling Definition 1.3.22 and suppressing the t_a-dependence):

$$U_\gamma(q_0, q_1) = \left(2\pi\hbar\sqrt{-1}\right)^{-d/2} \left(\sqrt{-1}\right)^{-\eta(\gamma)} \times \exp\left(-(\hbar\sqrt{-1})^{-1} J_\gamma(q_0, q_1)\right) \sqrt{\left|\det \frac{\partial^2[-J_\gamma]}{\partial q_0 \partial q_1}\right|} \sum_\Gamma \frac{(\hbar\sqrt{-1})^{-\chi(\Gamma)} \operatorname{ev}(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

For the Feynman diagrams appearing in U_γ, we will use doubled edges:  $= -\mathcal{S}^{(n)}(\gamma)$ and $= G_\gamma$.

On the left-hand side, we have:

$$\begin{aligned}
&\int_{\approx q_{\mathrm{cr}}(q_0,q_1)}^{\text{formal}} U_{\gamma_0}(q_0, q)\, U_{\gamma_1}(q, q_1)\, \mathrm{d}q \\
&= \int_{\approx q_{\mathrm{cr}}(q_0,q_1)}^{\text{formal}} \left(2\pi\hbar\sqrt{-1}\right)^{-d/2} \left(\sqrt{-1}\right)^{-\eta(\gamma_0)} e^{-(\hbar\sqrt{-1})^{-1} J_{\gamma_0}(q_0,q)} \sqrt{\left|\det \frac{\partial^2[-J_{\gamma_0}]}{\partial q_0 \partial q}\right|} \sum_\Gamma \frac{(i\hbar)^{-\chi(\Gamma)} \operatorname{ev}_0(\Gamma)}{|\operatorname{Aut}\Gamma|} \\
&\quad \times \left(2\pi\hbar\sqrt{-1}\right)^{-d/2} \left(\sqrt{-1}\right)^{-\eta(\gamma_1)} e^{-(\hbar\sqrt{-1})^{-1} J_{\gamma_1}(q,q_1)} \sqrt{\left|\det \frac{\partial^2[-J_{\gamma_1}]}{\partial q \partial q_1}\right|} \sum_\Gamma \frac{(i\hbar)^{-\chi(\Gamma)} \operatorname{ev}_1(\Gamma)}{|\operatorname{Aut}\Gamma|}\, \mathrm{d}q
\end{aligned}$$

Here $q_{\mathrm{cr}}(q_0, q_1)$ is the critical point of $J_{\gamma_0}(q_0, -) + J_{\gamma_1}(-, q_1)$, provided it can be chosen uniquely and is nondegenerate; part of the statement of the theorem (Lemmas 1.6.4 and 1.6.6) is that it is nondegenerate, and its value is $q_{\mathrm{cr}} = \gamma(t)$. The Feynman rules "$\operatorname{ev}_a$" are those for U_{γ_a}. So that we can consider both types of diagrams simultaneously, we will write little 0s and 1s next to components that are to be evaluated via ev_0 or ev_1.

Performing the formal integral following the second part of Definition 1.2.14, we arrive at an expression of the following shape:

$$\begin{aligned}\int_{\approx q_{\mathrm{cr}}}^{\mathrm{formal}} &= \big(2\pi\hbar\sqrt{-1}\big)^{d/2}\big(2\pi\hbar\sqrt{-1}\big)^{-d/2}\big(2\pi\hbar\sqrt{-1}\big)^{-d/2} \\ &\quad\times \big(\sqrt{-1}\big)^{-\eta(\gamma_0)-\eta(\gamma_1)-\eta(q_{\mathrm{cr}})} \times \exp\big(-(\hbar\sqrt{-1})^{-1}(J_{\gamma_0}(q_0,q)+J_\gamma(q,q_1))\big) \\ &\quad\times \left|\det\frac{\partial^2[-J_{\gamma_0}]}{\partial q_0\partial q}\right|^{1/2}\left|\det\frac{\partial^2[-J_{\gamma_1}]}{\partial q\partial q_1}\right|^{1/2}\left|\det\frac{\partial^2[J_{\gamma_0}+J_{\gamma_1}]}{\partial q^2}\right|^{-1/2} \times \sum_{\text{diagrams}}\end{aligned}$$

evaluated at $q = q_{\mathrm{cr}}(q_0, q_1)$. The sum of diagrams involves complicated Feynman rules, with pieces coming from ev_0 and ev_1, and also pieces coming from the finite-dimensional formal integral. We will use dashed lines like  and  for the contribution from the new formal integral, and we will describe all these Feynman rules in Definition 1.6.10.

Thus we can see how to proceed towards verifying Theorem 1.6.1. First, the powers of $2\pi\hbar\sqrt{-1}$ in U_γ and $\int^{\mathrm{formal}} U_{\gamma_0}U_{\gamma_1}$ match. Second, in Lemmas 1.6.4 and 1.6.6 we will show that $J_\gamma(q_0,q_1) = J_{\gamma_0}(q_0,q_{\mathrm{cr}})+J_{\gamma_1}(q_{\mathrm{cr}},q_1)$ and that $\det\frac{\partial^2[-J_\gamma]}{\partial q_0\partial q_1} = \det\frac{\partial^2[-J_{\gamma_0}]}{\partial q_0\partial q}\det\frac{\partial^2[-J_{\gamma_1}]}{\partial q\partial q_1}\big(\det\frac{\partial^2[J_{\gamma_0}+J_{\gamma_1}]}{\partial q^2}\big)^{-1}$, and in Lemma 1.6.8 we will show that $\eta(\gamma) = \eta(\gamma_0)+\eta(\gamma_1)+\eta(q_{\mathrm{cr}})$. Third, we will introduce a few more Feynman rules and perform a diagrammatic calculation to check the agreement of the sums.

1.6.4 Lemma (composition law for classical mechanics) Let $L : \mathbb{R}\times\mathrm{T}\mathbb{R}^d\to\mathbb{R}$ be a Lagrangian such that $\frac{\partial^2 L}{\partial v^2}$ is everywhere positive definite. Fix $t_0 < t < t_1$ and choose open neighborhoods $\mathcal{O}_0, \mathcal{O}, \mathcal{O}_1 \subset \mathbb{R}^d$. Suppose that we have families $\gamma_0 : \mathcal{O}_0\times\mathcal{O}\times[t_0,t]\to\mathbb{R}^d$ and $\gamma_1 : \mathcal{O}\times\mathcal{O}_1\times[t,t_1]\to\mathbb{R}^d$ of classical paths — i.e. for each $(q_0,q,q_1)\in\mathcal{O}_0\times\mathcal{O}\times\mathcal{O}_1$ the paths $\gamma_0(q_0,q;-)$ and $\gamma_1(q,q_1;-)$ are classical. Define the Hamilton principal functions $J_a = \mathcal{S}(\gamma_a)$ for $a = 0, 1$. Then the critical points of $J_0(q_0,-)+J_1(-,q_1)$ are precisely those points $q\in\mathcal{O}$ such that the "glued-together" path $\gamma(q_0,q,q_1;-) : [t_0,t_1]\to\mathbb{R}^d$ given by

$$\gamma(q_0,q,q_1;\tau) = \begin{cases}\gamma_0(q_0,q;\tau), & \tau\le t\\ \gamma_1(q,q_1;\tau), & \tau\ge t\end{cases}$$

is smooth and classical.

1.6.5 Proof of Lemma 1.6.4 The Euler-Lagrange equations are local and closed in τ, so $\gamma(q_0,q,q_1;-)$ is classical if it is smooth. Since the Euler-Lagrange equations are nondegenerate second-order, $\gamma(q_0,q,q_1;-)$ is smooth if and only if $\dot\gamma_0(q_0,q;t) = \dot\gamma_1(q,q_1;t)$. But recall that L is convex on fibers of $\mathrm{T}\mathbb{R}^d\to\mathbb{R}^d$, as $\frac{\partial^2 L}{\partial v^2}$ is positive-definite. Therefore:

$$\left.\frac{\partial L}{\partial v}\right|_{(\tau,v_0,q)} = \left.\frac{\partial L}{\partial v}\right|_{(\tau,v_1,q)} \text{ if and only if } v_0 = v_1.$$

However, $\frac{\partial L}{\partial v}\big(t, \dot\gamma_0(t), \gamma_0(t)\big) = \frac{\partial J_0}{\partial q}(q_0, q)$, and $\frac{\partial L}{\partial v}\big(t, \dot\gamma_1(t), \gamma_1(t)\big) = -\frac{\partial J_1}{\partial q}(q, q_1)$. Thus $\dot\gamma_0(q_0, q; t) = \dot\gamma_1(q, q_1; t)$ if and only if $\frac{\partial}{\partial q}[J_0(q_0, q) + J_1(q, q_1)] = 0$.

1.6.6 Lemma (nondegeneracy and the determinant) Suppose that $\gamma, \gamma_0, \gamma_1$ are as in the statement of Theorem 1.6.1, and define the corresponding Hamilton principal functions $J_\gamma(q_0, q_1) = J_\gamma(t_0, q_0, t_1, q_1)$, $J_0(q_0, q) = J_{\gamma_0}(t_0, q_0, t, q)$, and $J_1(q, q_1) = J_{\gamma_1}(t, q, t_1, q_1)$. Moreover, set $q_{\mathrm{cr}}(q_0, q_1) = \gamma(t_0, q_0, t_1, q_1; t)$. Then q_{cr} is a nondegenerate critical point for $J_0(q_0, -) + J_1(-, q_1)$, and:

$$\frac{\partial^2[-J_\gamma]}{\partial q_0^i \partial q_1^j} = \frac{\partial^2[-J_0]}{\partial q_0^i \partial q^l}\frac{\partial^2[-J_1]}{\partial q^k \partial q_1^j}\left(\left(\frac{\partial^2[J_0 + J_1]}{\partial q \partial q}\right)^{-1}\right)^{kl}\Bigg|_{q=q_{\mathrm{cr}}}$$

1.6.7 Proof of Lemma 1.6.6 This proof follows [DeW92].

The additivity of the action together with Lemma 1.6.4 implies:

$$J_\gamma(q_0, q_1) = \big[J_0(q_0, q) + J_1(q, q_1)\big]_{q=q_{\mathrm{cr}}(q_0, q_1)}$$

Differentiating with respect to q_0 and q_1 gives:

$$\frac{\partial^2 J_\gamma}{\partial q_0^i \partial q_1^j} = \left[\frac{\partial^2 J_0}{\partial q_0^i \partial q^l}\frac{\partial q_{\mathrm{cr}}^l}{\partial q_1^j} + \frac{\partial^2 J_1}{\partial q^k \partial q_1^j}\frac{\partial q_{\mathrm{cr}}^k}{\partial q_0^i} + \frac{\partial^2[J_0 + J_1]}{\partial q^k \partial q^l}\frac{\partial q_{\mathrm{cr}}^k}{\partial q_0^i}\frac{\partial q_{\mathrm{cr}}^l}{\partial q_1^j} + \frac{\partial J_0}{\partial q^l}\frac{\partial^2 q_{\mathrm{cr}}^l}{\partial q_0^i \partial q_1^j} + \frac{\partial J_1}{\partial q^k}\frac{\partial^2 q_{\mathrm{cr}}^k}{\partial q_0^i \partial q_1^j}\right]_{q=q_{\mathrm{cr}}}$$

But $\big[\frac{\partial J_0}{\partial q}(q_0, q) + \frac{\partial J_1}{\partial q}(q, q_1)\big]_{q=q_{\mathrm{cr}}} = 0$ from Lemma 1.6.4, and so the last two terms cancel each other, and we see:

$$\frac{\partial^2 J_\gamma}{\partial q_0^i \partial q_1^j} = \left[\frac{\partial^2 J_0}{\partial q_0^i \partial q^l}\frac{\partial q_{\mathrm{cr}}^l}{\partial q_1^j} + \frac{\partial^2 J_1}{\partial q^k \partial q_1^j}\frac{\partial q_{\mathrm{cr}}^k}{\partial q_0^i} + \frac{\partial^2[J_0 + J_1]}{\partial q^k \partial q^l}\frac{\partial q_{\mathrm{cr}}^k}{\partial q_0^i}\frac{\partial q_{\mathrm{cr}}^l}{\partial q_1^j}\right]_{q=q_{\mathrm{cr}}}$$

Moreover, we differentiate $\big[\frac{\partial J_0}{\partial q}(q_0, q)\big]_{q=q_{\mathrm{cr}}} = -\big[\frac{\partial J_1}{\partial q}(q, q_1)\big]_{q=q_{\mathrm{cr}}}$ with respect to q_0 to conclude:

$$\left[\frac{\partial^2 J_0}{\partial q_0^i \partial q^l}\right]_{q=q_{\mathrm{cr}}} = -\left[\frac{\partial^2[J_0 + J_1]}{\partial q^k \partial q^l}\frac{\partial q_{\mathrm{cr}}^k}{\partial q_0^i}\right]_{q=q_{\mathrm{cr}}}$$

Since γ_0 is nonfocal, the left-hand side is an invertible matrix, and hence so is each component of the right-hand side. In particular, q_{cr} is a nondegenerate critical point of $J_0(q_0, -) + J_1(-, q_1)$.

Moreover:

$$\frac{\partial q_{\mathrm{cr}}^k}{\partial q_0^i} = -\left[\frac{\partial^2 J_0}{\partial q_0^i \partial q^l}\left(\left(\frac{\partial^2[J_0 + J_1]}{\partial q \partial q}\right)^{-1}\right)^{kl}\right]_{q=q_{\mathrm{cr}}}$$

Differentiating $\big[\frac{\partial J_0}{\partial q}(q_0, q)\big]_{q=q_{\mathrm{cr}}(q_0, q_1)} = -\big[\frac{\partial J_1}{\partial q}(q, q_1)\big]_{q=q_{\mathrm{cr}}(q_0, q_1)}$ with respect to q_1 instead gives:

$$\frac{\partial q_{\mathrm{cr}}^l}{\partial q_1^j} = -\left[\frac{\partial^2 J_1}{\partial q^k \partial q_1^j}\left(\left(\frac{\partial^2[J_0 + J_1]}{\partial q \partial q}\right)^{-1}\right)^{kl}\right]_{q=q_{\mathrm{cr}}}$$

Substituting these two equations into the formula for $\frac{\partial^2 J_\gamma}{\partial q_0^i \partial q_1^j}$ completes the proof.

1.6.8 Lemma (Morse index agreement) Let $\gamma : [t_0, t_1] \to \mathbb{R}^d$ be classical and nonfocal with nonfocal restrictions $\gamma_0 : [t_0, t] \to \mathbb{R}^d$ and $\gamma_1 : [t, t_1] \to \mathbb{R}^d$, as in Theorem 1.6.1, and with Morse indexes $\eta(\gamma), \eta(\gamma_0), \eta(\gamma_1)$. Let $\eta(q_{\mathrm{cr}})$ be the Morse index of $q_{\mathrm{cr}} = \gamma(t)$ with respect to the function $q \mapsto J_0(q_0, q) + J_1(q, q_1)$. Then $\eta(\gamma) = \eta(\gamma_0) + \eta(q_{\mathrm{cr}}) + \eta(\gamma_1)$.

1.6.9 Proof of Lemma 1.6.8 Recall (Definition 1.3.16) that for any nonfocal classical path, its Morse index is the dimension of any maximal subspace of the space of based loops on which $\mathcal{S}^{(2)}$ is negative definite. On the other hand, $\eta(q_{\mathrm{cr}})$ is the dimension of any maximal subspace of $\mathbb{R}^d$ on which the Hessian $\frac{\partial^2[J_0+J_1]}{\partial q^2}$ is negative definite. For $a = 0, 1$, there are natural embeddings that extend based loops by 0:

$$\begin{aligned}\mathrm{ext}_0 &: \{\text{based loops with domain } [t_0, t]\} \hookrightarrow \{\text{based loops with domain } [t_0, t_1]\}\\ \mathrm{ext}_1 &: \{\text{based loops with domain } [t, t_1]\} \hookrightarrow \{\text{based loops with domain } [t_0, t_1]\}\end{aligned}$$

Define also the map $\mathrm{ext}_{\mathrm{cr}} : \mathbb{R}^d \to \{\text{based loops with domain } [t_0, t_1]\}$ by:

$$\mathrm{ext}_{\mathrm{cr}}(x)^i(\tau) = \begin{cases} x^j \frac{\partial \gamma_0^i}{\partial q^j}(\tau), & \tau \le t \\ x^j \frac{\partial \gamma_1^i}{\partial q^j}(\tau), & \tau \ge t \end{cases}$$

The continuity of $\exp_{\mathrm{cr}}(x)$ follows from the equality $\frac{\partial \gamma_a^i}{\partial q^j}(t) = \delta^i_j$, $a = 0, 1$.

Then for $a = 0, 1$, it's clear that $\mathcal{S}^{(2)}(\gamma) \cdot \mathrm{ext}_a(\xi)\ \mathrm{ext}_a(\zeta) = \mathcal{S}^{(2)}(\gamma_a) \cdot \xi\zeta$. Moreover, $\mathcal{S}^{(2)}(\gamma) \cdot \mathrm{ext}_{\mathrm{cr}}(x)\ \mathrm{ext}_{\mathrm{cr}}(z) = \frac{\partial^2[S_0+S_1]}{\partial q^i \partial q^j} x^i z^j$, by Lemma 1.4.7 or by direct calculation. On the other hand, the images of the various extension maps are orthogonal: $\mathcal{S}^{(2)}(\gamma) \cdot \mathrm{ext}_0(\xi)\ \mathrm{ext}_1(\zeta) = 0 = \mathcal{S}^{(2)}(\gamma) \cdot \mathrm{ext}_a(\xi)\ \mathrm{ext}_{\mathrm{cr}}(z)$. Let V_0 (resp. V_1) be some maximal subspace of the space of based loops with domain $[t_0, t]$ ($[t, t_1]$) on which $\mathcal{S}^{(2)}(\gamma_0)$ ($\mathcal{S}^{(2)}(\gamma_1)$) is negative definite, and let V_{cr} be a maximal subspace of $\mathbb{R}^d$ on which $\frac{\partial^2[S_0+S_1]}{\partial q^2}$ is negative definite. Then $\mathcal{S}^{(2)}(\gamma)$ is negative definite on $\mathrm{ext}_0(V_0) + \mathrm{ext}_{\mathrm{cr}}(V_{\mathrm{cr}}) + \mathrm{ext}_1(V_1)$. Thus $\eta(\gamma_0) + \eta(q_{\mathrm{cr}}) + \eta(\gamma_1) \le \eta(\gamma)$.

On the other hand, let

$$\begin{aligned}\mathrm{res}_0 &: \{\text{paths with domain } [t_0, t_1]\} \twoheadrightarrow \{\text{paths with domain } [t_0, t]\}\\ \mathrm{res}_1 &: \{\text{paths with domain } [t_0, t_1]\} \twoheadrightarrow \{\text{paths with domain } [t, t_1]\}\end{aligned}$$

be the natural restriction maps. If $\xi : [t_0, t_1] \to \mathbb{R}^d$, $\zeta_0 : [t_0, t] \to \mathbb{R}^d$, and $\zeta_1 : [t, t_1] \to \mathbb{R}^d$ are based loops, then $\mathcal{S}^{(2)}(\gamma) \cdot \xi\, \mathrm{ext}_a(\zeta_a) = \mathcal{S}^{(2)}(\gamma_a) \cdot \mathrm{res}_a(\xi)\zeta_a$, and if $z \in \mathbb{R}^d$, then $\mathcal{S}^{(2)}(\gamma) \cdot \xi\ \mathrm{ext}_{\mathrm{cr}}(z) = \frac{\partial^2[S_0+S_1]}{\partial q^i \partial q^j} \xi^i(t)\, z^j$. Indeed:

$$\xi = \mathrm{ext}_0\big(\mathrm{res}_0\big(\xi - \mathrm{ext}_{\mathrm{cr}}\big(\xi(t)\big)\big)\big) + \mathrm{ext}_{\mathrm{cr}}\big(\xi(t)\big) + \mathrm{ext}_1\big(\mathrm{res}_1\big(\xi - \mathrm{ext}_{\mathrm{cr}}\big(\xi(t)\big)\big)\big)$$

Suppose that $\xi : [t_0, t_1] \to \mathbb{R}^d$ is a based loop such that $\mathcal{S}^{(2)}(\gamma) \cdot \xi\zeta \le 0$ for every $\zeta \in \mathrm{ext}_0(V_0) + \mathrm{ext}_{\mathrm{cr}}(V_{\mathrm{cr}}) + \mathrm{ext}_1(V_1)$. Then $\mathrm{res}_a(\xi) \in V_a$ and $\xi(t) \in V_{\mathrm{cr}}$ by maximality. Thus

$\mathrm{ext}_0(V_0) + \mathrm{ext}_{\mathrm{cr}}(V_{\mathrm{cr}}) + \mathrm{ext}_1(V_1)$ is a maximal negative-definite subspace of the space of based loops with domain $[t_0, t_1]$. Therefore:

$$\eta(\gamma_0) + \eta(q_{\mathrm{cr}}) + \eta(\gamma_1) = \eta(\gamma)$$

1.6.10 Definition (Feynman rules for the formal integral in Theorem 1.6.1) Repeating some of Remark 1.6.3, we wish to compare the value of a certain formal integral with:

$$U_\gamma(q_0, q_1) = \left(2\pi\hbar\sqrt{-1}\right)^{-d/2} e^{\frac{\sqrt{-1}}{\hbar} J_\gamma} \left(\sqrt{-1}\right)^{-\eta(\gamma)} \sqrt{\left|\det \frac{\partial^2[-J_\gamma]}{\partial q_0 \partial q_1}\right|} \sum_\Gamma \frac{(\hbar\sqrt{-1})^{-\chi(\Gamma)} \operatorname{ev}(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

The sum ranges over unmarked diagrams, which we will draw with doubled edges:  $= -\mathcal{S}^{(n)}(\gamma)$ and  $= G_\gamma$.

After applying Lemmas 1.6.4, 1.6.6, and 1.6.8, and Definition 1.2.14, the formal integral is:

$$\begin{aligned}
&\int_{\approx q_{\mathrm{cr}}}^{\text{formal}} U_{\gamma_0}(q_0, q)\, U_{\gamma_1}(q, q_1)\, \mathrm{d}q \\
&\qquad = \left(2\pi\hbar\sqrt{-1}\right)^{-d/2} e^{\frac{\sqrt{-1}}{\hbar} J_\gamma} \left(\sqrt{-1}\right)^{-\eta(\gamma)} \sqrt{\left|\det \frac{\partial^2[-J_\gamma]}{\partial q_0 \partial q_1}\right|} \sum_{\Gamma \text{ marked}} \frac{(\hbar\sqrt{-1})^{-\chi(\Gamma)} \operatorname{ev}(\Gamma)}{|\operatorname{Aut}\Gamma|}
\end{aligned}$$

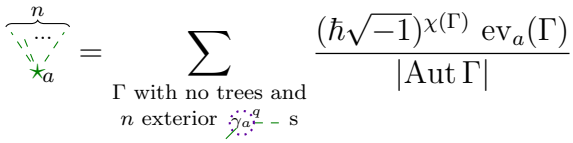
The sum ranges over marked diagrams with precisely two marked vertices $\star_0, \star_1$ of arbitrary valence. Since the integrated variable q ranges over $\mathbb{R}^d$, to be consistent with Definition 1.4.2 we will use dashed edges in these diagrams. The Feynman rules for this formal integral are:

 $= -\dfrac{\partial^n}{\partial q^n}\left[J_0(q_0, q) + J_1(q, q_1)\right]$  $= \left(\dfrac{\partial^2[J_0 + J_1]}{\partial q \partial q}\right)^{-1}$

 $= \dfrac{1}{\sqrt{\left|\det \frac{\partial^2[-J_a]}{\partial q_a \partial q}\right|}} \dfrac{\partial^n}{\partial q^n} \left[\sqrt{\left|\det \dfrac{\partial^2[-J_a]}{\partial q_a \partial q}\right|} \sum_\Gamma \dfrac{(\hbar\sqrt{-1})^{-\chi(\Gamma)} \operatorname{ev}_a(\Gamma)}{|\operatorname{Aut}\Gamma|}\right]$

Here $a = 0, 1$, ev_a denotes the Feynman rules used to define U_{γ_a}, and every expression is evaluated at $q = q_{\mathrm{cr}}(q_0, q_1)$.

Proposition 1.4.11 allows us to expand the new vertices in terms of old ones::

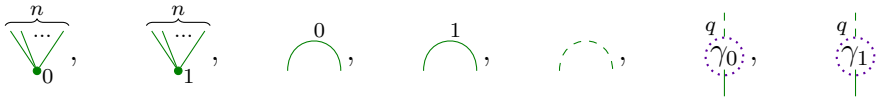


In the formula for $\star_a$, the diagrams in the sum may be disconnected, but no connected component can be a tree. In total, each diagram must have n occurrences of $\frac{\partial \gamma_a}{\partial q}$. The two sums in the formula for $\bullet$ are the same, but in one we evaluate each diagram with respect to the Feynman rules for U_{γ_0} and contract each leaf with $\frac{\partial \gamma_0}{\partial q}$, and in the other we use U_{γ_1} and $\frac{\partial \gamma_1}{\partial q}$.

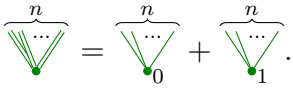
Finally, we modify the notation slightly so that we can drop the "ev_a" notation but still consider diagrams with both $\star_0, \star_1$ expanded out:

1.6.11 Remark (notational conflict with Section 1.8) We will use the above Feynman rules for the remainder of this section. In Section 1.8, it will be convenient to reuse the dashed arc  for something different.

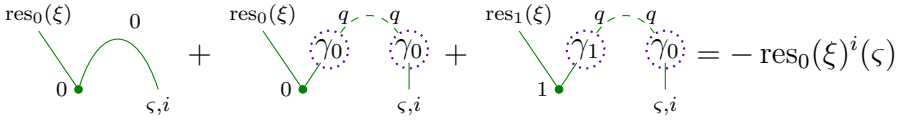
1.6.12 Remark (remainder of proof of Theorem 1.6.1) With this notation, we are left comparing two sums of diagrams. On the one hand, the sum of diagrams in $\int_{\approx q_{\mathrm{cr}}}^{\text{formal}} U_{\gamma_0} U_{\gamma_1}$ ranges over all diagrams made from the following ingredients (in the vertices, we require $n \geq 3$):



On the other hand, the sum of diagrams in U_γ ranges over diagrams built out of:

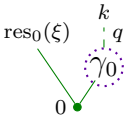
Since $\int_{t_0}^{t_1} = \int_{t_0}^{t} + \int_{t}^{t_1}$, the vertices in U_γ decompose as:  . To match the edges, we will show in Corollary 1.6.15 that a certain sum of six terms satisfies the defining relation of  . That will complete the proof of Theorem 1.6.1.

1.6.13 Lemma (an equality with res**)** We claim that if $\xi : [t_0, t_1] \to \mathbb{R}^d$ has $\xi(t_0) = 0$, then:



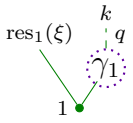
$= -\operatorname{res}_0(\xi)^i(\varsigma)$

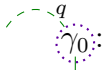
1.6.14 Proof of Lemma 1.6.13 Integrating by parts gives:

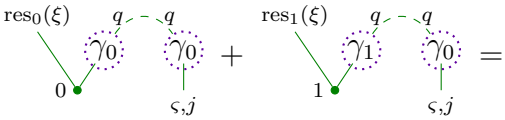


$$= 0 - \left(\xi^i \frac{\partial^2 L}{\partial v^i \partial v^j} \frac{\partial \dot{\gamma}_0^j}{\partial q^k} + \xi^i \frac{\partial^2 L}{\partial v^i \partial q^j} \frac{\partial \gamma_0^j}{\partial q^k} \right) \Bigg|_t =$$

$$= -\xi^i(t) \frac{\partial}{\partial q^k} \left[\frac{\partial L}{\partial v^i} \Big|_{\gamma_0(t_0,q_0,t,q;t)} \right] = \xi^i(t) \frac{\partial^2 [-J_0]}{\partial q^k \partial q^i}$$

Similarly 

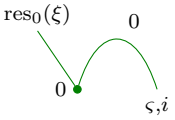
$= \xi^i(t) \frac{\partial^2 [-J_1]}{\partial q^k \partial q^i}$. We contract with 



$$= \left(\xi^i(t) \frac{\partial^2 [-J_0]}{\partial q^k \partial q^i} + \xi^i(t) \frac{\partial^2 [-J_1]}{\partial q^k \partial q^i} \right) \left(\left(\frac{\partial^2 [J_0 + J_1]}{\partial q \partial q} \right)^{-1} \right)^{kl} \frac{\partial \gamma_0^j}{\partial q^l}(\varsigma) =$$

$$= -\xi^i(t) \frac{\partial \gamma_0^j}{\partial q^i}(\varsigma)$$

On the other hand, by another integration by parts and recalling Proposition 1.3.14:



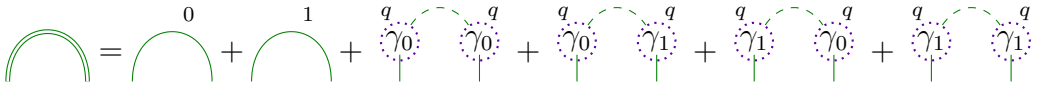
$$= \int_{t_0}^{t} \xi^k(\tau)\, \mathcal{D}_{1,jk}[G_0^{ij}(\varsigma,\tau)]\, d\tau - \left[\xi^k(\tau) \frac{\partial^2 L}{\partial v^j \partial v^k} \frac{\partial G_0^{ij}}{\partial \tau}(\varsigma,\tau) + \xi^k(\tau) \frac{\partial^2 L}{\partial q^j \partial v^k} \frac{\partial G_0^{ij}}{\partial \tau}(\varsigma,\tau) \right]_{\tau=t}$$

$$= -\int_{t_0}^{t} \xi^k(\tau)\, \delta_k^i \delta(\tau - \varsigma)\, d\tau - \xi^k(t) \frac{\partial^2 L}{\partial v^j \partial v^k} \frac{\partial G_0^{ij}}{\partial \tau}(\varsigma, t) =$$

$$= -\xi^i(\varsigma) + \xi^k(t) \frac{\partial \gamma_0^i}{\partial q^k}(\varsigma)$$

This proves Lemma 1.6.13.

1.6.15 Corollary (comparing the edges) Along with a similar formula for $\mathrm{res}_1(\xi)$ when $\xi(t_1) = 0$, Lemma 1.6.13 implies that:



as the right-hand side satisfies the defining relation of the left-hand side (the boundary conditions are clear). Because this sum comprises precisely the possible ways to connect vertices in the sum of diagrams in $\int^{\text{formal}} U_{\gamma_0} U_{\gamma_1}$, the sums of diagrams in U_γ and in $\int^{\text{formal}} U_{\gamma_0} U_{\gamma_1}$ match identically. This completes the proof of Theorem 1.6.1.

1.7 Nonrelativistic quantum mechanics is free of ultraviolet-divergences

Although Theorem 1.5.1 is formally true even when the formal path integral of Definition 1.3.22 has ultraviolet divergences, in Theorem 1.6.1 we required that all ultraviolet divergences cancel. Our goal in this section is to prove in Theorem 1.7.4 that the formal path-integral quantization of a nonrelativistic charged particle is free of ultraviolet divergences.

Ultraviolet divergences in quantum-mechanical path integrals have been addressed before. Notably, Manuel and Tarrach [MT94] consider the case of motion on flat space with a mildly divergent potential (too divergent and the problem is hopeless: the corresponding Schrödinger operator is not self-adjoint). They handle the corresponding path-integral divergences through a system of regularization and renormalization with counterterms, just as is standardly done in quantum field theory, and achieve finite physical results. Closer to our approach, Kleinert and Chervyakov [KC99, KC00a, KC00b, KC01, KC02, KC03] discuss the divergences that arise from using "the wrong coordinates" — as we will see in this section, divergences do not arise when using the "correct" volume form — within the framework of dimensional renormalization. In both approaches, Planck's constant $\hbar$ is set to unity from the beginning, forcing the authors to incorporate their perturbation parameters into the potential energy functions. Our semiclassical ($\hbar \approx 0$) approach allows us more freedom to consider quantizations of very nonlinear classical theories, and in particular Riemannian manifolds in which the metric is not flat.

1.7.1 Definition (divergence-free) Even in "divergence-free" path integrals, individual Feynman diagrams may represent divergent integrals. We say that a path integral is *free of ultraviolet divergences* if for each n, the divergent parts of the integrals that contribute to the coefficient of $\hbar^n$ in the path integral cancel. One can express this in several equivalent ways. By Lemma 1.7.3, the divergent part of any of our Feynman diagrams is always of the form "$\int \big(\delta(0)\big)^m [\dots]$", where "$\int$" represents some finite-dimensional integral and "$[\dots]$" some piecewise-continuous bounded function. Our informal approach will be to

write $\int \big(\delta(0)\big)^m [\dots] = \big(\delta(0)\big)^m \int [\dots]$, and assert that at the end of the day the polynomial in $\delta(0)$ is degree-zero.

Although somewhat obscured, our proof of Theorem 1.7.4 finds coordinates in which to write the sum of all diagrams with the same Euler characteristic as a single integral, and show that for this integral, the integrand is bounded. Alternately, one can introduce a small parameter ϵ, replace the Green's function by a smoothing that differs from the true Green's function only by ϵ, and prove that as $\epsilon \to 0$, the limit of the corresponding "regularized" formal path integral is a well-defined power series in $\hbar$.

1.7.2 Example (geodesic motion on $\mathbb{R}$ in the wrong coordinates) To begin, we illustrate that ultraviolet divergences really can be a problem.

Consider free motion on the line in exponential coordinates. Under the map $x \mapsto q = \exp x$, the usual Lagrangian $L = \frac{1}{2}\dot{x}^2$ transforms to:

$$L(v, q) = \frac{1}{2}\frac{v^2}{q^2}$$

Here q is the usual coordinate on $\mathcal{N} = \mathbb{R}_{>0}$ and v is the corresponding fiber coordinate — we can trivialize the tangent bundle as $\mathrm{T}\mathcal{N} = \mathbb{R} \times \mathbb{R}_{>0}$. The Euler–Lagrange equations of motion in these coordinates are:

$$\frac{\ddot{\gamma}}{\gamma^2} - 2\frac{\dot{\gamma}^2}{\gamma^3} = -\frac{\dot{\gamma}^2}{\gamma^3}$$

and so $\gamma(\tau) = q_0^{(t_1-\tau)/(t_1-t_0)} q_1^{(\tau-t_0)/(t_1-t_0)}$ is the unique solution with $\gamma(t_0) = q_0$, $\gamma(t_1) = q_1$. For notational convenience, we write $\ell = \frac{1}{t_1-t_0} \log \frac{q_1}{q_0}$. Then $\dot{\gamma}/\gamma = \ell$, and the Green's function $G(\varsigma, \tau)$ must satisfy:

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau} - \ell\right)^2 G = -\delta(\tau - \varsigma)\, \gamma(\tau)^2$$

The solution with $G(\varsigma, t_0) = 0 = G(\varsigma, t_1)$ is:

$$G(\varsigma, \tau) = \gamma(\varsigma)\, \gamma(\tau) \left(\frac{1}{2}(\varsigma + \tau) - \frac{\varsigma\tau}{t_1 - t_0} - \frac{1}{2}|\tau - \varsigma|\right)$$

Then G, $\frac{\partial G}{\partial \varsigma}$, $\frac{\partial G}{\partial \tau}$ are bounded, but:

$$\frac{\partial^2 G}{\partial \varsigma \partial \tau} = \ell^2\, G - \ell\, \gamma(\varsigma)\, \gamma(\tau) \left(\frac{\varsigma + \tau}{t_1 - t_0}\right) + \gamma(\varsigma)\, \gamma(\tau) \left(-\frac{1}{t_1 - t_0} + \delta(\tau - \varsigma)\right)$$

The derivatives of the Lagrangian are:

$$\left.\frac{\partial^n L}{\partial v^k \partial q^{n-k}}\right|_{(v,q)=(\dot{\gamma}(\tau),\gamma(\tau))} = (-1)^{n-k} \frac{(n+1-k)!}{(2-k)!} \frac{\ell^k}{\gamma(\tau)^n}$$

where by convention $(-m)! = \pm\infty$ for m a positive integer, so $1/(2-k)!$ vanishes for $k \geq 3$.

We can now (try to) evaluate any diagram we wish. There are no diagrams with one loop, and three diagrams with two loops:

"barbell" "theta" "infinity sign"

Whenever the Green's function is twice-differentiated, it contributes a δ-function to the integrand. These become problems for loops in diagrams, as then there can be as many δ-functions as integration variables. In particular, the values of the "infinity" and "theta" graphs are of the form [finite]$\delta(0)$ + [finite]. In the "barbell," the two loops do not overlap, and each one can diverge. Thus, in addition to terms of the form [finite]$\delta(0)$ + [finite], the barbell has a divergence equal to $\frac{(t_1-t_0)^3}{24}(\delta(0))^2$, after performing all integrals. This term will not be canceled by divergences from other diagrams.

1.7.3 Lemma (divergences in quantum mechanics are polynomials in $\delta(0)$, and live on loops in Feynman diagrams) We now explain why no Feynman diagram in Definition 1.3.22 evaluates to a divergence worse than a polynomial in $\delta(0)$. Since quantum mechanics is "one-dimensional" as a quantum field theory, the Green's function $G(\varsigma, \tau)$ is continuous, and has a singularity like the absolute value. In particular, the first derivative of G in either variable ς, τ is bounded, and has a discontinuity like Heaviside's step function Θ. Only when we take a second derivative do we meet Dirac's delta function δ:

$$\frac{\partial^2}{\partial\varsigma\partial\tau}\Big[\ \underset{\varsigma,i \quad \tau,j}{\frown}\ \Big] = \delta(\varsigma-\tau)\left(\left(\frac{\partial^2 L}{\partial v^2}\big(\tau,\dot\gamma(\tau),\gamma(\tau)\big)\right)^{-1}\right)^{ij} + \text{finite}$$

But a quick inspection of Definition 1.3.3 shows that each end of an arc $G =$  is differentiated at most once by its connecting vertex. (If the Lagrangian were allowed to depend on acceleration, we could have more differentiation of G.) Therefore the only way a non-bounded function can work its way into an evaluation of a Feynman diagram is if both ends of a Green's function are differentiated, and this introduces nothing worse than a δ-function.

By and large, Dirac-delta functions are not a problem in integrals: they simply identify integration variables. So the $\delta(\varsigma-\tau)$ above is a problem only when ς and τ are already identified. This can happen only when the Feynman diagram has a loop of Green's functions, all of which are differentiated twice: ultraviolet divergences live on loops in Feynman diagrams.

1.7.4 Theorem (Nonrelativistic quantum mechanics is divergence-free) Let L be a Lagrangian on $\mathbb{R}^d$ of the form $L(\tau, v, q) = \frac{1}{2}a_{ij}(\tau, q)\, v^i v^j + b_i(\tau, q)v^i + c(\tau, q)$, where $a_{ij}(\tau, q) = a_{ji}(\tau, q)$ and $\det a(\tau, q) = 1$ for all $(\tau, q) \in \mathbb{R}^{d+1}$. Then the formal path integral for L has no ultraviolet divergences.

In particular, the problem in Example 1.7.2 really is that the coordinates are wrong.

1.7.5 Remark (interpretation of a, b, c) In order for the Morse index in Definition 1.3.22 to be defined, we in fact need $a_{ij}(q)$ to be positive-definite for each q, so it provides some metric on $\mathbb{R}^d$, and $\frac{1}{2}a_{ij}(q)\,v^i v^j$ is a "kinetic energy" term. The one-form b is a "magnetic potential" on $\mathbb{R}^d$, and the function c is an "electric potential," so the Lagrangian in Theorem 1.7.4 describes the classical nonrelativistic motion of a charged particle moving through an external electromagnetic field on a curved background.

1.7.6 Proof of Theorem 1.7.4 Since the Lagrangian L is quadratic in velocity, no vertex differentiates more than two of its incoming edges. Thus, divergent loops in the same Feynman diagram cannot intersect. Our strategy, then, is as follows. For each Euler characteristic, we record all possible Feynman diagrams, expand the summations implicit in each vertex (Definition 1.3.3), and keep only the divergent diagrams, labeling individually the divergent loops. We can then grade each diagram by the multiset that records the number of external edges attached to each divergent loop. By "pulling the loops far away from each other," one can express the sum of divergent Feynman diagrams as essentially the exponential of a sum of individual divergent loops, contracted with some convergent parts.

In particular, to prove Theorem 1.7.4, it suffices to prove that for each n, we have:

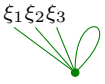
$$\sum_{\substack{\text{loops } \Gamma \text{ with} \\ n \text{ exterior edges}}} \frac{\operatorname{ev}(\Gamma)}{|\operatorname{Aut}\Gamma|} = \text{finite}$$

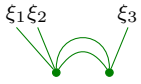
The n external edges are ordered and contracted with based loops $\xi_1, \ldots, \xi_n$. For example, the left-hand sides for $n = 1$, 2, and 3 are:

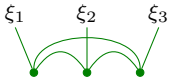
$n = 1$: $\frac{1}{2}$ 

$n = 2$: $\frac{1}{2}$ 

$+ \frac{1}{2}$ 

$n = 3$: $\frac{1}{2}$ 

$+ \frac{1}{2}$ 

$+$ 

We first prove the claim when $n = 1$. By Definition 1.3.3,

$$\frac{1}{2}\ \text{  }\ = \frac{1}{2}\int_0^t \delta(\tau - \tau) \times \frac{\partial a_{jk}}{\partial q^i}\left(a(\tau, q)^{-1}\right)^{jk}\Big|_{q=\gamma(\tau)} \xi(\tau)^i \,\mathrm{d}\tau + \text{finite}$$

But by assumption, $\det a = 1$, and so $0 = \frac{1}{\det a}\frac{\partial}{\partial q}\left[\det a(\tau, q)\right] = \frac{\partial a_{jk}}{\partial q^i}\left(a(\tau, q)^{-1}\right)^{jk}$. Therefore the divergent part of  vanishes.

Differentiating again shows that the $n = 2$ term is finite, and in general:

$$\sum_{\substack{\text{loops } \Gamma \text{ with} \\ n \text{ exterior edges} \\ \text{connected to } \xi_1, \dots, \xi_n}} \frac{\operatorname{ev}(\Gamma)}{|\operatorname{Aut} \Gamma|}$$

$$= \int_{n=0}^{t} \delta(\tau - \tau) \frac{1}{\sqrt{\det a(\tau, q)}} \frac{\partial^n \left[\sqrt{\det a(\tau, q)}\right]}{\partial q^{i_1} \cdots \partial q^{i_n}} \bigg|_{q=\gamma(\tau)} \xi_1^{i_1}(\tau) \cdots \xi_n^{i_n}(\tau) \, \mathrm{d}\tau + \text{finite}$$

$$= \text{finite}$$

This completes the proof of Theorem 1.7.4.

1.8 Schrödinger's equation on a manifold

Combining Theorems 1.5.1, 1.6.1, and 1.7.4, we have shown that for any Lagrangian $L : \mathbb{R} \times \mathrm{T}\mathcal{N} \to \mathbb{R}$ which is quadratic-plus-lower in velocity and for any nonfocal classical path γ, there is a well-defined formal path integral U_γ which is free of ultraviolet divergences and satisfies a gluing identity. In this section we will show that U_γ satisfies the appropriate Schrödinger's equation. In Section 1.9 we will show that U_γ satisfies the correct initial value problem.

1.8.1 Definition (Schrödinger operator) Let $\mathcal{N}$ be a smooth finite-dimensional manifold equipped with: a Riemannian metric a (determining the *mass* of an electron moving in $\mathcal{N}$); a one-form b (the *magnetic potential* the electron moves through); and a function c (the *electric potential*). All of these data may also depend on an external time parameter $t \in \mathbb{R}$.

Continue to set $\mathrm{dVol} = \sqrt{\det a}$, and choose volume-compatible coordinates q^i, so that $a = a_{ij}(t, q)$, $b = b_i(t, q)$, and $c = c(t, q)$. The *nonrelativistic Schrödinger operator* is the $\mathbb{R}[\![\hbar\sqrt{-1}]\!]$-valued second-order differential operator $\hat{H}$ on $\mathcal{N}$ given locally by:

$$\hat{H}_{t,q} = \frac{1}{2} \left(\hbar\sqrt{-1} \frac{\partial}{\partial q^i} + b_i(t, q) \right) \left(a^{-1}(t, q)\right)^{ij} \left(\hbar\sqrt{-1} \frac{\partial}{\partial q^j} + b_j(t, q) \right) - c(t, q)$$

The first term is a shifted version of the Laplace–Beltrami operator in the presence of a one-form. In coordinates that are not compatible with the volume form $\mathrm{dVol} = \sqrt{\det a}$, the formula for $\hat{H}_{t,q}$ is more complicated.

We will prove:

1.8.2 Theorem (Schrödinger's equation for formal-path-integral quantum mechanics) Let $\mathcal{N}$ be equipped with a metric a, a one-form b, and a function c, all of which may be time-dependent as in Definition 1.8.1. Choose a nonfocal classical path

$\gamma : [t_0, t_1] \to \mathcal{N}$ (sending $t_0 \mapsto q_0$ and $t_1 \mapsto q_1$) for the classical mechanics determined by the Lagrangian $L : \mathbb{R} \times \mathrm{T}\mathcal{N} \to \mathbb{R}$ given in local coordinates by:

$$L(t, v, q) = \frac{1}{2}\, a_{ij}(t, q)\, v^i v^j + b_i(t, q)\, v^i + c(t, q)$$

Following Definition 1.3.22, construct the formal path integral $U_\gamma(t_0, q_0, t_1, q_1)$ supported near γ. Then U_γ satisfies *Schrödinger's equation with a curvature term*:

$$\hbar\sqrt{-1}\,\frac{\partial}{\partial t_1}\big[U_\gamma(t_0, q_0, t_1, q_1)\big] = \hat{H}_{t_1, q_1}\big[U_\gamma(t_0, q_0, t_1, q_1)\big] - \frac{1}{8}\, R(t_1, q_1)\, U_\gamma(t_0, q_0, t_1, q_1)$$

The function R is given in local volume-compatible coordinates by the formula:

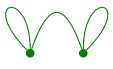
$$\begin{aligned} R(t, q) \;=\; & (a^{-1})^{ii'}\,(a^{-1})^{jj'}\,\frac{\partial^2 a_{i'j'}}{\partial q^i \partial q^j} \;-\; (a^{-1})^{ii'}\,(a^{-1})^{jj'}\,(a^{-1})^{kk'}\,\frac{\partial a_{ij}}{\partial q^{i'}}\,\frac{\partial a_{j'k}}{\partial q^{k'}} \\ & - \;(a^{-1})^{ii'}\,(a^{-1})^{jj'}\,(a^{-1})^{kk'}\,\frac{\partial a_{ij}}{\partial q^{k}}\,\frac{\partial a_{j'k'}}{\partial q^{i'}} \end{aligned}$$

1.8.3 Remark (curvature terms) The function R is related to the scalar curvature for the metric a. It has been recognized by many authors (e.g. [AD99]) that different equally reasonable approaches to the problem of defining the path integral lead to different "curvature" corrections to Schrödinger's equation — the factor of $\frac{1}{8}$ in Theorem 1.8.2 might be replaced by other factors depending on the definition of path integral used.

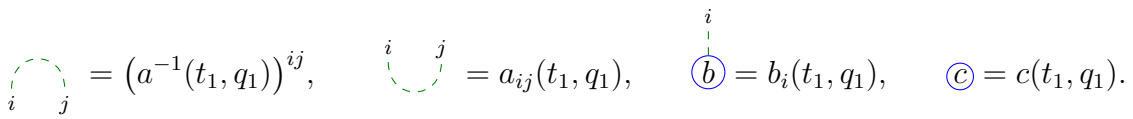
1.8.4 Remark (connected diagrams) It will be convenient to work not with U_γ but with $V_\gamma = \sqrt{-1}\hbar \log\Big(\big(2\pi\hbar\sqrt{-1}\big)^{\dim \mathcal{N}}\big(\sqrt{-1}\big)^{\eta(\gamma)} U_\gamma\Big)$. Recall Remark 1.2.18: the logarithm of a sum over all diagrams is a sum over connected diagrams. Given a *connected* diagram Γ, its *first Betti number* is $\beta(\Gamma) = 1 - \chi(\Gamma)$. Then:

$$V_\gamma = \frac{\hbar\sqrt{-1}}{2}\,\mathrm{tr}\log\left|\frac{\partial^2[-J_\gamma]}{\partial q_0 \partial q_1}\right| + \sum_{\substack{\text{connected Feynman diagrams } \Gamma \\ \text{with no bivalent vertices}}} \frac{(\hbar\sqrt{-1})^{\beta(\Gamma)}\,\mathrm{ev}(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

For the remainder of this section, we will work only with connected diagrams, and adopt the convention that all diagrams Γ are implicitly weighted by $\big(\hbar\sqrt{-1}\big)^{\beta(\Gamma)}$. Thus:

$$V_\gamma = \bullet + \frac{\hbar\sqrt{-1}}{2}\,\mathrm{tr}\log\left|\frac{\partial^2[-J_\gamma]}{\partial q_0 \partial q_1}\right| + \frac{1}{8}$$  $$+ \frac{1}{8}$$  $$+ \frac{1}{12}$$  $$+ \ldots$$

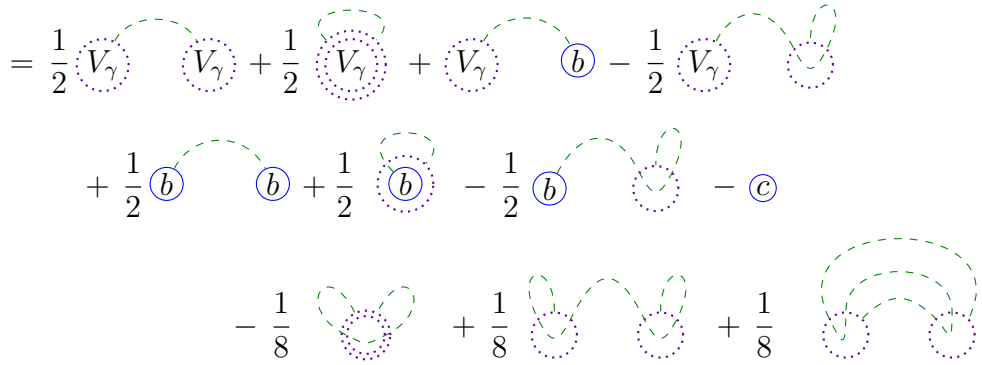
1.8.5 Definition (Feynman rules for $\hat{H}$) Set $d = \dim \mathcal{N}$. We will continue to use the notation of Section 1.4: solid edges carry paths in $\mathbb{R}^d$, and dashed edges carry vectors in $\mathbb{R}^d$. In order to write out the action of $\hat{H}$ in terms of Feynman diagrams, we introduce a few more Feynman rules. Note that the first of these conflicts with the notation from Section 1.6.



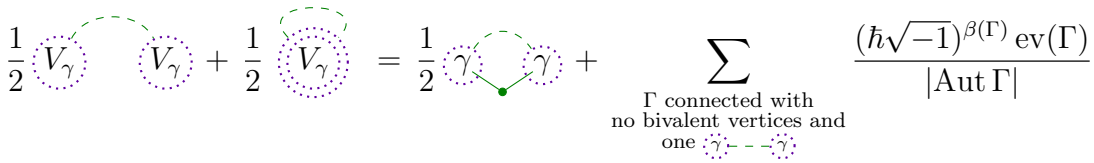
Recall the dotted-circle notation for derivatives from Definition 1.4.2. We will need derivatives with respect to both q_0 and q_1, but the latter will be much more common, and so we will occasionally leave off the q_1s to avoid visual clutter.

Working out the following (and taking advantage of Remark 1.8.4 to suppress all factors of $\hbar\sqrt{-1}$) is then straightforward:

$\big(U_\gamma(t_0, q_0, t_1, q_1)\big)^{-1} \hat{H}_{t_1,q_1}\big[U_\gamma(t_0, q_0, t_1, q_1)\big]$

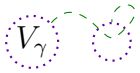


1.8.6 Proposition (simplifying $U^{-1}\hat{H}U$) The results of Section 1.4 allow us to quickly evaluate all derivatives of V_γ in terms of diagrams. In particular:

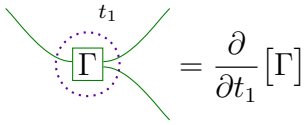


On the right-hand side, the first term is the second-derivative of • from the second term on the left-hand side. All derivatives are with respect to q_1, and of course the γs are connected to the rest of the diagram by solid edges. In the sum, the diagrams which would become disconnected if the dashed edge were cut are from the first term on the right-hand side, and the ones which would remain connected are from the second term. The combinatorics are as in Proposition 1.4.11.

Similarly, V_γ b is a sum over all connected diagrams with one γ b and no

bivalent vertices, and  is a sum of diagrams with one  . Note that in all cases, a univalent vertex can exist if it connects directly to a  (otherwise it would be 0). A zero-valent vertex cannot appear, as it would necessarily be disconnected from the  s.

1.8.7 Definition (derivatives with respect to t_1) Finally, we introduce a Feynman rule for differentiation with respect to t_1:



It satisfies a product rule like Lemma 1.4.4.

Continuing the convention from Remark 1.8.4 in which we only work with connected diagrams, and each diagram Γ is weighted by $\big(\hbar\sqrt{-1}\big)^{\beta(\Gamma)}$, the left-hand side of Schrödinger's equation is:

$$\big(U_\gamma(t_0,q_0,t_1,q_1)\big)^{-1}\,\hbar\sqrt{-1}\frac{\partial}{\partial t_1}\big[U_\gamma(t_0,q_0,t_1,q_1)\big] = \quad$$

Our strategy will be to evaluate this as a sum of diagrams. Thus, we turn now to understanding the derivatives of diagrammatic components with respect to t_1.

1.8.8 Fact (Hamilton–Jacobi equation) Given a Lagrangian $L : \mathbb{R}\times \mathrm{T}\mathcal{N} \to \mathbb{R}$ which is convex along fibers and increases faster than linearly in all fiber directions, the corresponding *Hamiltonian* is the function $H : \mathbb{R}\times \mathrm{T}^*\mathcal{N} \to \mathbb{R}$ given by the Legendre transform of L. When $L(t,v,q) = \frac{1}{2}a_{ij}(t,q)\,v^iv^j + b_i(t,q)\,v^i + c(t,q)$, we have $H(t,p,q) = \frac{1}{2}\big(a^{-1}(t,q)\big)^{ij}\,(p_i-b_i)\,(p_j-b_j) - c(t,q)$.

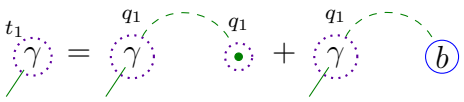
The Hamilton principal function $J_\gamma(t_0,q_0,t_1,q_1)$ satisfies the *Hamilton–Jacobi equations*, as can be checked directly by differentiating under the integral sign:

$$-\frac{\partial J_\gamma}{\partial t_1} = H\left(t_1, \frac{\partial J_\gamma}{\partial q_1}, q_1\right) = \frac{(a^{-1})^{ij}}{2}\left(\frac{\partial J_\gamma}{\partial q_1^i} - b_i\right)\left(\frac{\partial J_\gamma}{\partial q_1^j} - b_j\right) - c$$

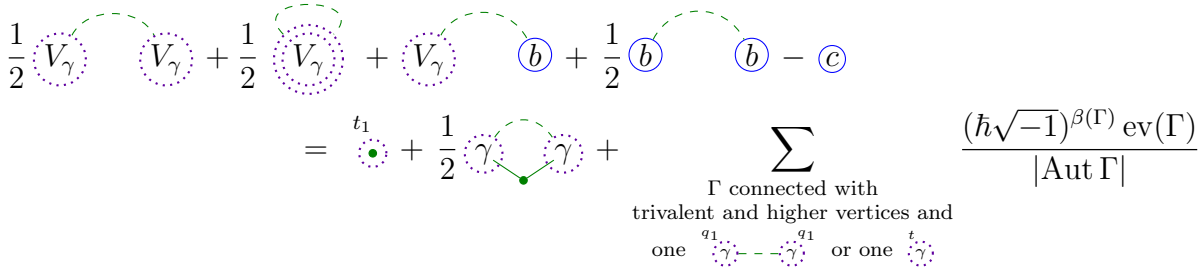
In terms of our diagrams:

1.8.9 Lemma ($\partial\gamma/\partial t_1$) It is straightforward to check that $\eta^j(\tau) = \frac{\partial\gamma^j}{\partial t_1}(\tau)$ is a solution to the equation $\mathcal{D}_{ij}[\eta^j] = 0$, where $\mathcal{D}_{ij}$ is the second-order operator form Definition 1.3.5. As in Lemma 1.3.7, the nonfocality of γ implies that any such solution is a linear combination of $\frac{\partial\gamma}{\partial q_0}$ and $\frac{\partial\gamma}{\partial q_1}$, determined by its boundary conditions. In this case, $\frac{\partial\gamma^j}{\partial t_1}(t_0) = 0$ and $\frac{\partial\gamma^j}{\partial t_1}(t_1) = -\dot\gamma^j(t_1)$. Thus $\frac{\partial\gamma^j}{\partial t_1} = -\dot\gamma^k(t_1)\,\frac{\partial\gamma^j}{\partial q_1^k}$.

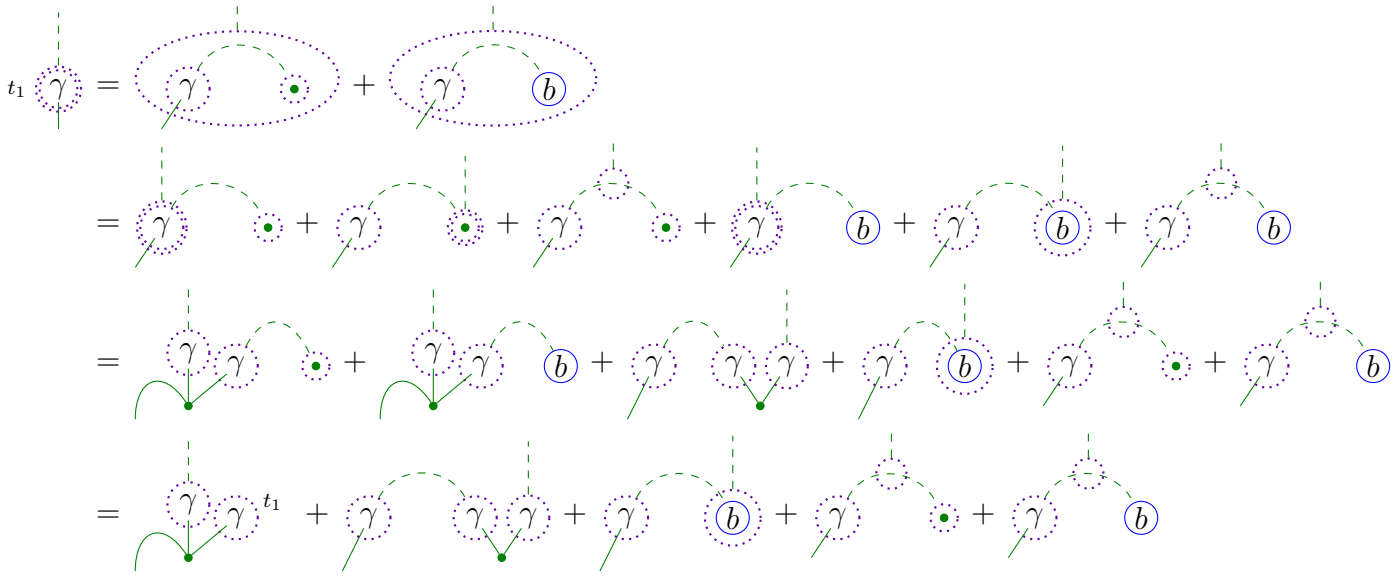
On the other hand, by Fact 1.3.13, $\frac{\partial J_\gamma}{\partial q_1^i} = a_{ij}(t_1, q_1)\,\dot\gamma^j(t_1) + b_i(t_1, q_1)$. Therefore $\frac{\partial \gamma^i}{\partial t_1} = -\frac{\partial \gamma^i}{\partial q_1^j}\,(a^{-1})^{jk}\,\frac{\partial J_\gamma}{\partial q_1^k} + \frac{\partial \gamma^i}{\partial q_1^j}\,(a^{-1})^{jk}\,b_k$, or in diagrams:



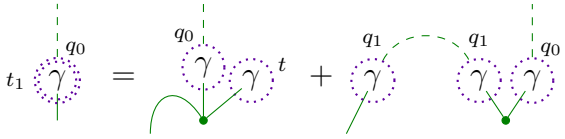
1.8.10 Corollary ($U^{-1}\hat H U$ again) Combining Proposition 1.8.6, Fact 1.8.8, and Lemma 1.8.9, we see:



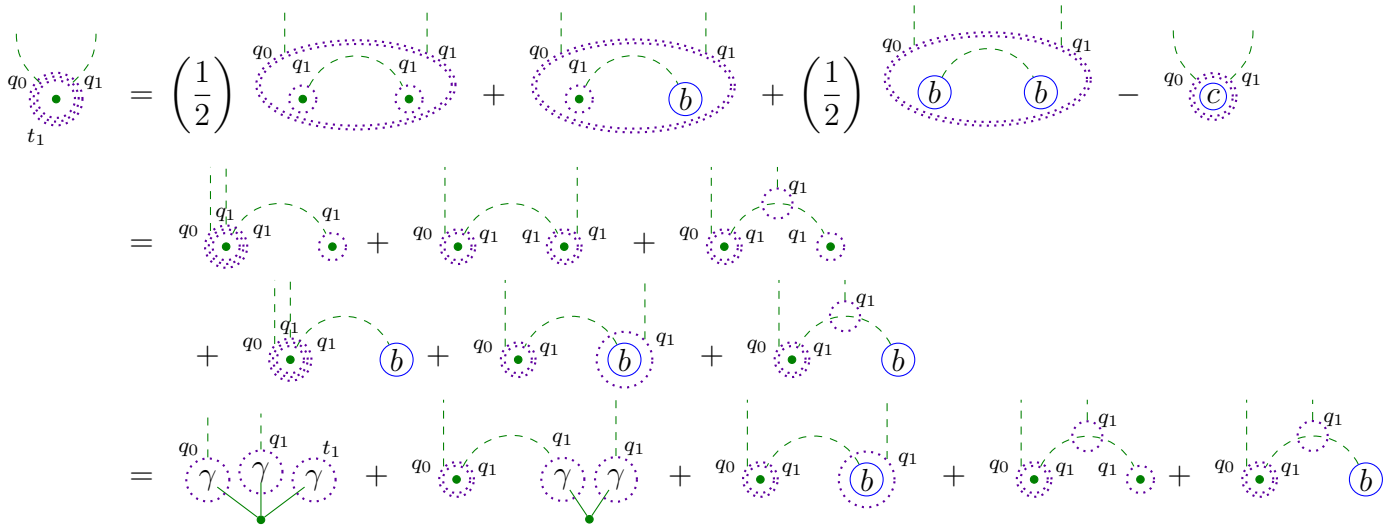
1.8.11 Lemma ($\partial^2\gamma/\partial t\partial q$) Differentiating the formula for $\frac{\partial \gamma}{\partial t}$ from Lemma 1.8.9 with respect to q_1 and applying the product rule and Lemma 1.4.6 gives:



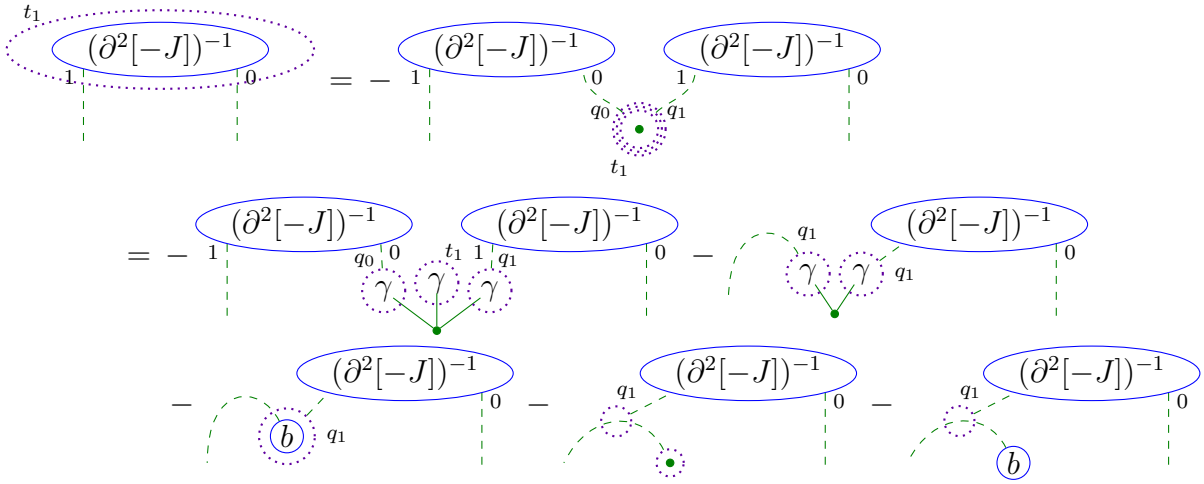
The analogous calculation with q_0 is much easier, since a, b, and c are independent of q_0. The result is:



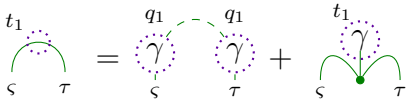
1.8.12 Lemma ($\partial^3 J/\partial t \partial q_0 \partial q_1$) We differentiate the expression for $^{t_1}\!\bullet$ from Fact 1.8.8 with respect to both q_0 and q_1. Note that only $\bullet$ depends on q_0.



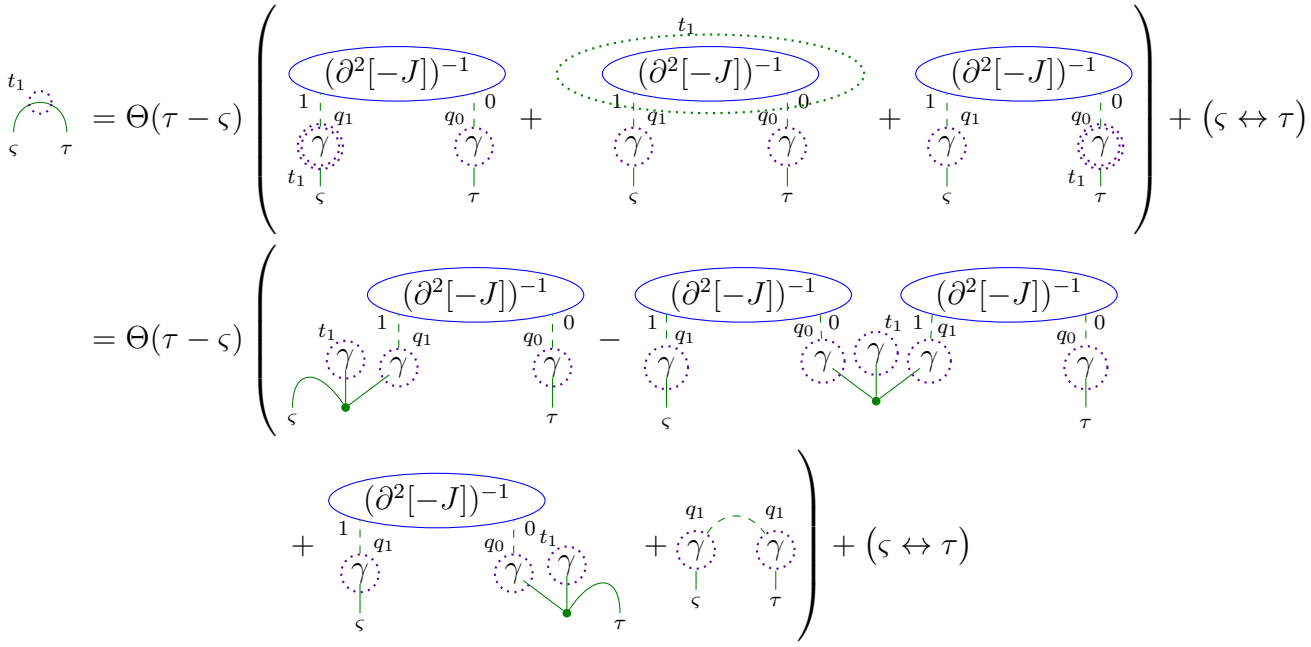
1.8.13 Corollary ($\partial(\partial^2[-J])^{-1}/\partial t$) By the quotient rule relating the derivatives of a matrix and its inverse, we have:



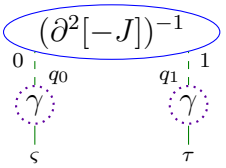
1.8.14 Proposition ($\partial G/\partial t$) We can now differentiate the Green's function $G =$  with respect to t_1:

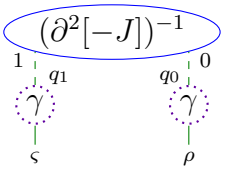


1.8.15 Proof of Proposition 1.8.14 We use the formula for G from Proposition 1.3.14, and apply Lemma 1.8.11 and Corollary 1.8.13. There are many cancelations, which the reader is invited to track.



The γ γ can be pulled out from within the parentheses, since $\Theta(\tau-\varsigma)+\Theta(\varsigma-\tau)=1$. What we're left with multiplying $\Theta(\tau-\varsigma)$ are three integrals, because of the interior vertex. Call that interior integration variable ρ. We can then combine all the integrals into one. We abbreviate by $f(\rho)$ the distribution made out of third derivatives of L and $\frac{\partial\gamma}{\partial t_1}$ that along with integration implements the trivalent vertex, as it depends only on ρ and not on ς and τ. We also use:

$G_{0,1}(\varsigma,\tau) =$ 

$G_{1,0}(\varsigma,\tau) =$ 

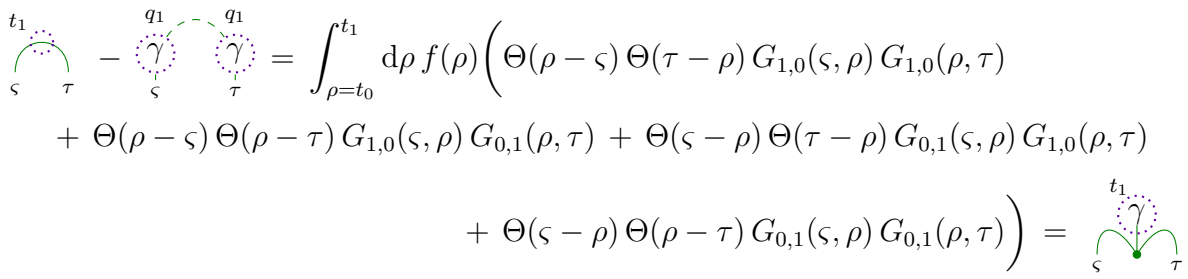
Then we have:

$$- \;\; \overset{q_1 \quad q_1}{\gamma \quad \gamma}_{\varsigma \quad \tau} = \int_{\rho=t_0}^{t_1} \mathrm{d}\rho\, f(\rho) \Big(\Theta(\tau-\varsigma)\,\Theta(\rho-\varsigma)\,G_{1,0}(\varsigma,\rho)\,G_{1,0}(\rho,\tau)$$
$$+ \Theta(\tau-\varsigma)\,\Theta(\varsigma-\rho)\,G_{0,1}(\varsigma,\rho)\,G_{1,0}(\rho,\tau) - \Theta(\tau-\varsigma)\,G_{1,0}(\varsigma,\rho)\,G_{1,0}(\rho,\tau)$$
$$+ \Theta(\tau-\varsigma)\,\Theta(\tau-\rho)\,G_{1,0}(\varsigma,\rho)\,G_{1,0}(\rho,\tau) + \Theta(\tau-\varsigma)\,\Theta(\rho-\tau)\,G_{1,0}(\varsigma,\rho)\,G_{0,1}(\rho,\tau)$$
$$+ \Theta(\varsigma-\tau)\,\Theta(\rho-\varsigma)\,G_{1,0}(\varsigma,\rho)\,G_{0,1}(\rho,\tau) + \Theta(\varsigma-\tau)\,\Theta(\varsigma-\rho)\,G_{0,1}(\varsigma,\rho)\,G_{0,1}(\rho,\tau)$$
$$- \Theta(\varsigma-\tau)\,G_{0,1}(\varsigma,\rho)\,G_{0,1}(\rho,\tau) + \Theta(\varsigma-\tau)\,\Theta(\tau-\rho)\,G_{0,1}(\varsigma,\rho)\,G_{1,0}(\rho,\tau)$$
$$+ \Theta(\varsigma-\tau)\,\Theta(\rho-\tau)\,G_{0,1}(\varsigma,\rho)\,G_{0,1}(\rho,\tau)\Big)$$

Combine like terms, using the following identity for Θ:

$$\Theta(\tau-\varsigma)\,\Theta(\varsigma-\rho)+\Theta(\varsigma-\tau)\,\Theta(\tau-\rho)=\Theta(\varsigma-\rho)\,\Theta(\tau-\rho).$$

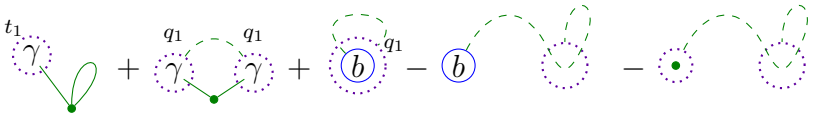
The result is:



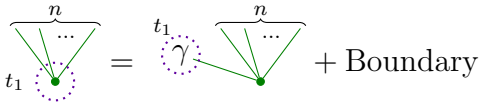
$$= \int_{\rho=t_0}^{t_1} \mathrm{d}\rho\, f(\rho)\Big(\Theta(\rho-\varsigma)\,\Theta(\tau-\rho)\,G_{1,0}(\varsigma,\rho)\,G_{1,0}(\rho,\tau) + \Theta(\rho-\varsigma)\,\Theta(\rho-\tau)\,G_{1,0}(\varsigma,\rho)\,G_{0,1}(\rho,\tau) + \Theta(\varsigma-\rho)\,\Theta(\tau-\rho)\,G_{0,1}(\varsigma,\rho)\,G_{1,0}(\rho,\tau) + \Theta(\varsigma-\rho)\,\Theta(\rho-\tau)\,G_{0,1}(\varsigma,\rho)\,G_{0,1}(\rho,\tau)\Big) =$$

1.8.16 Lemma ($\partial \det / \partial t$) We use Lemma 1.8.12, the formula for the logarithmic derivative of the determinant, Remark 1.4.10, and the quotient rule give to conclude:

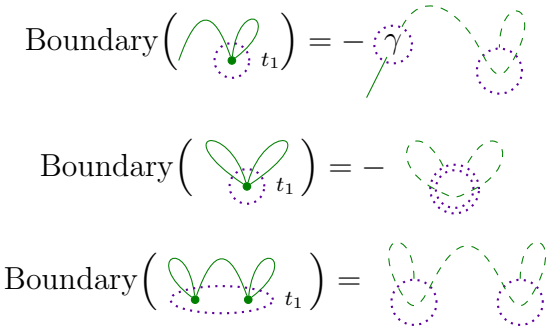
$$\frac{\partial}{\partial t_1}\left[\log\left|\det\frac{\partial^2[-J_\gamma]}{\partial q_1\partial q_2}\right|\right] =$$



1.8.17 Proposition (derivative of a vertex) In Lemma 1.4.4 we worked out the derivative of a vertex with respect to q_1. A little bit more care is required for the derivative with respect to t_1: each vertex includes an integral from t_0 to t_1, and so the derivative includes a boundary term. The *bulk* term is as in Lemma 1.4.4:



Understanding the boundary term requires some care. We studied the zero-valent vertex in Fact 1.8.8; thus, we need only to understand connected diagrams all of whose vertices are trivalent or higher. Of these, we claim the only nonzero boundary terms are:



$$\text{Boundary}\Big(\quad t_1\Big) = \frac{3}{2}$$

1.8.18 Proof of Proposition 1.8.17 By definition, $\text{Boundary}\big(\frac{\partial}{\partial t_1}\int_{t_0}^{t_1} F_{t_1}(\tau)\,\mathrm{d}\tau\big) = F_{t_1}(t_1)$. Each vertex contracts with a function which is a based loop in each variable, and so if there is no differentiation involved then the boundary term vanishes. Indeed, suppose that $\xi_1, \ldots, \xi_n$ are independent based loops. Then the only way that $\text{Boundary}\Big($ $\xi_1\ \xi_2\quad \xi_n$ / t $\Big)$ can be non-zero is if all the ξ_is are differentiated with respect to τ by the vertex. Since $\frac{\partial}{\partial\tau}$s correspond to $\frac{\partial L}{\partial v}$s and the Lagrangian L is quadratic in velocity, we never differentiate all incoming edges at a vertex that is trivalent or higher.

Unfortunately, things are not quite so simple when the Green's function $G(\varsigma,\tau) =$ $\varsigma\quad\tau$ is involved, as it is not an element of the *algebraic* tensor square of the space of based loops. In particular, $\frac{\partial G}{\partial\varsigma}(\varsigma,\tau)\big|_{\varsigma=\tau=t_1} \neq 0$. (If $\varsigma < t_1$ and we set $\tau = t_1$, we do get 0, but there is a discontinuity along the diagonal.) By Proposition 1.3.14,

$$\frac{\partial}{\partial\varsigma}\Big(\ \varsigma\quad\tau\ \Big) = \frac{\partial}{\partial\varsigma}\left(\begin{array}{l}(\partial^2[-J])^{-1}\\ 0\ q_0\quad q_1\ 1\\ \dot\gamma\qquad\dot\gamma\\ \varsigma\qquad\tau\end{array}\Theta(\varsigma-\tau) + \begin{array}{l}(\partial^2[-J])^{-1}\\ 1\ q_1\quad q_0\ 0\\ \dot\gamma\qquad\dot\gamma\\ \varsigma\qquad\tau\end{array}\Theta(\tau-\varsigma)\right)$$

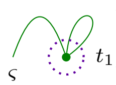
$$= \begin{array}{l}(\partial^2[-J])^{-1}\\ 0\ q_0\quad q_1\ 1\\ \dot\gamma\qquad\dot\gamma\\ \varsigma\qquad\tau\end{array}\Theta(\varsigma-\tau) + \begin{array}{l}(\partial^2[-J])^{-1}\\ 1\ q_1\quad q_0\ 0\\ \dot\gamma\qquad\dot\gamma\\ \varsigma\qquad\tau\end{array}\Theta(\tau-\varsigma) + 0.$$

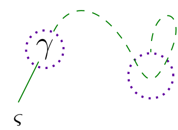
The derivatives of Θ cancel because G is continuous. We now evaluate at $\varsigma = \tau = t$. We have $\Theta(0) = \frac{1}{2}$, $\frac{\partial\gamma}{\partial q_0}(t_1) = 0$, $\frac{\partial\gamma^i}{\partial q_1^j}(t_1) = \delta^i_j$, $\frac{\partial\dot\gamma^i}{\partial q_1}(t_1) = 0$, and $\frac{\partial\dot\gamma^i}{\partial q_0^j}(t_1) = \frac{\partial^2 J_\gamma}{\partial q_0^j \partial q_1^k}\big(a^{-1}(t_1,q_1)\big)^{ik}$. Thus:

$$\frac{\partial}{\partial\varsigma}\Big(\ \varsigma,i\quad\tau,j\ \Big)\Big|_{\varsigma=\tau=t} = -\frac{1}{2}\ \ i\quad j$$

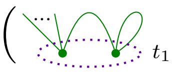
Consider, then, $\text{Boundary}\Big($ t_1 $\Big)$. The left-hand edge represents a based loop, and so must be differentiated at the vertex in question if the boundary term is to be non-zero. One end of the right-hand edge must also be differentiated. There are two ways to choose this edge. For each choice, the vertex contracts the incoming differentiated edges with

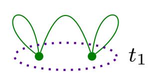
$-\frac{\partial^3 L}{\partial v^i \partial v^j \partial q^k} = -\frac{\partial}{\partial q^k}(a_{ij})$, and Boundary evaluates everything at t_1. Thus:

$$\text{Boundary}\Big($$ 

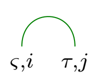
$$\Big) = 2 \times (-1) \times (-1) \times \left(-\frac{1}{2}\right) \times$$ 

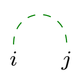
The 2 counts the number of ways the right-hand diagram can appear as a boundary term for the left-hand diagram. The first -1 comes from the left arc (and $\Theta(t_1 - \varsigma) = 1$). The second -1 is the sign at the vertex, and the $-\frac{1}{2}$ is from the right-hand edge.

Actually, there is one more thing that can happen. The left-hand Green's function can be differentiated in each of its variables resulting in a δ-function identifying the integrations at its two ends. Since the integrations are identified, we will treat their boundary terms together. So consider Boundary(

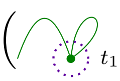
), assuming the middle edge is differentiated at each end resulting in a δ-function times a^{-1}. If any incoming edge to the left-hand vertex is an undifferentiated based loop, the boundary term vanishes. Since the left-hand vertex is at least trivalent, but can differentiate only one more of its incoming edges, the only way to have a non-zero result is if the diagram is precisely 

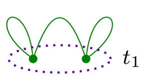
. Recall from Lemma 1.7.3:

$$\frac{\partial^2}{\partial\varsigma\partial\tau}\Big($$ 

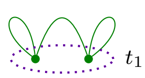
$$\Big) =$$ 

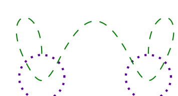
$$\delta(\tau - \varsigma) + \text{finite}$$

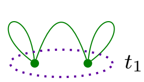
Thus, in addition to terms from Boundary(

), there is an additional boundary term for 

:

$$\text{Boundary}\Big($$ 

$$\Big) = 4 \times \left(-\frac{1}{2}\right) \times (-1) \times (1) \times (-1) \times \left(-\frac{1}{2}\right) \times$$ 

More generally, we can consider in a Feynman diagram a chain of δ-functions identifying the integrations at the vertices. Again since L is quadratic in velocity, δ-functions cannot branch — they either form a chain or a loop — and the loops of δ-functions vanish identically by Theorem 1.7.4. But if the corresponding boundary term is to be non-zero, then at each vertex in the chain, any undifferentiated incoming edges must connect back the the chain, where they must be differentiated. This can happen at most twice, as no vertex in the middle of the chain can differentiate any more edges, and each end can differentiate only one more edge. Thus, other than the case 

already considered, the only options are:

- A chain of zero δ-functions and one vertex, emitting two differentiated edges that each link back. The corresponding boundary term is:

$$\text{Boundary}\Big(\ \Big) = 4 \times \Big(-\frac{1}{2}\Big) \times (-1) \times \Big(-\frac{1}{2}\Big) \times$$

- A chain of one δ-function and two vertices, each emitting a differentiated edge that links to the other vertex. The corresponding boundary term is:

$$\text{Boundary}\Big(\ \Big) = 6 \times (-1) \times \Big(-\frac{1}{2}\Big) \times (1) \times \Big(-\frac{1}{2}\Big) \times (-1) \times$$

1.8.19 Proof of Theorem 1.8.2 Propositions 1.8.14 and 1.8.17 and Lemma 1.8.16 imply:

$$= + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2}$$

$$+ \sum_{\substack{\Gamma \text{ connected with} \\ \text{trivalent and higher vertices and} \\ \text{one } \gamma\text{---}\gamma \text{ or one } \gamma \\ \text{or one } \gamma}} \frac{(\hbar\sqrt{-1})^{\beta(\Gamma)}\,\mathrm{ev}(\Gamma)}{|\mathrm{Aut}\,\Gamma|}$$

$$- \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

We continue the convention that even when unspecified, each diagram Γ is multiplied by $\big(\hbar\sqrt{-1}\big)^{\beta(\Gamma)}$. In the sum, the diagram γ comes from the determinant term, as do all the diagrams Γ with first Betti number $\beta(\Gamma) = 1$ in the top line. All other diagrams in the sum come from the diagram formed by contracting γs (smooth a bivalent vertex if necessary), replacing $\gamma\text{---}\gamma$s with solid edges, and replacing γs with $\cap$s. The last line consists of the boundary terms from Proposition 1.8.17.

Comparing the above expression for V_γ with Corollary 1.8.10 and Definition 1.8.5 completes the proof.

1.9 Summing over all classical paths, and the initial value problem for Schrödinger's equation

Definition 1.3.22 along with Theorems 1.5.1 and 1.6.1 together define a local function U_γ given the following data: a manifold $\mathcal{N}$; a Riemannian metric a on $\mathcal{N}$, a one-form b, and a function c, all time-varying, combining into a Lagrangian $L(t,v,q) = \frac{1}{2}a_{ij}(t,q)v^iv^j + b_i(t,q)v^i + c(t,q) : \mathbb{R}\times\mathrm{T}\mathcal{N}\to\mathbb{R}$; and a nonfocal classical path $\gamma:[t_0,t_1]\to\mathcal{N}$. The function U_γ is defined in a neighborhood of $(t_0,\gamma(t_0),t_1,\gamma(t_1)) \in \mathbb{R}\times\mathcal{N}\times\mathbb{R}\times\mathcal{N}$, and valued in formal expressions of the form $\bigl(\hbar\sqrt{-1}\bigr)^{-\dim\mathcal{N}/2}\exp\Bigl(r_{-1}\bigl(\hbar\sqrt{-1}\bigr)^{-1}\Bigr)\sum_{n=0}^\infty r_n\bigl(\hbar\sqrt{-1}\bigr)^n$, $r_n\in\mathbb{R}$. (For want of a better word, we continue to refer to linear combinations of expressions of the form $\bigl(\hbar\sqrt{-1}\bigr)^{-\dim\mathcal{N}/2}\exp\Bigl(r_{-1}\bigl(\hbar\sqrt{-1}\bigr)^{-1}\Bigr)\sum_{n=0}^\infty r_n\bigl(\hbar\sqrt{-1}\bigr)^n$, $r_n\in\mathbb{R}$ simply as *formal expressions*.) The motivation for U_γ is that it represents the contribution to an ill-defined integral from paths that are "near" γ:

$$U_\gamma(t_0,q_0,t_1,q_1) = \int\limits_{\substack{\varphi:[t_0,t_1]\to\mathcal{N}\\ \varphi(t_a)=q_a\\ \varphi\approx\gamma}} \exp\left(-\bigl(\hbar\sqrt{-1}\bigr)^{-1}\int_{\tau=t_0}^{t_1} L\bigl(\tau,\dot\varphi(\tau),\varphi(\tau)\bigr)\,\mathrm{d}\tau\right)\,\mathrm{d}\varphi$$

where the "measure" is "$\mathrm{d}\varphi = \prod_{t_0<\tau<t_1}\mathrm{dVol}\bigl(\tau,\varphi(\tau)\bigr)$" and $\mathrm{dVol}(t,q) = \sqrt{\det a(t,q)}$ is the Riemannian volume form for the metric a. Moreover, in Theorem 1.8.2 we proved that U_γ satisfies Schrödinger's equation for the Lagrangian L, lending credence to the claim that it is the value of the path integral.

Of course, an oscillating integral in the limit as $\hbar\to 0$ receives contributions from any point that is near *some* critical point. Therefore to define the "true" path integral, we should sum over all classical paths:

$$U(t_0,q_0,t_1,q_1) = \sum_{\substack{\gamma:[t_0,t_1]\to\mathcal{N}\text{ classical and nonfocal}\\ \gamma(t_0)=q_0,\,\gamma(t_1)=q_1}} U_\gamma(t_0,q_0,t_1,q_1)$$

We call U *the formal path integral* for the quantum mechanics determined by L, and we will be more precise in Definition 1.9.1. The goal of this section is to study this sum, show it exists, and prove that as $t_1\to t_0$ it converges pointwise as a distribution to $\delta(t_1-t_0)$.

1.9.1 Definition (formal path integral U) For each open subset $\mathcal{O}\subseteq\mathbb{R}\times\mathcal{N}\times\mathbb{R}\times\mathcal{N}$, set:

$$U_\mathcal{O} = \sum_{\substack{\gamma\text{ a family of classical nondegenerate paths}\\ \text{with boundary values varying over }\mathcal{O}}} U_\gamma$$

Then U_γ has domain all of $\mathcal{O}$, provided the sum converges.

Let $\{\mathcal{O}_i\}$ be a locally-finite cover of $\mathbb{R}\times\mathcal{N}\times\mathbb{R}\times\mathcal{N}$, and set:

$$U_{\{\mathcal{O}_i\}} = \sum_i \mathcal{U}_{\mathcal{O}_i} - \sum_{i\neq j} \mathcal{U}_{\mathcal{O}_i\cap\mathcal{O}_j} + \sum_{i\neq j\neq k\neq i} \mathcal{U}_{\mathcal{O}_i\cap\mathcal{O}_j\cap\mathcal{O}_k} - \dots$$

We define the *formal path integral* to be the limit of $U_{\{\mathcal{O}_i\}}$s as the cover becomes finer and finer:

$$U = \lim_{\{\mathcal{O}_i\} \text{ very fine}} U_{\{\mathcal{O}_i\}}$$

1.9.2 Proposition (U exists) The limits in Definition 1.9.1 all converge pointwise as distributions.

More specifically, recall from Definition 1.2.1 that a *distribution* on $\mathcal{N}$ is defined by its integrals against all compactly-supported smooth functions on $\mathcal{N}$. Since U is valued in formal expressions, we will use the finite-dimensional formal integral from Definition 1.2.14, and test against compactly supported functions valued in formal expressions. Let $g = \exp\Big(\big(\hbar\sqrt{-1}\big)g_{-1}\Big)\sum_{i=0}^{\infty} g_i\left(\hbar\sqrt{-1}\right)^i$, where all $g_i\in\mathscr{C}^\infty(\mathcal{N})$, and for $i\geq 0$ each g_i is compactly supported. We will prove that for each n, there exists a finite collection $\{\gamma_j\}$ of families of nonfocal classical paths such that for any cover $\{\mathcal{O}_i\}$ finer than the domains of the γ_js, we have an equality:

$$\int^{\text{formal}} U_{\{\mathcal{O}_i\}}(t_0,q_0,t_1,q_1)\,g(q_0)\;\mathrm{dVol}(q_0) = \sum_j \int^{\text{formal}} U_{\gamma_j}(t_0,q_0,t_1,q_1)\,g(q_0)\;\mathrm{dVol}(q_0) + O(\hbar^n)$$

In particular, for each $\mathcal{O}_i$ the possibly-infinite sum implicit in $\int^{\text{formal}} U_{\mathcal{O}_i}\,g$ truncates modulo $\hbar^n$.

1.9.3 Proof of Proposition 1.9.2 It suffices to assume that all g_i share the same compact domain C, in which case the sum will simply truncate to a finite one at all orders in $\hbar$.

Recall that $U_\gamma = \exp\Big(-\big(\hbar\sqrt{-1}\big)^{-1}J_\gamma\Big)\times O(1)$. Since $g = \exp\Big(\big(\hbar\sqrt{-1}\big)^{-1}g_{-1}\Big)\times O(1)$, the formal integral $\int^{\text{formal}} U_\gamma g$ is supported only at those points q_0 where $\frac{\partial}{\partial q_0}\big(J_\gamma(t_0,q_0,t_1,q_1) - g_{-1}(q_0)\big) = 0$. But $\frac{\partial}{\partial q_0}J_\gamma = -\frac{\partial L}{\partial v}(t_0,\dot\gamma(t_0),q_0) = -a(t_0,q_0)\cdot\dot\gamma(t_0) - b(t_0,q_0)$, by Fact 1.3.13, and so the formal integral is supported only at those points q_0 where the classical path with initial conditions determined by q_0 and $\frac{\partial g}{\partial q_0}$ ends at q_1.

Put another way, the space of classical paths with domain $[t_0,t_1]$ is isomorphic to (an open subset of) the tangent bundle $\mathrm{T}\mathcal{N}$ under the isomorphism $\gamma\mapsto\big(\dot\gamma(t_0),\gamma(t_0)\big)$, and $\mathrm{T}\mathcal{N}\cong\mathrm{T}^*\mathcal{N}$ under the isomorphism $(v,q)\mapsto\big(\frac{\partial L}{\partial v}(t_0,v,q),q\big)$. Thus we can talk about the graph of $-\mathrm{d}g_{-1}$ as a section of $\mathrm{T}\mathcal{N}$, and consider the compact set $-\mathrm{d}g_{-1}(C)\subseteq\mathrm{T}\mathcal{N}$. The

formal integral $\int^{\text{formal}} U_\gamma$ is supported near those classical paths γ with initial conditions $(\dot\gamma(t_0), \gamma(t_0))$ in $-\mathrm{d}g_{-1}(C)$ and final location q_1.

From this perspective, a family of nonfocal classical paths with domain $[t_0, t_1]$ is among other things an open set in $\mathrm{T}\mathcal{N}$. Since $\mathrm{d}g_{-1}(C)$ is compact, we can choose a finite set $\{\gamma_j\}$ of families of nonfocal classical paths such that the corresponding open sets in $\mathrm{T}\mathcal{N}$ cover $\mathrm{d}g_{-1}(C)$. Let $\mathcal{O}_i \subseteq \mathbb{R}\times\mathcal{N}\times\mathbb{R}\times\mathcal{N}$ be an open set which is either contained in or disjoint from the boundary conditions of each γ_j. Then the possibly-infinite sum in $\int^{\text{formal}} U_{\mathcal{O}_i} g$ truncates to a finite sum:

$$\int^{\text{formal}} U_{\mathcal{O}_i} g = \sum_{j \text{ s.t. } \mathcal{O}_i \subseteq \text{ boundary conditions}(\gamma_i)} \int^{\text{formal}} U_{\gamma_j} g$$

Moreover, for any open cover $\{\mathcal{O}_i\}$ for which each $\mathcal{O}_i$ is either disjoint from or included in the boundary conditions of each γ_j, we have the equality claimed in Proposition 1.9.2.

1.9.4 Theorem ($\lim_{t_1\to t_0} U = \delta$) Given a metric a, a one-form b, and a function c as in Definition 1.8.1, let $U(t_0, q_0, t_1, q_1)$ be the formal path integral from Definition 1.9.1 they determine. Fix $t_0 \in \mathbb{R}$ and $q_1 \in \mathcal{N}$. Then pointwise as a distribution in q_0, we have:

$$\lim_{t_1\to t_0} U(t_0, q_0, t_1, q_1) = \delta(q_1 - q_0).$$

1.9.5 Remark (outline of proof) As in Proposition 1.9.2, we will test against compactly supported functions valued in formal expressions. Let $g = \exp\Big(\big(\hbar\sqrt{-1}\big)^{-1} g_{-1}\Big) \sum_{n=0}^{\infty} \big(\hbar\sqrt{-1}\big)^n g_n$ have compact domain C. By Proposition 1.9.2, the formal integral $\int^{\text{formal}} Ug$ truncates to a finite sum of integrals of the form $\int^{\text{formal}} U_\gamma g$.

Choose $\mathcal{O}_0 \supseteq C$ open with compact closure. In Proof 1.9.6 we will find $\epsilon_3 > 0$ and a family γ of classical paths with boundary conditions (t_1, q_0, q_1) ranging in $(t_0, t_0+\epsilon)\times\mathcal{O}_0\times\mathcal{O}_0$, such that for $t_1 \in (t_0, t_0+\epsilon)$,

$$\int^{\text{formal}} U(t_0, q_0, t_1, q_1)\, g(q_0)\ \mathrm{dVol}(q_0) = \int^{\text{formal}} U_\gamma(t_0, q_0, t_1, q_1)\, g(q_0)\ \mathrm{dVol}(q_0).$$

Then, in Proof 1.9.7 we will evaluate the limit $\lim_{t_1\to t_0} U_\gamma$ for this family γ.

1.9.6 Proof of Theorem 1.9.4 part I: short-duration classical paths can be almost geodesic For each $\epsilon \in \mathbb{R}$, consider the following Lagrangian on $\mathcal{N}$:

$$L_\epsilon(t, v, q) = \frac{1}{2}\, a_{ij}(t, q)\, v^i v^j + \epsilon\, b_i(t, q)\, v^i + \epsilon^2\, c(t, q).$$

Recall the partial function Flow from Definition 1.3.11. We define the partial function $\mathrm{Flow}_\epsilon : \mathbb{R}\times\mathbb{R}\times\mathrm{T}\mathcal{N} \to \mathbb{R}\times\mathcal{N}\times\mathbb{R}\times\mathcal{N}$ by declaring that for each path $\gamma : [t_0, t_1] \to \mathcal{N}$ which is classical for L_ϵ,

$$\mathrm{Flow}_\epsilon\big(t_0, t_1, \dot\gamma(t_0), \gamma(t_0)\big) = \big(t_0, \gamma(t_0), t_1, \gamma(t_1)\big).$$

Thus Flow_0 describes geodesic motion, and Flow_ϵ is smooth in its domain and depends smoothly on ϵ.

Let $\pi : \mathbb{R} \times \mathcal{N} \times \mathbb{R} \times \mathcal{N} \to \mathcal{N} \times \mathcal{N}$ denote projection onto the second and last factors, and define the partial function:

$$\phi_\epsilon(-) = \pi \circ \mathrm{Flow}_\epsilon(t_0, t_0 + 1, -) : \mathrm{T}\mathcal{N} \to \mathcal{N} \times \mathcal{N}$$

Then $\phi_{[-]}$ is smooth with domain an open set containing $\{0\} \times (\text{zero section } \mathcal{N} \hookrightarrow \mathrm{T}\mathcal{N}) \subseteq \mathbb{R} \times \mathrm{T}\mathcal{N}$.

Let $\mathcal{O}_0$ have compact closure $\overline{\mathcal{O}_0}$, and let $\mathcal{O}_1 \supseteq \overline{\mathcal{O}_0}$ be open with compact closure $\overline{\mathcal{O}_1}$. Let $\mathcal{P} = \phi_0^{-1}\big(\overline{\mathcal{O}_1} \times \overline{\mathcal{O}_1}\big)$. It is a compact subset of $\mathrm{T}\mathcal{N}$ containing $\{0\} \times \overline{\mathcal{O}_1}$. Then we can find $\epsilon_0 > 0$ such that $\mathcal{P} \times (-\epsilon_0, \epsilon_0)$ is contained in the domain of $\phi_{[-]}$. Choose $(v, q) \in \mathcal{P}$ such that $q \in \mathcal{O}_1$. For $\epsilon \in (-\epsilon_0, \epsilon_0)$, let $\gamma_\epsilon : [t_0, t_0 + 1] \to \mathcal{N}$ be the classical path for the Langrangian L_ϵ with initial conditions $\big(\dot\gamma_\epsilon(0), \gamma_\epsilon(0)\big) = (v, q)$. Then γ_0 is nonfocal, and the nonfocality condition depends smoothly on ϵ. Thus for each (v, q) there is some number $0 < \epsilon_1(v, q) < \epsilon_0$ so that for $\epsilon \in (-\epsilon_1, \epsilon_1)$, γ_ϵ is nonfocal. By nonfocality, ϵ_1 can be taken to depend lower-semicontinuously on (v, q). Thus it has a minimum value ϵ_2 on the compact set $\{(v, q) \in \mathcal{P} \text{ s.t. } q \in \overline{\mathcal{O}_0}\}$.

Then $\epsilon_2 > 0$ satisfies the following: for each $\epsilon \in (-\epsilon_2, \epsilon_2)$ and for each $(q_0, q_1) \in \mathcal{O}_0 \times \mathcal{O}_0$, we have chosen a nondegenerate classical path γ for the mechanics controlled by L_ϵ and satisfying $\gamma(t_0) = q_0$ and $\gamma(t_0 + 1) = q_1$. Moreover, the set of all these chosen paths for a given ϵ is precisely the set of L_ϵ-classical paths γ_ϵ with $\big(\dot\gamma_\epsilon(0), \gamma_\epsilon(0)\big) \in \mathcal{P}$ and $\gamma(0), \gamma(1) \in \mathcal{O}_0$. Thus this family $\gamma_{[-]}$ is smooth.

However, if $\epsilon > 0$ and $\gamma_\epsilon : [t_0, t_0 + 1] \to \mathcal{N}$ is an L_ϵ-classical path, then $\tau \mapsto \gamma_\epsilon\big(t_0 + \epsilon^{-1}(\tau - t_0)\big)$ is a classical path for $L = L_1$ with domain $[t_0, t_0 + \epsilon]$. In particular, we have constructed a smooth family γ of classical (for the original Lagrangian L) paths with boundary values ranging in $\{t_0\} \times \mathcal{O}_0 \times (t_0, t_0 + \epsilon_2) \times \mathcal{O}_0$.

Moreover, suppose that $\gamma : [t_0, t_0 + \epsilon] \to \mathcal{N}$ is a classical path with initial conditions $\gamma(t_0) \in \mathcal{O}_0$ and fixed $\dot\gamma(t_0) = \mathrm{d}g_{-1}(\gamma(0))$, where $-\mathrm{d}g_{-1}$ is the section of the tangent bundle $\mathrm{T}\mathcal{N}$ defined in Proof 1.9.3 (it corresponds to the usual section $-\mathrm{d}g_{-1}$ of $\mathrm{T}^*\mathcal{N}$ under the isomorphism $v \mapsto \frac{\partial L}{\partial v}$). Then if ϵ is sufficiently small depending on $\gamma(t_0)$, we have $\big(\epsilon\dot\gamma(0), \gamma(0)\big) \in \mathcal{P}$, and so γ is a member of our family. Again we can use compactness of C to choose a global bound ϵ_3 on these ϵs. Then our family γ contains all paths $[t_0, t_0 + \epsilon] \to \mathcal{N}$ for $\epsilon < \epsilon_3$ with initial conditions in $-\mathrm{d}g_{-1}(C)$.

1.9.7 Proof of Theorem 1.9.4 part II: the limit as $t_1 \to t_0$ We henceforth let $\epsilon = t_1 - t_0$. By Proofs 1.9.3 and 1.9.6, to prove Theorem 1.9.4 it suffices to consider

$$\lim_{\epsilon \to 0} \int^{\text{formal}} U_\gamma(t_0, q_0, t_0 + \epsilon, q_1)\, g(q_0)\, \mathrm{d}q_0$$
$$= \int^{\text{formal}} e^{-(\hbar\sqrt{-1})^{-1} J_\gamma(t, q_0, q_1)} \sqrt{\left|\det \frac{\partial^2[-J_\gamma]}{\partial q_0 \partial q_1}\right|} \Big(\sum(\text{diagrams})\Big)\, g(q_0)\ \mathrm{dVol}(q_0)$$

where γ is the family of almost-geodesics constructed in Proof 1.9.6. In particular, for $q_1 \notin \mathcal{O}_0$ and $\epsilon < \epsilon_3$, the formal integral has no support, and for $q_1 \in \mathcal{O}_0$ there is a unique point $q_0 \in \mathcal{O}_0$ depending on ϵ at which the formal integral is supported.

Moreover, $q_0^i - q_1^i = \epsilon \big(a^{-1}(q_1,t_0)\big)^{ij}\big(b_j(q_1,t_0) + \frac{\partial g_{-1}}{\partial q_0^j}\big) + O(\epsilon^2)$, by tracing bounds in Proof 1.9.6. Using Fact 1.8.8, we have:

$$\frac{\partial^2[-J_\gamma]}{\partial q_1^l \partial q_0^k} = \frac{\partial}{\partial q_1^l}\big(a_{jk}(t_0,q_0)\, v_0^j + b_k(t_0,q_0)\big) = a_{jk}(t_0,q_0)\,\frac{\partial v_0^j}{\partial q_1^l}$$

and therefore:

$$\left(\left(\frac{\partial^2[-J_\gamma]}{\partial q_1 \partial q_0}\right)^{-1}\right)^{kl} = \big(a^{-1}(t_0,q_0)\big)^{kj}\,\frac{\partial q_1^l}{\partial v_0^j} = \epsilon\big(a^{-1}(t_0,q_1)\big)^{kl} + O(\epsilon^2)$$

Here $v_0 = \dot\gamma(t_0)$, $q_0 = \gamma(t_0)$, and $q_1 = \gamma(t_0+\epsilon)$, for some classical path γ. In $\frac{\partial v_0}{\partial q_1}$, we hold q_0 fixed and consider v_0 as a function of q_0 and q_1; in $\frac{\partial q_1}{\partial v_0}$ we hold q_0 fixed, considering q_1 as a function of v_0 and q_0. We observed already that $q_0 = q_1 + \epsilon v_0 + O(\epsilon^2)$. The estimate $\frac{\partial q_1^l}{\partial v_0^j} = \epsilon\delta_j^l + O(\epsilon^2)$ does not necessarily follow from this, but it does follow from integrating the equations of motion and noting that a bunch of terms are smooth and hence bounded on any compact domain.

A similar argument gives:

$$\left(\left(\frac{\partial^2 J_\gamma}{(\partial q_0)^2}\right)^{-1}\right)^{kl} = \epsilon\big(a^{-1}(t_0,q_1)\big)^{kl} + O(\epsilon^2)$$

Since $\frac{\partial^2 g_{-1}}{\partial q_0}$ is independent of ϵ, it follows that:

$$\left(\left(\frac{\partial^2[J_\gamma - g_{-1}]}{(\partial q_0)^2}\right)^{-1}\right)^{kl} = \epsilon\big(a^{-1}(t_0,q_1)\big)^{kl} + O(\epsilon^2)$$

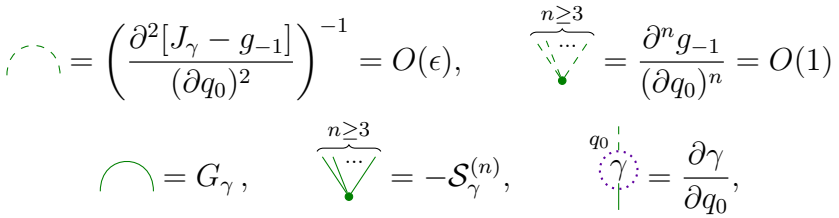
Finally, note that very short classical paths have trivial Morse index. Combining these observations, Definition 1.2.14 gives:

$$\begin{aligned}
&\int^{\text{formal}} U_\gamma(t_0,q_0,t_0+\epsilon,q_1)\, g(q_0)\ \mathrm{dVol}(q_0)\\
&= \big(2\pi\hbar\sqrt{-1}\big)^{\dim\mathcal{N}/2-\dim\mathcal{N}/2}\,\exp\Big(-\big(\hbar\sqrt{-1}\big)^{-1} J_\gamma\big(t_0, q_1+\epsilon a^{-1}b(t_0,q_1)+O(\epsilon^2), t_0+\epsilon, q_1\big)\Big)\\
&\qquad\times \det\big(\epsilon a^{-1}(t_0,q_1)+O(\epsilon^2)\big)^{-1/2}\times\det\big(\epsilon a^{-1}(t_0,q_1)+O(\epsilon^2)\big)^{1/2}\\
&\qquad\qquad\times g\big(q_1+\epsilon a^{-1}b(t_0,q_1)+O(\epsilon^2)\big)\Big)\times\big(1+O(\hbar)\big)
\end{aligned}$$

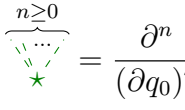
Finally, $J_\gamma\big(t_0, q_1 + \epsilon a^{-1}\big(b(t_0,q_1) - \mathrm{d}g_{-1}(q_1)\big) + O(\epsilon^2), t_0+\epsilon, t_1\big) = \int_{t_0}^{t_0+\epsilon} L = O(\epsilon)$, since the classical path γ does not leave the compact set $\overline{\mathcal{O}_0}$, and $\dot\gamma$ is never more than $O(\epsilon)$ from $-a^{-1}\big(b(t_0,q_1) + \mathrm{d}g_{-1}(q_1)\big) + O(\epsilon)$. All together, we have proven:

$$\lim_{\epsilon\to 0} \int^{\text{formal}} U_\gamma(t_0, q_0, t_0+\epsilon, q_1)\, g(q_0)\ \mathrm{dVol}(q_0) = g(q_1)\big(1 + O(\hbar)\big).$$

To complete the proof, we must handle the Feynman-diagrammatic terms arising in this finite-dimensional formal integral. Following arguments similar to those from the proof of Theorem 1.6.1, and taking advantage of Proposition 1.4.11, we arrive at a sum of all diagrams built out of the following pieces:



and precisely one vertex of the form:



The diagrams consisting of one 0-valent $\star$ and nothing else are already accounted for in the previous calculations. Our goal is to show that all other diagrams vanish as $\epsilon \to 0$.

To begin, we note that for $a = 0, 1$, the matrix-valued function $\frac{\partial\gamma}{\partial q_a}(\varsigma)$ is bounded as $\epsilon \to 0$, but its derivative $\frac{\partial\dot\gamma}{\partial q_a}$ grows as $O(\epsilon^{-1})$. Recalling Proposition 1.3.14, it follows that $G_\gamma(\varsigma,\tau) = O(\epsilon)$ but $\frac{\partial G_\gamma}{\partial\varsigma} = O(1)$ and $\frac{\partial^2 G_\gamma}{\partial\varsigma\partial\tau} = O(\epsilon^{-1})$. Each vertex  consists of an integral $\int_{t_0}^{t_0+\epsilon}$ against some derivatives of L evaluated along the classical path γ. But the γ has bounded initial conditions, and hence bounded velocity and position, so the integrand is bounded, and the act of integrating contributes a factor of $O(\epsilon)$.

Now recall that a vertex differentiates at most two of its incoming edges. Thus we can change the count: *every* edge, whether  or  , contributes a factor of $O(\epsilon)$, and we count the factors of $O(\epsilon^{-1})$ coming from differentiating  and  with the  vertex that does the differentiating. Since no vertex differentiates more than two of its incoming edges, we see that every non-$\star$ vertex is $O(\epsilon) \times O(\epsilon^{-2}) = O(\epsilon^{-1})$.

Finally, every  vertex must be trivalent or higher. It follows that in any diagram which is not solely a zero-valent $\star$, there are strictly more edges than such vertices. Thus the whole diagram scales as some positive power of ϵ, and hence vanishes as $\epsilon \to 0$. This completes the proof of Theorem 1.9.4.

Chapter 2

Homological methods: BV complex and homological perturbation theory

The goal of this chapter is to describe how homological perturbation theory can be used to study even *nonperturbative* integrals of the type appearing in quantum field theory. The main punchline is a nonperturbative variation of the method of Feynman diagrams: algebra alone is enough to "integrate out the higher degrees of freedom," resulting in an integral over the (scheme-theoretic, hence imaginary points contribute) Euler–Lagrange locus.

2.1 Introduction

The method of Feynman diagrams computes, in a totally algebraic fashion, the asymptotics of compactly supported oscillating integrals: the input consists of the power-series expansions of the "action" and "observable" functions in question; the Feynman diagrams encode rational functions in these Taylor coefficients; only at one step is any transcendental input required, and it is only to know the volume of a Gaussian distribution (some power of π).

This chapter presents analogous exact formulas for non-asymptotic integrals, in an algebrogeometric setting. We will use the homological perturbation lemma to reduce any "oscillating" integral to an integral over the scheme-theoretic critical locus of the "action" function. This reduction step consists of completely explicit rational functions of the coefficients of the action, and results in a linear combination of finitely many integrals that seem to be irreducibly transcendental. We will allow critical points with high-order degeneracy, although we will not work in the most general setting — we focus instead on the simplest case of bosonic integrals in finitely many variables, and the reader is invited to adapt our techniques to whatever problem is at hand.

This chapter is based on [GJF12, JF12]. Contrary to the conventions in Chapter 1, in this chapter all indices are lowered and we will not use Einstein's summation convention.

2.1.1 Outline of the chapter In Section 2.2, we give some motivation for the Batalin–Vilkovisky approach to "oscillating" integrals of the shape $\int_X f\, e^s$ dVol, and overview the construction of the (quantum) BV chain complex. Most of the section is devoted to Example 2.2.5, in which we review from [GJF12] the derivation of Feynman diagrams from the BV complex. In Example 2.2.6 we introduce the main topic of this chapter, namely integrals in which the functions f and s are complex polynomials, and we briefly discuss contours of integration.

The meat of the chapter is in Section 2.3. For the impatient readers who skipped Section 2.2, we begin by precisely defining the classical and quantum Batalin–Vilkovisky complexes that we will be concerned with. The classical BV complex for a function $s \in \mathcal{O}(X)$ is the Koszul resolution of the critical locus of s, and the quantum BV complex is a version of the twisted de Rham complex. The basic question is to have explicit control over the quantum BV complex: the algebraic part of "integration" is the problem of computing the homology class represented by a given $f \in \mathcal{O}(X)$. In Theorem 2.3.4, we give explicit formulas identifying the quantum BV homology with the classical critical locus, thereby providing an algebraic way to "integrate out the higher modes," analogous to the method of Feynman diagrams used in asymptotic integration. Our basic tool is the Homological Perturbation Lemma 2.3.10, which is a well-known formula-full version of spectral sequences.

We conclude the chapter with Section 2.4, which discusses at a non-rigorous level to what extent our techniques can be applied to the infinite-dimensional integrals appearing in quantum field theory, and we include some brief comments on the Volume Conjecture in Chern–Simons Theory.

2.2 Encoding oscillating integral problems in the Batalin–Vilkovisky chain complex

The close relationship between homological algebra and integration theory has been known since at least the time of de Rham (a good history is available in [Wei99]). In this section, we describe in general terms the shape of integrals appearing in quantum field theory. The values of such integrals are controlled by a *twisted de Rham complex* (for various reasons, we will use instead the term "Batalin–Vilkovisky complex"), which is the universal recipient for an "integral" satisfying Stokes' formula. Specializing to the formal asymptotic case gives a complex for which a straightforward analysis results in the method of Feynman diagrams; details are in Example 2.2.5. In Section 2.3 we will apply a similar analysis to the polynomial case, which we will briefly introduce in Example 2.2.6, where we also comment on the space of contours.

2.2.1 "Oscillating" integrals A basic tenet of quantum field theory, as was illustrated in Chapter 1, holds that the predicted values of physical measurements should occur as the values of definite integrals [SS09]. Specifically, in quantum field theory there is a space X of

"fields" (which is usually an infinite-dimensional stack) equipped with a "volume form" dVol (often dVol is the unique-up-to-scale "volume form" compatible with some symmetry). The "physics" is controlled by an *action function* $s \in \mathcal{O}(X)$, where $\mathcal{O}(X)$ denotes some distinguished algebra of ($\mathbb{C}$-valued, say) functions on X. The data of a measurement is encoded in an *observable* $f \in \mathcal{O}(X)$, and the predicted expectation value of the measurement is the ratio of definite integrals $\langle f \rangle_s = I_s(f)/I_s(1)$, where $I_s(f) = \int_X f\, e^s$ dVol. One special case is when X is a real manifold and s is pure-imaginary. Then the integrand is oscillatory, and we will abuse the language a bit and refer to any integral of the shape $I_s(f) = \int_X f\, e^s$ dVol as an *oscillating integral* by analogy.

The most physically interesting question is the "inverse problem": one measures the values of I_s for various inputs f, and tries to extract from this data information about the action s. To get off the ground, however, one must begin by understanding how to compute $I_s(f)$ given the data of s and f. Or perhaps one should say "one must begin by understanding how to *define* $I_s(f)$," as the spaces appearing in quantum field theory tend not to be of the type that support analytic definitions of integration. Whether the problem is one of computation or definition, the best situation is when the functional I_s depends entirely algebraically on s: algebraic definitions can more easily be transported to infinite dimensions, and are more readily computable.

2.2.2 Approximate Definition (BV Complex) Fix (X, dVol) and s. Any reasonable theory of integration should satisfy a *Stokes formula*, computing the value of $I_s(-)$ on "total derivatives" in terms of a boundary term. In many situations, these boundary terms can be made to vanish identically. If so, $I_s : \mathcal{O}(X) \to \mathbb{C}$ factors through the quotient vector space $\mathcal{O}(X)/\{\text{total derivatives}\}$. Often this quotient is finite-dimensional, and so our approach to understanding I_s will be to understand algebraically the projection $\mathcal{O}(X) \to \mathcal{O}(X)/\{\text{total derivatives}\}$ with respect to some distinguished bases, so that the only non-algebraic part of the functional I_s consists of finitely many data describing the map $\mathcal{O}(X)/\{\text{total derivatives}\} \to \mathbb{C}$.

The *quantum Batalin–Vilkovisky (or BV) complex* is a chain complex that resolves the quotient $\mathcal{O}(X)/\{\text{total derivatives}\}$. It is constructed as follows. Let V denote the Lie algebra (over $\mathbb{C}$) of vector fields on X that are divergence-free for the volume form dVol. Then a *total derivative* is a function of the form $\vec{v}(g\, e^s) = \big(\vec{v}(g) + g\, \vec{v}(s)\big)e^s$ for $g \in \mathcal{O}(X)$ and $\vec{v} \in V$. Thus the vector space $\{\text{total derivatives}\}$ is the image of the map $\partial_s : V \otimes \mathcal{O}(X) \to \mathcal{O}(X)$ given by $(\vec{v}, g) \mapsto \vec{v}(g) + g\, \vec{v}(s)$, and so we can compute $\mathcal{O}(X)/\{\text{total derivatives}\}$ as the degree-0 homology of a two-term chain complex

$$\frac{\mathcal{O}(X)}{\{\text{total derivatives}\}} = \mathrm{H}_0\Big(V \otimes \mathcal{O}(X) \xrightarrow{\partial_s} \mathcal{O}(X)\Big),$$

where $V \otimes \mathcal{O}(X)$ is in homological degree 1 and $\mathcal{O}(X)$ is in degree 0. This two-term complex has extra homology in degree 1, and so we construct the complete BV complex in the usual

"Koszul" way by taking an exterior power:

$$\mathrm{BV}_\bullet(X, \mathrm{dVol}, s) = V^{\wedge\bullet} \otimes \mathcal{O}(X),$$
$$\partial_{\mathrm{BV}}(\vec{v}_0 \wedge \cdots \wedge \vec{v}_{k-1} \otimes g) = \sum_{i=0}^{k-1} (-1)^i \, \vec{v}_0 \wedge \cdots \widehat{\vec{v}_i} \cdots \wedge \vec{v}_{k-1} \otimes (\vec{v}_i(g) + g\, \vec{v}_i(s))$$

The "hat" denotes removing the ith term in the wedge product.

This definition is only approximate, because we haven't made precise what we mean by "total derivatives," and other details might depend on one's application. Rather than trying to give a completely precise universal definition, we will illustrate the notion of Batalin–Vilkovisky complex with Examples 2.2.3, 2.2.5, and 2.2.6.

2.2.3 Example (smooth manifolds) Suppose that X is an n-dimensional compact oriented smooth manifold equipped with a nowhere-vanishing top-form $\mathrm{dVol} \in \Omega^n(X)$, where $\Omega^\bullet(X)$ denotes the smooth de Rham complex. Consider the graded vector space $\mathrm{MV}_\bullet(X) = \Gamma\big(\mathrm{T}^{\wedge\bullet}X\big)$ of antisymmetric multivector fields, constructed from the tangent bundle in analogy to the construction of $\Omega^\bullet(X) = \Gamma\big((\mathrm{T}^*)^{\wedge\bullet}X\big)$ from the cotangent bundle. The choice of dVol determines an isomorphism $\mathrm{MV}_\bullet(X) \cong \Omega^{n-\bullet}(X)$ given by contraction with dVol. The differential on $\mathrm{MV}_\bullet$ given by pulling the de Rham differential d across this isomorphism is the *divergence* operator on multivector fields; on vector fields, it is defined by $\mathrm{div}(\vec{v})\,\mathrm{dVol} = \mathcal{L}_{\vec{v}}(\mathrm{dVol})$, where $\mathcal{L}_{\vec{v}}$ denotes the Lie derivative in the $\vec{v}$ direction. Given a function $s \in \mathscr{C}^\infty(X)$, we get an exact one-form $\mathrm{d}s$ and a corresponding map $\iota_{\mathrm{d}s} : \mathrm{MV}_\bullet(X) \to \mathrm{MV}_{\bullet-1}(X)$ given by contraction with $\mathrm{d}s$. The BV complex is:

$$\mathrm{BV}_\bullet(X, \mathrm{dVol}, s) = \mathrm{MV}_\bullet(X), \quad \partial_{\mathrm{BV}} = \mathrm{div} + \iota_{\mathrm{d}s}$$

Note that the data of div and $\iota_{\mathrm{d}s}$ are invariant under rescaling dVol or shifting s by any locally constant function, so that the *BV differential* $\mathrm{div} + \iota_{\mathrm{d}s}$ depends only on the projective data of $e^s\,\mathrm{dVol}$; on the other hand, one can recover $e^s\,\mathrm{dVol}$ up to locally-constant rescaling from $\mathrm{div} + \iota_{\mathrm{d}s}$.

If X is not compact, then the most natural functions f to integrate are the compactly-supported ones, and in order to assure that there are no boundary terms we should request compact-support in higher degrees as well. So we take $\mathrm{BV}_\bullet$ to consist of the compactly-supported multivector fields, with the same differential. If X is not oriented, dVol is not a section of $\Omega^n(X)$ but rather of the density line bundle, and so the interpretation in terms of the de Rham complex must be corrected by an orientation bundle; we can still define $\mathrm{BV}_\bullet$ in terms of multivector fields and their divergences.

This example is elaborated upon in [Wit90].

2.2.4 Remark (twisted de Rham complex) When X is an n-dimensional compact oriented manifold, Example 2.2.3 describes a Batalin–Vilkovisky complex $\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big)$ that is closely related to the de Rham complex $\big(\Omega^\bullet(X), \mathrm{d}\big)$, where as always d denotes the de

Rham differential. In fact, by contracting not with dVol but with e^s dVol, one can construct an isomorphism of chain complexes $\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big) \cong \big(\Omega^{n-\bullet}(X), \mathrm{d}\big)$. The isomorphism "contract with dVol" used in Example 2.2.3 instead relates ∂_{BV} to the differential $\mathrm{d} + \mathrm{d}s\wedge$ on $\Omega^{n-\bullet}(X)$, where $\mathrm{d}s\wedge$ is the operation of multiplication by the exact one-form $\mathrm{d}s$.

The cochain complex $\big(\Omega^\bullet(X), \mathrm{d} + \mathrm{d}s\wedge\big)$ is called the *twisted de Rham complex* for s. Of course, the differential $\mathrm{d}+\mathrm{d}s\wedge$ is the result of conjugating d by the operation of multiplication by e^s, and so when for smooth or analytic manifolds the twisted and untwisted de Rham complexes are isomorphic. For most of this chapter we will be interested in the polynomial situation when $e^s \notin \mathcal{O}(X)$. Then the twisted and untwisted de Rham complexes are not in general isomorphic. Up to a grading shift, our Batalin–Vilkovisky complex will remain isomorphic to the twisted de Rham complex.

Nevertheless, we prefer not to use the language of twisted de Rham complexes. There are two main reasons for this:

1. In the Batalin–Vilkovisky complex, the homology group of most interest is in degree 0, whereas in the de Rham complex the most interesting homology group is in "top" degree. For finite-dimensional spaces, this is a mild æsthetic difference, but it becomes important when trying to generalize to the infinite-dimensional spaces appearing in quantum field theory: the natural infinite-dimensional de Rham complex includes functions, one-forms, two-forms, etc., and has no "top," whereas the natural infinite-dimensional Batalin–Vilkovisky complex has no "bottom."

   A related issue is that when X fails to be both compact and oriented, the most natural definitions of the twisted de Rham complex and the Batalin–Vilkovisky complex diverge.

2. Although it won't play a major rôle in this chapter, Batalin–Vilkovisky complexes have important algebraic structures that are obscured by thinking of them in terms of twisted de Rham complexes. Temporarily ignoring the differentials, both $\mathrm{MV}_\bullet(X)$ and $\Omega^\bullet(X)$ are graded-commutative rings under wedge multiplication. The de Rham differential d is a derivation of the algebra structure on $\Omega^\bullet(X)$, making $\big(\Omega^\bullet(X), \mathrm{d}\big)$ into a dga; the twisted differential $\mathrm{d} + \mathrm{d}s\wedge$ is not a derivation, but is a first-order differential operator. On the other hand, on $\mathrm{MV}_\bullet(X)$ the differential $\iota_{\mathrm{d}s}$ is a derivation, and for the wedge multiplication the "BV" differential div is a second-order differential operator. (Recall that an endomorphism of a graded-commutative algebra is a *kth-order differential operator* if its graded-commutator with multiplication by any element of the algebra is a $(k-1)$th-order differential operator.) Moreover, the failure of div to be a derivation for $\wedge$ is measured exactly by the *Schouten–Nijenhuis bracket* on $\mathrm{MV}_\bullet(X)$, which is an important piece of classical geometry.

   When X is infinite-dimensional, the graded-commutative algebra $\mathrm{MV}_\bullet(X)$ is relatively straightforward to define, and comes equipped with its Schouten–Nijenhuis bracket. A *BV projective volume form* on X is by definition a differential on $\mathrm{MV}_\bullet(X)$ whose failure

to be a derivation is measured by the Schouten–Nijenhuis bracket. Our techniques for writing down explicit formulas should extend well to many infinite-dimensional settings.

What will be important in this chapter is to relate the BV complex $\big(\mathrm{MV}_\bullet(X), \mathrm{div} + \iota_{\mathrm{d}s}\big)$ with the *derived critical locus* $\big(\mathrm{MV}_\bullet(X), \iota_{\mathrm{d}s}\big)$, which is a dga whose degree-0 homology is the algebra of functions on the critical locus of s. In this way, the algebraic structure on the Batalin–Vilkovisky complex will be directly important.

2.2.5 Extended Example (Feynman diagrams) The Batalin–Vilkovisky formalism packages neatly the method of Feynman diagrams, as we now explain. More details are available in [GJF12].

Suppose that f is a compactly supported complex-valued smooth function on a manifold X equipped with volume form dVol, and that s is smooth and real-valued. We denote the critical locus of s by $\{\mathrm{d}s = 0\}$. Suppose that $\{\mathrm{d}s = 0\}$ is *clean*, in the sense that it is an embedded submanifold of X and that the Hessian of s is nondegenerate in directions transverse to $\{\mathrm{d}s = 0\}$. For simplicity, suppose furthermore that s takes constant-value 0 on $\{\mathrm{d}s = 0\}$. Then as $\hbar \to 0$ along positive real numbers, the values of the integrals $\int_X f \exp\big(\sqrt{-1}s/\hbar\big)\,\mathrm{dVol}$ have an asymptotic expansion as a power series in $\hbar$. Moreover, this expansion depends only on the value of f near $\{\mathrm{d}s = 0\}$. For details on these estimates, see [Zwo12].

Thus, if we are interested only in the $\hbar \to 0$ asymptotics of $\int_X f \exp\big(\sqrt{-1}s/\hbar\big)\,\mathrm{dVol}$, we can choose a tubular neighborhood $X_{\approx\{\mathrm{d}s=0\}}$ of $\{\mathrm{d}s = 0\}$ and multiply f by a bump function supported in this neighborhood. We can then choose a contraction making $X_{\approx\{\mathrm{d}s=0\}}$ into a fiber bundle over $\{\mathrm{d}s = 0\}$. Pushing the density $f \exp\big(\sqrt{-1}s/\hbar\big)\,\mathrm{dVol}$ forward along $X_{\approx\{\mathrm{d}s=0\}} \to \{\mathrm{d}s = 0\}$ produces a (smooth!) density $\int_{\mathrm{fiber}} f \exp\big(\sqrt{-1}s/\hbar\big)\,\mathrm{dVol}$ on $\{\mathrm{d}s = 0\}$, which we can then integrate over $\{\mathrm{d}s = 0\}$ to produce $\int_X f \exp\big(\sqrt{-1}s/\hbar\big)\,\mathrm{dVol}$. As $\hbar \to 0$, the $\int_{\mathrm{fiber}} f \exp\big(\sqrt{-1}s/\hbar\big)\,\mathrm{dVol}$ makes sense as a $\mathbb{C}[\![\hbar]\!]$-valued measure on $\{\mathrm{d}s = 0\}$. In physics, one would consider this pushed-forward measure as an "effective action" achieved by "integrating out the high-energy modes."

Let $s^{(2)}$ denote the Hessian of s along $\{\mathrm{d}s = 0\}$. Since we assume that $\{\mathrm{d}s = 0\}$ is clean, $s^{(2)}$ is nondegenerate in the fiber directions of $X_{\approx\{\mathrm{d}s=0\}} \to \{\mathrm{d}s = 0\}$. We can restrict $s^{(2)}$ to these fibers, take its determinant, take a square root, and take a ratio with dVol to product a volume form $\mathrm{dVol}\,/\sqrt{|\det s^{(2)}|}$ on $\{\mathrm{d}s = 0\}$. In fact, this volume form is independent of the choice of trivialization. Using $\mathrm{dVol}\,/\sqrt{|\det s^{(2)}|}$ as a reference allows us to turn the $\mathbb{C}[\![\hbar]\!]$-valued measure $\int_{\mathrm{fiber}} f \exp\big(\sqrt{-1}s/\hbar\big)\,\mathrm{dVol}$ into a function on $\{\mathrm{d}s = 0\}$. By the stationary phase approximation, it begins

$$\frac{\int_{\mathrm{fiber}} f \exp\big(\sqrt{-1}s/\hbar\big)\ \mathrm{dVol}}{\mathrm{dVol}\,/\sqrt{|\det s^{(2)}|}} = f + O(\hbar).$$

The method of Feynman diagrams computes the higher-order coefficients of the above $\mathbb{C}[\![\hbar]\!]$-valued function. Choose a way to identify the fiber bundle $X_{\approx\{\mathrm{d}s=0\}} \to \{\mathrm{d}s = 0\}$ with

a neighborhood of the zero-section of a vector bundle over $\{\mathrm{d}s = 0\}$. Note that each fiber has a unique volume form $\mathrm{dVol}_{\mathrm{fiber}}$ that multiplies with the pull-back of $\mathrm{dVol}/\sqrt{|\det s^{(2)}|}$ to give dVol; we can always choose the identification of $X_{\approx\{\mathrm{d}s=0\}} \to \{\mathrm{d}s = 0\}$ with a vector bundle in such a way that this volume form $\mathrm{dVol}_{\mathrm{fiber}}$ is the pull-back of the Lebesgue measure. Having done all this, we can define the fiberwise Taylor coefficients of f and s. The higher-order corrections in "$f + O(\hbar)$" are rational functions of these Taylor coefficients. In this sense, the method of Feynman diagrams is totally algebraic.

To derive these rational functions, it suffices now to restrict attention to the case when $X \cong \mathbb{R}^n$, dVol is the Lebesgue measure, and s has a nondegenerate critical point at $0 \in X$, with $s(0) = 0$. Since it suffices to consider only the Taylor expansions of f and s near this critical point, we take $\mathcal{O}(X)$ to be the formal power series ring $\mathbb{C}_\hbar[\![x_1, \ldots, x_n]\!]$, where $\mathbb{C}_\hbar = \mathbb{C}[\hbar^{-1}][\![\hbar]\!]$ denotes the field of Laurent series, and all power series rings are completed for the power-series topology. Stokes' formula will never produce boundary terms, because we can always extend any Taylor series to a compactly-supported smooth function, and so we can control the integral with a Batalin–Vilkovisky complex. Let ξ_i be an odd variable corresponding to the divergence-free (for the Lebesgue measure) vector field $\frac{\partial}{\partial x_i}$. As a graded algebra, our BV complex will be:

$$\mathrm{MV}_\bullet(X) = \mathbb{C}_\hbar[\![x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]\!]$$

The ξ_i variables are in homological degree $+1$.

The differential, as in Example 2.2.3, has two terms, one corresponding to "divergence with respect to the Lebesgue measure," and the other corresponding to "contraction with $\mathrm{d}\big(\sqrt{-1}s/\hbar\big)$." In terms of the graded power-series algebra, these differentials together are:

$$\partial_{\mathrm{BV}} = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i \partial \xi_i} + \frac{\sqrt{-1}}{\hbar} \frac{\partial s}{\partial x_i} \frac{\partial}{\partial \xi_i} \right)$$

By construction, the map $\mathbb{C}[\![x_1, \ldots, x_n]\!] \to \mathbb{C}[\![\hbar]\!]$ taking f to the formal integral $I_{\sqrt{-1}s/\hbar}(f) = \int_X f \exp\big(\sqrt{-1}s/\hbar\big)\,\mathrm{dVol}$ factors through the degree-0 homology of this complex, and so we are primarily interested in the following question: *Given* $f \in \mathrm{MV}_0(X) = \mathbb{C}_\hbar[\![x_1, \ldots, x_n]\!]$, *what class does it represent in* $\mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big)$*?* The answer to this question is invariant under rescalings of ∂_{BV}. Since $\hbar$ is an infinitesimally small parameter, we will work with $-\hbar\partial_{\mathrm{BV}}$, as this avoids having to divide by $\hbar$.

By assumption, $s(0) = 0$ and s has a critical point 0. We replace s by its Taylor series, and make a mild change of variables:

$$s(x) = \sqrt{-1} \sum_{i,j=1}^{n} \frac{1}{2} a_{i,j}\, x_i x_j - \sqrt{-1} \sum_{\ell \geq 3} \frac{1}{\ell!} \sum_{\vec{\imath} \in \{1,\ldots,n\}^\ell} b_{\vec{\imath}}^{(\ell)}\, x_{i_1} x_{i_2} \cdots x_{i_\ell}$$

where the symmetric matrix $\{a_{i,j}\}_{i,j=1}^n$ is invertible since we supposed that $0 \in X$ was

nondegenerate as a critical point. Put together, we are interested in the following differential:

$$-\hbar\partial_{\mathrm{BV}} = \sum_{i,j=1}^{n} a_{i,j}\, x_i\, \frac{\partial}{\partial \xi_j} - \sum_{\ell\geq 2} \frac{1}{\ell!} \sum_{\substack{\vec{\imath}\in\{1,\dots,n\}^\ell \\ j\in\{1,\dots,n\}}} b^{(\ell+1)}_{\vec{\imath},j}\, x_{i_1} x_{i_2} \cdots x_{i_\ell}\, \frac{\partial}{\partial \xi_j} - \hbar \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial \xi_i}$$

The leading term of this differential removes a factor of ξ in exchange for a factor of x. The terms with a "b" in them are "smaller" in the power-series topology, in that they produce more xs, and the final term is "small" in that it produces an $\hbar$. This suggests that we understand $-\hbar\partial_{\mathrm{BV}}$ as a perturbation of its leading term $\sum_{i,j=1}^{n} a_{i,j}\, x_i\, \frac{\partial}{\partial \xi_j}$. There is a general theory of how to understand differentials as perturbations of their leading terms, which we will review in Lemma 2.3.10, but this differential is sufficiently simple as to be amenable to direct analysis.

If the differential were just $\sum_{i,j=1}^{n} a_{i,j}\, x_i\, \frac{\partial}{\partial \xi_j}$, then the exact elements in degree 0 would consist of all those power series that are first-order and higher in the x_is. Indeed, suppose that $f_{\vec{\imath}}$ is an arbitrary (not necessarily symmetric) m-tensor, so that the index $\vec{\imath}$ runs over m-tuples $\vec{\imath}\in\{1,\dots,n\}^m$. Then $f_{\vec{\imath}}$ determines a homogeneous polynomial $\sum_{\vec{\imath}} f_{\vec{\imath}}\, x_{i_1}\cdots x_{i_m}$, which satisfies:

$$\sum_{\vec{\imath}\in\{1,\dots,n\}^m} f_{\vec{\imath}}\, x_{i_1}\cdots x_{i_m} = \sum_{i,j=1}^{n} a_{i,j}\, x_i\, \frac{\partial}{\partial \xi_j} \left(\sum_{i',j'=1}^{n} \xi_{j'}\, (a^{-1})_{i',j'} \sum_{\vec{\imath}\in\{1,\dots,n\}^{m-1}} f_{\vec{\imath},i'}\, x_{i_1}\cdots x_{i_{m-1}} \right)$$

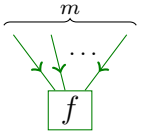
Here $(a^{-1})_{i',j'}$ denotes the (i',j')th entry of the inverse of the matrix with entries $a_{i,j}$.

If we instead apply $-\hbar\partial_{\mathrm{BV}}$, we conclude that:

$$\begin{aligned}\sum_{\vec{\imath}\in\{1,\dots,n\}^m} f_{\vec{\imath}}\, x_{i_1}\cdots x_{i_m} = &\sum_{\substack{i\in\{1,\dots,n\} \\ \vec{\imath}\in\{1,\dots,n\}^{m-1}}} f_{\vec{\imath},i}\, x_{i_1}\cdots x_{i_{m-1}} \sum_{\ell\geq 2} \frac{1}{\ell!} \sum_{\substack{\vec{\jmath}\in\{1,\dots,n\}^\ell \\ j\in\{1,\dots,n\}}} (a^{-1})_{i,j}\, b^{(\ell+1)}_{\vec{\jmath},j}\, x_{j_1} x_{j_2}\cdots x_{j_\ell} \\ &+ \hbar \sum_{\substack{i\in\{1,\dots,n\} \\ \vec{\imath}\in\{1,\dots,n\}^{m-1}}} \sum_{k=1}^{m-1} (a^{-1})_{i,i_k}\, f_{\vec{\imath},i}\, x_{i_1}\cdots \widehat{x_{i_k}} \cdots x_{i_{m-1}} \quad \text{modulo } \partial_{\mathrm{BV}}\text{-exact terms}\end{aligned}$$

The hat denotes a term left out of the product.

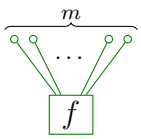
This formula has a graphical interpretation. The tensor $f_{\vec{\imath}}$ represents a multilinear functional of m variables, each ranging over $\mathbb{C}^n$; we can draw such an operator as a box with m ordered inputs:

$f_{\vec{\imath}} =$ 

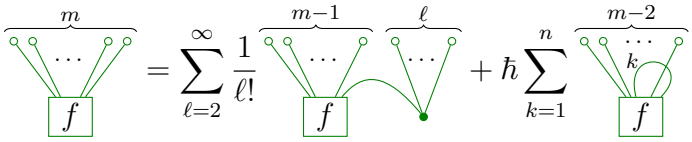
We henceforth suppress the arrows denoting inputs and outputs to multilinear functions, using instead the convention that "time goes down the page," so that inputs are at the

top and outputs are at the bottom. Tensor products are denoted by placing diagrams side-by-side, and composition (i.e. contraction of tensors) is denoted by connecting input and output strands. We use an open circle ⚲ with one output to denote the variable vector $(x_1, \dots, x_n) \in \mathbb{C}^n$, so that the polynomial $\sum_{\vec{\imath}} f_{\vec{\imath}} x_{i_1} \cdots x_{i_m}$ is given in pictures by:

$$\sum_{\vec{\imath} \in \{1,\dots,n\}^m} f_{\vec{\imath}} x_{i_1} \cdots x_{i_m} =$$



We introduce a solid circle  with ℓ incoming edges (which are unordered) to denote the the tensor $b^{(\ell)}_{\vec{\jmath}}$. Finally, we let a cap  denote the matrix a^{-1} thought of as an element of $\mathbb{C}^n \otimes \mathbb{C}^n$. With this notation, the result of the computation above reads:



modulo ∂_{BV}-exact terms.

It follows that any homogeneous polynomial $\sum_{\vec{\imath}} f_{\vec{\imath}} x_{i_1} \cdots x_{i_m} \in \mathbb{C}_\hbar[\![x_1, \dots, x_n]\!]$ is cohomologous to an element of $\mathbb{C}_\hbar$. Indeed, think of diagrams with ⚲s as *many-headed Hydra*, and invite Hercules to kill one. He chops off its right-most head, and either fuses it with one of the other heads, increasing degree in $\hbar$, or the Hydra grows at least two more new ones. Hercules thereby produces a sequence of Hydra, which converges in the power-series topology. The limit consists of all diagrams with no heads at all — these are the only Hydra that Hercules cannot attack — and each one appears with a factor of $\hbar^\beta$, where β is the first Betti number of the diagram, and also a factor coming from the $\frac{1}{\ell!}$s counting the number of symmetries of the diagram. This is the usual sum of Feynman diagrams.

We have not proved that $1 \in \mathbb{C}_\hbar$ is not ∂_{BV}-exact. One can prove this by arguing that any ∂_{BV}-primitive of 1 must satisfy a differential equation, and the solutions to this differential equation all involve terms of the form $\exp\bigl(g(x)/\hbar\bigr)$ for non-zero polynomials g, and hence do not have power-series expansions in $\hbar$. We will leave the details to the reader.

At the end of the computation, there is the difficulty of evaluating one particular integral. The homological calculation translates into a calculation of the ratio $I_{\sqrt{-1}s/\hbar}(f)/I_{\sqrt{-1}s/\hbar}(1)$, where $I_{\sqrt{-1}s/\hbar}(f) \in \mathbb{C}[\![\hbar]\!]$ is the formal asymptotics of $\int_X f \, \exp\bigl(\sqrt{-1}s/\hbar\bigr) \, \mathrm{dVol}$, and all integrals are regulated by multiplying the integrand by a compactly-supported bump function. We have not, however, computed the asymptotics of $I_{\sqrt{-1}s/\hbar}(1)$. Similar algebraic techniques allow one to reduce this computation to the computation of the volume of a Gaussian distribution, which is some power of π and not an algebraic number.

2.2.6 Example (complex polynomials) We come now to our main example, which will occupy the remainder of this chapter. Suppose that $X = \mathbb{C}^n$ with algebra of functions

$\mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n]$. Given a function $s \in \mathcal{O}(X)$, we can construct a Batalin–Vilkovisky complex by mimicking the construction from Example 2.2.5. Namely, we include (anticommuting) variables $\xi_1, \dots, \xi_n$ in homological degree 1, and construct:

$$\mathrm{MV}_\bullet(X) = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n], \quad \partial_{\mathrm{BV}} = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i \partial \xi_i} + \frac{\partial s}{\partial x_i} \frac{\partial}{\partial \xi_i} \right)$$

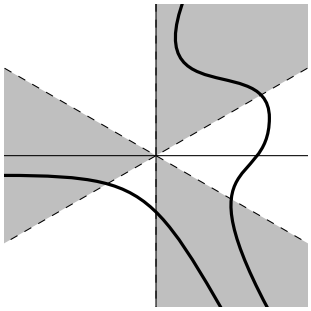
Since we are working over complex numbers and hope to work nonperturbatively, we have absorbed the factor of $\frac{\sqrt{-1}}{\hbar}$ into the function s. As in Example 2.2.3 and Remark 2.2.4, the BV complex is isomorphic to some shift of the perhaps-more-familiar twisted de Rham complex. Our goal in Section 2.3 will be to give explicit formulas for the homology classes represented by elements of $\mathrm{MV}_0(X) = \mathbb{C}[x_1, \dots, x_n]$, thereby giving a nonperturbative version of Feynman diagrams.

We have motivated Batalin–Vilkovisky complexes as a way to study "oscillating" integrals. The choice of polynomials $f, s \in \mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n]$ is not enough to define an integral "$\int_{\mathbb{C}^n} f\, e^s \,\mathrm{dVol}$." We must, of course, choose a "volume form" dVol, which we take to be the canonical holomorphic n-form $\mathrm{dVol} = \mathrm{d}x_1 \cdots \mathrm{d}x_n$ on $\mathbb{C}^n$. (This is justified because it does determine a good notion of "divergence" of a vector field.) But we must also choose a contour for integration.

By definition, a *contour* γ is a properly immersed n-real-dimensional submanifold $\gamma \looparrowright \mathbb{C}^n$. We do not demand that γ be compact (indeed, if γ is compact, then $\int_\gamma f\, e^s \,\mathrm{dVol} = 0$), and so we must assure that the integral $\int_\gamma f\, e^s \,\mathrm{dVol}$ converges. Moreover, since the BV complex encodes the integration by parts formula, we must assure that all boundary terms vanish. Roughly speaking, a contour γ is *allowable for* s if s has very-negative real part $\Re(s)$ near the ends of γ, as then $|f\, e^s|$ enjoys exponential decay for any polynomial s. (Recall that the *end* of a non-compact space is "the part of the space that is outside any compact subspace," so that our condition is that for every $r \in \mathbb{R}$, there is a compact subset $C \subseteq \gamma$ such that $\Re(s) < r$ on $\gamma \smallsetminus C$.) By a theorem usually named after either Stokes or Cauchy, provided convergence is maintained, homotopies of allowable contours do not change the values of integrals. Thus we can consider the space of (linear combinations of) allowable contours up to homotopy, and in short-hand we will represent this space as a relative homology group:

$$\{\text{allowable contours}\}/(\text{homotopy}) = \mathrm{H}_n\big(\mathbb{C}^n; \{\Re(s) \ll 0\}\big)$$

In fact, asking that $\Re(s) \ll 0$ near the ends of γ is not sufficient to assure convergence of the integral, because γ might have wild behavior at its ends. But every contour with ends in $\{\Re(s) \ll 0\}$ is homotopy equivalent to one for which all integrals of the form $f\, e^s \,\mathrm{dVol}$ with polynomial f converge, and in fact the homotopy types of these spaces of contours are equivalent [Pha83, Mal80].



A basis for {allowable contours}/(homotopy) when $s = x^3$.

The usual way to study the relative homology group $\mathrm{H}_n\big(\mathbb{C}^n; \{\Re(s) \ll 0\}\big)$ is through the theory of Lefschetz thimbles. This theory is well-developed, and so we will give only a summary account, referring the interested reader to [Pha83, Wit11]. Suppose that s has only isolated critical points with no critical point at ∞, and suppose furthermore that the Hessian of s is nondegenerate at each critical point. (These conditions hold for generic polynomials s.) Then $\Re(s)$ is a Morse function on $\mathbb{C}^n = \mathbb{R}^{2n}$ for which all critical points have Morse index n. The *Lefschetz thimble* for a critical point p of s is the n-dimensional disk of points in $\mathbb{C}^n$ that can be reached by downward gradient flow from p. By general Morse theory, the Lefschetz thimbles form a $\mathbb{Z}$-basis of the relative homology group $\mathrm{H}_n\big(\mathbb{C}^n; \{\Re(s) \ll 0\}\big)$, and moreover witness that:

$$\text{For generic } s,\ \mathrm{H}_k\big(\mathbb{C}^n; \{\Re(s) \ll 0\}\big) = \begin{cases} 0, & k \neq n, \\ \mathbb{Z}^{\#\{\mathrm{d}s=0\}}, & k = n, \end{cases}$$

where $\#\{\mathrm{d}s = 0\}$ is the number of critical points of s. If s is generic of maximum total degree d in n variables, then $\#\{\mathrm{d}s = 0\} = (d-1)^n$.

Another important situation is when s is required to be homogenous. Suppose that $s \in \mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n]$ is homogeneous of total degree d, and that the corresponding projective hypersurface $\{s = 0\} \subseteq \mathbb{CP}^{n-1}$ is smooth. Then the critical locus $\{\mathrm{d}s = 0\}$ consists of the origin with multiplicity $(d-1)^n$. The relative homology group $\mathrm{H}_n\big(\mathbb{C}^n; \{\Re(s) \ll 0\}\big)$ in this case can be seen directly to be free on $(d-1)^n$ generators [Goo12].

These results suggest that there is in general a close relationship between the relative homology group $\mathrm{H}_n\big(\mathbb{C}^n; \{\Re(s) \ll 0\}\big)$ and the critical locus $\{\mathrm{d}s = 0\}$. There are also many close relationships between the BV complex and the algebra of functions $\mathcal{O}(\{\mathrm{d}s = 0\})$ on the critical locus. Indeed, as we will construct in Theorem 2.3.4, in many cases there are isomorphisms $\mathrm{H}_0\big(\mathrm{BV}_\bullet(X, s), \partial_{\mathrm{BV}}\big) \cong \mathcal{O}(\{\mathrm{d}s = 0\})$, and of course $\dim \mathcal{O}(\{\mathrm{d}s = 0\}) = \#\{\mathrm{d}s = 0\}$, counted with multiplicity.

All together, we have a topologically-defined space $\mathrm{H}_n\big(\mathbb{C}^n; \{\Re(s) \ll 0\}\big)$ of contours γ, and an algebraically-defined space $\mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big)$ of observables f, which have a canonical analytically-defined pairing $(\gamma, f) \mapsto \int_\gamma f\, e^s\, \mathrm{dVol} \in \mathbb{C}$. In many situations these two vector spaces have the same dimension — this is strong evidence that the pairing is always perfect. However, I am not aware of a universal statement in this direction.

2.3 A non-asymptotic analog of Feynman diagrams

In this section we study homological-algebraic aspects of the Batalin–Vilkovisky complex introduced in Example 2.2.6. We begin by reviewing notation and definitions from Section 2.2. We then state our main Theorem 2.3.4 constructing explicit isomorphisms between the quantum and classical BV homologies; the formulas presenting such isomorphisms are a non-asymptotic version of the method of Feynman diagrams, and allow the "higher modes" of any "oscillating" integral to be integrated out in a totally algebraic fashion, resulting in an integral over the scheme-theoretic critical locus. Theorem 2.3.8 and Corollary 2.3.17 give a complete list of integrals (for generic action s) that cannot be computed algebraically, analogous to the situation in Wick's formula (Example 2.3.6) in which the volume of a Gaussian measure is transcendental, but all other integrals are determined by this volume and pure algebra.

2.3.1 Definitions and notation As analytic aspect of integration will not appear in the remainder of this chapter, we have many choices of ring over which to work. Nevertheless, we will continue to call our ground ring $\mathbb{C}$, and occasionally we will use that $\mathbb{C}$ is a field of characteristic 0. We fix a positive integer n, and write $\mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n]$ for the polynomial ring in n variables. We set $\mathrm{MV}_\bullet(X) = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ to be the $\mathbb{Z}$-graded ring of antisymmetric polynomial multivector fields on $X = \mathbb{A}^n$, where the ξ variables have homological degree $+1$ and anticommute with each other.

Choose $s \in \mathcal{O}(X)$. We will denote by d the maximal homogeneous degree of s. Write $s = \sum_{k=0}^{d} s^{(k)}$ where each $s^{(k)}$ is homogeneous of degree k; then $s^{(d)}$ is the *top part* of s. Every homogeneous polynomial in n variables defines a hypersurface in $\mathbb{CP}^{n-1}$. We say that s has *nonsingular top part* if the hypersurface defined by $s^{(d)}$ is smooth, or equivalently if the discriminant of $s^{(d)}$ is non-zero.

The *scheme-theoretic critical locus* of s is the subscheme of X with ring of functions $\mathcal{O}(\{\mathrm{d}s = 0\}) = \mathcal{O}(X)/\left(\sum_i \frac{\partial s}{\partial x_i}\mathcal{O}(X)\right)$, which is also known as the *Jacobian ring* of s. This ring of functions appears as the degree-0 homology of the *Koszul resolution* of $\{\mathrm{d}s = 0\}$, which is the differential graded algebra with underlying graded algebra $\mathrm{MV}_\bullet(X)$ and differential

$$\partial_{\mathrm{cl}} = \sum_{i=1}^{n} \frac{\partial s}{\partial x_i} \frac{\partial}{\partial \xi_i}.$$

The complex $\left(\mathrm{MV}_\bullet(X), \partial_{\mathrm{cl}}\right)$ is also known as the *classical BV complex for* s and as the *derived critical locus of* s.

The *divergence* operator corresponding to the volume form $\mathrm{dVol} = \mathrm{d}x_1 \cdots \mathrm{d}x_n$ is the differential on $\mathrm{MV}_\bullet(X)$ given by:

$$\mathrm{div} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial \xi_i}$$

The *(quantum) BV complex for s* is:

$$\mathrm{BV}_\bullet(X,s) = \big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}} = \partial_{\mathrm{cl}} + \mathrm{div}\big).$$

2.3.2 Remark (complete intersections) The critical locus $\{\mathrm{d}s = 0\}$ is *zero-dimensional* if $\dim \mathcal{O}(\{\mathrm{d}s = 0\}) < \infty$, in which case we define $\#\{\mathrm{d}s = 0\} = \dim \mathcal{O}(\{\mathrm{d}s = 0\})$. If the critical locus is zero-dimensional and there are no critical points at infinity, then the critical locus is a complete intersection and a theorem of Serre's implies that the Koszul resolution $\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{cl}}\big)$ has homology entirely in degree zero.

If s is of degree d and has nonsingular top part, then the critical locus $\{\mathrm{d}s = 0\}$ is zero-dimensional and Bezout's theorem implies that $\#\{\mathrm{d}s = 0\} = (d-1)^n$. Indeed, the top part $s^{(d)}$ cannot have critical points at infinity if it is nonsingular, and so $\{\mathrm{d}s^{(d)} = 0\}$ is a complete intersection with $\#\{\mathrm{d}s^{(d)} = 0\} = (d-1)^n$; the corresponding statement for s will follow, for example, from our Proof 2.3.14 of Theorem 2.3.4.

2.3.3 Remark (strategy to study the quantum BV complex) The classical and quantum BV complexes for s are closely related. For example, by filtering $\mathcal{O}(X)$ by polynomial degree, one can prove from a spectral sequence argument that there exists a differential $\tilde{\partial}$ on the classical BV homology $\mathrm{H}_\bullet\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{cl}}\big)$ and isomorphisms

$$\mathrm{H}_\bullet\big(\mathrm{H}_\bullet\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{cl}}\big), \tilde{\partial}\big) \cong \mathrm{H}_\bullet\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big).$$

In particular, if the critical locus $\{\mathrm{d}s = 0\}$ is zero-dimensional, then the classical BV homology is concentrated in degree 0 and so $\tilde{\partial}$ must vanish. Therefore there exist isomorphisms $\mathrm{H}_0\big(\mathrm{MV}_\bullet, \partial_{\mathrm{BV}}\big) \cong \mathrm{H}_0\big(\mathrm{MV}_\bullet, \partial_{\mathrm{cl}}\big) = \mathcal{O}(\{\mathrm{d}s = 0\})$.

(When $\mathbb{C}$ is not a semisimple ring, we cannot immediately guarantee such an isomorphism. The correct statement filters each homology group by declaring that a class is in the kth filtered piece if it is represented by a degree-k polynomial, and then compares $\mathrm{H}_\bullet\big(\mathrm{gr}\,\mathrm{H}_\bullet\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{cl}}\big), \tilde{\partial}\big)$ with $\mathrm{gr}\,\mathrm{H}_\bullet\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big)$, where gr denotes the associated-graded functor from filtered modules to graded modules. Over a field, or more generally over a semisimple ring, there are always non-canonical isomorphisms between filtered modules and their associated gradeds.)

The problem with the usual spectral-sequence approach is that it does not pick out explicit formulas. The differential $\tilde{\partial}$ is not canonical, because the identification of a filtered vector space with its associated graded is not canonical. Moreover, the classical and quantum BV complexes each have algebraic structure which is lost by the spectral sequence.

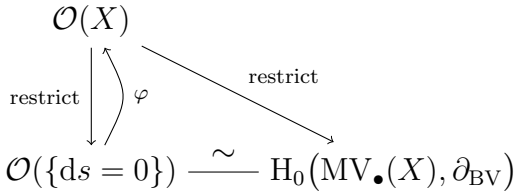
Most importantly, our goal is to produce explicit formulas for integrals against e^s, and this goal translates into the problem of producing an explicit basis of $\mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big)$ and giving an explicit formula for the map taking each element of $\mathrm{MV}_0(X) = \mathcal{O}(X)$ to its class in the quantum BV homology.

Our strategy to construct such explicit formulas will be to use a formula-full analog of spectral sequences called the Homotopy Perturbation Lemma 2.3.10. We will focus on the

simplest case when s has nonsingular top part, because it is already rich enough to provide an illustrative example: the point is not primarily the results themselves, but the techniques we use to do the calculations, which the reader is invited to generalize to whatever problems are at hand. We will prove:

2.3.4 Theorem (integrating out the higher modes) Let s be a degree-d polynomial with nonsingular top part. Filter $\mathcal{O}(\{\mathrm{d}s = 0\})$ by declaring that a class $[f] \in \mathcal{O}(\{\mathrm{d}s = 0\})$ is of degree at most k if it is the restriction to $\{\mathrm{d}s = 0\}$ for a polynomial $f \in \mathcal{O}(X)$ of total degree at most k. Denote the map that restricts functions to the critical locus by $\tau : \mathcal{O}(X) \to \mathcal{O}(\{\mathrm{d}s = 0\})$. Also denote the restriction map from zero-chains to their BV-homology classes by $\tilde{\tau} : \mathcal{O}(X) = \mathrm{MV}_0(X) \to \mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big)$.

Let $\varphi : \mathcal{O}(\{\mathrm{d}s = 0\}) \to \mathcal{O}(X)$ be any linear map splitting τ which is filtration-non-increasing. Said another way, if $[f]$ is a function on $\{\mathrm{d}s = 0\}$ of degree at most k, then $\varphi([f])$ should be an extension of $[f]$ to all of X which is also of degree at most k. Then there exists a unique isomorphism $\mathcal{O}(\{\mathrm{d}s = 0\}) \cong \mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big)$ of vector spaces such that φ also splits $\tilde{\tau}$. The homology group $\mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big)$ can be similarly filtered by declaring that the classes of degree at most k are the restrictions of functions of degree at most k; with this filtration, the unique isomorphism is an isomorphism of filtered vector spaces.



For every filtered $\nwarrow$ such that $\swarrow = \mathrm{id}$,
there exists a unique $\simeq$ such that $\searrow = \mathrm{id}$.

Recall that $\mathcal{O}(\{\mathrm{d}s = 0\})$ appears as the degree-0 homology of the classical BV complex, i.e. $\mathcal{O}(\{\mathrm{d}s = 0\}) = \mathrm{MV}_0(X)/\partial_{\mathrm{cl}}\big(\mathrm{MV}_1(X)\big)$. We give $\mathrm{MV}_1(X) = \bigoplus_i \mathcal{O}(X)\xi_i$ a filtration generated by the polynomial degree in $\mathcal{O}(X)$ and the declaration that each ξ_i is of degree $d-1$. Then there exists a filtration-non-increasing *homotopy* $\eta : \mathrm{MV}_0(X) \to \mathrm{MV}_1(X)$ such that $\partial_{\mathrm{cl}} \circ \eta = \mathrm{id} - \varphi \circ \tau$. Thus the composition $\mathrm{div} \circ \eta = \sum_i \frac{\partial^2}{\partial \xi_i \partial x_i} \circ \eta : \mathrm{MV}_0(X) \to \mathrm{MV}_0(X)$ acts locally nilpotently, as it lowers filtration-degree by at least d, and so any power series in $\mathrm{div} \circ \eta$ converges. With respect to the isomorphism $\mathcal{O}(\{\mathrm{d}s = 0\}) \cong \mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big)$ determined by φ, the restriction map $\tilde{\tau} : \mathrm{MV}_0(X) \to \mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big)$ is given by the formula:

$$\tilde{\tau}(f) = \tau(\mathrm{id} - \mathrm{div} \circ \eta)^{-1}(f) = \sum_{k=0}^{\deg(f)/d} \tau \circ \big(\mathrm{div} \circ \eta\big)^k(f).$$

The proof will occupy Proof 2.3.14. Note that, similar to the situation in Remark 2.3.13, $\tilde{\tau}$ is uniquely determined by the requirements that $\tilde{\tau} \circ \varphi = \mathrm{id}$ and $\tilde{\tau}$ vanishes on ∂_{BV}-exact elements of $\mathcal{O}(X)$. It is straightforward to check that for any filtration-non-increasing homotopy η satisfying $\partial_{\mathrm{cl}} \circ \eta = \mathrm{id} - \varphi \circ \tau$, the above formula for $\tilde{\tau}$ satisfies the required conditions.

2.3.5 Remark (inaccurate geometric interpretation) The critical locus $\{\mathrm{d}s = 0\}$ is a subscheme of $X = \mathbb{A}^n$. In Example 2.2.5, after restricting attention $X \rightsquigarrow X_{\approx\{\mathrm{d}s=0\}}$ to a tubular neighborhood of the critical locus, we chose a fibration of X over $\{\mathrm{d}s = 0\}$, and integrated over the fibers to produce a function on $\{\mathrm{d}s = 0\}$. In our present setting of polynomial functions and schemes, we cannot retract X to the critical locus in any geometric way, so there are no fibers to integrate out over.

Nevertheless, that is how Theorem 2.3.4 should be interpreted. The choice of linear map φ is acting as if it were a retraction of X onto $\{\mathrm{d}s = 0\}$ — if φ were an algebra homomorphism, then it would induce a retraction $X \to \{\mathrm{d}s = 0\}$, but in general the map φ cannot be chosen to be an algebra homomorphism. The composition $\tilde{\tau} : \mathcal{O}(X) \to \mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big) \cong \mathcal{O}(\{\mathrm{d}s = 0\})$ is the map that takes a function on X and "integrates it over the fibers" of this φ to produce a function on the critical locus $\{\mathrm{d}s = 0\}$. Comparing further with the Feynman diagrams in Example 2.2.5, the choice of homotopy η plays the role of a propagator, and the operation $\mathrm{div} \circ \eta$ corresponds to playing one round of "many-headed Hydra."

2.3.6 Example (Wick's formula) An important example is when s is quadratic with nonsingular top part. (This example is too simple to illustrate many of the phenomena present when $\deg(s) > 2$.) Then the critical locus $\{\mathrm{d}s = 0\}$ is a single point, and there is a unique filtration-non-increasing map $\varphi : \mathbb{C} = \mathcal{O}(\{\mathrm{d}s = 0\}) \to \mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n]$, and it happens to be an algebra homomorphism.

Expand s in coordinates:

$$s = \sum_{i,j=1}^{n} \frac{1}{2}\, s^{(2)}_{i,j}\, x_i x_j + \sum_{i=1}^{n} s^{(1)}_i\, x_i + s^{(0)}$$

Then $\{\mathrm{d}s = 0\}$ consists of the point at coordinates $x_i = -\sum_j \big(s^{(2)}\big)^{-1}_{i,j}\, s^{(1)}_j$ where $\big(s^{(2)}\big)^{-1}_{i,j}$ is the (i,j)th entry in the inverse matrix to $s^{(2)}_{i,j}$. The classical BV complex has differential:

$$\partial_{\mathrm{cl}} = \sum_{i,j=1}^{n} s^{(2)}_{i,j}\, x_i\, \frac{\partial}{\partial \xi_j} + \sum_{i=1}^{n} s^{(1)}_i\, \frac{\partial}{\partial \xi_i}$$

Because of the simplicity of the example, there is a unique filtration-non-increasing homotopy $\eta : \mathcal{O}(X) \to \mathrm{MV}_1(X)$ satisfying $\partial_{\mathrm{cl}} \circ \eta = \mathrm{id} - \varphi \circ \tau$. It is easiest to describe after changing coordinates to $y_i = x_i + \sum_j \big(s^{(2)}\big)^{-1}_{i,j}\, s^{(1)}_j$. Then $\partial_{\mathrm{cl}} = \sum_{i,j} s^{(2)}_{i,j}\, y_i\, \frac{\partial}{\partial \xi_j}$, and:

$$\eta(f) = \begin{cases} 0 & \text{if } f \text{ is constant} \\ \frac{1}{\ell} \sum_{i,j} \big(s^{(2)}\big)^{-1}_{i,j}\, \xi_i\, \frac{\partial f}{\partial y_j} & \text{if } f \text{ is homogeneous in the } y \text{ variables of degree } \ell > 0. \end{cases}$$

Composing with $\mathrm{div} = \sum_i \frac{\partial^2}{\partial x_i \partial \xi_i} = \sum_i \frac{\partial^2}{\partial y_i \partial \xi_i}$ gives:

$$\mathrm{div} \circ \eta = \frac{1}{\ell} \sum_{i,j=1}^{n} \big(s^{(2)}\big)^{-1}_{i,j}\, \frac{\partial^2 f}{\partial y_i \partial y_j} \text{ on functions that are homogeneous in } y \text{ of degree } \ell > 0.$$

In particular, if f is homogeneous of degree ℓ in the y variables, then $\operatorname{div}(\eta(f))$ is homogeneous of degree $\ell-2$. Since τ evaluates a function at $y=0$, we see that $\tau\circ\big(\operatorname{div}\circ\eta\big)^k(f)=0$ for every k if f is homogeneous in y of odd degree. If f is homogeneous in y of even degree ℓ, then $\tau\circ\big(\operatorname{div}\circ\eta\big)^k(f)\neq 0$ only when $k=\ell/2$, in which case:

$$\tau\circ\big(\operatorname{div}\circ\eta\big)^k(f)=\frac{1}{2k(2k-2)(2k-4)\cdots 2}\left(\sum_{i,j=1}^{n}\big(s^{(2)}\big)^{-1}_{i,j}\,\frac{\partial^2}{\partial y_i\partial y_j}\right)^k f$$

Summing over the possible values of k, and restoring to the x variables, this implies the following version of Wick's formula:

$$\tilde\tau(f)=\exp\left(\frac{1}{2}\sum_{i,j=1}^{n}\big(s^{(2)}\big)^{-1}_{i,j}\,\frac{\partial^2}{\partial x_i\partial x_j}\right) f\Bigg|_{x_i=-\sum_j(s^{(2)})^{-1}_{i,j}\,s^{(1)}_j}$$

"Wick's" formula is originally due to Isserlis [Iss18].

2.3.7 Remark (the inverse problem) Fixing a quadratic s with nonsingular top part and a contour, let $I_s(f)=\int f\,e^s$. Example 2.3.6 says that $I_s(f)=\tilde\tau(f)\,I_s(1)$, where $\tilde\tau(f)$ is determined algebraically in terms of f and s. Moreover, it implies:

$$\frac{I_s(x_i)}{I(1)}=-\sum_j(s^{(2)})^{-1}_{i,j}\,s^{(1)}_j$$

$$\frac{I_s(y_iy_j)}{I_s(1)}=(s^{(2)})^{-1}_{i,j}\text{ where }y_i=x_i-\frac{I_s(x_i)}{I_s(1)}$$

Most importantly, Example 2.3.6 implies that $I_s(f)$ is a polynomial in the values of $I_s(1)$, $I_s(x_i)$, and $I_s(x_ix_j)$ just for those variables x_i,x_j appearing in f, and s is a rational function in the values of $I_s(1)$, $I_s(x_i)$, and $I_s(x_ix_j)$ for all i and j.

When s is not quadratic, the ability to ignore variables that do not appear in f probably is not possible. But we can ask whether there exists a finite list of functions f such that the values of $I_s(f)$ for that list, along with *a priori* knowledge of the degree of s, is enough to determine s and the contour. A partial result in this direction is the following:

2.3.8 Theorem (towards a higher Wick's formula) Let $s\in\mathcal{O}(X)=\mathbb{C}[x_1,\dots,x_n]$ have maximum total degree $\deg(s)=d$, and suppose that the top part $s^{(d)}$ is generic. Then the quantum BV homology $\mathrm{H}_0\big(\mathrm{MV}_\bullet(X),\partial_{\mathrm{BV}}\big)$ has a basis consisting of the classes of the $(d-1)^n$ monomials $x_1^{m_1}\cdots x_n^{m_n}$ for which $m_i<d-1$ for all $i=1,\dots,n$. With respect to this basis and the monomial basis of $\mathcal{O}(X)$, in Corollary 2.3.17 we will give an explicit formula for the restriction map $\tilde\tau$ as a rational function of the coefficients of s when $s=s^{(d)}$ is homogeneous. We will prove Theorem 2.3.8 in Proof 2.3.16, and Corollary 2.3.17 is an easy consequence of our proof.

There are $(d-1)^n$ degrees of freedom in the choice of contour, and $\binom{n+d}{d}$ degrees of freedom in the choice of s. Since $\tilde{\tau}$ depends rationally on the coefficients of s, each function $f \in \mathcal{O}(X)$ determines an explicit rational equation satisfied by these $(d-1)^n + \binom{n+d}{d}$ variables, of the following form: for each $\vec{m} \in \{0, \dots, d-2\}^n$, the map $\tilde{\tau}$ produces a coefficient $\tilde{\tau}(f)_{\vec{m}}$ such that $\tilde{\tau}(f) = \sum_{\vec{m}} \tilde{\tau}(f)_{\vec{m}}\, x_1^{m_1} \cdots x_n^{m_n}$; to f we associate the equation $I(\tilde{\tau}(f)) = \sum_{\vec{m}} \tilde{\tau}(f)_{\vec{m}}\, I(x_1^{m_1} \cdots x_n^{m_n})$. One generically expects, therefore, that the system constructed by testing $(d-1)^n + \binom{n+d}{d}$ functions f has a finite number of solutions, and these solutions are distinguished by testing one more function.

2.3.9 Remark There are much more general results concerning bases for Jacobian rings like $\mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{cl}}\big)$ (e.g. [Kou76, Dou05, Sch05, Sab06]), and by Theorem 2.3.4 such results translate directly to the quantum BV homology. What we would like to emphasize in this chapter are the techniques used to prove Theorems 2.3.4 and 2.3.8, as they can be generalized to physically-interesting infinite-dimensional settings whereas classical singularity theoretic techniques seem more finite-dimensionally bound.

We turn now to the main ingredient in Proofs 2.3.14 and 2.3.16:

2.3.10 The Homological Perturbation Lemma Homological perturbation has a long history. Although not named until [GM70], the Homological Perturbation Lemma first appeared in [Bro65], and it is implicit in [Shi62] and explicit in unpublished work by Barrat. More detailed history is available in [Hue11]. A particularly good write-up of the Homological Perturbation Lemma is [Cra04]. The following definitions and result apply in any category enriched over abelian groups. We will intentionally use many of the same letters $(\varphi, \tau, \eta, \dots)$ as we used in the statement of Theorem 2.3.4.

A *retraction* consists of two chain complexes $(V_\bullet, \partial_V)$ and $(H_\bullet, \partial_H)$, chain maps $\varphi : H \to V$ and $\tau : V \to H$, and a *homotopy* (of homological degree $+1$) $\eta : V \to V$. These maps are required to satisfy that $\tau \circ \varphi = \mathrm{id}_H$ and $\varphi \circ \tau = \mathrm{id}_V + [\partial_V, \eta]$.

$$(H_\bullet, \partial_H) \underset{\varphi}{\overset{\tau}{\rightleftarrows}} (V_\bullet, \partial_V) \circlearrowright \eta \qquad \begin{aligned} \tau \circ \varphi &= \mathrm{id}_H \\ \varphi \circ \tau &= \mathrm{id}_V - [\partial_V, \eta] \end{aligned}$$

It follows that φ and τ are quasi-isomorphisms. The commutator is to be understood with the appropriate signs: since ∂_V is of homological degree -1 and η is of degree $+1$, both of which are odd, we have $[\partial_V, \eta] = \partial_V \circ \eta + \eta \circ \partial_V$. It is standard but unnecessary to also impose *side conditions* that $\eta^2 = 0$, $\eta \circ \varphi = 0$, and $\tau \circ \eta = 0$.

A *deformation* of a chain complex $(V_\bullet, \partial_V)$ is a degree-(-1) map $\delta : V \to V$ such that $(\partial_V + \delta)^2 = 0$. Equivalently, δ is a *Maurer–Cartan element* of $\mathrm{End}(V)$. A deformation δ is *small* with respect to a given retraction $(V_\bullet, \partial_V, H_\bullet, \partial_H, \varphi, \tau, \eta)$ if the degree-0 map $(\mathrm{id}_V - \delta\eta)$ is invertible. Note that then $(\mathrm{id}_V - \eta\delta)^{-1} = \mathrm{id}_V + \eta(\mathrm{id}_V - \delta\eta)^{-1}\delta$ also exists.

Suppose we are given a retraction $(V_\bullet, \partial_V, H_\bullet, \partial_H, \varphi, \tau, \eta)$ and a small deformation δ of $(V_\bullet, \partial_V)$. Then the deformed complex $(V_\bullet, \partial_V + \delta)$ is part of a deformed retraction:

$$\big(H_\bullet, \tilde{\partial}_H = \partial_H + \tau \circ (\mathrm{id} - \delta\eta)^{-1}\delta \circ \varphi\big) \underset{\tilde{\varphi} = (\mathrm{id}-\eta\delta)^{-1}\circ\varphi}{\overset{\tilde{\tau} = \tau\circ(\mathrm{id}-\delta\eta)^{-1}}{\rightleftarrows}} (V_\bullet, \partial_V + \delta) \circlearrowright \tilde{\eta} = \eta(\mathrm{id}-\delta\eta)^{-1}$$

The graded vector spaces $V_\bullet$ and $H_\bullet$ do not change, but their differentials do.

The proof consists simply of checking some equations, and we leave it to the reader. In fact, at the cost of working harder at the proof one can even drop the condition that $\tau \circ \varphi = \mathrm{id}_H$, replacing it only with the condition that τ and φ be quasi-isomorphisms [Cra04], but we will not need such generality.

2.3.11 Remark (Maurer–Cartan element as an algebraic structure) The Homological Perturbation Lemma 2.3.10 is an example of a much more general homotopical perturbation theory allowing to move (strongly homotopy) algebraic structures across quasi-isomorphisms. In the case of the Lemma 2.3.10, the algebraic structure to be moved is "the choice of a Maurer–Cartan element": we turn δ into the Maurer–Cartan element $\tau \circ (\mathrm{id} - \delta\eta)^{-1}\delta \circ \varphi$ on $(H_\bullet, \partial_H)$.

2.3.12 Corollary (Betti numbers are upper-semicontinuous and Euler characteristic is locally constant) Work in the category of vector spaces. Given a complex $(V_\bullet, \partial_V)$, set $H_\bullet = \mathrm{H}_\bullet(V_\bullet, \partial_V)$ its homology, with $\partial_H = 0$. Then it is possible to choose a retraction of $V_\bullet$ onto $H_\bullet$.

It follows that the dimension of the jth homology group of $(V_\bullet, \partial_V)$ is an upper semicontinuous function of ∂_V. Suppose that $(V_\bullet, \partial_V)$ is *elliptic*, in the sense that $\dim H_\bullet < \infty$. Then the Euler characteristic of $(V_\bullet, \partial_V)$ is constant under small deformations of ∂_V.

In particular, if $\mathrm{H}_\bullet(V_\bullet, \partial_V)$ is supported entirely in degree 0, then (up to isomorphism) it cannot change under small deformations of ∂_V. This holds more generally if $(V_\bullet, \partial_V)$ has no consecutive non-zero homology groups.

2.3.13 Remark (often $\tilde{\tau}$ is independent of η) Suppose that $H_\bullet$ is supported entirely in homological degree 0 and that $V_\bullet$ is supported entirely in nonnegative degrees. Then the map τ is entirely determined by the map φ and the condition that $\tau \circ \varphi = \mathrm{id}_H$. Usually the homotopy η is not uniquely determined.

In this situation, for any small deformation δ, we have $\tilde{\varphi} = \varphi$, as $\delta \circ \varphi = 0$. Our ultimate goal will be to compute the deformed $\tilde{\tau}$. To write explicit formulas, we must choose a homotopy η, but this choice doesn't matter provided it can be made such that $(\mathrm{id}_V - \delta\eta)$ is invertible.

2.3.14 Proof of Theorem 2.3.4 Let $V_\bullet = \mathrm{MV}_\bullet(X)$ and $H_\bullet = \mathrm{H}_\bullet\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{cl}}\big)$. Since s has nonsingular top part, the critical locus $\{\mathrm{d}s = 0\}$ is a complete intersection, and therefore $H_\bullet$ is concentrated in degree 0 by Remark 2.3.2. Therefore the natural projection $\tau : V_0 \to H_0$ extends by zero to a chain map $\tau : V_\bullet \to H_\bullet$. The choice of $\varphi : H_0 \to V_0$ in the statement of Theorem 2.3.4 extends by zero to a chain map $\varphi : H_\bullet \to V_\bullet$. We consider deforming $\partial_V = \partial_{\mathrm{cl}}$ by $\delta = \mathrm{div}$. We can thus apply Lemma 2.3.10 and Remark 2.3.13 provided a homotopy $\eta : V_\bullet \to V_{\bullet+1}$ can be found such that $(\mathrm{id}_V - \delta\eta)$ is invertible, and the conclusions of Theorem 2.3.4 would follow.

We give the algebra $\mathrm{MV}_\bullet(X) = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ a (bosonic) grading by declaring that $\deg(x_i) = 1$ and $\deg(\xi_i) = d-1$. Consider the differential $\partial_{(d)} = \sum_i \frac{\partial s^{(d)}}{\partial x_i} \frac{\partial}{\partial \xi_i}$ on $\mathrm{MV}_\bullet(X)$. Since $s^{(d)}$ is nonsingular, by Remark 2.3.2 the homology $\mathrm{H}_\bullet\big(\mathrm{MV}_\bullet(X), \partial_{(d)}\big)$ is concentrated in degree 0, where its homology is $\mathcal{O}(\{\mathrm{d}s^{(d)} = 0\})$. Moreover, $\partial_{(d)}$ has weight zero with respect to the grading on $\mathrm{MV}_\bullet(X)$, and $\big(\mathrm{MV}_\bullet(X), \partial_{(d)}\big)$ breaks into a direct sum of chain complexes indexed by the weight for the new grading. Similarly, $\mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{(d)}\big)$ is graded by polynomial degree. By considering each graded piece individually, we can therefore choose a grading-preserving splitting $\varphi_{(d)} : \mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{(d)}\big) \to \mathrm{MV}_0(X)$ of the projection $\tau_{(d)} : \mathrm{MV}_0(X) \to \mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{(d)}\big)$, and for any such splitting we can iteratively choose a grading-preserving homotopy $\eta_{(d)} : \mathrm{MV}_\bullet(X) \to \mathrm{MV}_{\bullet+1}(X)$ satisfying $\varphi_{(d)}\tau_{(d)} = \mathrm{id} - [\partial_{(d)}, \eta_{(d)}]$.

Let $\varphi' : \mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{(d)}\big) \to \mathrm{MV}_0(X)$ be some other linear splitting of the projection $\tau_{(d)}$ which does not necessarily preserve degree but does not increase it. Then φ' has a decomposition as $\varphi' = \sum_{w \geq 0} \varphi'_w$, where φ'_w is homogeneous of weight $-w$. The top part φ'_0 preserves degree, and necessarily also splits $\tau_{(d)}$. Henceforth we identify $\varphi'_0 = \varphi_{(d)}$, and choose a compatible grading-preserving homotopy $\eta_{(d)}$ as in the previous paragraph. Set $\eta' = \eta_{(d)} + \eta_{(d)} \circ (\varphi' - \varphi) \circ \tau$. By construction, $\eta' - \eta$ strictly lowers degree, and so η' is degree-non-increasing. A straightforward calculation implies that $\varphi' \circ \tau = \mathrm{id} - [\partial, \eta']$.

We now apply the Homological Perturbation Lemma 2.3.10 to the contraction

$$\mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{(d)}\big) \underset{\varphi'}{\overset{\tau_{(d)}}{\leftrightarrows}} \big(\mathrm{MV}_\bullet(X), \partial_{(d)}\big) \circlearrowright \eta'$$

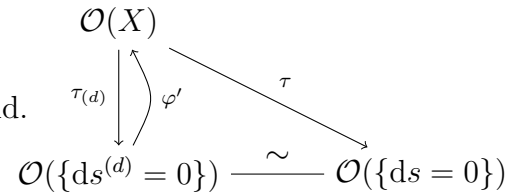
with deformation $\partial_{\mathrm{cl}} - \partial_{(d)}$, which is small because η' is degree-non-increasing and $\partial_{\mathrm{cl}} - \partial_{(d)}$ strictly lowers degree. By Remark 2.3.13, we get a contraction of the form:

$$\mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{(d)}\big) \underset{\varphi'}{\overset{\widetilde{\tau_{(d)}}}{\leftrightarrows}} \big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{cl}}\big) \circlearrowright \widetilde{\eta'}$$

The formulas for $\widetilde{\tau_{(d)}}$ and $\widetilde{\eta'}$ guarantee that they are degree-non-increasing. Let $\tau : \mathrm{MV}_\bullet(X) \to \mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{cl}}\big)$ denote the natural projection. Then $\tau \circ \varphi'$ is an isomorphism of filtered vector spaces $\mathcal{O}(\{\mathrm{d}s^{(d)} = 0\}) = \mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{(d)}\big) \cong \mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{cl}}\big) = \mathcal{O}(\{\mathrm{d}s = 0\})$, and

we have proven:

For every filtered $\wr$ such that $\Downarrow = \mathrm{id}$,
there exists a unique $\simeq$ such that $\searrow = \mathrm{id}$.



Put another way, we have given a construction turning any filtered splitting φ' of $\tau_{(d)}$ into a filtered splitting φ of τ, and moreover shown that for any filtered splitting φ of τ so constructed, there is a compatible filtered homotopy $\widetilde{\eta'}$. Moreover, by using $\widetilde{\eta'}$ as our homotopy, we can run the Homological Perturbation Lemma 2.3.10 in reverse to reconstruct φ' from φ. Thus the map $\big\{\text{filtered maps } \varphi' : \mathcal{O}(\{\mathrm{d}s^{(d)} = 0\}) \to \mathcal{O}(X) \text{ splitting } \tau_{(d)}\big\} \to \big\{\text{filtered maps } \varphi : \mathcal{O}(\{\mathrm{d}s = 0\}) \to \mathcal{O}(X) \text{ splitting } \tau\big\}$ is an inclusion of finite-dimensional affine spaces. Since we have constructed a filtered isomorphism $\mathcal{O}(\{\mathrm{d}s^{(d)} = 0\}) \cong \mathcal{O}(\{\mathrm{d}s = 0\})$ intertwining $\tau_{(d)}$ with τ, the two spaces of possible splittings are isomorphic. By finite-dimensionality, we conclude that every splitting φ of τ comes from some splitting φ' of $\tau_{(d)}$.

But our construction $\varphi' \mapsto \varphi$ produced a filtered homotopy $\eta = \widetilde{\eta'}$ compatible with φ. Thus we can run the Homological Perturbation Lemma 2.3.10 again with deformation div. This completes the proof of Theorem 2.3.4.

2.3.15 Remark (dropping the condition that $s^{(d)}$ is nonsingular) The conclusion of Theorem 2.3.4 ought to follow only from the condition that the critical locus $\{\mathrm{d}s = 0\}$ is zero-dimensional with no points at infinity. If one adds the condition that the scheme $\{\mathrm{d}s = 0\}$ is reduced (i.e. that the Hessian of s is nondegenerate at every critical point), then related results follow from the technique of Lefschetz thimbles (see e.g. [Pha83]). We prefer our more algebraic approach, as it has a better chance of applying in infinite-dimensional situations, but without the nonsingularity of $s^{(d)}$ we are not aware of a general way to construct homotopies η for which the deformation $\delta = \mathrm{div}$ is small.

2.3.16 Proof of Theorem 2.3.8 By Proof 2.3.14, it suffices to prove that $\mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{(d)}\big)$ has the specified monomial basis.

A homogeneous degree-d polynomial $s = s^{(d)}$ is *diagonal* if $s = \sum_i a_i \frac{(x_i)^d}{d!}$. It is clear that any homogeneous polynomial s decomposes as $s = s_{\mathrm{diag}} + s_{\mathrm{mix}}$, where s_{diag} is diagonal and every monomial in s_{mix} contains at least two different variables. We similarly decompose $\partial_{\mathrm{cl}} = \sum_i \frac{\partial s}{\partial x_i}\frac{\partial}{\partial \xi_i}$ as $\partial_{\mathrm{cl}} = \partial_{\mathrm{diag}} + \partial_{\mathrm{mix}}$. Our strategy will be to consider ∂_{mix} as a (hopefully small) perturbation to $\partial_{\mathrm{diag}} = \sum_i a_i \frac{(x_i)^{d-1}}{(d-1)!}\frac{\partial}{\partial \xi_i}$ in the Homological Perturbation Lemma 2.3.10.

For s generic, all a_i are non-zero. Then the complex $\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{diag}}\big)$ factors as a tensor product over $\mathbb{C}$ of 2-term complexes:

$$\left(\mathbb{C}[x_1,\dots,x_n,\xi_1,\dots,\xi_n], \sum_i a_i \frac{(x_i)^{d-1}}{(d-1)!}\frac{\partial}{\partial \xi_i}\right) = \bigotimes_{i=1}^{N}\left(\mathbb{C}[x_i,\xi_i], a_i\frac{(x_i)^{d-1}}{(d-1)!}\frac{\partial}{\partial\xi_i}\right)$$

Each tensorand has an obvious retraction onto its homology $H = \frac{\mathbb{C}[x_i]}{(x_i)^{d-1}}$. Namely, we set $\varphi(x_i^m) = x_i^m$ for $m < d-1$, and choose the homotopy to be

$$\eta(x_i^m) = \begin{cases} 0, & m < d-1, \\ \frac{(d-1)!}{a_i}\,\xi_i\, x^{m-(d-1)}, & m \geq d-1. \end{cases}$$

We can tensor together the splittings to get a splitting $\varphi_{\mathrm{diag}} : H = \frac{\mathbb{C}[x_1,\dots,x_n]}{(x_1^{d-1},\dots,x_n^{d-1})} \to \mathbb{C}[x_1,\dots,x_n]$ of the projection τ_{diag}. There is no functorial way to tensor together the homotopies of a retraction, but we can make an arbitrary choice for η, which by Remark 2.3.13 won't matter much. A good choice for $\eta_{\mathrm{diag}} : V_0 \to V_1$ is:

$$\eta_{\mathrm{diag}}\big(x_1^{m_1}\cdots x_n^{m_n}\big) = \begin{cases} 0, & \text{all } m_i < d-1, \\ \dfrac{\sum_i \frac{\xi_i}{a_i}\big(\frac{\partial}{\partial x_i}\big)^{d-1}}{\sum_i \binom{m_i}{d-1}}\big(x_1^{m_1}\cdots x_n^{m_n}\big), & \text{some } m_i \geq d-1. \end{cases}$$

In no formula will the choice of $\eta_{\mathrm{diag}} : V_k \to V_{k+1}$ for $k \geq 1$ appear, and we can always extend η_{diag} iteratively to the components of higher homological degree, while preserving the extra grading for which $\deg(x_i) = 1$ and $\deg(\xi_i) = d-1$.

We can now ask whether the perturbation $\delta = \partial_{\mathrm{mix}}$ is small with respect to the retraction $(V_\bullet, \partial_{\mathrm{diag}}, \frac{\mathbb{C}[\vec{x}]}{(x_i^{d-1})}, 0, \varphi_{\mathrm{diag}}, \tau_{\mathrm{diag}}, \eta_{\mathrm{diag}})$; i.e. is the operator $(\mathrm{id} - \partial_{\mathrm{mix}}\eta_{\mathrm{diag}})$ invertible? By construction, this operator preserves the grading, and so we decompose the retraction into a direct sum over weights $w \in \mathbb{Z}$, and ask for every w whether $(\mathrm{id} - \partial_{\mathrm{mix}}\eta_{\mathrm{diag}})^{(w)} : V_\bullet^{(w)} \to V_\bullet^{(w)}$ is invertible, where $V_\bullet^{(w)}$ is the weight-w piece of $\mathrm{MV}_\bullet(X)$.

For each w, this latter question is about the invertibility of a finite-dimensional matrix. Hence it is answered by whether the determinant of that matrix is or is not 0, and this determinant is some polynomial in the coefficients of s_{mix}. On the other hand $(\mathrm{id} - \partial_{\mathrm{mix}}\eta_{\mathrm{diag}})^{(w)}$ is definitely invertible when $s_{\mathrm{mix}} = 0$. Therefore, for each $w \in \mathbb{Z}$, $(\mathrm{id} - \partial_{\mathrm{mix}}\eta_{\mathrm{diag}})^{(w)}$ is invertible for generic s_{mix}.

At this point, we make an aside about vocabulary. We have been using the word "generic," which has a technical meaning in algebraic geometry. A property holds *generically* if it holds on a dense Zariski-open subset, and *very generally* if it holds on a countable intersection of dense Zariski-open subsets. Since we are working over an uncountable field, such an intersection is still uncountable and dense. For example, we have shown that for very general s_{mix}, and hence for a dense uncountable set, ∂_{mix} is a small deformation with

respect to the retraction $(V_\bullet, \partial_{\text{diag}}, \frac{\mathbb{C}[\vec{x}]}{(x_i^{d-1})}, 0, \varphi_{\text{diag}}, \tau_{\text{diag}}, \eta_{\text{diag}})$, as this smallness holds in the intersection of the countably many Zariski-open sets for which $(\text{id} - \partial_{\text{mix}}\eta_{\text{diag}})^{(w)}$ is invertible.

On the other hand, for generic $s = s^{(d)}$ it follows from Remark 2.3.2 that $\text{H}_\bullet(V_\bullet, \partial_{\text{diag}} + \partial_{\text{mix}})$ is $(d-1)^n$-dimensional and concentrated in degree 0, and retains its $\mathbb{Z}$-grading by polynomial degree. Let M denote the highest degree of any non-zero class in $\frac{\mathbb{C}[\vec{x}]}{(\partial_i s)}$. Consider the deformation ∂_{mix} to the differential ∂_{diag} on the complex $\bigoplus_{w=0}^M V_\bullet^{(w)}$. The smallness of ∂_{mix} on $\bigoplus_{w=0}^M V_\bullet^{(w)}$ requires just the invertibility of $M+1$ finite-dimensional matrices, and hence holds for generic s. On the other hand, if ∂_{mix} is small, it follows from the Homological Perturbation Lemma 2.3.10 that $\text{H}_0\big(\bigoplus_{w=0}^M V_\bullet^{(w)}, \partial_{\text{diag}} + \partial_{\text{mix}}\big) \subseteq \text{H}_0(V_\bullet, \partial_{\text{cl}}) = \frac{\mathbb{C}[\vec{x}]}{(\partial_i s)}$ has a basis consisting of the representatives of the monomials $\{x_1^{m_1} \cdots x_n^{m_n}\}$ for which all $m_i < d-1$. But this basis, having the same size as the dimension of $\frac{\mathbb{C}[\vec{x}]}{(\partial_i s)}$, must be a basis for the whole space. This completes the proof.

2.3.17 Corollary (explicit formulas from an *ad hoc* choice of basis) For very general homogeneous $s = \sum_{i_1,\dots,i_d=1}^n s_{i_1\dots i_d} \frac{x_{i_1}\cdots x_{i_d}}{d!}$, the projection $\tilde{\tau} : \mathcal{O}(X) \to \text{H}_0\big(\text{MV}_\bullet(X), \partial_{\text{BV}}\big)$ is given by:

$$\tilde{\tau} = \tau_{\text{diag}}\big(\text{id} - \partial_{\text{mix}}\eta_{\text{diag}}\big)^{-1} \sum_{\ell \geq 0} \Big(\operatorname{div} \eta_{\text{diag}}\big(\text{id} - \partial_{\text{mix}}\eta_{\text{diag}}\big)^{-1}\Big)^\ell$$

where:

$$\partial_{\text{mix}}\eta_{\text{diag}}(x_1^{m_1}\cdots x_n^{m_n}) = \begin{cases} 0, & \text{all } m_i < d-1, \\ \dfrac{1}{\sum_i \binom{m_i}{d-1}} \displaystyle\sum_{\substack{i_1,\dots,i_{d-1},j \\ \text{not all equal}}} \frac{s_{i_1\dots i_{d-1}j}}{s_{j\dots j}} \frac{x_{i_1}\cdots x_{i_{d-1}}}{(d-1)!} \left(\frac{\partial}{\partial x_j}\right)^{d-1} (x_1^{m_1}\cdots x_n^{m_n}). \end{cases}$$

$$\operatorname{div} \eta_{\text{diag}}\big(x_1^{m_1}\cdots x_n^{m_n}\big) = \begin{cases} 0, & \text{all } m_i < d-1, \\ \dfrac{1}{\sum_i \binom{m_i}{d-1}} \displaystyle\sum_i \frac{1}{s_{i\dots i}} \left(\frac{\partial}{\partial x_i}\right)^d \big(x_1^{m_1}\cdots x_n^{m_n}\big), & \text{else.} \end{cases}$$

Note that $\partial_{\text{mix}}\eta_{\text{diag}}$ preserves polynomial degree, and $\operatorname{div}\eta_{\text{diag}}$ reduces it by d, so the sum over ℓ converges. Similar but longer formulas apply when s is allowed to be inhomogeneous. If one is only interested in the values of $\tilde{\tau}$ on polynomials of fixed maximal degree, then the above formulas hold for generic s.

2.3.18 Example (a case when Theorem 2.3.8 fails) The above formulas do not hold for all s. For s a generic quartic in two variables x and y, Theorem 2.3.8 implies that $\mathcal{O}(\{\text{d}s = 0\})$ has as a basis the set $\{1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2\}$. But for $s(x,y) = x^4 + 2x^3y + 2xy^3 + y^4$, for example, $x^2y^2 = \frac{1}{12}\big((2y-x)\partial_x s + (2x-y)\partial_y s\big)$ is 0 in $\mathcal{O}(\{\text{d}s = 0\})$ and thus cannot be a basis element.

2.3.19 Remark (as far as algebra can go?) Provided the integration pairing $\mathrm{H}_n\big(\mathbb{C}^n; \{\Re(s) \ll 0\}\big) \otimes \mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big) \to \mathbb{C}$ between contours and observables is perfect, for a general contour γ and fixed action s Theorems 2.3.4 and 2.3.8 and Corollary 2.3.17 are as much as pure algebra can say about the values of integrals. In special cases, there is often more that can be said by studying the symmetry of the problem. Moreover, for fixed $f \in \mathcal{O}(X)$ and contour γ, one can write differential equations describing how $I_s(f) = \int_\gamma f\, e^s \,\mathrm{dVol}$ varies as a function of the coefficients of s. (If s is changed by a small amount, γ remains allowable.) For instance, there is a one-parameter family interpolating between s and $s^{(d)}$, and (provided $s^{(d)}$ is nonsingular) Proof 2.3.14 identifies the quantum BV homologies for all members of this family with $\mathcal{O}(\{\mathrm{d}s^{(d)} = 0\}) \cong \mathbb{C}^{(d-1)^n}$; different members of the one-parameter family give different integration maps out of $\mathcal{O}(\{\mathrm{d}s^{(d)} = 0\})$, which are related by an explicit differential equation.

A related question is to understand the values of $I_s(1) = \int_\gamma e^s \,\mathrm{dVol}$. For fixed γ, $I_s(1)$ is an $\mathfrak{sl}(n,\mathbb{C})$-invariant of s, transforming in a specific weight space for the center of $\mathfrak{gl}(n,\mathbb{C})$. Indeed, $I_s(1)$ solves a differential equation making it a branch of a hypergeometric function of the polynomial invariants of s. These questions have been pursued in [MS09, Sha10].

2.4 So, can we compute nonperturbative path integrals?

In this section we address to what extent the techniques we have developed so far apply to the infinite-dimensional integrals that appear in the path-integral approach to quantum field theory. The Feynman-diagrammatics described in Example 2.2.5 have proven immensely useful in high-energy physics and mathematics [Sta97, Fio03, ABF10, Res10a, Res10b, Cos11, CG11], so we will focus on the question of translating into infinite dimensions our approach to nonperturbative integrals. We will not prove any results, but simply outline some directions for further research. As providing all details would dramatically increase the length of this chapter, we will assume that the reader is already familiar with the types of path integrals that arise in quantum field theory, and some of the problems with defining them — our goal here is to mention some of these problems, and comment on whether and how they appear in the homological algebra formulation we have described.

We do not expect there to be enough patterns in the algebraization of finite-dimensional integral problems to be able to take the limit as $n \to \infty$ at the end of the problem. Rather, we can hope to begin with a complex playing the role of $\mathrm{BV}_\bullet(X) = \big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big)$ and study it with the Homological Perturbation Lemma 2.3.10. Doing so would give the algebraic part of the integral; then we could *define* an allowable contour for the infinite-dimensional integral problem as a map $\mathrm{H}_0\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{BV}}\big) \to \mathbb{C}$.

2.4.1 Remark (choices and ultraviolet divergences) The new feature in infinite-dimensional problems is that one must make choices where none were necessary in finite

dimensions. Rather than giving a general story, we focus on a simplified picture, in which the space replacing $X = \mathbb{C}^n$ to be integrated over is an infinite-dimensional vector space of sections of some vector bundle. The first step in generalizing the construction above is to come up with a reasonable version of the graded algebra $\mathrm{MV}_\bullet(X)$. Recall from Example 2.2.3 that when X is finite-dimensional, $\mathrm{MV}_\bullet(X) = \Gamma\big(\mathrm{T}^{\wedge\bullet}X\big)$, and for $X = \mathbb{A}^n$ we took polynomial sections in Example 2.2.6. For finite-dimensional vector spaces X, we thus have $\mathrm{MV}_\bullet(X) = \Lambda^\bullet(X) \otimes \mathrm{Sym}(X^*)$, where X^* is the dual vector space to X. When X is infinite-dimensional, we can try to take this as a definition, but now two choices must be made: first, which dual space to take, and second, how to complete the myriad tensor products present in the symmetric and antisymmetric powers.

Unfortunately, it is rare to find such choices so that both "∂_{cl}" and "div" are defined on $\mathrm{MV}_\bullet(X)$. In general, to define ∂_{cl} requires that the tensor products be completed appropriately. On the other hand, the invariant definition of div is as an extension of the map that pairs X with X^*, and this pairing is generally defined on the algebraic tensor product $X \otimes X^*$ but not on whatever completion is required to define ∂_{cl}.

This problem arises when defining perturbative integrals as well, and in that context it is called the problem of *ultraviolet divergences*. In the perturbative context the solution is reasonably understood, and goes by the name *renormalization theory*. The idea is to define div on the algebraic tensor product and then somewhat arbitrarily choose an extension of it to whatever tensor completion is necessary. One can instead or simultaneously choose a way to project from whatever completion naturally receives the map ∂_{cl} to a smaller tensor product. Almost certainly, these choices produce versions of div and ∂_{cl} that do not commute, and so the naive guess for ∂_{BV} will not be a differential. But in the perturbative integral, the hoped-for differential is (after rescaling by $\hbar$) $\partial_{\mathrm{BV}} = \partial_{\mathrm{cl}} + \hbar\,\mathrm{div}$, and so the failure to square to zero is order $\hbar$. One then modifies ∂_{cl} by a term which is order $\hbar$ in such a way that ∂_{cl} no longer squares to 0, but so that $\partial_{\mathrm{BV}}^2 = O(\hbar^2)$. After another modification, $\partial_{\mathrm{BV}}^2 = O(\hbar^3)$. In good situations, one can repeat this process infinitely, so that for a modified ∂_{cl} (or, what is equivalent, a modified action s) one can define a differential ∂_{BV}. Once defined, one can use the Homological Perturbation Lemma 2.3.10 to study ∂_{BV} in terms of the unmodified ∂_{cl}, and the answer is given by Feynman diagrams. This understanding of renormalization theory underpins Costello's work [Cos11, CG11, Gwi12].

2.4.2 Example (Chern–Simons Theory) The presence of ultraviolet divergences is a major obstruction to transporting our homological understanding of integration to the infinite-dimensional setting. But it seems to be the only one. Provided that one still works with an algebra $\mathrm{MV}_\bullet(X)$ that deserves to be thought of as an algebra of "polynomial multivector fields," one should still have the local nilpotence necessary in Theorems 2.3.4 and 2.3.8 to reduce the "quantum" problem of understanding the homology of ∂_{BV} to the "classical" problem of understanding the homology of ∂_{cl}. As an illustration, we discuss the well-studied example of quantum Chern–Simons Theory, the path integral for which (conjecturally) computes Reshetikhin–Turaev invariants of knots and three-manifolds [Wit89, RT90, Res10a].

Again we present only a very simplified version. Focusing on knot invariants, we fix our spacetime manifold to be the three-sphere $M = S^3$, and we choose a compact Lie group G with Lie algebra $\mathfrak{g}$. Then the naive space of fields to integrate over is $\Gamma = \Omega^1(M) \otimes \mathfrak{g}$. The *Chern–Simons action functional* s has as its critical locus the *flat* sections $\gamma \in \Gamma$, i.e. those satisfying the Maurer–Cartan equation $\mathrm{d}\gamma(w_1, w_2) = \frac{1}{2}[\gamma(w_1), \gamma(w_2)]$ for all $x \in M$ and $w_1, w_2 \in \mathrm{T}_x M$. There is a group $\mathscr{G} = \hom(M, G)$ acting nonlinearly on Γ. It does not preserve s, but it does preserve the one-form $\mathrm{d}s$, which is to say the $\mathscr{G}$ preserves the differential ∂_{cl} = "contract with $\mathrm{d}s$." Thus $\mathrm{d}s$ makes sense as a closed (but not exact) one-form on $\Gamma/\mathscr{G}$, and with the correct normalization it has integer periods. It is really over $\Gamma/\mathscr{G}$ that Chern–Simons Theory integrates.

Homological algebra is well-adapted to make sense of quotient spaces. Just like the classical BV complex $\big(\mathrm{MV}_\bullet(X), \partial_{\mathrm{cl}}\big)$ is a "derived intersection" computing certain Tor groups, *derived quotients* can be defined as the chain complexes computing certain Ext groups. In infinite dimensions to do this technically requires much work, mostly of the "making choices" form discussed above. Then derived quotients can be combined with the formation of odd cotangent bundles in a certain homological version of Marsden–Weinstein reduction.

When $\mathscr{G}$ is replaced by the Lie algebra $\Omega^0(M) \otimes \mathfrak{g}$ (thought of as an infinitesimal group), the result of these derived operations is reasonably well-known. The answer is that

$$\mathrm{MV}_\bullet(\Gamma/\mathscr{G}) = \mathrm{Sym}\Big(\big(\Omega^0(M)[0] \oplus \Omega^1(M)[1] \oplus \Omega^2(M)[2] \oplus \Omega^3(M)[3]\big) \otimes \mathfrak{g}[-1]\Big)$$

where the numbers in brackets denote shifts in homological grading, and the symmetric algebra construction is interpreted in the graded sense. In writing this, we have made specific choices of the type described in Remark 2.4.1: we used the orientation on M to identify $\big(\Omega^k(M)\big)^* = \Omega^{3-k}(M)$ and a choice of Killing form on $\mathfrak{g}$ to identity $\mathfrak{g}^* = \mathfrak{g}$. More generally, one could reasonable replace the de Rham complex by some other model of chains. The differential ∂_{cl} combines a "de Rham" part and a "Chevalley–Eilenberg" part. A more natural origin of this infinite-dimensional derived space is described in [AKSZ97].

Of course, the homology of the complex with infinitesimal $\mathscr{G}$ is the wrong one. Up to the action of $\mathscr{G}$, since $M = S^3$ is simply connected there is a unique flat connection. Its stabilizer under the $\mathscr{G}$ action is a copy of G. For comparison, the action of $\Omega^1(M) \otimes \mathfrak{g}$ on $\{\mathrm{d}s = 0\} = \{\text{flat connections}\}$ has countably many orbits. In any case, the correct quotient should satisfy that

$$\big(\mathrm{MV}(\Gamma/\mathscr{G}), \partial_{\mathrm{cl}}\big) \simeq \mathrm{MV}(\{\mathrm{pt}\}/G)$$

for a suitable definition of the right-hand side. Since G is compact, the derived quotient $\mathcal{O}(\{\mathrm{pt}\}/G) = \mathrm{Ext}^\bullet_G(1,1)$ is computed by the Chevalley–Eilenberg cochain complex of $\mathfrak{g}$, and has homology only in degree 0 where it is one-dimensional. Since we are working in homological gradings, the generators of this algebra are in homological degree -1. After keeping track of the grading shifts, a good guess for $\mathrm{MV}(\{\mathrm{pt}\}/G)$ is that it should be the Chevalley–Eilenberg complex for G acting by the coadjoint action on the manifold $\mathfrak{g}^*$, except with the generators for functions on the manifold $\mathfrak{g}^*$ placed in degree $+2$. Thus one expects that $\big(\mathrm{MV}(\Gamma/\mathscr{G}), \partial_{\mathrm{cl}}\big)$ will have a retraction onto a chain complex whose underlying graded

algebra is $\mathrm{Sym}(\mathfrak{g}[2] \oplus \mathfrak{g}^*[-1])$. Note that without the Chevalley–Eilenberg differential, this algebra of functions is precisely the homology of $\big(\mathrm{Sym}(\Omega^\bullet(M) \otimes \mathfrak{g}[-1]), \partial_{\text{de Rham}}\big)$, at least after identifying $\mathfrak{g}$ with its dual.

The algebra $V'_\bullet = \mathrm{Sym}(\mathfrak{g}[2] \oplus \mathfrak{g}^*[-1])$ is infinite-dimensional, but *locally finite*: for each k, $\dim V'_k < \infty$. If we can write down an algebra $\mathrm{MV}_\bullet(\Gamma/\mathscr{G})$ with differentials ∂_{cl} and div, then the Homological Perturbation Lemma 2.3.10 should allow to transfer the deformation div (plus possible ultraviolet corrections from 2.4.1) to a deformation of the Chevalley–Eilenberg differential on $V'_\bullet$. This deformation cannot break the local finiteness by Corollary 2.3.12, and we would be able to conclude that $\dim \mathrm{H}_0\big(\mathrm{MV}_\bullet(\Gamma/\mathscr{G}), \partial_{\mathrm{BV}}\big) < \infty$.

Moreover, we could get in this way explicit control over the "algebraic integration" map from ∂_{BV}-closed elements $f \in V_0$ to their homology classes. The most important such f measure the holonomy of a connection around a knot or link, and take the trace of the corresponding element of G in some representation R. If the program we have outlined can be carried out, then doing so would yield knot invariants, presumably equal to the Reshetikhin–Turaev invariants. Probably some modifications will be necessary, but there is reason for optimism that some nonperturbative path-integral problems can be completely solved using homological machinery.

2.4.3 Remark (Volume Conjecture and analytic continuation) One final remark is in order. There has been continuing interest in the analytic continuations of Chern–Simons theory to non-compact gauge groups [Wit11]. One motivation comes from the "Volume Conjecture" [Kas97]: in Witten's original path-integral description of the Jones polynomial [Wit89], one imagines integrating over connections valued in $\mathrm{SU}(2)$, but the volume conjecture predicts that in a certain limit the Jones polynomial is dominated by certain connections valued instead in $\mathrm{SL}(2,\mathbb{R})$.

The method of steepest descent implies that many integrals of the form $\int f e^s$ can be dominated by imaginary critical points of s, and Theorem 2.3.4 says that this is a purely algebraic result, and so should apply even to as-yet-to-be-defined infinite-dimensional integrals. (If s has degenerate critical points, the usual method of steepest descent does not give an answer, but Theorem 2.3.4 allows one to work with the non-reduced scheme-theoretic critical locus.) The reason we have such an isomorphism is that our observables are required to be polynomial: applying the same techniques but with $\mathscr{C}^\infty$ observables would give dramatically different (and much less algebraic) answers. Similarly, it is also interesting to consider Chern–Simons theory with non-compact gauge group and not restrict to polynomials everywhere. To a polynomial, $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ are indistinguishable, but to a smooth function they are very different. Note that in [Wit11], Witten considers $\mathrm{SL}(2,\mathbb{C})$, for example, both as a real smooth manifold and as a complex algebraic variety.

Chapter 3

Modern methods: factorization algebras, Poisson AKSZ theory, and manifestly rational universal quantization

The primary goal of this chapter is to construct a manifestly rational universal $\star$-quantization of formal Poisson manifolds; see Section 3.1 for definitions and background, as well as a more detailed chapter overview. A secondary goal of this chapter is to present some techniques that could be applied to other quantization and formality problems. This is not as much a departure from out story of perturbative integration in the previous chapters as it might seem: as we will explain in Remark 3.1.4, our construction corresponds to a certain one-dimensional quantum field theory (just like quantum mechanics), and has many features similar to the Feynman-diagrams-plus-renormalization-theory approach to path integrals. To emphasize the universality and manifest rationality of the construction, we build in Section 3.2 a universal formal Poisson manifold defined over $\mathbb{Q}$. Sections 3.3 and 3.4 construct a combinatorial version of classical field theory, which we quantize in Sections 3.5 and 3.6. In Section 3.7 we use this quantized field theory to build the universal $\star$-product.

3.1 Overview of the construction

In this section we introduce the basic goal of the chapter: to construct a rational universal $\star$-product. We then outline our construction. We conclude the section with a comparison, written for experts, between our construction and the AKSZ approach to path integrals.

3.1.1 Definition (manifestly rational universal $\star$-quantization of formal Poisson manifolds) Fix a field $\mathbb{K}$ of characteristic 0. The data of a *formal manifold over* $\mathbb{K}$ is that of a $\mathbb{K}$-vector space V, which we think of as the "linear functions" (in a choice of

coordinates) on the manifold. The whole algebra of functions is the power-series algebra $A = \prod_{n\geq 0} V^{\otimes n}/S_n$, where S_n denotes the symmetric group acting on $V^{\otimes n}$ in the obvious way, and $V^{\otimes n}/S_n$ is the quotient or coinvariant space. (Since we are in characteristic 0, we will regularly identify the coinvariant and invariant spaces.) A morphism of formal manifolds is (opposite to) a continuous algebra homomorphism.

A formal manifold is *Poisson* if the topological commutative algebra $A = \prod_{n\geq 0} V^{\otimes n}/S_n$ is given a continuous Poisson structure π; such a structure is determined by its Taylor coefficients $\pi^{(n)} : V^{\otimes 2} \to V^{\otimes n}$. If (A, π) is any Poisson algebra over $\mathbb{K}$, a *$\star$-quantization* of A is a continuous $\mathbb{K}[\![\hbar]\!]$-linear associative multiplication $\star$ on $A[\![\hbar]\!]$ for which $1 \in A \hookrightarrow A[\![\hbar]\!]$ is the unit and satisfying $f \star g = fg + \frac{\hbar}{2}\{f, g\}_\pi \mod \hbar^2$. It is common also to request that each $\hbar$-coefficient of $(f, g) \mapsto f \star g$ is a differential operator in each variable f, g. A quantization is *universal* (a better word would be "uniform") if each $\hbar$-coefficient of $\star$ is given by a universal polynomial in the Taylor coefficients of π. A universal $\star$-quantization is *rational* if all coefficients are rational; i.e. if we can take $\mathbb{K} = \mathbb{Q}$.

3.1.2 Remark (comparison with related results) For more than a decade, the state of the art in the search for universal $\star$-quantizations has been the celebrated formulas by Kontsevich [Kon03] (appearing on the arXiv in 1997 at `arXiv:q-alg/9709040v1`). Kontsevich's $\star$-quantization is defined over $\mathbb{K} = \mathbb{R}$ (actually, over the countable subfield generated by the real periods), and is given in terms of certain explicit definite integrals. Rationality of the Kontsevich $\star$-product is an open question, but one should not be too optimistic [FW10]. From one point of view, the underlying reason for Kontsevich's formulas is the Poisson Sigma Model [CF00, CF01], which is a two-dimensional path-integral topological quantum field theory for which the Kontsevich formulas comprise the Feynman diagram expansion in a certain choice of gauge fixing.

Tamarkin has also given an impressive homological algebraic argument assuring the existence of rational universal formulas [TT00, Hin03]. Tamarkin's arguments depend on the data of a Drinfel′d associator, and while rational associators are known to exist [Dri90] (a deeper result than the existence of $\star$-quantizations!), there seems to be no algorithm known for computing one, and even with one it is hard to see how to extract an algorithm to actually compute the sought-after $\star$-quantization from the Tamarkin-style proofs.

Thus our goal is to present a construction of a rational universal $\star$-quantization which is sufficiently explicit as to be amenable to direct computation. Our construction depends on "profinitely many" choices: to compute the coefficients up to $\hbar^n$ in our $\star$-product, *finitely* many choices must be made. We will not address here the extent to which these choices matter. Nor will we address any questions about the behavior of our $\star$-quantization under changes of coordinates, and in particular we will ignore the (very important) question of gluing local quantizations on coordinate patches into a quantization of a Poisson manifold.

3.1.3 Outline The structure of the chapter is as follows. To begin, in Section 3.2 we set up a category $\textsc{Vect}[X, \pi]$ containing a Poisson algebra $\mathcal{A}$, which we think of as "the

universal formal Poisson manifold": $\textsc{Vect}[X, \pi]$ is sufficiently rich as to allow all the usual constructions from linear algebra, but is constructed in such a way so that deformation quantizations of $\mathcal{A}$ correspond precisely to universal $\star$-products over $\mathbb{Q}$.

The first step in the construction of the $\star$-product on $\mathcal{A}$ is to "smear $\mathcal{A}$ out over $\mathbb{R}$." To complete this first step, we build in Section 3.3 a particular model of $\text{Chains}_\bullet(\mathbb{R}^n)$ which is tailored to our application; other models would also work, and we comment on the essential features of our model at the end of that section. We then describe the actual "smearing" in Section 3.4, when we construct from $\mathcal{A}$ and our model of Chains a $\text{DG}\textsc{Vect}[X, \pi]$-valued prefactorization algebra $\mathcal{A}(-)$ on $\mathbb{R}$ (i.e. an assignment to each open $U \subseteq \mathbb{R}$ a chain complex $\mathcal{A}(U)$ in $\textsc{Vect}[X, \pi]$, satisfying some conditions; see Definition 3.4.5). Elements of $\mathcal{A}(-)$ "live" on chains in $\mathbb{R}^n$ for $n \in \mathbb{N}$.

At the end of Section 3.4 we pose the problem of "deformation quantizing" the prefactorization algebra $\mathcal{A}(-)$. Achieving this quantization is the second and most important step of our overall construction of a universal $\star$-product. The Jacobi identity for the Poisson structure implies that a certain natural operation Δ on $\mathcal{A}(-)$ is almost, but not quite, a differential: its square does not vanish identically, but does vanish on the part of $\mathcal{A}(-)$ living over chains that are transverse to all diagonals. Thus, the central tool is to construct a system of homotopies that move every chain to be transverse to the diagonals, and thereby parameterize the necessary "higher algebraic structures"; we choose such a system of "transversalizing homotopies" in Section 3.5. The final deformation quantization of the prefactorization algebra $\mathcal{A}(-)$ is built in Section 3.6, which is the real heart of the chapter, and involves a combinatorial case-by-case analysis.

The third and final step of our construction of the universal $\star$-product occurs in Section 3.7. We use some homotopy perturbation theory to reconstruct the desired $\star$-product on the universal formal Poisson algebra $\mathcal{A}$ and to prove it satisfies the requisite properties. We end that section with a too-brief comparison with Kontsevich's sum-of-diagrams construction of a universal $\star$-product, and some questions that will have to wait until future work.

3.1.4 Remark (relation with topological field theory and AKSZ) This chapter is not written in the language of quantum field theory, but it is certainly related. As such, it might be valuable to provide a short sketch of these relations, written primarily for experts. These ideas will be further developed in future work, and in this sketch no attempt will be made at mathematical rigor. Another reason to include this sketch is that the ideas herein were the what motivated the construction in this chapter. An impatient reader should skip to Section 3.2.

Pois_n and $\mathbf{E}_n$ algebras. One description of our construction is that given a formal Poisson manifold M, we build a classical 1-dimensional topological field theory with target M, and then quantize it. The dimension "1" corresponds to the fact that the Poisson structure poses a deformation problem to associative algebras, and associative algebras are controlled by the topology of the spaces of configurations of distinct marked points in $\mathbb{R}$.

Similarly, an E_n *algebra* has k-ary operations parameterized by the configuration space of k distinct marked points in $\mathbb{R}^n$. (E.g. an E_0 algebra is a vector space with a distinguished vector, which is the image of the 0-ary operation corresponding to the unique way to embed no points into $\mathbb{R}^0$.) Just like an associative algebra gives rise to a Lie algebra, and so can degenerate to a Poisson algebra, E_n algebras give rise to L_∞ algebras with the role of the commutator played by the action of the fundamental class of S^{n-1} in the space of pairs of points in $\mathbb{R}^n$. A *Pois$_n$ algebra* is a (graded or dg) commutative algebra equipped with a degree-$(n-1)$ Poisson structure (or Gerstenhaber structure, when n is even); see [CFL], which uses different grading conventions. Thus E_n algebras can degenerate to Pois_n algebras, and in particular Pois_n manifolds pose E_n quantization problems.

We have indexed so that Poisson manifolds are Pois_1, and use *homological* conventions wherein any differential should have degree -1. We can more generally consider a *strongly homotopy Pois$_n$ manifold*, which is a (graded or dg) manifold M along with a function S on the shifted cotangent bundle $\mathrm{T}^*[n]\,M$ whose Hamiltonian vector field is homological (i.e. of degree -1 and self-commuting). A strongly homotopy Pois_n manifold is actually (strict) Pois_n if S is linear and quadratic in the fibers. More generally it is *flat* if $S|_M = 0$, where $M \hookrightarrow \mathrm{T}^*[n]\,M$ along the zero section.

Constraining the problem: a Poisson version of AKSZ–BV theory. Deformation problems can be hard to solve because the space of initial choices might be too big, and this is the generic situation for $\mathrm{Pois}_n \rightsquigarrow \mathrm{E}_n$ quantization. The *first AKSZ trick*, based on the groundbreaking work by Alexandrov, Kontsevich, Schwarz, and Zaboronsky [AKSZ97], converts a Pois_n manifold M into a more constrained deformation problem whose solution would give rise to an E_n algebra. Let U be an oriented m-dimensional manifold, and $U_{\mathrm{dR}} = \mathrm{spec}(\Omega^{-\bullet}U, \partial_{\mathrm{dR}})$ the dg manifold whose underlying graded manifold is the odd tangent bundle $\mathrm{T}[1]\,U$. The Koszul duality between the dg algebra $\Omega^{-\bullet}U$ of differential forms on U and the associative algebra $\mathcal{D}_U$ of differential operators on U has a geometric interpretation: U_{dR} is the "homotopy quotient" of U in which nearby points are identified. In particular, when U is a disk, there is a dg manifold equivalence $U_{\mathrm{dR}} \sim \{\mathrm{pt}\}$.

Consider the infinite-dimensional dg manifold $\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, M)$. On the one hand, when U is a disk, $U_{\mathrm{dR}} \sim \{\mathrm{pt}\}$ and so one should have $\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, M) \sim \underline{\mathrm{Maps}}(\{\mathrm{pt}\}, M) = M$. On the other hand, if M is Pois_n and U is m-dimensional, then $\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, M)$ should be Pois_{n-m}. The quickest way to see this is as follows. The orientation on U equips U_{dR} with a degree-m volume form. By integrating against this volume form, we can see $\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, \mathrm{T}^*[n]\,M)$ as a dense subbundle of $\mathrm{T}^*[n-m]\ \underline{\mathrm{Maps}}(U_{\mathrm{dR}}, M)$. If these were honestly isomorphic, then the strongly homotopy Poisson structure S on M would pull back to a strongly homotopy Poisson structure $\int_U S$ on $\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, M)$. The problem is that the embedding $\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, \mathrm{T}^*[n]\,M) \to \mathrm{T}^*[n-m]\ \underline{\mathrm{Maps}}(U_{\mathrm{dR}}, M)$ merely has dense image, and is not an isomorphism, and $\int_U S$ does not extend continuously to the rest of the bundle. This is the essential origin of "ultraviolet divergences." It is worth emphasizing that at this level, the symplectic case provides no simplification. If one fixes the asymptotic behavior of the maps $U_{\mathrm{dR}} \to M$ at the topological ends of U and if M is symplectic, then it is possible to give $\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, M)$ a nondegenerate symplectic form, but as with any nondegenerate pairing in infinite dimensions,

this symplectic form does not have a well-defined inverse Poisson structure.

But let's suppose that $\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, M)$ can be made Pois_{n-m}, and take $m = n$. Then the Poisson structure and the differential are both of degree -1. Let U range over open sets inside $\mathbb{R}^n$, and consider the precosheaf of Pois_0-algebras $\mathcal{A}(-) : U \mapsto \mathscr{C}^\infty(\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, M))$, assuming that a useful stand-in for $\mathscr{C}^\infty$ of an infinite-dimensional manifold is chosen. This precosheaf in fact has the structure of a factorization algebra valued in Pois_0 algebras. Factorization algebras were introduced by Beilinson and Drinfel′d [BD04] and developed by Costello and Gwilliam [CG11, Cos11, Gwi12], among others, as a formalism for (part of) quantum field theory; we recall the definition (of a slightly weaker structure that is sufficient for the purposes of this chapter) in Definition 3.4.5. Then one can ask to deform this Pois_0-valued factorization algebra to an E_0-valued factorization algebra. Since we had homotopy equivalences $M \sim \underline{\mathrm{Maps}}(U_{\mathrm{dR}}, M)$ when U was a disk, homotopy perturbation theory assures that we still have a homotopy equivalence $\mathscr{C}^\infty(M) \sim \mathcal{A}^{\mathrm{deformed}}(U)$. Allowing now U to be a disjoint union of open disks, we get a structure parameterized by a space that is homotopy equivalent to the configuration space of points in $\mathbb{R}^n$. So it is not surprising that homotopy perturbation theory allows to define an E_n algebra structure on $\mathscr{C}^\infty(M)$ out of $\mathcal{A}^{\mathrm{deformed}}(U)$. In general, "topological" factorization algebras on $\mathbb{R}^n$ are equivalent to E_n algebras [Lur09a, Lur09b]. This approach to quantization is essentially contained in the work by Costello and Gwilliam, although they stay a bit closer to traditional field theory, working primarily with symplectic objects and retaining more functional analysis than we will use.

Whence the model of $\mathrm{Chains}_\bullet$. Suppose that M is a vector space. Then $\underline{\mathrm{Maps}}(U_{\mathrm{dR}}, M) \cong \Omega^{-\bullet}(U) \otimes M$, and to form an algebra of functions on it, we can take (possibly a completion of) a symmetric algebra on some dual vector space. Since $\Omega^{-\bullet}(U)$ is infinite-dimensional, choices must be made about which dual and which tensor products in defining the symmetric algebra. In general, one could choose any normalized model of $\mathrm{Chains}_\bullet(U)$ as the dual space to $\Omega^{-\bullet}(U)$, and define $\mathrm{Chains}_\bullet(U) \otimes \mathrm{Chains}_\bullet(V) = \mathrm{Chains}_\bullet(U \times V)$ when constructing the symmetric algebra. In the construction in this chapter, we do this for a particular combinatorial model of chains chosen to make later steps manageable (Sections 3.3 and 3.4) — our choice corresponds to a certain "lattice approximation" of the theory. The presence of ultraviolet divergences and corresponding necessity of a renormalization scheme manifest themselves in the requirement to choose a transversalizing homotopy in Section 3.5.

Cotangent bundles and Hochschild cohomology. Although it plays no role in this chapter, the *second AKSZ trick* does explain a conjectural relationship between our construction and the Poisson Sigma Model of [CF00, CF01]. As mentioned above, a (strongly homotopy) Pois_n manifold M determines a homological Hamiltonian vector field on the shifted cotangent bundle $\mathrm{T}^*[n]\, M$, which is a Pois_{n+1} manifold with the canonical symplectic structure. Thus $\mathrm{T}^*[n]\, M$ is a potential target of an $(n+1)$-dimensional AKSZ theory. Coisotropic submanifolds provide boundary conditions for Poisson AKSZ theories, and so we can try to construct and then deform a factorization algebra on the half space $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ with bulk target $\mathrm{T}^*[n]\, M$ and boundary target the zero section $M \hookrightarrow \mathrm{T}^*[n]\, M$, which is a coisotropic

submanifold if and only if the strongly homotopy Pois_n structure on M is flat.

Given a topological factorization algebra on a manifold with boundary, one can restrict attention just to those opens that contract onto their intersection with the boundary, and thereby construct a factorization algebra on the boundary. Thus the deformed AKSZ-with-boundary system will consist of an E_{n+1} algebra quantizing $\mathrm{T}^*[n]\, M$, an E_n algebra quantizing M, and a compatibility condition that when $n = 0$ gives an associative algebra acting on a module with a distinguished vector (which need not be fixed by the action). In general, any E_n algebra A has a universal E_{n+1} algebra $\mathrm{End}(A)$ acting on it, given by a version of the Hochschild cochain complex. Conjecturally, given an E_n deformation A of M, we can form a perturbative version of $\mathrm{End}(A)$ that deforms the AKSZ-with-boundary theory for $M \hookrightarrow \mathrm{T}^*[n]\, M$. When $n = 1$ and M is a usual Poisson manifold, this should be the relationship between our construction and the usual Poisson Sigma Model.

Cotangent deformation = integration, and we "know" how to do integrals. To bring this nonrigorous story to a close, it is worth recalling the relationship between cotangent bundle deformations and oscillating integrals; see also Chapter 2 of this dissertation.

Suppose that the Pois_n manifold M is symplectic. One should then expect that any E_n deformation of M is Morita equivalent to the ground field thought of as an E_n module. This is particularly apparent in the case of cotangent bundles, where the deformed zero section provides such a Morita equivalence. On the other hand, E_0 deformations of cotangent bundles are closely related to integrals, as follows. If a manifold M is equipped with a function s, then the derived critical locus of s makes sense as a dg manifold structure on the symplectic Pois_0 manifold $\mathrm{T}^*[-1]\, M$. Any choice of nowhere-vanishing volume form μ on M determines an E_0 deformation A of $\mathrm{T}^*[-1]\, M$, and in good situations the homology of the deformation computes the de Rham homology of the critical locus of s. In particular, when s has a single nondegenerate critical point, then the homology $\mathrm{H}_\bullet A$ is a one-dimensional E_0 algebra whose distinguished vector is nonzero. Thus there is a unique E_0 isomorphism between $\mathrm{H}_\bullet A$ and the ground field, and the homology class of a closed element $a \in A$ is precisely the expectation value of (the restriction to M of) $a \in \mathscr{C}^\infty(\mathrm{T}^*[-1]\, M)$ for the measure $e^{-s/\hbar}\mu$. (The preceding construction only uses the flat connection on the line bundle of densities determined by μ. A *Morita equivalence* of E_0 algebras is a linear isomorphism that need not preserve the distinguished elements. An actual volume form μ determines a nontrivial Morita equivalence between $\mathrm{H}_\bullet A$ and the ground field, and the image of $1 \in A$ under this equivalence is the value of $\int e^{-s}\mu$.) This is the homological interpretation of the Batalin–Vilkovisky approach to integrals [BV83, BV84], and from this perspective the homological perturbation theory in Section 3.7 can be thought of as "doing the integral."

Thus deformations of *symplectic* Pois_n manifolds can be understood in terms of Feynman-style integrals. The Poisson Sigma Model solves a one-dimensional algebra problem (E_1 deformation), but it is necessarily two-dimensional in order to have a path-integral interpretation. By working with factorization algebras and homological perturbation theory, we are freed from path integrals, and can handle directly (and purely algebraically) various Poisson deformation problems in the dimensions in which they should reside.

3.2 The universal formal Poisson manifold

In this section we will describe a category $\textsc{Vect}[X, \pi]$ (and an algebra $\mathcal{A}$ therein) that we think of as "the universal formal Poisson manifold" — another name is *the PROP presenting a formal Poisson manifold*.

3.2.1 Definition ($\textsc{Vect}[X]$) We write S_n for the permutation group on n objects, and $\mathbb{Q}[S_n]$ for its group algebra. By "vector space" we always mean "possibly-infinite-dimensional vector space over $\mathbb{Q}$."

Consider the category whose objects are sequences $\{V_n\}_{n\geq 0}$ of vector spaces, which we think of as formal direct sums "$\bigoplus V_n X^{\otimes n}$." We set

$$\mathrm{Hom}\big(\bigoplus V_m X^{\otimes m}, \bigoplus W_n X^{\otimes n}\big) = \prod_n \mathrm{Hom}(V_n, W_n) \otimes \mathbb{Q}[S_n].$$

This is the free symmetric monoidal category generated by an object X. The monoidal structure should be obvious; the symmetric structure involves the permutation groups in a natural way.

Recall that for any category $\mathcal{C}$, an object of its *idempotent splitting* or *Karoubi envelope* is a pair (A, e) for $A \in \mathcal{C}$ and $e = e^2 \in \mathrm{End}(A)$, and a morphism $(A, e) \to (A', e')$ is a morphism $\{f : A \to A'\} \in \mathcal{C}$ satisfying $f = e' \circ f \circ e$. We define $\textsc{Vect}[X]$ to be the idempotent splitting of the category defined above. $\textsc{Vect}[X]$ is tensored over $\textsc{Vect}$, and can be considered "the free symmetric-monoidal $\textsc{Vect}$-module on one $\otimes$-generator."

3.2.2 Definition ($\textsc{Vect}[X, \pi]$ and $\mathcal{A} \in \textsc{Vect}[X, \pi]$) We introduce more morphisms to $\textsc{Vect}[X]$ via generators and relations. For each n, we include a generator $\pi^{(n)} : X^{\otimes 2} \to X^{\otimes n}$. The relations are the following. We require that π be antisymmetric under the flip $X^{\otimes 2} \to X^{\otimes 2}$, and to be invariant under the S_n action on $X^{\otimes n}$. The interesting relation that we impose is the *Jacobi identity*. Let $\mathrm{ave}_n : X^{\otimes n} \to X^{\otimes n}$ denote the operation $\frac{1}{n!}\sum_{\sigma \in S_n} \sigma$ which averages for the S_n action. Let $A_n \subset S_n$ denote the "alternating group," i.e. the kernel of the sign representation $S_n \to \{\pm 1\}$, and $\mathrm{alt}_n = \frac{2}{n!}\sum_{\sigma \in A_n} \sigma$ the average for this subgroup. Then we ask that, for each n:

$$0 = \mathrm{ave}_n \circ \sum_{m \geq 1} m \cdot \left(\pi^{(n-m+1)} \otimes (\mathrm{id}_X)^{\otimes (m-1)}\right) \circ \left(\mathrm{id}_X \otimes \pi^{(m)}\right) \circ \mathrm{alt}_3 : X^{\otimes 3} \to X^{\otimes n}$$

(The multiplication by m counts the number of ways to contract an output of $\pi^{(m)}$ with the second input of $\pi^{(n-m+1)}$.) The idempotent splitting of this new category is the category $\textsc{Vect}[X, \pi]$.

We will primarily work with certain pro-objects in $\textsc{Vect}[X, \pi]$, all of which are "infinite formal products" of actual objects. Let $\mathcal{C}$ be a linear category, and A_m and B_n two families of objects. One can make sense of the formal products $\prod A_m$ and $\prod B_n$ by declaring

that $\mathrm{Hom}(\prod A_m, \prod B_n) = \prod_n \bigoplus_m \mathrm{Hom}(A_m, B_n)$; the composition is via $(\infty \times \infty)$ matrix multiplication. We will often abuse notation and refer to pro-objects as if they were objects.

Denote by $X^{\otimes n}/S_n$ the splitting of the projection ave_n. Then $\odot = \mathrm{ave} \circ \otimes$ defines a commutative algebra structure on $\bigoplus_{n\geq 0}(X^{\otimes n}/S_n)$, and also on the pro-object $\mathcal{A} = \prod_{n\geq 0}(X^{\otimes n}/S_n)$. We can extend $\pi^{(n)}$ to the map $(X^{\otimes \ell}/S_\ell) \otimes (X^{\otimes m}/S_m) \to (X^{\otimes \ell+m+n-2}/S_{\ell+m+n-2})$ defined by:

$$\ell\, m\, \mathrm{ave}_{\ell+m+n-2} \circ (\mathrm{id}_X)^{\otimes \ell-1} \otimes \pi^{(n)} \otimes (\mathrm{id}_X)^{\otimes m-1} \circ \mathrm{ave}_\ell \otimes \mathrm{ave}_m\,.$$

This is an antisymmetric biderivation on $\mathcal{A}$, and the sum of these defines a Poisson bracket on $\mathcal{A}$. The pro-object $\mathcal{A} \in \textsc{Vect}[X, \pi]$ is the algebra of functions on *the universal formal Poisson manifold* in the following sense. Let V be any vector space over a field $\mathbb{K}$ of characteristic 0, and choose a continuous Poisson structure on the power series algebra $\prod(V^{\otimes n}/S_n)$. Then there is a unique (up to unique isomorphism) symmetric monoidal functor from $\textsc{Vect}[X, \pi]$ to the category of $\mathbb{K}$-vector spaces which takes X to V and $\pi^{(n)}$ to the nth Taylor coefficient of the chosen Poisson structure.

In the language of properads, our category $\textsc{Vect}[X, \pi]$ can be described succinctly as the Karoubi envelope of the symmetric monoidal extension of the properad that describes a vector space X and a Poisson structure on its completed symmetric algebra.

3.2.3 Notation (diagrams for $\textsc{Vect}[X, \pi]$) We will adopt a graphical notation for working with the category $\textsc{Vect}[X, \pi]$. An early version of this notation was introduced by Penrose [Pen71], motivated by Feynman's diagrams, and its foundational underpinnings were established by Joyal and Street [JS91]. By now, such graphical notation is quite standard among category theorists, although due to the difficulty in typesetting it perhaps doesn't appear as much as it should.

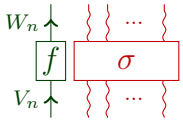
We denote the object $X \in \textsc{Vect}[X, \pi]$ by a vertical squiggly line $\wr$, and tensor products by placing diagrams next to each other horizontally:

$$X^{\otimes n} = \underbrace{\wr\, \wr \cdots \wr}_{n}$$

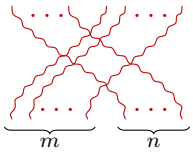
Let $V_n \in \textsc{Vect}$. Then we denote $V_n \otimes X^{\otimes n}$ by $\overset{V_n}{\uparrow}\wr \cdots \wr$. The arrow on the line corresponding to V_n is not really necessary, but is traditional to remind that objects might be nonisomorphic to their duals. We do not put an arrow on $\wr$ to save visual space; we will never refer to a dual object of it, and so no confusion should arise.

We read morphisms from bottom to top ("time goes up"). (On its face, this is the opposite convention from what we used in Chapters 1 and 2, but it will give more closely related diagrams, as in those chapters we thought of the points of the manifold, rather than the linear functions thereon, as most basic.) Let $f \in \mathrm{Hom}(V_n, W_n)$ and $\sigma \in S_n \subseteq \mathbb{Q}[S_n]$.

Then the morphism $f \otimes \sigma \in \mathrm{Hom}(V_n \otimes X^n, W_n \otimes X^n)$ in $\textsc{Vect}[X]$ is:

$f \otimes \sigma =$ 

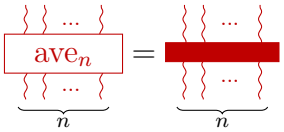
where $\boxed{\sigma}$ is a picture of the permutation. For example, the symmetric structure on $\textsc{Vect}[X]$ is based on the isomorphism $X^{\otimes(m+n)} \to X^{\otimes(n+m)}$ given by:

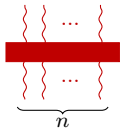


Composition is given by vertical concatenation of pictures.

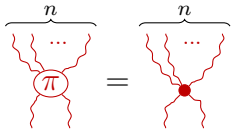
Note that we have used the same picture for an object and its identity morphism. Similarly, if we have a projection e on an object A, we denote the object (A, e) in the idempotent splitting by 

. The projection ave_n, which projects any S_n-module onto its space of (co)invariants, will be particularly important, and so we denote it by a solid horizontal bar. With this notation, we have:

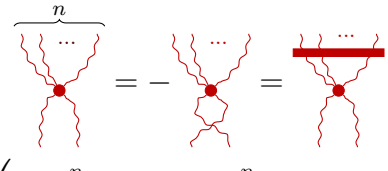
$X^{\otimes n}/S_n =$ 

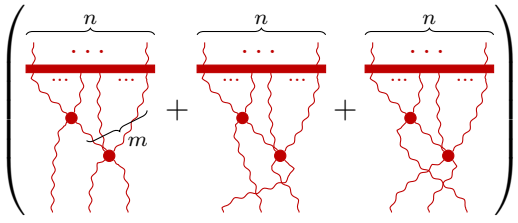
$\qquad \mathcal{A} = \prod_{n\geq 0}$ 

We denote the generators $\pi^{(n)}$ of $\textsc{Vect}[X, \pi]$ by solid dots:

$\pi^{(n)} =$ 

Their required relations are:

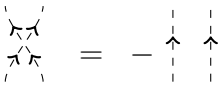


$$\sum_m m \cdot$$ 

$= 0$

3.2.4 Remark (chain complexes in $\textsc{Vect}[X,\pi]$) The category $\textsc{Vect}[X,\pi]$ is completely adequate to support a theory of "chain complexes." We denote by $\textsc{GVect}[X,\pi]$ the category of $\mathbb{Z}$-graded objects in $\textsc{Vect}[X,\pi]$, and by $\textsc{DGVect}[X,\pi]$ the category of chain complexes. These are both symmetric-monoidal in the usual way — odd things anticommute — and can be equivalently constructed by repeating the construction of $\textsc{Vect}[X,\pi]$ with $\textsc{Vect}$ replaced by $\textsc{GVect}$ or $\textsc{DGVect}$. We adopt "homological" grading conventions, whereby the differential on any dg object has $\mathbb{Z}$-grading -1.

There is a particularly important object in $\textsc{GVect}$ (and hence in $\textsc{DGVect}$ and $\textsc{DGVect}[X,\pi]$), namely the one-dimensional vector space "in homological degree 1." In the graphical notation, we will denote this object by a dashed directed line:  . Here the arrow is very important, as this object is dualizable; its dual is the one-dimensional vector space supported in degree -1, and is denoted  . For example, for any chain complex $(M,\partial_M)\in\textsc{DGVect}$, the map ∂_M is best defined as a morphism $\partial_M : \dashv \to \mathrm{End}(M)$ in $\textsc{GVect}$ (and, in fact, in $\textsc{DGVect}$). The Koszul sign conventions are summed up by demanding:



Many signs follow from this. We mention a few in Remark 3.5.8.

3.2.5 Remark (strategy for the construction) Our strategy to construct a universal $\star$-quantization is to find a sequence of morphisms that package together to an associative algebra structure on $\mathcal{A}[\![\hbar]\!] = \big(\prod(X^{\otimes n}/S_n)\big)[\![\hbar]\!] \in \textsc{Vect}[X,\pi]$. Our request is that this $\star$-product satisfy $\star = \odot + \frac{\hbar}{2}\pi + O(\hbar^2)$. By the construction of our category $\textsc{Vect}[X,\pi]$, the morphisms defining the $\star$-product are necessarily rational polynomials in the $\pi^{(n)}$. By universality, any formal Poisson manifold defines a functor from this category, and pushing our $\star$-product forward along this functor defines the $\star$-quantization of the particular formal manifold.

3.3 A model of $\mathrm{Chains}_\bullet(\mathbb{R}^n)$

Our construction of the universal $\star$-quantization will involve taking the commutative algebra $\bigoplus(X^{\otimes n}/S_n)$ in $\textsc{Vect}[X,\pi]$ and "smearing it out over $\mathbb{R}$": we will assign to each open interval $U\subseteq\mathbb{R}$ an object $\mathcal{A}(U)\in\textsc{DGVect}[X,\pi]$ that is quasi-isomorphic to $\mathcal{A}=\prod(X^{\otimes n}/S_n)$. This "smearing" depends on a choice of model of "$\mathrm{Chains}_\bullet(\mathbb{R}^n)$," with rational coefficients. In this section we will describe a particular choice which is tailored to our application. At the end of the section we will summarize some of the features of our model.

3.3.1 Definition ($\mathrm{C}_\bullet(U)$) Denote by $t_1,\dots,t_n$ the coordinates on $\mathbb{R}^n$. We begin by decomposing $\mathbb{R}^n$ into cells by dividing it along the hyperplanes $\{t_i=z\}$ for $z\in\mathbb{Z}$ and along $\{t_i=t_j\}$. In the standard way we can turn a cell complex into a dg vector space: the

degree-k part of the dg vector space is spanned by the set of oriented k-dimensional cells, modulo the relation that switching an orientation is the same as multiplication by -1. This defines a dg vector space we will call $\mathrm{C}^0_\bullet(\mathbb{R}^n)$.

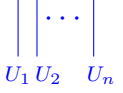
For ℓ a nonnegative integer, we define $\mathrm{C}^\ell_\bullet(\mathbb{R}^n)$ analogously to $\mathrm{C}^0_\bullet(\mathbb{R}^n)$, by dividing along the hyperplanes $\{t_i = t_j\}$ and $\{t_i = \frac{z}{2^\ell}\}$ with $z \in \mathbb{Z}$. There is then a natural inclusion $\mathrm{C}^\ell_\bullet(\mathbb{R}^n) \hookrightarrow \mathrm{C}^{\ell+1}_\bullet(\mathbb{R}^n)$. We set $\mathrm{C}_\bullet(\mathbb{R}^n)$ to be the direct limit (union) of the C^ℓs along these inclusions.

More generally, let $U \subseteq \mathbb{R}^n$ be an open region. Then we can define $\mathrm{C}^\ell_\bullet(U)$ as the subcomplex of $\mathrm{C}^\ell_\bullet(\mathbb{R}^n)$ spanned by those chains that are contained within U. One can form a similar inductive limit to define $\mathrm{C}_\bullet(U)$. Although we will not need it, it should be clear that $\mathrm{C}_\bullet(U)$ does compute the homology of U. In fact, we will be interested only in the situation when U is *rectangular*: $U = U_1 \times \cdots \times U_n$ for $U_i \subseteq \mathbb{R}$ open.

One should think of $\mathrm{C}^\ell_\bullet$ as a "lattice approximation" to $\mathrm{C}_\bullet$, with a "mesh spacing" controlled by ℓ. Note that for fixed U and "low" ℓ, $\mathrm{C}^\ell_\bullet(U)$ may not compute the homology of U.

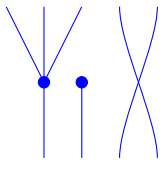
3.3.2 Definition (diagonal map) A linear map $\mathbb{R}^m \to \mathbb{R}^n$ is *diagonal* if it is given by an $m \times n$ matrix which is all 0s except for precisely one 1 in each of the n rows. The diagonal maps $\mathbb{R}^m \to \mathbb{R}^n$ correspond precisely to the maps $\{1, \ldots, n\} \to \{1, \ldots, m\}$ of finite sets. By convention, when $n = 0$, the unique map $\mathbb{R}^m \to \mathbb{R}^0$, corresponding to the unique inclusion $\emptyset \to \{1, \ldots, m\}$, is diagonal. Diagonal maps restrict to maps of the ℓth cell complexes. It follows that $\mathrm{C}^\ell_\bullet$ and $\mathrm{C}_\bullet$ are functorial for diagonal maps. A particular case of diagonal maps are the permutations of the coordinates.

3.3.3 Notation (diagrams for $\mathrm{C}_\bullet$) It is convenient to adopt a graphical notation for the action of diagonal maps on $\mathrm{C}_\bullet$. As in 3.2.3, we will read our diagrams from bottom to top ("time goes up"). Choose opens $U_1, \ldots, U_n \subseteq \mathbb{R}$. Then write $\mathrm{C}_\bullet(U_i) =$  and:

$$\mathrm{C}_\bullet(U_1 \times \cdots \times U_n) =$$ 

This notation is justified by Remark 3.3.4. We will often leave off the U_is.

Then we can denote diagonal maps with natural pictures. For example:

 $= \begin{pmatrix} 1 & & & \\ 1 & & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} : \mathrm{C}_\bullet(\mathbb{R}^4) \to \mathrm{C}_\bullet(\mathbb{R}^5)$

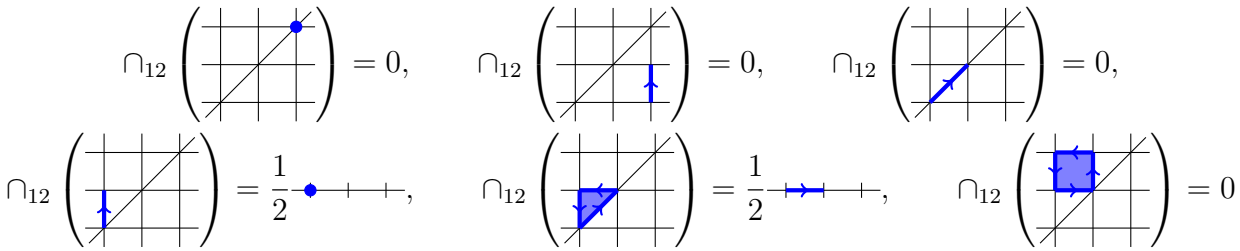
The set of diagonal maps is generated by the maps  ,  , and  . They satisfy many relations, and the reader is invited to write out a generating set: the relations express that the crossing squares to identity, that vertices can be pulled across crossings, and so on. The most

important relations say that  and  are the comultiplication and counit of a coassociative cocommutative coalgebra. All of these relations follow from the functoriality of $\mathrm{C}_\bullet$ for diagonal maps: the diagonal map makes any space into a coassociative cocommutative coalgebra in the category of spaces.

3.3.4 Remark (justification for Notation 3.3.3) If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, then $\mathrm{C}_\bullet(U \times V)$ will be our stand-in for $\mathrm{C}_\bullet(U) \otimes \mathrm{C}_\bullet(V)$. This is justified for two reasons. First, there is a natural inclusion of chain complexes $\mathrm{C}_\bullet(U) \otimes \mathrm{C}_\bullet(V) \hookrightarrow \mathrm{C}_\bullet(U \times V)$. Second, if $U \times V \subseteq \mathbb{R}^{m+n}$ does not intersect any diagonal of the form $\{t_i = t_j\}$ with $i \in \{1, \dots, m\}$ and $j \in \{m+1, \dots, m+n\}$, then in fact the natural inclusion is an isomorphism of chain complexes. This inclusion is compatible with the diagonal maps. In particular, consider the permutation map $\mathbb{R}^{m+n} \to \mathbb{R}^{n+m}$ that switches the first m coordinates with the last n coordinates. The inclusion $\mathrm{C}_\bullet(U) \otimes \mathrm{C}_\bullet(V) \hookrightarrow \mathrm{C}_\bullet(U \times V)$ intertwines this permutation with the (signed) symmetry map $\mathrm{C}_\bullet(U) \otimes \mathrm{C}_\bullet(V) \to \mathrm{C}_\bullet(V) \otimes \mathrm{C}_\bullet(U)$ in DGVect.

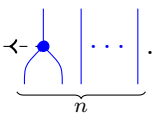
3.3.5 Definition (intersection) We now define the operation of "intersection with the diagonal $\{t_1 = t_2\}$," which is to be a map $\cap_{12} : \mathrm{C}_\bullet(\mathbb{R}^n) \to \mathrm{C}_{\bullet-1}(\mathbb{R}^{n-1})$. We will define it on $\mathrm{C}^\ell_\bullet$, and it will be clear that it behaves well under the inclusions $\mathrm{C}^\ell_\bullet \hookrightarrow \mathrm{C}^{\ell+1}_\bullet$. Given a basic k-dimensional (oriented) cell c in the ℓth cell-decomposition of $\mathbb{R}^n$, we send c to $0 \in \mathrm{C}^{k+1}_\bullet(\mathbb{R}^{n-1})$ if it does not intersect the diagonal. If c does intersect the diagonal, it does so along a cell which is of dimension at most k. If the intersection is of dimension not equal to $k-1$ then $\cap_{12}(c) = 0$. Otherwise, there is a unique $(k-1)$-dimensional cell in the ℓth decomposition of $\mathbb{R}^{n-1}$ whose image under the diagonal embedding is the intersection of c with $\{t_1 = t_2\}$. We send our k-dimensional cell c to $\frac{1}{2}\times$ this $(k-1)$-dimensional cell, oriented so that the product orientation of the diagonal embedding of the $(k-1)$-dimensional cell with the oriented interval in the $(-1, 1, 0, 0, 0, 0, \dots)$ direction gives the orientation of c.

For example:



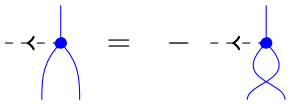
The factor of $\frac{1}{2}$ is included so that sums of cells that together intersect the diagonal transversally map to their full intersections.

It is convenient to draw the intersection also via a diagram:

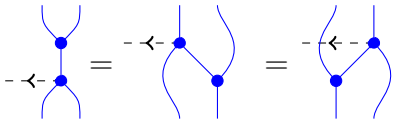
$\cap_{12} : \mathrm{C}_\bullet(\mathbb{R}^n) \to \mathrm{C}_{\bullet-1}(\mathbb{R}^{n-1}) \quad =$ 

The dashed line reminds that $\cap_{12}$ is a degree-(-1) map. The graphical notation also allows to quickly define the intersections along other diagonals.

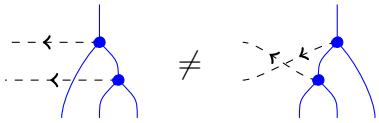
3.3.6 Remark (some identities and non-identities for $\cap$) Under a reflection, orientation rules switch, and so $\cap_{12}$ transforms as -1 under permuting t_1 with t_2:



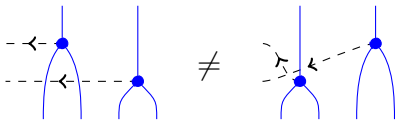
Moreover, we invite the reader to check the following *Frobenius-algebra-like axiom*:



However, this intersection pairing is defective in one way: it does not satisfy the associative law.



Consider, for example, the action on the square $[0,1]\times[0,1]\times\{0\}\in \mathrm{C}_2(\mathbb{R}^3)$. Indeed, the graphical notation is a bit dangerous because intersections that look like they don't interact nevertheless do not commute:



This is witnessed by, for example, the 2-cell $\{t_1\in[0,1],\ t_2\in[0,1],\ t_3=t_2,\ t_4=0\}$ in $\mathbb{R}^4$.

Since $\cap$ is an odd operator, its natural self-commutator is $[\cap,\cap]=\cap\circ\cap+\cap\circ\cap=2\cap^2$ rather than $\cap^2-\cap^2=0$. Keeping track of such signs is the primary reason to record the dashed lines; we always follow the convention from Remark 3.2.4. Note also that when working with homogeneous elements some signs enter when and  lines cross.

3.3.7 Remark (what makes our construction tick?) The most important features of our model of chains are: (i) It is functorial for diagonal maps. (ii) If U and V are disjoint open regions, then $\mathrm{C}_\bullet(U\times V)=\mathrm{C}_\bullet(U)\otimes\mathrm{C}_\bullet(V)$. (iii) It is sufficiently small as to be amenable to direct computation, as well as to making explicit universal choices. (iv) It has a reasonable intersection theory, which along with the diagonal maps makes it almost a dg Frobenius algebra (only the associativity is missing). (v) It is defined with coefficients in $\mathbb{Q}$. Probably any model with similar features would work in our construction.

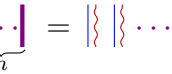
3.4 A cosheaf on $\mathbb{R}$ that "smears out" the universal formal Poisson manifold

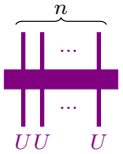
In this section we will describe what we mean by "smearing out" the universal formal Poisson manifold $\mathcal{A} = \prod(X^{\otimes n}/S_n)$ across $\mathbb{R}$. We will end up assigning to each open set $U \subseteq \mathbb{R}$ a pro-object $\mathcal{A}(U) \in \mathrm{DGVECT}[X, \pi]$. If U is an interval, then $\mathcal{A}(U)$ will be homotopic to $\mathcal{A}$. The precosheaf $\mathcal{A}(-)$ will satisfy the conditions of a *prefactorization algebra* (Definition 3.4.5). At the end of the section we will discuss the problem of deforming or "quantizing" this prefactorization algebra.

3.4.1 Definition ($\mathcal{A}(-)$) For each open $U \subseteq \mathbb{R}$ and for each n, in the previous section we defined a chain complex $\mathrm{C}_\bullet(U^n) \in \mathrm{DGVECT}$, with an action of S_n permuting the n copies of U. Functoriality for diagonal maps implies that $U \mapsto \mathrm{C}_\bullet(U^n)$ is a precosheaf of S_n-modules. We consider then the object $\mathrm{C}_\bullet(U^n) \otimes X^{\otimes n} \in \mathrm{DGVECT}[X, \pi]$, and give it the diagonal S_n action, and define:

$$\mathcal{A}(U) = \prod_{n \geq 0} \big(\mathrm{C}_\bullet(U^n) \otimes X^{\otimes n}\big)/S_n$$

Experts may recognize this as a particular model for factorization homology with coefficients in $\mathcal{A}$.

Combining the graphical notations of the previous sections, we set  $= \mathrm{C}_\bullet(U) \otimes X$, and  $= \mathrm{C}_\bullet(U^n) \otimes X^{\otimes n}$. As in 3.2.3, denote the map that averages for the S_n action by a horizontal bar, and identify objects with the projections that pick them out. Then:

$$\mathcal{A}(U) = \prod_{n \geq 0}$$ 

3.4.2 Remark ($\mathcal{A}(-)$ is valued in commutative algebras) Recall that $\mathrm{C}_\bullet$ comes equipped with canonical maps $\otimes : \mathrm{C}_\bullet(U^m) \otimes \mathrm{C}_\bullet(U^n) \hookrightarrow \mathrm{C}_\bullet(U^{m+n})$, compatible with the symmetric group actions. We define

$$\odot : \big((\mathrm{C}_\bullet(U^m) \otimes X^{\otimes m})/S_m\big) \otimes \big((\mathrm{C}_\bullet(U^n) \otimes X^{\otimes n})/S_n\big) \to \big((\mathrm{C}_\bullet(U^{m+n}) \otimes X^{\otimes m+n})/S_{m+n}\big)$$

by $\odot = \mathrm{ave}_{m+n} \circ \otimes$. Summing the various maps $\odot$ gives a map $\mathcal{A}(U) \otimes \mathcal{A}(U) \to \mathcal{A}(U)$. In this way we make $\mathcal{A}(U)$ into a (pro, dg) commutative algebra, and $\mathcal{A}(-)$ into a precosheaf of commutative algebras.

Before describing further structure, it is worth pointing out the behavior of $\mathcal{A}(-)$ on disjoint unions. Let U and V be disjoint open sets in $\mathbb{R}$. Then $\mathrm{C}_\bullet(U^m) \otimes \mathrm{C}_\bullet(V^m) \hookrightarrow$

$\mathrm{C}_\bullet(U^m \times V^n)$ is an isomorphism. By the binomial theorem, $(U \cup V)^n \cong \bigcup_k \binom{n}{k} U^{n-k} \times V^k$, where the binomial coefficient $\binom{n}{k}$ means to take the disjoint union of that many copies. Thus $\mathrm{C}_\bullet((U \cup V)^n) \cong \bigoplus \binom{n}{k} \mathrm{C}_\bullet(U^{n-k}) \otimes \mathrm{C}_\bullet(V^k)$. After taking coinvariants, we find a canonical isomorphism:

$$\big((\mathrm{C}_\bullet((U \cup V)^n) \otimes X^{\otimes n})/S_n\big) \cong \bigoplus_k \big((\mathrm{C}_\bullet(U^{n-k}) \otimes X^{\otimes n-k})/S_{n-k}\big) \otimes \big((\mathrm{C}_\bullet(V^k) \otimes X^{\otimes k})/S_k\big)$$

if U and V are disjoint. It follows that for U and V disjoint we have a canonical isomorphism $\mathcal{A}(U) \otimes \mathcal{A}(V) \xrightarrow{\sim} \mathcal{A}(U \cup V)$. In fact, $\mathcal{A}(-)$ is not just a precosheaf but a cosheaf — the distribution over disjoint unions is an example of this, because the coproduct of commutative algebras is their tensor product.

Then it is straightforward to see that the isomorphism $\mathcal{A}(U) \otimes \mathcal{A}(V) \xrightarrow{\sim} \mathcal{A}(U \cup V)$ factors through

$$\mathcal{A}(U) \otimes \mathcal{A}(V) \hookrightarrow \mathcal{A}(U \cup V) \otimes \mathcal{A}(U \cup V) \xrightarrow{\odot} \mathcal{A}(U \cup V).$$

Because of this, for any $W \subseteq \mathbb{R}$ containing both U and V, we will refer to the composition $\mathcal{A}(U) \otimes \mathcal{A}(V) \xrightarrow{\sim} \mathcal{A}(U \cup V) \hookrightarrow \mathcal{A}(W)$ by the name $\odot$.

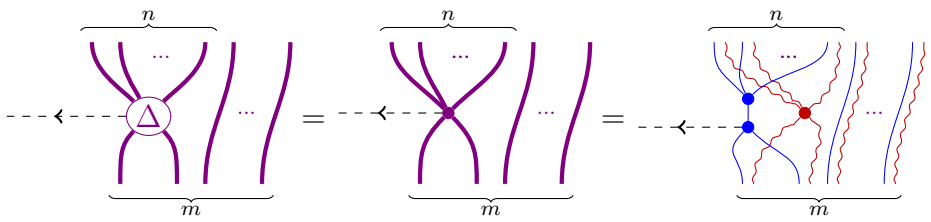
3.4.3 Lemma ($\mathrm{H}_\bullet\big(\mathcal{A}(-), \partial\big) = \mathcal{A}$) For any open set $U \subseteq \mathbb{R}$, there is an *integration* map $\int : \mathrm{C}_\bullet(U^n) \to \mathbb{Q} = \mathrm{C}_\bullet(U^0)$ corresponding to the unique map $U^n \to U^0$ (which happens to be diagonal). This map $\int$ is invariant for the S_n action. Tensoring with the identity on $X^{\otimes n}$ gives a map $\int : (\mathrm{C}_\bullet(U^n) \otimes X^{\otimes n})/S_n \to (X^{\otimes n}/S_n)$. We claim that when U is an interval then $\int$ is a quasi-isomorphism. Indeed, choose a 0-chain $u \in \mathrm{C}_0(U)$ with $\int u = 1$. Its diagonal embedding gives $u^n \in \mathrm{C}_0(U^n)$. We can choose a homotopy $\eta_u : \mathrm{C}_\bullet(U^n) \to \mathrm{C}_{\bullet+1}(U^n)$ such that $[\partial, \eta_u](c) = c - (\int c)u^n$ for any chain $c \in \mathrm{C}_\bullet(U^n)$. (Recall that since ∂ and η_u both have odd homological degree, $[\partial, \eta_u] = \partial \circ \eta_u + \eta_u \circ \partial$.) By averaging for the induced S_n-action on the space of homotopies, we can assume that η_u is equivariant for the S_n action. Then $\eta_u \otimes \mathrm{id}_X^{\otimes n}$ descends to the quotient $(\mathrm{C}_\bullet(U^n) \otimes X^{\otimes n})/S_n$, and witnesses the quasi-isomorphism.

Thus we have a cosheaf of algebras on $\mathbb{R}$ which assigns to contractable intervals something quasi-isomorphic to $\mathcal{A}$. Although we will not need it, it is worth mentioning that the colimits appearing in the cosheaf axiom for $\mathcal{A}(-)$ happen also to be homotopy colimits, and so $\mathcal{A}(-)$ is "the" homotopy cosheaf on $\mathbb{R}$ that assigns $\mathcal{A}$ to an interval. This is a particularly easy example of a rich structure called *factorization homology*. We refer the reader to [GTZ10] for more details.

3.4.4 Definition (Δ) The algebra $\mathcal{A} = \prod X^{\otimes n}/S_n$ was more than a commutative algebra: it was the universal formal Poisson algebra, with Poisson structure given by the Taylor coefficients $\pi^{(n)} : X^{\otimes 2} \to X^{\otimes n}$. We will now discuss the manifestation of this structure on the precosheaf $\mathcal{A}(-)$.

Recall that we defined a degree-(-1) "intersection" map $\cap_{12} : \mathrm{C}_\bullet(U^m) \to \mathrm{C}_\bullet(U^{m-1})$. We compose this with a diagonal map $U^{m-1} \to U^{n+m-2}$, and tensor the result with $\pi^{(n)}$ to get

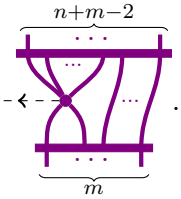
a map $\mathrm{C}_\bullet(U^m) \otimes X^{\otimes m} \to \mathrm{C}_\bullet(U^{n+m-2}) \otimes X^{\otimes n+m-2}$:



The most important observation is that this map is *symmetric* under the S_2 acting diagonally to switch the first two Us and also the first two Xs. This is because both $\pi^{(n)}$ and $\cap$ are independently antisymmetric.

Thinking of $\mathrm{C}(U^n)$ as roughly the same as $\mathrm{C}(U)^{\otimes n}$, we can think of this operator with $m = 2$ as a tensor determining a second-order differential operator Δ on $\mathcal{A}(U)$. We sum over m and n with the appropriate combinatorial factors, and define $\Delta : \mathcal{A}(U) \to \mathcal{A}(U)$ to be:

$$\Delta = \sum_{m,n} \binom{m}{2}$$



Then Δ is of degree-(-1). It is clear that Δ is equivariant for inclusions of one open into another, and so $U \mapsto \mathcal{A}(U)$ is a precosheaf on $\mathbb{R}$ valued in chain complexes equipped with a degree-(-1) operator.

3.4.5 Definition (prefactorization algebra) There is a general theory of (pre)factorization algebras, most of which we will not need; we refer the reader to [CG11, GTZ10, Gwi12] for more details. Our goal is simply to axiomatize the structure present in the precosheaves $\mathcal{A}(-)$ and $(\mathcal{A}(-), \Delta)$.

Let $\mathcal{C}$ be a symmetric monoidal category and T a topological space. A *$\mathcal{C}$-valued prefactorization algebra on T* is a $\mathcal{C}$-valued precosheaf $\mathcal{O}$ on T with an additional algebraic structure. Namely, given disjoint opens U and V in T, the prefactorization algebra comes equipped with a distinguished isomorphism $\mathcal{O}(U) \otimes \mathcal{O}(V) \xrightarrow{\sim} \mathcal{O}(U \cup V)$. (We also demand that $\mathcal{O}(\emptyset) \in \mathcal{C}$ be the monoidal unit.) This isomorphism should be compatible with the symmetric monoidal structure on $\mathcal{C}$ and with the precosheaf structure in the following strong sense: let $U_1, \ldots, U_n$ be pairwise-disjoint opens in T and $V \supseteq \bigcup_i U_i$. Then using the precosheaf structure (and possibly calling on other opens in T), and the symmetry and associativity in $\mathcal{C}$, and the isomorphisms $\mathcal{O}(U_i) \otimes \mathcal{O}(U_j) \cong \mathcal{O}(U_i \cup U_j)$, there are many ways to build morphisms $\bigotimes \mathcal{O}(U_i) \to \mathcal{O}(V)$, and we demand that all such maps be equal.

A *factorization algebra* is required to satisfy an additional locality axiom that we will not need.

3.4.6 Lemma ($(\mathcal{A}(-),\Delta)$ is a prefactorization algebra on $\mathbb{R}$) Any cosheaf of commutative algebras is a prefactorization algebra (in fact, a factorization algebra): the isomorphism $\mathcal{O}(U)\otimes\mathcal{O}(V)\xrightarrow{\sim}\mathcal{O}(U\cup V)$ for $U\cap V=\emptyset$ is an example of the cosheaf axiom, as for commutative algebras $\otimes$ is the categorical coproduct. In particular, $\mathcal{A}(-)$ is a prefactorization algebra on $\mathbb{R}$ valued in the category of dg commutative algebra (pro-)objects in $\mathrm{VECT}[X]$. Since forgetting $\mathrm{DGCA}\to\mathrm{DGVECT}$ is symmetric monoidal, $\mathcal{A}(-)$ is also a prefactorization algebra valued in $\mathrm{DGVECT}[X]$, and thus in $\mathrm{DGVECT}[X,\pi]$.

Consider the category of {chain complexes with a degree-(-1) endomorphism}. Suppressing sign conventions, it can be equipped with a symmetric monoidal structure by setting $(A,e)\otimes(A',e')=(A\otimes A',e\otimes\mathrm{id}_{A'}+\mathrm{id}_A\otimes e')$. Note that in the case when the endomorphism is required to square to 0 this is precisely the usual tensor product of double chain complexes. With this tensor structure, the symmetric multiplication $\odot:(\mathcal{A}(U),\Delta)\otimes(\mathcal{A}(U),\Delta)\to(\mathcal{A}(U),\Delta)$ is not a morphism. This is related to the fact that if M is a Poisson manifold (with nontrivial Poisson structure), then the diagonal map $M\to M\times M$ is not a Poisson map. However, let U and V be disjoint opens in $\mathbb{R}$. Then $\odot:(\mathcal{A}(U),\Delta)\otimes(\mathcal{A}(V),\Delta)\to(\mathcal{A}(U\cup V),\Delta)$ is a morphism of complexes-with-endomorphism: since Δ involves an intersection, it vanishes on the "mixed" summands of $\mathcal{A}(U\cup V)=\prod(\mathrm{C}_\bullet(U^m)\otimes X^{\otimes m})/S_m\otimes\prod(\mathrm{C}_\bullet(V^n)\otimes X^{\otimes n})/S_n$. It follows that $(\mathcal{A}(-),\Delta)$ is a prefactorization algebra valued in {chain complexes in $\mathrm{VECT}[X,\pi]$ with a degree-(-1) endomorphism}.

3.4.7 Corollary ($\mathcal{A}(-)\rightsquigarrow\big(\mathcal{A}(-)[\hbar]/(\hbar^2),\partial+\hbar\Delta\big)$ is a first-order deformation of $\mathrm{DGVECT}[X,\pi]$-valued prefactorization algebras) By construction, Δ commutes with the differential ∂ on $\mathcal{A}(U)$. Temporarily let $\hbar$ be a formal variable that squares to 0. Then we can deform ∂ to $\partial+\hbar\Delta$, and get a new differential on the underlying graded vector space $\mathcal{A}(U)$. Since the underlying $\mathbb{Z}$-graded algebraic object hasn't changed, $\big(\mathcal{A}(-)[\hbar]/(\hbar^2),\partial+\hbar\Delta\big)$ is a prefactorization algebra valued in $\mathrm{DGVECT}[X,\pi]$ (and in fact a factorization algebra). Note that $\hbar\Delta$ is not a derivation of the algebra structure on $\mathcal{A}(U)$, so $\big(\mathcal{A}(-)[\hbar]/(\hbar^2),\partial+\hbar\Delta\big)$ is not valued in dgas. Thus we can interpret Δ as a first-order deformation of the $\mathrm{DGVECT}[X,\pi]$-valued prefactorization algebra $\mathcal{A}$.

3.4.8 Remark (quantization goal) Our goal is to lift this deformation to all orders: we would like to find a differential on $\mathcal{A}(-)[\![\hbar]\!]$ that is equal to $\partial+\hbar\Delta+O(\hbar^2)$ and which is compatible with the (undeformed) factorization-algebra structure on the underlying graded vector spaces of $\mathcal{A}(-)[\![\hbar]\!]$. Performing this lift is the central "quantization" step needed to find a $\star$-product on $\mathcal{A}[\![\hbar]\!]$. We remark that the first-order deformation $\big(\mathcal{A}(-)[\hbar]/(\hbar^2),\partial+\hbar\Delta\big)$ exists regardless of whether the Poisson structure π satisfies the Jacobi identity. Now enforcing the Jacobi identity for π, if $\cap$ were associative then we would have $[\Delta,\Delta]=0$, and so $\partial+\hbar\Delta$ would be a differential on $\mathcal{A}(-)[\hbar]$. Our strategy will be to find homotopies parameterizing the failure of associativity.

3.5 A transversalizing homotopy

The intersection $\cap$ is associative on chains that are transverse to diagonals. In this section (specifically, Construction 3.5.6) we will describe a carefully chosen homotopy that moves every chain to a transverse chain.

3.5.1 Definition ($\mathrm{T}_\bullet^{i\dots j}(U^n)$) We fix an open $U \subseteq \mathbb{R}$, and choose at least two distinct indices $i, \dots, j \in \{1, \dots, n\}$. We write $\{t_i = \cdots = t_j\}$ for the corresponding diagonal in U^n. It has codimension one less than the number of chosen indices. We will build a subcomplex $\mathrm{T}_\bullet^{i\dots j}(U^n) \subseteq \mathrm{C}_\bullet(U^n)$ as a union of complexes $\mathrm{T}_\bullet^{i\dots j;\ell} \subseteq \mathrm{C}_\bullet^\ell(U^n)$. We describe it this way because for each ℓ, $\mathrm{C}_\bullet^\ell(U^n)$ has (almost) a distinguished basis.

We set $\mathrm{T}_0^{i\dots j;\ell}(U^n)$ to be spanned by those zero-cells in $\mathrm{C}_0^\ell(U^n)$ that do not lie on $\{t_i = \cdots = t_j\}$. Consider now the span of those one-cells in $\mathrm{C}_1^\ell(U^n)$ whose intersection with the diagonal is of dimension at most (1 − codimension of the diagonal). Note that the only such one-cells are totally off the diagonal if the diagonal is of codimension 2 or higher, but if the codimension is 1 then these cells might end on this diagonal but may not be contained in the diagonal. We set $\mathrm{T}_1^{i\dots j;\ell}(U^n)$ to be the intersection of {linear combinations of such one-cells} with $\partial^{-1}(\mathrm{T}_0^{i\dots j;\ell}(U^n))$. We continue by induction:

$$\mathrm{T}_k^{i\dots j;\ell}(U^n) = \partial^{-1}(\mathrm{T}_{k-1}^{i\dots j;\ell}(U^n)) \cap \{\text{linear combinations of } k\text{-dimensional cells whose intersection with the diagonal is of dimension} \leq (k - \text{codimension of diagonal})\}$$

By construction $\mathrm{T}_\bullet^{i\dots j;\ell}(U^n)$ is a subcomplex of $\mathrm{C}_\bullet^\ell(U^n)$. A moment's thought shows that the inclusion $\mathrm{C}_\bullet^\ell(U^n) \hookrightarrow \mathrm{C}_\bullet^{\ell+1}(U^n)$ maps $\mathrm{T}_\bullet^{i\dots j;\ell}(U^n)$ into $\mathrm{T}_\bullet^{i\dots j;\ell+1}(U^n)$, and so we set $\mathrm{T}_\bullet^{i\dots j}(U^n) = \bigcup_\ell \mathrm{T}_\bullet^{i\dots j;\ell}(U^n)$. It is a subcomplex of $\mathrm{C}_\bullet(U^n)$, and is invariant under those permutations in S_n that preserve (although may act nontrivially on) the diagonal $\{t_i = \cdots = t_j\}$.

When there are fewer than two chosen indices, then we set $\mathrm{T}_\bullet^i(U^n) = \mathrm{T}_\bullet^\emptyset(U^n) = \mathrm{C}_\bullet(U^n)$.

3.5.2 Lemma (preservation of transversality) We will need in Section 3.6 the following facts, which the reader can check directly. Suppose that $c \in \mathrm{T}_\bullet^{12\cdots j}(\mathbb{R}^n)$. Then $\cap_{12}(c) \in \mathrm{T}_{\bullet-1}^{12\dots(j-1)}(\mathbb{R}^{n-1})$. Denote by $\curlyvee_{12} : \mathrm{C}_\bullet(\mathbb{R}^n) \to \mathrm{C}_\bullet(\mathbb{R}^{n+1})$ the diagonal map duplicating the first coordinate (i.e. $(t_1, t_2, \dots, t_n) \mapsto (t_1, t_1, t_2, \dots, t_n)$). Then $\curlyvee_{12}(c) \in \mathrm{T}_\bullet^{23\dots(j+1)}(\mathbb{R}^{n+1})$ (although generically $\curlyvee_{12}(c) \notin \mathrm{T}_\bullet^{123\dots(j+1)}(\mathbb{R}^{n+1})$).

3.5.3 Lemma ($\mathrm{T}_\bullet \hookrightarrow \mathrm{C}_\bullet$ is a quasi-isomorphism) It is straightforward (and so we leave to the reader) that $\mathrm{T}_\bullet^{i\dots j}(U^n)$ computes the homology of U^n, and therefore the inclusion $\mathrm{T}_\bullet^{i\dots j}(U^n) \hookrightarrow \mathrm{C}_\bullet(U^n)$ is a quasi-isomorphism. We remark that for fixed ℓ, we do not always have a quasi-isomorphism $\mathrm{T}_\bullet^{i\dots j;\ell}(U^n) \hookrightarrow \mathrm{C}_\bullet^\ell(U^n)$. Indeed, if U is "short" and ℓ is "low" then there may be no zero-dimensional cells in C_0^ℓ that are off the diagonal, and so $\mathrm{T}_\bullet^{i\dots j;\ell}(U^n)$ could be 0. For fixed U, as ℓ increases the mesh-spacing becomes sufficiently fine as to

include zero-cells off the diagonal, and once this happens then we do have the desired quasi-isomorphism.

It follows from the zig-zag lemma that the quotient complex $\mathrm{C}_\bullet(U^n)/\,\mathrm{T}_\bullet^{i\dots j}(U^n)$ is exact.

3.5.4 Remark (hands-on description of $\mathrm{C}_\bullet\,/\,\mathrm{T}_\bullet$) The quotient $\mathrm{C}_\bullet(U^n)/\,\mathrm{T}_\bullet^{i\dots j}(U^n)$ can be described quite geometrically. It includes the chain complex spanned by those cells completely contained in the diagonal $\{t_i = \cdots = t_j\}$. But it contains a little more. At each point in this diagonal, there are also basis vectors in $\mathrm{C}_1\,/\,\mathrm{T}_1$ corresponding to all the directions along our mesh leaving from that point into the region off the diagonal. Similarly, we have higher-dimensional cells in the quotient which are "the intersection of a cell in U^n with the infinitesimal neighborhood of the diagonal." Finally, at the codimension of the diagonal and higher, we impose a further relation that kills any sum of these "infinitesimal cells" that transversely intersects the diagonal.

3.5.5 Lemma ($\mathrm{T}_\bullet^{i\dots j} \subseteq \mathrm{T}_\bullet^{i'\dots j'} + \mathrm{D}_\bullet^{i\dots j}$) Denote by $\mathrm{D}_\bullet^{i\dots j}(U^n)$ the subcomplex of $\mathrm{C}_\bullet(U^n)$ that is spanned by cells with absolutely no intersection with the diagonal $\{t_i = \cdots = t_j\}$. Then we clearly have $\mathrm{D}_\bullet^{i\dots j} \subseteq \mathrm{T}_\bullet^{i\dots j}$, with equality at all dimensions less than the codimension of the diagonal.

Choose two sets of indices $\{i \dots j\}$ and $\{i' \dots j'\}$, with $\{i' \dots j'\} \subseteq \{i \dots j\}$. The central fact, which is clear after a moment's thought, is that:

$$\mathrm{T}_\bullet^{i\dots j}(U^n) \subseteq \mathrm{T}_\bullet^{i'\dots j'}(U^n) + \mathrm{D}_\bullet^{i\dots j}(U^n).$$

In words: Suppose that a chain is transverse to the smaller diagonal (the one in which we impose more equations). Then the part of the chain that is near the smaller diagonal intersects the larger diagonal transversely. The rest of the chain is disjoint from the smaller diagonal.

3.5.6 Construction ($h^{i\dots j;n}$) The meat of our construction of a $\star$-product on $\mathcal{A}$ is to cleverly choose a system of homotopies witnessing the exactness of the complexes $\mathrm{C}_\bullet(U^n)/\,\mathrm{T}_\bullet^{i\dots j}(U^n)$. We write $h_U^{i\dots j;n}$ for our desired homotopy $h : \mathrm{C}_\bullet\,/\,\mathrm{T}_\bullet \to \mathrm{C}_{\bullet+1}\,/\,\mathrm{T}_{\bullet+1}$; we will also write h for the composition $\mathrm{C}_\bullet \to \mathrm{C}_\bullet\,/\,\mathrm{T}_\bullet \xrightarrow{h} \mathrm{C}_{\bullet+1}\,/\,\mathrm{T}_{\bullet+1}$. By "$h$ witnesses the exactness" we mean the request that $[\partial, h] = \mathrm{id}_{\mathrm{C}_\bullet\,/\,\mathrm{T}_\bullet}$. We furthermore ask the following properties:

permutation invariance We ask that $h^{i\dots j;n}$ is invariant under conjugations by the subgroup of S_n permuting the chosen set of indices $\{i, \dots, j\}$ and under the subgroup permuting $\{1, \dots, n\} \smallsetminus \{i, \dots, j\}$.

We can always assure this by averaging.

dependence only on chosen indices Suppose given disjoint open subsets $V, V' \subseteq U$ and a chain $c \in \mathrm{C}_\bullet(V^n)$ and a chain $c' \in \mathrm{C}_\bullet((V')^m)$. We write also c, c' for their images in $\mathrm{C}_\bullet\,/\,\mathrm{T}_\bullet$. Choose indices $i, \dots, j \in \{1, \dots n\}$. We ask that $h^{i\dots j;n+m}(c \otimes c') = h^{i\dots j;n}(c) \otimes c'$.

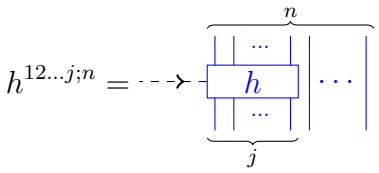
We assure this by: first, choosing $h^{1\cdots n;n}$ for each n; second, extending to $h^{1\cdots n;n+m}$ consecutively to satisfy the request; and third, extending to the rest of $\mathrm{C}_\bullet(U^{n+m})/(\mathrm{C}_\bullet(U^n)\otimes \mathrm{C}_\bullet(U^m))$. By the hands-on description in Remark 3.5.4 we can always do this, and in a way compatibly with the permutation-invariance request.

locality We ask that h_U be compatible with inclusions $U \subseteq U'$. Equivalently, suppose given a cell c in the ℓth cell-decomposition of $\mathbb{R}^n$ which is contained within U^n. Then it intersects the diagonal $\{t_i = \cdots = t_j\}$ along some cells of various dimensions. We ask that for a sufficiently large ℓ' there exists some cells $c'_1, \ldots, c'_k$ in the ℓ'th decomposition of $\mathbb{R}^n$ such that: the span of their images in $\mathrm{C}_\bullet(U^n)/\mathrm{T}_\bullet^{i\cdots j}(U^n)$ includes $h(c)$; and the intersection of the cells $c'_1, \ldots, c'_k$ with the diagonal $\{t_i = \cdots = t_j\}$ is contained within the intersection of c and the diagonal. We remark that both $\mathrm{C}_\bullet(-)$ and $\mathrm{T}_\bullet^{i\cdots j}(-)$ are DGVect-valued precosheaves on $\mathbb{R}$, and thus so is their quotient.

We can assure this by constructing $h^{1\cdots n;n}$ as follows. For each 0-cell c on the diagonal $\{t_1 = \cdots = t_n\}$ we choose a 1-chain $h(c)$ leaving it in a direction not along the main diagonal; in the quotient $\mathrm{C}_\bullet / \mathrm{T}_\bullet$ only the direction matters. Given a 1-cell c, we get a 1-chain $c - h(\partial c)$, and we choose a way to complete it to a cycle without intersecting the diagonal and then to realize this cycle as a boundary without increasing the intersection with the diagonal. We continue to higher-dimensional cells. In dimensions 1 and higher, we first make the definition with cells in the $\ell = 0$th decomposition of $\mathbb{R}^n$; then we choose what to do with cells in the $\ell = 1$st decomposition whose left end is at an integer (what to do with the other half of the cells is determined by linearity); then we choose what to do with cells in the $\ell = 2$nd decomposition whose left end is at a half-integer. And so on.

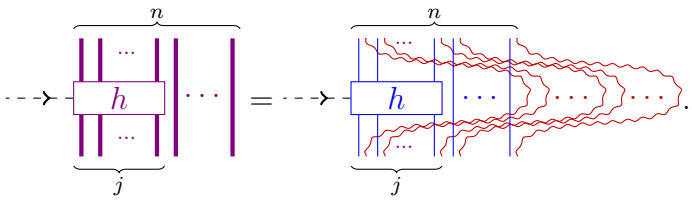
We remark that it is vital to the locality axiom that we are working with the quotient $\mathrm{C}_\bullet / \mathrm{T}_\bullet$. If we were to try to lift h to have codomain $\mathrm{C}_\bullet$, then we would get stuck right away: h of a point should be a 1-chain, but any particular 1-chain cannot be confined to too small a region.

We will denote $h^{12\cdots j;n}$ in the graphical notation by:

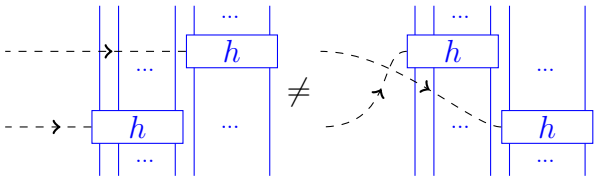


The dashed line records the shift in homological degree. Other $h^{i\cdots j;n}$ can be equally easily drawn by conjugating by some permutation $\sigma \in S_n$. Our diagram for h is an abuse of notation, as it suggests that we have chosen a lift of $h^{i\cdots j}$ back to $\mathrm{C}_\bullet$. But no ambiguity will be present, because (except cautiously in Proposition 3.5.10, Case VII of Proof 3.6.6 of Theorem 3.6.3, and Example 3.7.14) we will only ever compose h with morphisms that vanish on $\mathrm{T}_\bullet^{i\cdots j}$.

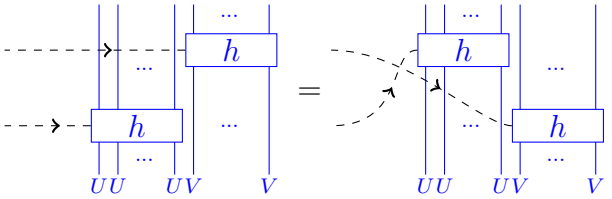
Tensoring with $X^{\otimes n}$ gives:



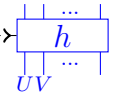
3.5.7 Remark (dangerous notation) As in Remark 3.3.6, there is some danger in the above notation for h. Namely, although it looks like homotopies on disjoint collections of strands should commute, in general they do not:



On the other hand, the request we called "dependence only on chosen indices" in Construction 3.5.6 assures that if U and V are disjoint opens in $\mathbb{R}$, then they do commute:

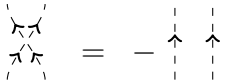


if $U \cap V = \emptyset$.

Moreover, if $U \cap V = \emptyset$, then  $= 0$, as $\mathrm{T}_\bullet^{1\dots j}(U \times V \times \dots) = \mathrm{C}_\bullet(U \times V \times \dots)$.

3.5.8 Remark (conventions for degree shifts) In Construction 3.5.6, we used the language of homogeneous elements to describe the relationship between h and ∂. As in Remark 3.2.4, denote by $\uparrow$ the one-dimensional graded vector space in homological degree $+1$ and by $\downarrow$ the the one-dimensional graded vector space in degree -1. From the perspective of homogeneous elements, the most natural way to choose an isomorphism $\uparrow \otimes \downarrow \xrightarrow{\sim} \mathbb{Q}$ is to choose a basis vector for each object. The same choice of basis vectors also yields an isomorphism $\downarrow \otimes \uparrow \xrightarrow{\sim} \mathbb{Q}$, but these isomorphisms are not the same: instead, we should use the Koszul braiding on the category to switch the two tensorands.

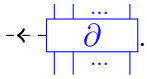
The graphical notation conveniently takes care of all such issues, provided one uses the convention from Remark 3.2.4:

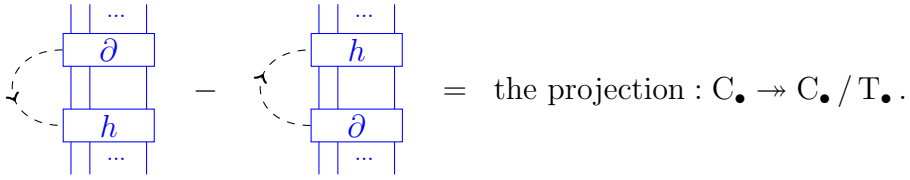


It follows from this convention that:

Then the various isomorphisms $\uparrow \otimes \downarrow \xrightarrow{\sim} \mathbb{Q}$ correspond to cups and caps, and we never need to choose homogeneous elements.

Thus there is the following notational conflict. The differential $\partial : \mathrm{C}_\bullet \to \mathrm{C}_{\bullet-1}$ is best understood as a map $\partial : \downarrow \to \mathrm{End}(\mathrm{C}_\bullet)$, and the homotopy is a map $h : \uparrow \to \mathrm{End}(\mathrm{C}_\bullet / \mathrm{T}_\bullet)$. Then the commutator is a map $[\partial, h] : \downarrow \otimes \uparrow \to \mathrm{End}(\mathrm{C}_\bullet / \mathrm{T}_\bullet)$. We asked that "$[\partial, h] = \mathrm{id}_{\mathrm{C}_\bullet / \mathrm{T}_\bullet}$," but the right-hand side is a map $\mathbb{Q} \to \mathrm{End}(\mathrm{C}_\bullet / \mathrm{T}_\bullet)$, and so to make sense of this equation requires choosing an isomorphism $\downarrow \otimes \uparrow \xrightarrow{\sim} \mathbb{Q}$, and thereby introducing some sign considerations.

Denote the differential by  . Our sign convention is:



The minus sign is correct: with respect to homogeneous elements, $[\partial, h] = \partial h + h\partial$, but the graphical notation already accounts for this sign. Further discussion of the differences between working with homogeneous elements and working more categorically is available in [DM99].

3.5.9 Example ($h^{123;n}$) We will never use $h^{12;n}$, which is anyway entirely determined by the locality axiom. To illustrate that our model is sufficiently finite, we will describe a particular choice for $h^{123;n}$. Similar choices can be made to define $h^{1\cdots j;n}$ for arbitrarily large j, and it will follow from our construction of the $\star$-product that to compute finitely many coefficients, only finite many such choices must be made. Since we are now interested only in an illustrative example, will take advantage of the fact that $\{t_1 = t_2 = t_3\} \subseteq \mathbb{R}^n$ is of codimension 2. For the remainder of the example we abbreviate $h^{123;n}$ by h.

As a warm-up, we describe how h acts on 0-chains. We can identify $\mathrm{C}_0(\mathbb{R}^n)/\mathrm{T}_0^{123}(\mathbb{R}^n)$ with the 0-chains on the $\{t_1 = t_2 = t_3\}$ diagonal, and thus the basis of $\mathrm{C}_0(\mathbb{R}^n)/\mathrm{T}_0^{123}(\mathbb{R}^n)$ consists of points $(t_1, \ldots, t_n) \in \mathbb{Z}[\frac{1}{2}, \frac{1}{4}, \ldots]^n$ with $t_1 = t_2 = t_3$. Pick one such point c, and suppose that all the coordinates t_i have denominator at most 2^ℓ. Then consider the set of "basic" one-cells in the ℓth decomposition of $\mathbb{R}^n$ with boundary this point, modulo points off the diagonal — in particular, we are disallowing any one-cell that intersects the diagonal anywhere except at this point. This is a finite set, and we set $h(c)$ to be the image in $\mathrm{C}_1 / \mathrm{T}_1$ of the average of the set — this image is independent of the choice of ℓ. By construction, $[\partial, h](c) = \partial hc + h\partial c = \partial hc = c$.

We now describe the k-dimensional case. We assume by induction that we have defined $h(c)$ for all chains c of dimension less than k, satisfying the above axioms and in particular satisfying locality. We choose a mesh size ℓ and a basic k-cell c in the ℓth decomposition of $\mathbb{R}^n$. We proceed with the definition in two steps:

Case 1: c is not contained in $\{t_1 = t_2 = t_3\}$. Then the basic cell c intersects $\{t_1 = t_2 = t_3\}$ along a region that is at most $(k-1)$-dimensional, and ∂c intersects the diagonal along the same region. By induction, we have defined $h\partial c$. Then $c - h(\partial c)$ intersects the diagonal along at most this same region of dimension $\leq (k-1)$, and $\partial(c - h\partial c) = \partial c - \partial h(\partial c) = \partial c - (\partial c) + h\partial(\partial c) = 0$.

Suppose that f is a $(k+1)$-dimensional chain intersecting $\{t_1 = t_2 = t_3\}$ in a region that is at most $(k-1)$-dimensional. As this is the expected dimension of the intersection, f is transverse to the diagonal if ∂f is. Thus for any $(k-1)$-dimensional region in the diagonal, the map

$$\{(k+1)\text{-dimensional chains intersection the diagonal along this region}\} \xrightarrow{\partial} \{k\text{-dimensional chains intersection the diagonal along this region}\}$$

is injective modulo $\mathrm{T}^{123}_\bullet$. Its image consists of $(\ker \partial) \cap \{\text{the codomain}\}$. It follows that in $\mathrm{C}_\bullet / \mathrm{T}_\bullet$, $c - h(\partial c)$ has a unique antiderivative $\partial^{-1}(c - h(\partial c))$ with not-larger intersection with the diagonal. We set $h(c) = \partial^{-1}(c - h\partial c)$.

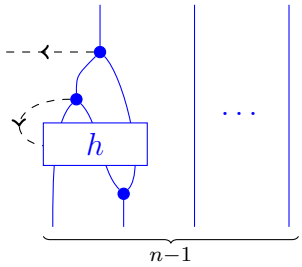
Case 2: c is contained in $\{t_1 = t_2 = t_3\}$. If so, then there are finitely many basic $(k+1)$-cells f in the ℓth cell decomposition of $\mathbb{R}^n$ with $\partial f = c$ modulo $\{$cells intersecting the diagonal at a $(k-1)$-dimensional region$\}$. Any such f necessarily intersects the diagonal precisely at c. We set $\tilde{h}(c)$ to be their average. Then $c - \partial\tilde{h}c$ is a linear combination of cells covered by Case 1, and so $h(c - \partial\tilde{h}c)$ is already defined. We set $h(c) = h(c - \partial\tilde{h}c) + \tilde{h}c$.

We must now prove that this is well-defined. Write $\tilde{h}_\ell$ for the map $\tilde{h}$ defined for the ℓth cell decomposition, and $\tilde{h}_{\ell+1}$ for the corresponding construction starting with a finer mesh, extended by linearity to all k-chains contained in the diagonal. Then we want to compute $\big(h(c - \partial\tilde{h}_{\ell+1}c) + \tilde{h}_{\ell+1}c\big) - \big(h(c - \partial\tilde{h}_\ell c) + \tilde{h}_\ell c\big)$, with the understanding that until proven otherwise h is only defined in situations covered by Case 1. Some thought shows that the $(k+1)$-dimensional chain $\tilde{h}_{\ell+1}c - \tilde{h}_\ell c$ intersects the diagonal at a $(k-1)$-dimensional region. Then $\partial(\tilde{h}_{\ell+1}c - \tilde{h}_\ell c)$ does as well, and so $h\partial(\tilde{h}_{\ell+1}c - \tilde{h}_\ell c)$ is defined by Case 1 to be the unique solution f to $\partial f = \partial(\tilde{h}_{\ell+1}c - \tilde{h}_\ell c)$ with the requirement that f intersects the diagonal in a $(k-1)$-dimensional region. But $f = \tilde{h}_{\ell+1}c - \tilde{h}_\ell c$ satisfies both requirements, and so $h\partial(\tilde{h}_{\ell+1}c - \tilde{h}_\ell c) = \tilde{h}_{\ell+1}c - \tilde{h}_\ell c$. On the other hand, the expression we want to compute is:

$$\big(h(c - \partial\tilde{h}_{\ell+1}c) + \tilde{h}_{\ell+1}c\big) - \big(h(c - \partial\tilde{h}_\ell c) + \tilde{h}_\ell c\big) = \tilde{h}_{\ell+1}c - \tilde{h}_\ell c + h(c - \partial\tilde{h}_{\ell+1}c - c + \partial\tilde{h}_\ell c) = 0.$$

It is worth emphasizing that Case 1 is completely forced by the locality condition and $[\partial, h] = \mathrm{id}$. It is in Case 2 that we have made a choice. Our ability to describe this choice depends essentially on the fact that we chose a model of $\mathrm{C}_\bullet(\mathbb{R}^n)$ that was sufficiently finite.

3.5.10 Proposition (a diagram that vanishes on $\mathrm{T}^{12}_\bullet$) In the next section, we will need the following fact. Fixing an open $U \subseteq \mathbb{R}$, choose a lift of $h^{123} : \mathrm{C}_\bullet(U^n) \to \mathrm{C}_{\bullet+1}(U^n)/\,\mathrm{T}^{123}_{\bullet+1}(U^n)$ back to $\mathrm{C}_{\bullet+1}(U^n)$. By averaging, we can assure that the lift satisfies permutation invariance. We claim that the morphism

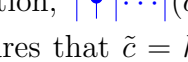


vanishes on $\mathrm{T}^{12}_\bullet(\mathbb{R}^{n-1})$.

3.5.11 Remark (independence of choice of lift) The claim in Proposition 3.5.10 is in fact independent of the choice of permutation-invariant lift $\mathrm{C}_{\bullet+1}\,/\,\mathrm{T}_{\bullet+1} \to \mathrm{C}_{\bullet+1}$. Two different lifts will differ on any particular input only by something in $\mathrm{T}^{123}_{\bullet+1}$, where $\cap$ is associative. On the other hand, the input to h is symmetric under permuting the second and third strands, whereas  is antisymmetric under permuting the second and third strands.

3.5.12 Proof of Proposition 3.5.10 The proof is more or less a straightforward calculation, and turns on the fact that h is invariant under permutations of its participating strands. Choose $c \in \mathrm{T}^{12;\ell}_k(\mathbb{R}^{n-1})$. Then c intersects the diagonal $\{t_1 = t_2\}$ along a region R of dimension at most $k-1$. By locality, the output of the above morphism is also supported in this region, and by degree-counting the output is a $(k-1)$-dimensional chain, and so we are done unless $\dim R = k-1$.

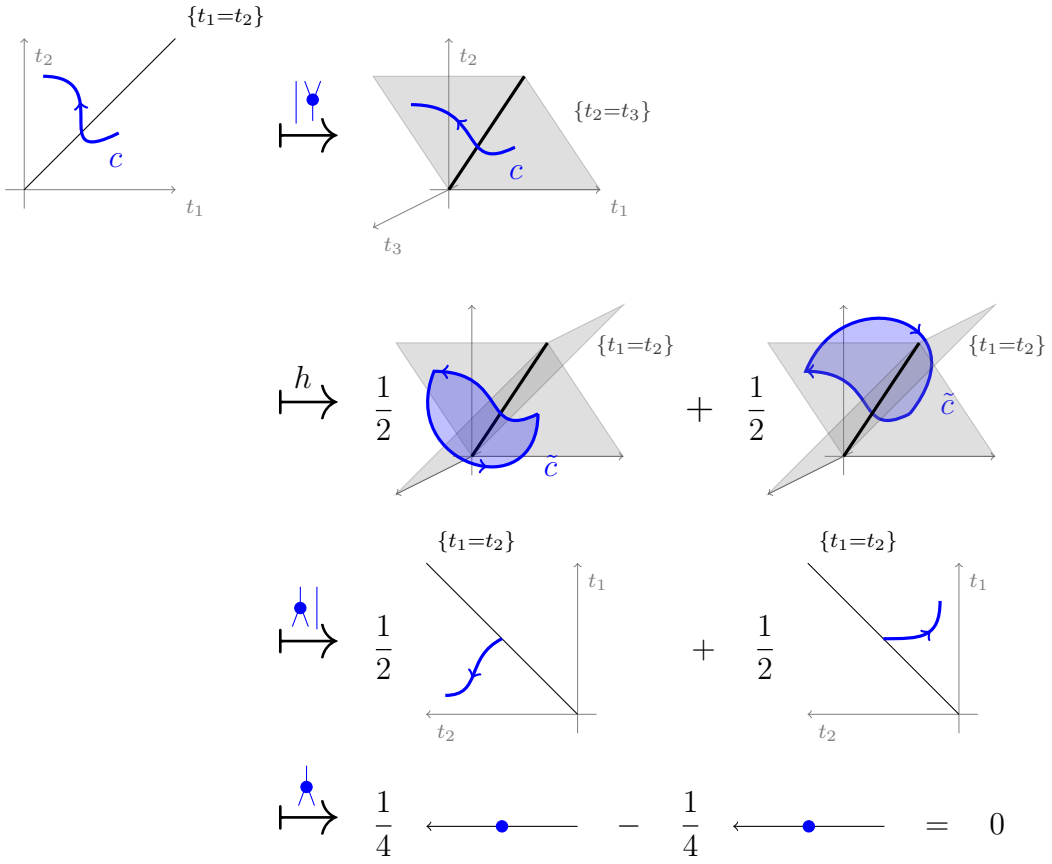
We identify $\mathbb{R}^{n-2} \cong \{t_1 = t_2\}$ via the diagonal pushforward, and then R consists of a collection of basic $(k-1)$-dimensional cells in the ℓth mesh approximation $\mathrm{C}^\ell_\bullet(\mathbb{R}^{n-2})$. At each basic cell $r \in R$, for any chain $c \in \mathrm{C}_k(\mathbb{R}^{n-1})$ we can record two numbers, which we will call $\lambda_{21,r}(c)$ and $\lambda_{12,r}(c)$. We set $\lambda_{12,r}$ to be the coefficient of r in $\cap_{12}(c \cap \{t_1 < t_2\})$, where $c \cap \{t_1 < t_2\} \in \mathrm{C}_k(\mathbb{R}^{n-1})$ denotes the linear combination of cells in $\{t_1 < t_2\} \subseteq \mathbb{R}^{n-1}$ with the same coefficients as their contributions to c; similarly, we set $\lambda_{21,r}$ to be the coefficient of r in $\cap_{12}(c \cap \{t_1 > t_2\})$. But if c is transverse to $\{t_1 = t_2\}$, then $\lambda_{12,r}(c) = \lambda_{21,r}(c) = \lambda_r(c)$.

After the first comultiplication,  $(c) \in \mathrm{C}^\ell_k(\mathbb{R}^n)$ intersects $\{t_1 = t_2 = t_3\} \cong \mathbb{R}^{n-2}$ along R, and locality also assures that $\tilde{c} = h^{123;n} \circ$ $(c) \in \mathrm{C}^\ell_{k+1}(\mathbb{R}^n)$ intersects $\{t_1 = t_2 = t_3\}$ along R. We now play a similar labeling game as in the previous paragraph. Given $\tilde{c} \in \mathrm{C}_{k+1}(\mathbb{R}^n)$, we record near each basic $(k-1)$-dimensional cell $r \in R \subseteq \{t_1 = t_2 = t_3\}$ six numbers $\lambda_{123,r}(\tilde{c})$, $\lambda_{132,r}(\tilde{c})$, $\lambda_{213,r}(\tilde{c})$, $\lambda_{231,r}(\tilde{c})$, $\lambda_{312,r}(\tilde{c})$, and $\lambda_{321,r}(\tilde{c})$. By definition, $\lambda_{123,r}(\tilde{c})$ counts the "weight" of $\tilde{c} \cap \{t_1 < t_2 < t_3\}$ near r, and similarly for permutations of $\{1,2,3\}$.

Now, since h is symmetric under $t_2 \leftrightarrow t_3$ and $\tilde{c} = h^{123;n} \circ$ (c) for some $c \in \mathrm{C}_k(\mathbb{R}^n)$, we have $\lambda_{123,r}(\tilde{c}) = -\lambda_{132,r}(\tilde{c})$, $\lambda_{213,r}(\tilde{c}) = -\lambda_{312,r}(\tilde{c})$, and $\lambda_{231,r}(\tilde{c}) = -\lambda_{321,r}(\tilde{c})$; the minus

signs correspond to the orientation switch induced by $t_2 \leftrightarrow t_3$. Moreover, since $\lambda_{12,r}(c) = \lambda_{21,r}(c)$, we have $\lambda_{123,r}(\tilde{c}) = \lambda_{213,r}(\tilde{c}) = \lambda_{231,r}(\tilde{c})$.

On the other hand, for arbitrary $\tilde{c} \in \mathrm{C}_k^\ell(\mathbb{R}^n)$, the coefficient of r in  $(\tilde{c})$ is $\pm\big(\lambda_{123,r}(\tilde{c}) + \lambda_{213,r}(\tilde{c}) + \lambda_{312,r}(\tilde{c}) + \lambda_{321,r}(\tilde{c})\big)$. With the equalities of the previous paragraph, this number is 0. This completes the proof, but we provide a picture of the $n = 3$ case for the reader's convenience:



3.6 Quantization

Recall that in Section 3.4 we defined a $\mathrm{DGVect}[X,\pi]$-valued prefactorization algebra $\mathcal{A}(-)$ on $\mathbb{R}$ that "smears out" the commutative algebra $\mathcal{A}$ (the algebra of functions on the universal formal Poisson manifold). Recall also that $\mathcal{A}(-)$ is equipped with a degree-(-1) operator Δ encoding the Poisson structure on $\mathcal{A}$, but that $\Delta^2 \neq 0$. Our goal in this section is to deform the differential ∂ to a differential $\partial + \delta$ on $\mathcal{A}(-)[\![\hbar]\!]$ without changing the underlying $\mathrm{GVect}[X,\pi]$-valued factorization algebra, with the request that $\delta = \hbar\Delta \mod \hbar^2$. Compatibility with the underlying prefactorization algebra structure amounts to two conditions: δ should be equivariant for inclusions $\mathcal{A}(U) \to \mathcal{A}(V)$ when $U \subseteq V$; and if U and V are

disjoint opens then $\delta_{U\cup V} = \delta_U \otimes \mathrm{id}_{\mathcal{A}(V)} + \mathrm{id}_{\mathcal{A}(U)} \otimes \delta_V$ on $\mathcal{A}(U \cup V) = \mathcal{A}(U) \otimes \mathcal{A}(V)$. Our construction of δ will use a choice of transversalizing homotopy h satisfying the requests of Construction 3.5.6. We construct the second-order deformation in Proposition 3.6.1 (and Proof 3.6.2), state the induction step in Theorem 3.6.3, and explain the details in Proof 3.6.6.

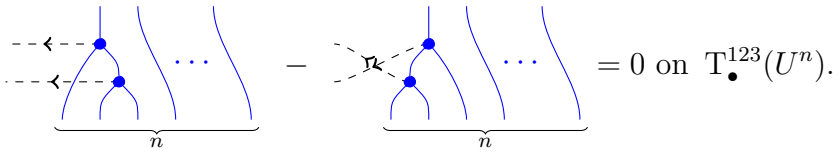
3.6.1 Proposition (existence of second-order deformation) For any antisymmetric bivector π (not necessarily satisfying the Jacobi identity), we constructed in Corollary 3.4.7 a first-order deformation of the prefactorization algebra $\mathcal{A}(-)$ on $\mathbb{R}$, given by the precosheaf $\big(\mathcal{A}(-)[\hbar]/(\hbar^2), \partial + \hbar\Delta\big)$ with the same algebraic structure as in $\mathcal{A}(-)$. To do this, what we needed was that $[\partial, \Delta] = 0$ and that if $U \cap V = \emptyset$ are disjoint opens, then

$$\Delta|_{\mathcal{A}(U\times V)} = \Delta|_{\mathcal{A}(U)} \otimes \mathrm{id}_{\mathcal{A}(V)} + \mathrm{id}_{\mathcal{A}(U)} \otimes \Delta|_{\mathcal{A}(V)}$$

This equation holds because Δ involves an intersection with the diagonal, and so vanishes on the "mixed" parts of $\mathcal{A}(U \times V)$.

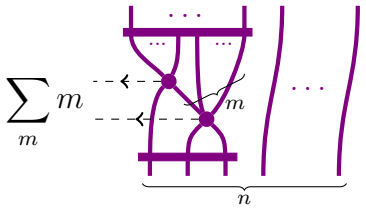
We claim that if π satisfies the Jacobi identity, then this factorization algebra can be lifted to second-order in $\hbar$: we can choose $\delta^{(2)}$ so that $\big(\mathcal{A}(-)[\hbar]/(\hbar^2), \partial + \hbar\Delta + \hbar^2\delta^{(2)}\big)$ is a factorization algebra.

3.6.2 Proof of Proposition 3.6.1 Although $\cap$ fails to be associative (Remark 3.3.6), this failure vanishes on transverse chains:

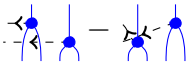


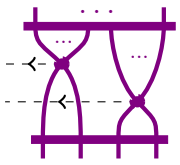
$= 0$ on $\mathrm{T}^{123}_\bullet(U^n)$.

It follows from the Jacobi identity (Notation 3.2.3) and the Frobenius axiom (Remark 3.3.6) that:



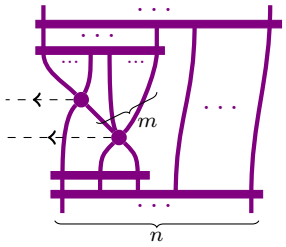
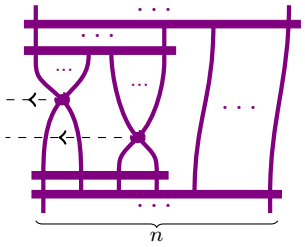
$= 0$ on $\mathrm{T}^{123}_\bullet(U^n) \otimes X^{\otimes n}$.

Moreover,  $= 0$ on $\mathrm{T}^{1234}_\bullet$, and therefore:



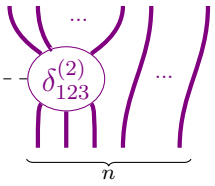
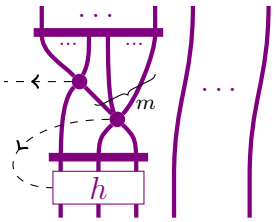
$= 0$ on $\mathrm{T}^{1234}_\bullet$

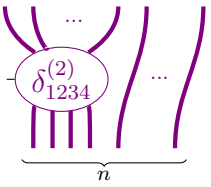
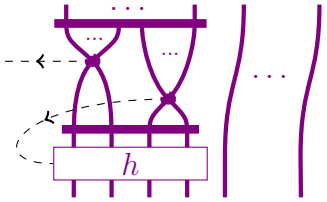
We now write out the commutator $\frac{1}{2}[\Delta,\Delta] = \Delta^2$ on the nth summand of $\mathcal{A}(U) = \prod_n \big(\mathrm{C}_\bullet(U^n)\otimes X^{\otimes n}\big)/S_n = \prod_n$  . Either the second Δ connects to the first or it doesn't:

$$\frac{1}{2}[\Delta,\Delta] = \binom{n}{3} 3 \sum m$$  $$+ \binom{n}{4} 6 \sum$$ 

The binomial prefactors count the number of ways such a diagram can arise. The sum is over the numbers of outputs of each Δ vertex. Actually, a third thing could happen: both inputs of the top Δ could connect to outputs of the bottom Δ. But in the output of any Δ, the ≀ and | terms each transform symmetrically under the S_m action, whereas the inputs separately transform antisymmetrically under the S_2 action, and so such terms vanish.

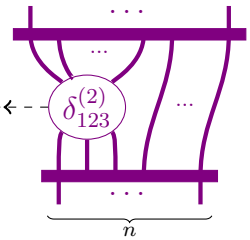
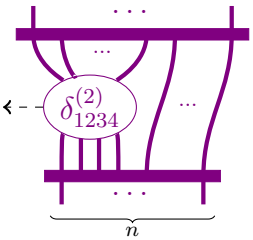
Thus we define:

$$\delta^{(2)}_{123;n} =$$  $$= 3 \sum m$$ 

$$\delta^{(2)}_{1234;n} =$$  $$= 6 \sum$$ 

These are well-defined because the homotopies h are followed by terms that vanish on transverse chains.

Combining these, we set:

$$\delta^{(2)} = \binom{n}{3} \sum$$  $$+ \binom{n}{4} \sum$$ 

The sums are over the number of output strands. We immediately observe that each of $\delta^{(2)}_{123;n}$ and $\delta^{(2)}_{1234;n}$ vanish on $\mathrm{C}_\bullet(U\times V\times\dots)$ if $U\cap V=\emptyset$. Indeed, $\delta^{(2)}_{123;n}$ vanishes on $\mathrm{T}^{123}_\bullet \supseteq \mathrm{D}^{123}_\bullet$

and $\delta^{(2)}_{1234;n}$ vanishes on $\mathrm{T}^{1234}_\bullet \supseteq \mathrm{D}^{1234}_\bullet$. Along with the dependence only on the chosen indices from Construction 3.5.6, it follows that $\delta^{(2)}$ vanishes on the "mixed" parts of $\mathcal{A}(U \cup V)$, and hence is compatible with the prefactorization algebra structure.

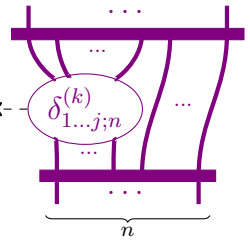
Finally, we must check that $\partial + \delta = \partial + \hbar\Delta + \hbar^2\delta^{(2)}$ is a differential on $\mathcal{A}(U)[\hbar]/(\hbar^3)$. This follows from: (i) $[\partial, \Delta] = 0$, (ii) $[\Delta, \Delta] = 0$ on $\mathrm{T}_\bullet$, and (iii) the sign conventions in Remark 3.5.8. As checking this is routine but provides good practice in manipulating diagrams, we leave it to the reader.

We are now ready to prove:

3.6.3 Theorem (existence of higher-order deformations) Suppose that we have built an mth-order deformation of the prefactorization algebra $\mathcal{A}(-)$, in the sense that we have built a differential $\partial + \hbar\Delta + \sum_{k=2}^m \hbar^k \delta^{(k)}$ making $\left(\mathcal{A}(-)[\hbar]/(\hbar^{m+1}), \partial + \hbar\Delta + \sum_{k=2}^m \hbar^k \delta^{(k)}\right)$ into a prefactorization algebra, where the prefactorization structure is the undeformed symmetric product $\odot : \mathcal{A}(U) \otimes \mathcal{A}(V) \xrightarrow{\sim} \mathcal{A}(U \cup V)$ on the $\mathrm{GVect}[X, \pi]$-valued prefactorization algebra $\mathcal{A}(-)[\hbar]/(\hbar^{m+1})$.

Suppose furthermore that for $k = 2, \ldots, m$, the operation $\delta^{(k)}$ has the form:

$$\delta^{(k)} = \sum_{j \geq 3} \binom{n}{j} \sum$$



where the $\delta^{(k)}_{1\ldots j;n}$ vertex has j input strands, and the second sum is over the number of output strands. Implicit in the notation is that the $\wr$ components of the $(n-j)$ strands pass through untouched. (Because of the subtleties relating $\mathrm{C}_\bullet(U \times V)$ with $\mathrm{C}_\bullet(U) \otimes \mathrm{C}_\bullet(V)$, we continue with our conventions from Remarks 3.3.6 and 3.5.7 that even when the $|$ strands look like they go straight through, in fact any operation may impact all components of a chain.)

We also suppose that the $\delta^{(k)}_{1\ldots j;n}$ satisfy the following axioms:

no new ingredients We suppose that $\delta^{(k)}_{1\ldots j;n}$ is built as a sum of compositions of transversalizing homotopies $\boxed{h}$ (Construction 3.5.6) and operators $\textcircled{\Delta}$ (Definition 3.4.4), and that the way that $\delta^{(k)}$ is built out of $\textcircled{\Delta}$s and $\boxed{h}$s satisfies a few restrictions. First, we will suppose that if two strands "interact" (via an $\boxed{h}$ or a $\textcircled{\Delta}$) in some part of a diagram, and if one of the strands passes through an $\boxed{h}$ lower in the diagram, then the other strand also passes through the same $\boxed{h}$. Second, we suppose that, reading from bottom to top, $\delta^{(k)}_{1\ldots j;n}$ begins with an $\boxed{h^{1\ldots j;n}}$, and if $k > 2$ then $\delta^{(k)}$ does not begin with an $\boxed{h}$ followed by two $\textcircled{\Delta}$s.

permutation invariance At no cost, we might as well assume that $\delta^{(k)}_{1\dots j;n}$ is invariant under the S_j action on the j incoming strands, and also under permutations of the outgoing strands.

dependence only on chosen indices If U and V are disjoint opens in $\mathbb{R}$, and $c \in \mathrm{C}_\bullet(U^m) \otimes X^{\otimes m}$ and $c' \in \mathrm{C}_\bullet(V^{n-m}) \otimes X^{\otimes(n-m)}$ for $m \geq j$, then we ask that $\delta^{(k)}_{1\dots j;n}(c \otimes c') = \delta^{(k)}_{1\dots j;m}(c) \otimes c'$.

locality We ask that $\delta^{(k)}_{1\dots j;n}$ vanish identically on $\mathrm{T}^{1\dots j}_\bullet(\mathbb{R}^n)$, and in particular on $\mathrm{D}^{1\dots j}_\bullet(\mathbb{R}^n)$. The prefactorization algebra condition implies that the maps $\delta^{(k)} : \mathcal{A}(U) \to \mathcal{A}(U)$ are compatible with inclusions $U \subseteq U'$; we ask that this be true for each $\delta^{(k)}_{1\dots j}$ individually, for each number of output strands.

final request Our final axiom deserves a short preamble. Let $\mathfrak{g}$ be any graded Lie algebra, and $Q \in \mathfrak{g}$ any *homogeneous* element. Then it follows from the Jacobi identity that $[Q,[Q,Q]] = 0$. This can be seen directly in the case when $\mathfrak{g}$ is an endomorphism algebra: if Q is even, then $[Q,Q] = 0$ already, and if Q is odd, then $[Q,[Q,Q]] = [Q,2Q^2] = 2Q^3 - 2Q^3 = 0$. In particular, consider $Q = \partial + \hbar\Delta + \sum_{k=2}^m \hbar^k \delta^{(k)}$ acting on $\mathcal{A}(U)[\![\hbar]\!]$. Since we assume that $[Q,Q] = O(\hbar^{m+1})$, the first nontrivial term in the equation $0 = [Q,[Q,Q]]$ reads:

$$0 = \hbar^{m+1}\Big[\partial, 2[\Delta,\delta^{(m)}] + \textstyle\sum_{k=2}^{m-1}\big[\delta^{(k)},\delta^{(m-k)}\big]\Big] + O(\hbar^{m+2}).$$

Put another way, $2[\Delta,\delta^{(m)}] + \sum_{k=2}^{m-1}[\delta^{(k)},\delta^{(m-k)}] : \mathcal{A}(U) \to \mathcal{A}(U)$ is a degree-(-2) chain map.

As with any operator $\mathcal{A}(U) \to \mathcal{A}(U)$, the operator $2[\Delta,\delta^{(m)}] + \sum_{k=2}^{m-1}[\delta^{(k)},\delta^{(m-k)}]$ is naturally a direct sum of operators, corresponding to the decomposition of $\mathcal{A}(U)$ as $\prod_n \big(\mathrm{C}_\bullet(U^n) \otimes X^{\otimes n}\big)/S_n$. In terms of diagrams, $2[\Delta,\delta^{(m)}] + \sum_{k=2}^{m-1}[\delta^{(k)},\delta^{(m-k)}]$ is a direct sum indexed by the pair (number of input strands, number of output strands). Now consider expanding each direct summand as a sum of diagrams. In each diagram, some of the | strands interact with the Δ and $\delta^{(k)}$ vertices, and some strands run straight through from bottom to top (possibly permuting).

Then our final request is as follows. Fix a choice of number of input and output strands and of which strands will interact and which will run through (and with which permutations). Then form the sum over diagrams contributing to $2[\Delta,\delta^{(m)}] + \sum_{k=2}^{m-1}[\delta^{(k)},\delta^{(m-k)}]$ that satisfy these choices. We request that this sum commute with ∂. Note that this almost, but not quite, follows from locality and dependence only on chosen indices.

Then we claim that there exist a collection of vertices $\delta^{(m+1)}_{1\dots j;n}$ for $j \geq 3$ satisfying the same conditions, and combining as above into a map $\delta^{(m+1)}$ such that $\big(\mathcal{A}(-)[\hbar]/(\hbar^{m+2}), \partial + \hbar\Delta + \sum_{k=2}^{m+1}\hbar^k\delta^{(k)}\big)$ is a $\mathrm{DGVect}[X,\pi]$-valued prefactorization algebra. In particular, $\big(\partial + \hbar\Delta + \sum_{k=2}^{m+1}\hbar^k\delta^{(k)}\big)^2 = 0 \mod \hbar^{m+2}$.

3.6.4 Corollary (deformed prefactorization algebra structure on $\mathcal{A}(-)[\![\hbar]\!]$) A quick inspection of the proof of Proposition 3.6.1 reveals that the $\delta^{(2)}$ constructed therein satisfies all five axioms in the statement of Theorem 3.6.3. Therefore we can proceed to all orders: there exists a deformation $\delta = \hbar\Delta + O(\hbar^2)$ such that $\big(\mathcal{A}(-)[\![\hbar]\!], \partial + \delta\big)$ is a $\mathrm{DGV}\textsc{ect}[X,\pi]$-valued prefactorization algebra.

3.6.5 Lemma (strong version of locality) By combining dependence only on chosen indices and locality, it follows that the support of $\delta^{(k)}_{1\dots j;n}(c)$ is contained within the diagonal pushforward of the intersection (support of c) $\cap\, \{t_1 = \cdots = t_j\}$.

3.6.6 Proof of Theorem 3.6.3

Last things first: the axioms for $\delta^{(m+1)}$ imply prefactorization algebra compatibility. We must construct the collection of maps $\delta^{(m+1)}_{1\dots j;n}$ with the highlighted axioms such that $\partial + \hbar\Delta + \sum_{k=2}^{m+1} \hbar^k \delta^{(k)}$ is a differential on $\mathcal{A}(-)$. We claim that the rest of the prefactorization algebra axioms are immediate. Indeed, since forgetting to $\mathrm{GV}\textsc{ect}[X,\pi]$ is faithful and symmetric monoidal, the only requirement to check towards being a prefactorization algebra is that for U and V disjoint opens, the map $\odot : \mathcal{A}(U)[\hbar]/(\hbar^{m+1}) \otimes \mathcal{A}(V)[\hbar]/(\hbar^{m+1}) \to \mathcal{A}(U \cup V)[\hbar]/(\hbar^{m+1})$ is a map of chain complexes when equipped with the differential $\partial + \hbar\Delta + \sum_{k=2}^{m+1} \hbar^k \delta^{(k)}$. For this, it suffices for $\delta^{(m+1)}$ to vanish on the "mixed" terms in $\mathcal{A}(U \cup V)$, and this follows as soon as $\delta^{(m+1)}_{1\dots j;n}$ vanishes on $\mathrm{D}^{1\dots j}_\bullet(\mathbb{R}^n)$; in particular, it follows from the locality axiom.

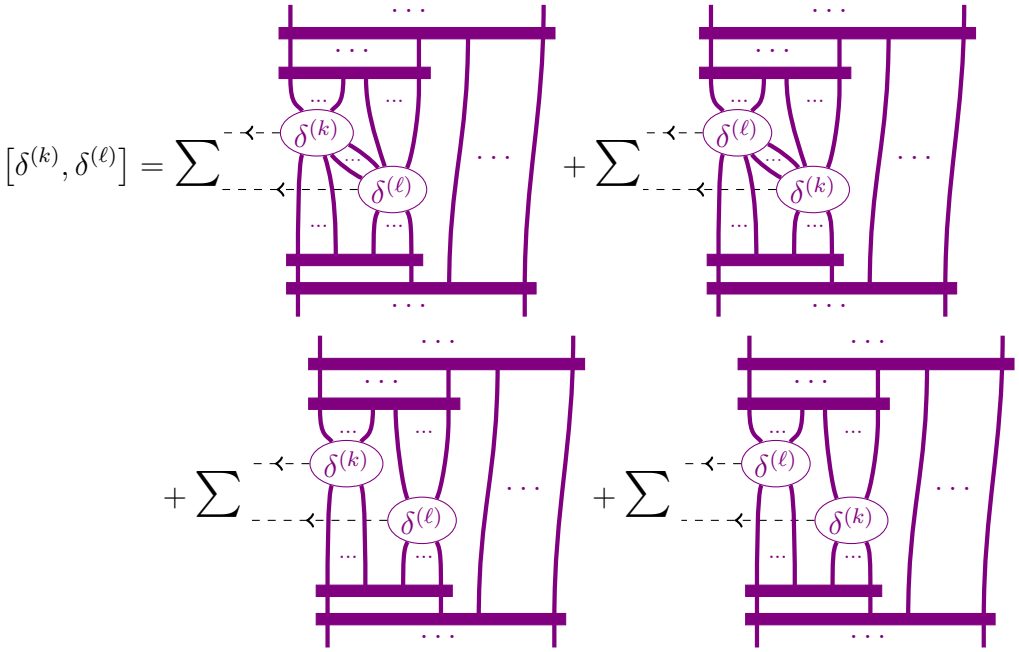
Overview of the construction. To construct $\delta^{(m+1)}$, we study $\big(\partial + \hbar\Delta + \sum_{k=2}^{m} \hbar^k \delta^{(k)}\big)^2$. By assumption, this vanishes mod $\hbar^{m+1}$; we are interested in the $\hbar^{m+1}$ term, and can ignore all higher terms. We get:

$$\left(\partial + \hbar\Delta + \sum_{k=2}^{m} \hbar^k \delta^{(k)}\right)^2 = \hbar^{m+1}\left(\big[\Delta, \delta^{(m)}\big] + \frac{1}{2}\sum_{k=2}^{m-1}\big[\delta^{(k)}, \delta^{(m+1-k)}\big]\right) + O(\hbar^{m+2})$$

We study these terms in more detail using the diagrammatics we have developed. Our goal is to show that if S is the sum of all contributing diagrams for which strands indexed $\{i,\dots,j\}$ interact, then S vanishes on $\mathrm{T}^{i\dots j}_\bullet$ and so $S \circ h^{i\dots j}$ is defined. We consider seven cases (some of which could be combined; c.f. the aside below), of which only the seventh is hard. Afterwards, we explain why the corresponding sum of diagrams $\delta^{(m+1)} =$ "$\Big(\big[\Delta, \delta^{(m)}\big] + \frac{1}{2}\sum_{k=2}^{m-1}\big[\delta^{(k)}, \delta^{(m+1-k)}\big]\Big) \circ h$" satisfies all required properties.

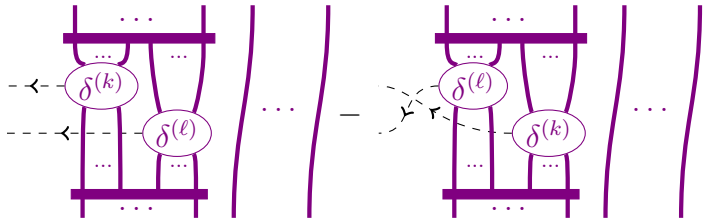
Preamble to Cases I and II: no Δs. Consider first the commutator $\frac{1}{2}\big[\delta^{(k)}, \delta^{(m+1-k)}\big]$ for $2 \le k \le m-1$. Abbreviate $\ell = m+1-k$. Expanding as a sum of diagrams, we see two

types of terms: the two δ-vertices might be connected, or they might be disconnected:



The sums are over all diagrams of the given type, and we have suppressed the numerical factors counting the number of ways a diagram can occur.

Case I: two disconnected δs. We discuss the disconnected terms first. Suppose the $\delta^{(k)}$ vertex has i input strands and the $\delta^{(\ell)}$ vertex has j input strands. Using Lemma 3.6.5 and the dependence only on chosen indices axiom, we conclude:

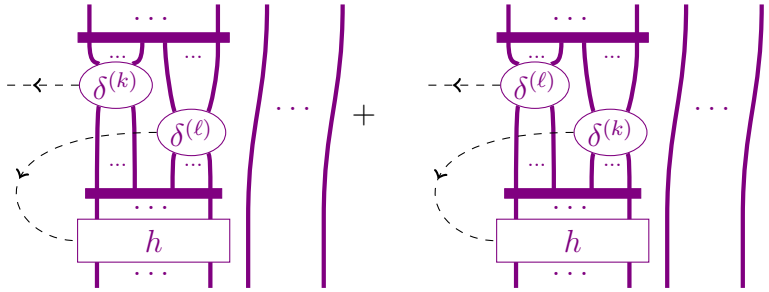


vanishes on $\mathrm{D}_\bullet^{12\cdots(i+j)}$

In more detail: because of the averagings, the difference between the second and first terms is simply to switch the relative heights of the $\delta^{(k)}$ and $\delta^{(\ell)}$ vertices; by dependence only on chosen indices, such switching commutes on $\mathrm{C}_\bullet(U^i \times V^j \times \mathbb{R}^{n-(i+j)}) \otimes X^{\otimes n}$ if $U \cap V = \emptyset$; we can write $\mathrm{D}_\bullet^{1\cdots(i+j)}(\mathbb{R}^n)$ as a (not-direct) sum $\mathrm{D}_\bullet^{1\cdots i}(\mathbb{R}^n) + \mathrm{D}_\bullet^{(i+1)\cdots j}(\mathbb{R}^n) + \sum \mathrm{C}_\bullet(\text{products of disjoint intervals})$; by Lemma 3.6.5, the $\delta^{(k)}$ vertex will vanish on $\mathrm{D}_\bullet^{1\cdots i}(\mathbb{R}^n)$, and the $\delta^{(\ell)}$ vertex will vanish on $\mathrm{D}_\bullet^{(i+1)\cdots(i+j)}(\mathbb{R}^n)$.

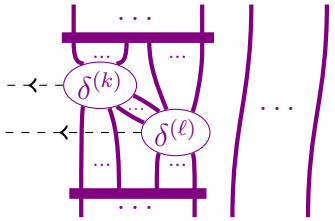
On the other hand, the locality axiom implies that the lower of the two δ vertices vanishes on $\mathrm{T}_\bullet^{\cdots}$, where $\cdots$ records the indices of the input strands on the lower vertex. Since the commutator vanishes on $\mathrm{D}_\bullet^{12\cdots(i+j)}$, it follows from Lemma 3.5.5 that the commutator vanishes on $\mathrm{T}_\bullet^{12\cdots(i+j)}$. In particular, the following sum of diagrams describes a well-defined degree-

(-1) operator on $\mathcal{A}(-)$:



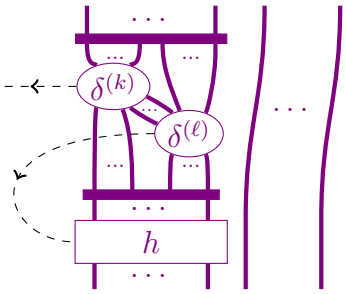
The above operator will contribute to $\delta^{(m+1)}$.

Case II: two connected δs. We consider next the diagrams contributing to $[\delta^{(k)}, \delta^{(\ell)}]$ in which the two δ vertices are connected by at least one |-edge. Suppose that the input edges connecting directly to the $\delta^{(k)}$ vertex are indexed $1, \ldots, i$, and those inputs heading to the $\delta^{(\ell)}$ vertex are $i+1, \ldots, i+j$. Then it follows from locality and Lemma 3.6.5 that



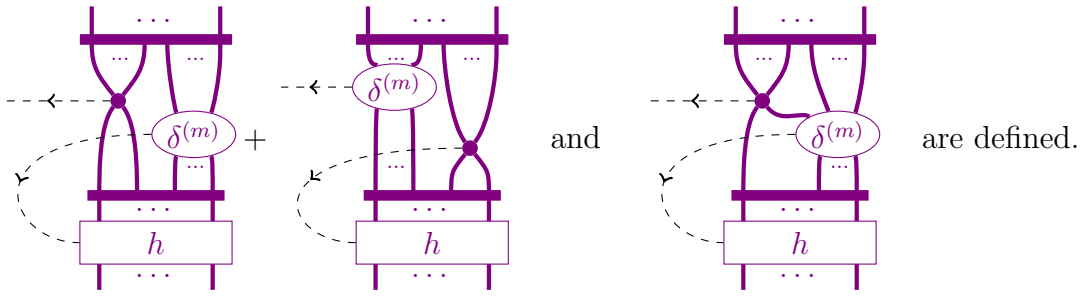
vanishes on $\mathrm{D}_\bullet^{1\cdots(i+j)}$ and on $\mathrm{T}_\bullet^{(i+1)\cdots(i+j)}$,

and therefore on $\mathrm{T}_\bullet^{1\cdots(i+j)}$ by Lemma 3.5.5. Thus the following operator is also well-defined, and will contribute to $\delta^{(m+1)}$:



Cases III and IV: disconnected Δ and δ, or connected with δ below Δ. We turn now to the commutator $[\Delta, \delta^{(m)}]$, which we again consider as a sum of diagrams. In case there are no |-edges connecting the Δ and $\delta^{(m)}$ vertices, we can proceed as in Case I. If there are connecting edges and the $\delta^{(m)}$ vertex is below the Δ vertex, we can use the Case

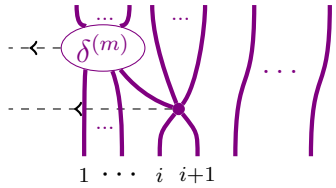
II. Together, we see that:



are defined.

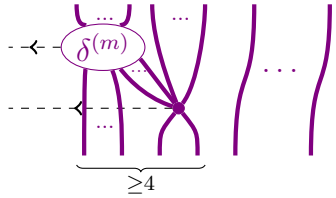
However, we cannot immediately repeat the arguments from above when the Δ vertex is below the $\delta^{(m)}$ vertex and they are connected by at least one strand, as Δ does not vanish on chains that are transverse to the diagonal.

Case V: δ above Δ, connected by exactly one strand. If the Δ and $\delta^{(m)}$ vertices are connected by exactly one strand, there is no problem.



Label the participating $|$-strands by indices $1, \dots, i+1$, so that strands i and $i+1$ hit the Δ vertex. If we input a chain in $\mathrm{T}_{\bullet}^{1\dots(i+1)}$, then by Lemma 3.5.2 the Δ vertex takes it to something in $\mathrm{T}_{\bullet-1}^{1\dots i}$, which then dies upon hitting $\delta^{(m)}$. Thus we can precompose with the homotopy h as we'd like.

Case VI: δ above Δ with at least four inputs. What if the vertices are connected by multiple edges, and $i+1 \geq 4$?



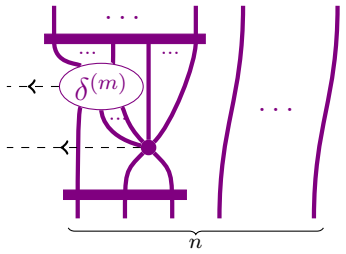
This case is not too bad, provided we make heavy use of Lemma 3.6.5. Let $c \in \mathrm{T}_k^{1\dots(i+1)}(\mathbb{R}^n)$ be a k-dimensional chain transverse to the diagonal. By definition, c intersects the diagonal $\{t_1 = \cdots = t_{i+1}\}$ along cells of dimension $\leq k-i$. By Lemma 3.6.5, if we feed c into the above diagram, it outputs a sum of cells with no increase in the dimension of their support: i.e. the output is in $\mathrm{C}_{k'}$ with $k' \leq k-i$. On the other hand, the above diagram describes a degree-(-2) map, and so $k' = k-2$. If $i \geq 3$, the only possibility is that the output is 0.

Aside: Case II redux. Actually, the argument of the previous paragraph could have also taken care of all cases when $\delta^{(k)}$ and $\delta^{(\ell)}$ are connected by at least one strand. The

permutation invariance axiom assures that the output of the lower vertex is invariant under permutations of its output $\big|$-strands, but Lemma 3.6.5 implies among other things that the output is also invariant under permutations just of the $|$-strands, where we decompose $\big| = |\,\wr$, and therefore also under permutations just of the $\wr$-strands. But on the $\wr$ side, each $\delta^{(k)}$ is a polynomial in the morphisms $\pi^{(n)}$, which transform strictly antisymmetrically under permuting their two inputs. It is therefore impossible for all inputs for the higher δ vertex to connect from the lower vertex — or rather, any such composition would be identically 0. Thus the only diagrams to consider have at least one input strand that runs straight to the upper vertex; but also the lower vertex has at least three input strands.

Note that in particular, we will never create a non-zero diagram in which only two incoming strands participate.

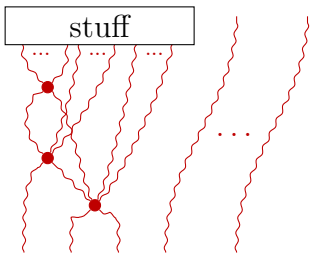
Case VII: δ above Δ, with precisely three participating incoming strands. We are thus left with one remaining case to consider:



Fixing an open $U \subseteq \mathbb{R}$, we choose permutation invariant lifts of $h : \mathrm{C}_\bullet(U^n) \to \mathrm{C}_{\bullet+1}(U^n)/\,\mathrm{T}_{\bullet+1}(U^n)$ back to $\mathrm{C}_{\bullet+1}(U^n)$.

By the construction of $\mathrm{DGVect}[X, \pi]$, such a diagram is necessarily a sum of diagrams that each factor as (operator on $\mathrm{C}_\bullet$)$\otimes$(morphism in $\mathrm{Vect}[X, \pi]$). The "no new ingredients" axiom assures that in each such summand, the operator and morphism are closely related, by the requirement that each generator π entering into the morphism in $\mathrm{Vect}[X, \pi]$ corresponds to an intersection-and-diagonal-pushforward on the $\mathrm{C}_\bullet$ side.

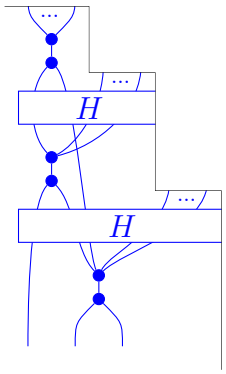
Choosing such a summand and reading up from the bottom, on the $\mathrm{Vect}[X, \pi]$ side we necessarily see:



As in Notation 3.2.3, each vertex corresponds to a generator π; the big box labeled "stuff" represents some arbitrary morphism in $\mathrm{Vect}[X, \pi]$. The lowest vertex corresponds to the Δ vertex in the composition we care about. The second-lowest vertex cannot have both its inputs connecting to the lowest vertex (as then the composition would be identically zero),

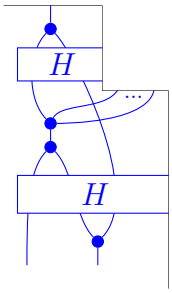
and cannot have more than one which does not connect (since only three inputs to the whole diagram participate in any vertices). Similarly, the third-from-lowest vertex cannot have any new inputs, and cannot have both its inputs connecting to the same lower vertex.

Thus the corresponding diagram on the $\mathrm{C}_\bullet$ side must begin:



The $\boxed{H}$s denote unknown compositions of homotopies $\boxed{h}$ with various input strands, and we have suppressed all $\uparrow$ edges and hidden behind a wall parts of the diagram that won't concern us.

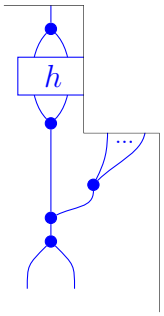
Now consider feeding in a transverse chain $c \in \mathrm{T}^{123}_\bullet$; we want to show that the output is 0. Using Lemma 3.5.2 and coassociativity of $\curlyvee$, it suffices to consider the value of



on some chain $c' \in \mathrm{T}^{12}_{\bullet-1}$. Suppose that c' is k-dimensional. Then it intersects $\{t_1 = t_2\}$ along a region with dimension at most $k-1$. Since intersections $\cap$ have degree -1, and homotopies h have degree $+1$, the output of the piece of diagram shown will have dimension $k - 2 + \#h$, where $\#h$ is the number of homotopies $\boxed{h}$ appearing in the two $\boxed{H}$s. Thus $\#h \leq 1$, or the output would be 0 by locality.

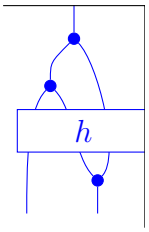
The bottom $\boxed{H}$ cannot consist of 0 homotopies, by the restrictions from the "no new ingredients" axiom. Even if it did, then coassociativity and the Frobenius-like axiom (Re-

mark 3.3.6) would imply that the diagram factors as:

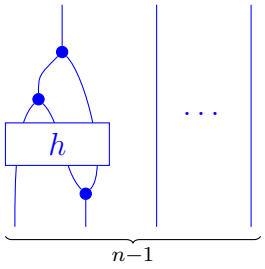


$= 0$, by a symmetry/antisymmetry argument.

Thus, after using the Frobenius-like axiom, the only potentially non-zero diagram is:

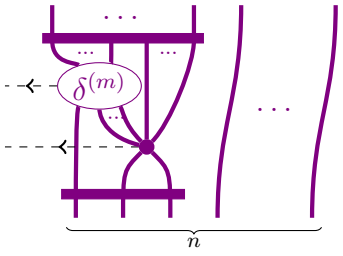


By the restrictions in the "no new ingredients" axiom, this subdiagram occurs only when $m = 2$, whence the whole diagram in question includes as a subdiagram:



That this morphism vanises on $\mathrm{T}^{12}_{\bullet}(\mathbb{R}^{n-1})$ is the content of Proposition 3.5.10.

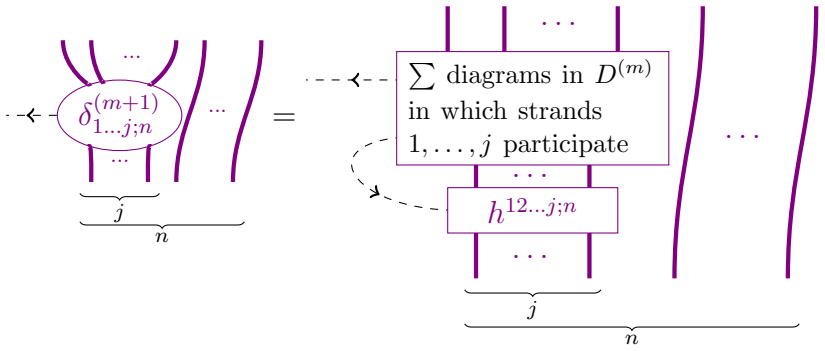
All together, we have:



vanishes on $\mathrm{T}^{123}_{\bullet}(\mathbb{R}^n)$

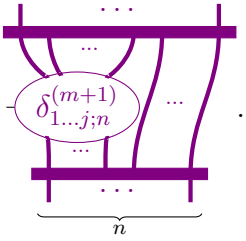
Summary and conclusion of the proof. Set $D^{(m)} = \big[\Delta, \delta^{(m)}\big] + \frac{1}{2}\sum_{k=2}^{m-1}\big[\delta^{(k)}, \delta^{(m+1-k)}\big]$. As in the final request in the statement of Theorem 3.6.3, consider expanding $D^{(m)}$ in diagrams and sorting the resulting sum by which strands participate and which pass through

untouched. For each collection of indices $\{i, \dots, j\}$, the sum of those diagrams in which strands indexed $\{i, \dots, j\}$ all participate and no other strands do vanishes on $\mathrm{T}^{i\cdots j}_{\bullet}$. The case-by-case analysis shows that it is well-defined to compose that sum with the transversalizing homotopy $h^{i\cdots j}$. We set:



and:

$$\delta^{(m+1)} = \sum_j \binom{n}{j} \sum$$



.

The "no new ingredients" request for $\delta^{(m+1)}$ is immediately satisfied, and locality, permutation invariance, and dependence only on chosen indices follow from the same axioms for the $\delta^{(k)}$s and hs. The only thing to check is that

$$\left(\partial + \hbar\Delta + \sum_{k=2}^{m} \hbar^k \delta^{(k)} + \hbar^{m+1}\delta^{(m+1)}\right)^2 = O(\hbar^{m+2}),$$

in the diagrammatic sense of the final request. By assumption, the left-hand side is:

$$\hbar^{m+1}\left(\left[\Delta, \delta^{(m)}\right] + \frac{1}{2}\sum_{k=2}^{m-1}\left[\delta^{(k)}, \delta^{(m+1-k)}\right]\right) + \hbar^{m+1}\left[\partial, \delta^{(m+1)}\right] + O(\hbar^{m+2})$$
$$= \hbar^{m+1}\left(D^{(m)} + \left[\partial, \delta^{(m+1)}\right]\right) + O(\hbar^{m+2})$$

On the other hand, the final request (and the sign conventions of Remark 3.5.8) assure that $\left[\partial, \delta^{(m+1)}\right] = -D^{(m)}$, by a calculation in diagrams that can be succinctly summarized as "$\left[\partial, \delta^{(m+1)}\right] = \left[\partial, D^{(m)}h\right] = \left[\partial, D^{(m)}\right]h - D^{(m)}\left[\partial, h\right] = 0 - D^{(m)}$." This completes the proof.

3.7 Homological perturbation theory and the $\star$-product

In the previous section we explained how to deform the differential on the prefactorization algebra $\mathcal{A}(-)[\![\hbar]\!]$ to $\partial+\delta = \partial+\hbar\Delta+O(\hbar^2)$. The main results in this section are Theorem 3.7.2 and Propositions 3.7.4 and 3.7.11, which explain how to use this deformation to construct a universal $\star$-product on the algebra $\mathcal{A} \in \textsc{Vect}[X,\pi]$. Our primary tool is the Homological Perturbation Lemma 3.7.5. At the end of the section, we relate our diagrams to Kontsevich's, and calculate our $\star$-product to order $\hbar^2$.

3.7.1 Notation (generalized elements) We could work in this section with the graphical notation used previously, but it is more convenient to work instead with *generalized elements*. Let $\mathcal{C}$ be a symmetric monoidal category with unit object $\mathbf{1}$. Given $A \in \mathcal{C}$, a *global element* of A is a map $a : \mathbf{1} \to A$. Since $\mathrm{Hom}(\mathbf{1}, -) : \mathcal{C} \to \textsc{Set}$ usually is not faithful, we do not expect to get much data about an object from knowing its global elements. On the other hand, a *generalized element* of A is a morphism with codomain A and arbitrary domain, and Yoneda's lemma says that any object is determined by its generalized elements. Generalized elements can be manipulated as if they were ordinary elements of ordinary sets, with one restriction: it is not allowed to duplicate or delete any generalized element in any equation, i.e. the only allowed equations are homogeneous linear in each generalized element.

For example, suppose that $(A, \cdot)$ is an algebra object in $\mathcal{C}$ and that $f : F \to A$ and $g : G \to A$ are generalized elements. Then an expression like "$f \cdot g \in A$" means the generalized element $F \otimes G \xrightarrow{f\otimes g} A \otimes A \xrightarrow{\cdot} A$. An expression like "$f \cdot g - g \cdot f$" means two isomorphic things. A priori, $f \cdot g$ has domain $F \otimes G$, and $g \cdot f$ has domain $G \otimes F$, and so their difference is not well-defined. But the symmetric monoidal structure on $\mathcal{C}$ picks out a distinguished isomorphism $G \otimes F \cong F \otimes G$, and "$f \cdot g - g \cdot f$" makes sense as a generalized element of A with domain $F \otimes G$ (or with domain $G \otimes F$) via composing with this distinguished isomorphism. For more details on working with generalized elements, we refer the reader to [DM99]. Henceforth, when we write "$f \in A$" we mean that f is a generalized element of A, with unspecific domain.

3.7.2 Theorem (prefactorization algebras give associative algebras) Continue to denote by $\big(\mathcal{A}(-), \partial\big)$ the $\mathrm{DGV}\textsc{ect}[X,\pi]$-valued factorization algebra constructed in Section 3.4. Consider deforming the differential (but not the underlying $\mathrm{GV}\textsc{ect}[X,\pi]$-valued prefactorization algebra structure) on $\mathcal{A}(-)[\![\hbar]\!]$ to a new differential $\partial + \delta$ with $\delta = O(\hbar)$, which is arbitrary except that we require that $\big(\mathcal{A}(-)[\![\hbar]\!], \partial + \delta\big)$ satisfies Definition 3.4.5 to be a $\mathrm{DGV}\textsc{ect}[X,\pi]$-valued prefactorization algebra, and that δ vanishes on the image of the diagonal pushforward $\mathrm{C}_\bullet(U) \otimes X^{\otimes n} \to \mathrm{C}_\bullet(U^n) \otimes X^n \to \mathcal{A}(U)$.

Let $U \subseteq \mathbb{R}$ be an open interval, and choose a point $u \in U \cap \mathbb{Z}[\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots]$. Denote by $u^n \in \mathrm{C}_0(U^n)$ the pushforward of u along the diagonal map $U \to U^n$. Then the map $\mathcal{A} \to \mathcal{A}(U)$ sending $f \in X^{\otimes n}/S_n$ to $u^n \otimes f \in \big(\mathrm{C}_0(U^n) \otimes X^n\big)/S_n$ extends to a $\mathbb{Q}[\![\hbar]\!]$-

linear chain map $\varphi_u : \big(\mathcal{A}[\![\hbar]\!], 0\big) \to \big(\mathcal{A}(U)[\![\hbar]\!], \partial + \delta\big)$, and there is a unique chain map $\tilde{\tau}_u : \big(\mathcal{A}(U)[\![\hbar]\!], \partial + \delta\big) \to \big(\mathcal{A}[\![\hbar]\!], 0\big)$ such that $\tilde{\tau}_u \circ \varphi_u = \mathrm{id}_{\mathcal{A}[\![\hbar]\!]}$.

Choose non-intersecting intervals $U, V \subseteq \mathbb{R}$, with $u < v$ for all $u \in U$ and $v \in V$, and choose an interval $W \supseteq U \cup V$. Also choose points $u \in U$, $v \in V$, and $w \in W$. Define a map $\star^w_{u,v} : \mathcal{A}[\![\hbar]\!] \otimes \mathcal{A}[\![\hbar]\!] \to \mathcal{A}[\![\hbar]\!]$ via:

$$\mathcal{A}[\![\hbar]\!] \otimes \mathcal{A}[\![\hbar]\!] \xrightarrow{\varphi_u \otimes \varphi_v} \mathcal{A}(U)[\![\hbar]\!] \otimes \mathcal{A}(V)[\![\hbar]\!] \xrightarrow{\odot} \mathcal{A}(W)[\![\hbar]\!] \xrightarrow{\tilde{\tau}_w} \mathcal{A}[\![\hbar]\!]$$

Then $\star^w_{u,v}$ is independent of the choices u, v, w, U, V, W provided $u < v$ (and $u \in U \subseteq W$ and $v \in V \subseteq W$ and $U \cap V = \emptyset$). Moreover, $\star = \star^w_{u,v}$ defines an associative algebra structure on $\mathcal{A}[\![\hbar]\!]$ with unit 1.

The proof will occupy Lemmas 3.7.6 and 3.7.7 and Propositions 3.7.8 and 3.7.9.

3.7.3 Remark (comments on Theorem 3.7.2) A version of Theorem 3.7.2 appears in [Lur12]. We include a complete proof specialized to our situation for two reasons. First, the more general statement in [Lur12] is in the language of homotopy operads, and can produce objects that are equivalent to associative algebras in a not quite canonical way: for example, if the perturbation δ does not vanish on the diagonal pushforward $\mathrm{C}_\bullet(U) \otimes X^{\otimes n} \to \mathrm{C}_\bullet(U^n) \otimes X^n \to \mathcal{A}(U)$, then it determines a family of nontrivial isomorphisms $\mathcal{A} \to \mathcal{A}$ parameterized by $\mathbb{Z}[\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots]^2$, and the $\star$-product given in Theorem 3.7.2 must be twisted for it to be strictly associative. Second, in Propositions 3.7.4 and 3.7.11 and Example 3.7.14 we will be interested in computing explicit formulas for the $\star$-product.

3.7.4 Proposition ($\star$ is a universal $\star$-quantization of $(\mathcal{A}, \pi)$) Continuing the notation in 3.7.2, we have $\star = \odot + O(\hbar)$, where $\odot : \mathcal{A}[\![\hbar]\!] \otimes \mathcal{A}[\![\hbar]\!] \to \mathcal{A}[\![\hbar]\!]$ is its commutative multiplication as defined in Definition 3.2.2. Suppose that $\delta = \hbar\Delta + O(\hbar)$, as in Corollary 3.6.4. Then $\star = \odot + \frac{\hbar}{2}\pi + O(\hbar^2)$, where π is the extension of $\bigoplus \pi^{(n)} : X^{\otimes 2} \to \mathcal{A}$ to a Poisson bracket on $\mathcal{A}$. By the universality of $\mathcal{A}$, $\star$ defines a (manifestly rational) universal $\star$-quantization of any formal Poisson manifold.

Since we have constructed such a deformation δ in Theorem 3.6.3, we conclude that manifestly rational universal $\star$-products exist. We gave an explicit description of the deformation δ in Proof 3.6.6 of Theorem 3.6.3; the construction of δ requires choosing transversalizing homotopies h as in Construction 3.5.6. In Lemma 3.7.7 we give an explicit formula for τ_w (in terms of a homotopy η_w chosen in Lemma 3.4.3, although τ_w does not in fact depend on this data), and hence for $\star = \star^w_{u,v}$.

We will verify Proposition 3.7.4 in Proof 3.7.10. In Proposition 3.7.11, we will show furthermore that for the deformation δ constructed by Theorem 3.6.3, all higher order terms in $\star$ are also bidifferential operators.

3.7.5 The Homological Perturbation Lemma We recall now an important piece of homotopy perturbation theory. In general, homotopy perturbation theory describes how

to move (strongly homotopy) algebraic structures across quasi-isomorphisms. The special case that we will present moves the structure of "a choice of a Maurer–Cartan element," and provides explicitly formulas for problems that might be implicitly solved via spectral sequences. The Homological Perturbation Lemma was named in [GM70], and appeared first in [Bro65] (it is also implicit in [Shi62] and explicit in unpublished work by Barrat); more detailed history is available in [Hue11], and a particularly good write-up is in [Cra04]. We have described the Homological Perturbation Lemma once already in this dissertation (Fact 2.3.10), but include it again for convenience.

The following definitions and result apply in any category $\mathcal{C}$ enriched in abelian groups; we will apply them in the case of $\mathcal{C} = \textsc{Vect}[X, \pi]$. The notions of *$\mathbb{Z}$-graded $\mathcal{C}$-object* and *chain complex in $\mathcal{C}$* should be clear, and we continue to use homological grading conventions (differentials have homological degree -1).

A *retraction* in $\mathcal{C}$ consists of two chain complexes $(A_\bullet, \partial_A)$ and $(B_\bullet, \partial_B)$, chain maps $\varphi : A \to B$ and $\tau : B \to A$, and a *homotopy* $\eta : B_\bullet \to B_{\bullet+1}$. These maps are required to satisfy that $\tau \circ \varphi = \mathrm{id}_A$ and $\varphi \circ \tau = \mathrm{id}_B + [\partial_B, \eta]$. It follows that φ and τ are quasi-isomorphisms. The commutator is to be understood with the appropriate signs: since ∂_B is of homological degree -1 and η is of degree $+1$, both of which are odd, we have $[\partial_B, \eta] = \partial_B \circ \eta + \eta \circ \partial_B$ (for more precise sign conventions, see Remark 3.5.8). It is standard but unnecessary to also impose *side conditions* that $\eta^2 = 0$, $\eta \circ \varphi = 0$, and $\tau \circ \eta = 0$.

$$\big(A, \partial_A\big) \underset{\varphi}{\overset{\tau}{\leftrightarrows}} \big(B, \partial_B\big) \circlearrowright \eta$$

A *deformation* of a chain complex $(B_\bullet, \partial_B)$ is a degree-(-1) map $\delta : B \to B$ such that $(\partial_B + \delta)^2 = 0$. Equivalently, δ is a *Maurer–Cartan element* of $\mathrm{End}(B_\bullet)$. A deformation δ is *small* with respect to a given retraction $A \leftrightarrows B \circlearrowleft$ if the degree-0 map $(\mathrm{id}_B - \delta\eta)$ is invertible. Note that then $(\mathrm{id}_B - \eta\delta)^{-1} = \mathrm{id}_V + \eta(\mathrm{id}_V - \delta\eta)^{-1}\delta$ also exists.

Suppose given a retraction as above, and a small deformation δ of $(B_\bullet, \partial_B)$. Then the deformed complex $(B_\bullet, \partial_B + \delta)$ is part of a deformed retraction:

$$\big(A, \partial_A + \tilde{\delta}\big) \underset{\tilde{\varphi}}{\overset{\tilde{\tau}}{\leftrightarrows}} \big(B, \partial_B + \delta\big) \circlearrowright \tilde{\eta} \qquad \begin{aligned} \tilde{\delta} &= \tau \circ (\mathrm{id}_B - \delta\eta)^{-1}\delta \circ \varphi = \tau \circ \delta(\mathrm{id}_B - \eta\delta)^{-1} \circ \varphi \\ \tilde{\varphi} &= (\mathrm{id}_B - \eta\delta)^{-1} \circ \varphi \\ \tilde{\tau} &= \tau \circ (\mathrm{id}_B - \delta\eta)^{-1} \\ \tilde{\eta} &= \eta(\mathrm{id}_B - \delta\eta)^{-1} = (\mathrm{id}_B - \eta\delta)^{-1}\eta \end{aligned}$$

The $\mathbb{Z}$-graded $\mathcal{C}$-objects $A_\bullet$ and $B_\bullet$ do not change, but their differentials do.

The proof consists simply of checking some equations, and we leave it to the reader. In fact, at the cost of working harder at the proof one can drop the condition that $\tau \circ \varphi = \mathrm{id}_A$, replacing it only with the condition that τ and φ be quasi-isomorphisms [Cra04], but we will not need such generality.

3.7.6 Lemma ($\mathrm{H}_\bullet\big(\mathcal{A}(-)[\![\hbar]\!], \partial+\delta\big) = A[\![\hbar]\!]$) Let $U \subseteq \mathbb{R}$ be an open interval. For each $u \in U \cap \mathbb{Z}[\frac{1}{2}, \frac{1}{4}, \dots]$, we constructed in Lemma 3.4.3 a retraction

$$\big(\mathcal{A}, 0\big) \underset{\varphi_u}{\overset{\tau=\int}{\leftrightarrows}} \big(\mathcal{A}(U), \partial\big) \circlearrowleft \eta_u$$

where $\varphi_u = u^n\otimes : X^n/S_n \to (\mathrm{C}_\bullet(U^n)\otimes X^n)/S_n$, and τ corresponds to the "integration" map $\mathrm{C}_0(U) \to \mathbb{Q}$. We now extend coefficients to $\mathbb{Q}[\![\hbar]\!]$ and apply the Homological Perturbation Lemma 3.7.5 with $A = \mathcal{A}$ and $B = \mathcal{A}(U)$, where $\delta = O(\hbar)$ is some (formal, hence small) deformation. The output includes a differential on $\mathcal{A}[\![\hbar]\!]$, which necessarily vanishes because $\mathcal{A}[\![\hbar]\!]$ is supported entirely in homological degree 0. Thus the complex $\big(\mathcal{A}(U)[\![\hbar]\!], \partial+\delta\big) \in \mathrm{DGVect}[X,\pi]$ is quasi-isomorphic to $\mathcal{A}[\![\hbar]\!]$.

3.7.7 Lemma ($\tilde\varphi_u = \varphi_u$**, and** $\tilde\tau_u$ **is independent of** η_u) Continue the notation of Lemma 3.7.6. Since $\mathcal{A}(U)[\![\hbar]\!]$ is supported entirely in nonnegative degrees and δ drops degree by 1, the composition $\delta\circ\varphi_u$ must vanish. Therefore $\tilde\varphi_u = (\mathrm{id}_{\mathcal{A}(U)} - \eta_u\delta)^{-1}\circ\varphi_u = \varphi_u + \sum_{n\geq 1}(\eta_u\delta)^n\varphi_u = \varphi_u$ does not deform.

By construction, $\tilde\tau_u$ is a quasi-isomorphism. Using again that $\mathcal{A}(U)[\![\hbar]\!]$ is supported in nonnegative degrees, we see that $\tilde\tau_u$ vanishes on the image of $\partial+\delta$, and is an isomorphism on the quotient of the homological-degree-0 part of $\mathcal{A}(U)[\![\hbar]\!]$ modulo the image of $\partial+\delta$. Hence $\tilde\tau_u$ is determined by the equation $\tilde\tau_u\circ\varphi_u = \mathrm{id}_{\mathcal{A}[\![\hbar]\!]}$. For any homotopy η_u, by the Homological Perturbation Lemma $\tilde\tau_u$ is given by the formula

$$\tilde\tau_u = \textstyle\int \circ \sum_{m\geq 0}(\delta\eta_u)^m = \int + \hbar(\int\circ\Delta\circ\eta_u) + O(\hbar^2)$$

where $\int : (\mathrm{C}_\bullet(U^n)\otimes X^{\otimes n})/S_n \to X^{\otimes n}/S_n$ corresponds to the integration map $\mathrm{C}_\bullet \to \mathbb{Q}$.

3.7.8 Proposition ($\star^w_{u,v}$ **is independent of** u, v, w) We suppose now that δ satisfies the conditions of Theorem 3.7.2: $\big(\mathcal{A}(-)[\![\hbar]\!], \partial+\delta\big)$ is a $\mathrm{DGVect}[X,\pi]$-valued prefactorization algebra on $\mathbb{R}$ and δ vanishes on complete diagonals $\{t_1 = \cdots = t_n\}$. We will prove that $\star^w_{u,v}$ is independent of the choices of u, v, and w, provided $u < v$.

Suppose that $U \subseteq \mathbb{R}$ is an open interval, and choose $u_1, u_2 \in U \cap \mathbb{Z}[\frac{1}{2}, \frac{1}{4}, \dots]$. Then there is a unique chain $\overrightarrow{u_1u_2} \in \mathrm{C}_1(U)$ with $\partial(\overrightarrow{u_1u_2}) = u_2 - u_1$. Let $(\overrightarrow{u_1u_2})^n \in \mathrm{C}_1(U^n)$ denote the diagonal pushforward of $\overrightarrow{u_1u_2}$ along $U \to U^n$; it satisfies $\partial((\overrightarrow{u_1u_2})^n) = (u_2)^n - (u_1)^n$. Consider the maps $(\overrightarrow{u_1u_2})^n\otimes : X^{\otimes n} \to \mathrm{C}_1(U^n)\otimes X^{\otimes n}$. They package together to a homological-degree-$(+1)$ map $\Phi_{u_1,u_2} : \mathcal{A} \to \mathcal{A}(U)$. Since δ vanishes on complete diagonals, $(\partial+\delta)\circ\Phi_{u_1,u_2} = \partial\circ\Phi_{u_1,u_2} = \varphi_{u_2} - \varphi_{u_1}$. In particular, φ_{u_1} and φ_{u_2} are homotopic.

We now consider the map $\star^w_{u,v} = \tilde\tau_w\circ\odot\circ(\varphi_u\otimes\varphi_v) : \mathcal{A}[\![\hbar]\!]\otimes\mathcal{A}[\![\hbar]\!] \to \mathcal{A}[\![\hbar]\!]$, where $u \in U$, $v \in V$, $w \in W$, and U and V are disjoint intervals (with $U < V$), and both are contained in an interval W. By the prefactorization algebra axioms, $\star^w_{u,v}$ does not depend on

the choices of intervals U, V, and W provided $\star^w_{u,v}$ is defined. Thus we might as well assume that $W = \mathbb{R}$, and that U extends to $-\infty$ and that V extends to $+\infty$. Recall that the chain maps that are homotopic to 0 are an "ideal" among all chain maps, both for composition and for tensor product (if α and β are chain maps such that α is homotopic to α', then $\alpha \otimes \beta$ is homotopic to $\alpha' \otimes \beta$, and $\alpha \circ \beta$ is homotopic to $\alpha' \circ \beta$ provided the compositions make sense). Therefore, we change $\star^w_{u,v}$ to a homotopic map by moving u to be very negative and v to be very positive.

On the other hand, $\mathcal{A}[\![\hbar]\!]$ is supported entirely in homological degree 0, and so homotopic maps are necessarily equal. Thus $\star^w_{u,v}$ is independent of u and v provided $u < v$. We cannot move u past v — at the moment they pass each other, there will not exist intervals $U \ni u$ and $V \ni v$ that are disjoint, and $\odot : \big(\mathcal{A}(U)[\![\hbar]\!], \partial + \delta\big) \otimes \big(\mathcal{A}(V)[\![\hbar]\!], \partial + \delta\big) \to \big(\mathcal{A}(U \cup V)[\![\hbar]\!], \partial + \delta\big)$ is not a chain map unless U and V are disjoint.

We now consider the dependence on w. Recall that $\tilde{\tau}_w$ is the unique chain map $\big(\mathcal{A}(W)[\![\hbar]\!], \partial + \delta\big) \to \mathcal{A}[\![\hbar]\!]$ such that $\tilde{\tau}_w \circ \varphi_w = \mathrm{id}_{\mathcal{A}[\![\hbar]\!]}$. Consider changing w to some w'; then φ_w is homotopic to $\varphi_{w'}$, and hence $\tilde{\tau}_w \circ \varphi_{w'}$ is homotopic to $\tilde{\tau}_w \circ \varphi_w = \mathrm{id}$, and hence equal. Therefore $\tilde{\tau}_w = \tilde{\tau}_{w'}$ by Lemma 3.7.7, as they satisfy the same defining relation.

3.7.9 Proposition ($\star$ is associative) By Proposition 3.7.8, we are justified in writing the map $\star^w_{u,v} : \mathcal{A}[\![\hbar]\!] \otimes \mathcal{A}[\![\hbar]\!] \to \mathcal{A}[\![\hbar]\!]$ simply as $\star$. We prove now that it defines an associative multiplication with unit $1 \in \mathcal{A}[\![\hbar]\!]$.

Pick $f_1, f_2, f_3 \in \mathcal{A}[\![\hbar]\!]$. We would like to compare $(f_1 \star f_2) \star f_3$ and $f_1 \star (f_2 \star f_3)$. Since $\star$ is independent of the choices of points (provided they come in the specified order), we are under no obligation to use the same points for different occurances of $\star$ in a computation. Thus we choose the specific points $1, 2, 3 \in \mathbb{Z}[\frac{1}{2}, \dots]$, and we want to compare:

$$(f_1 \star^2_{1,2} f_2) \star^2_{2,3} f_3 \quad \text{versus} \quad f_1 \star^2_{1,2} (f_2 \star^2_{2,3} f_3).$$

We claim that both sides are equal to:

$$\tilde{\tau}_2\big(\varphi_1(f_1) \odot \varphi_2(f_2) \odot \varphi_3(f_3)\big).$$

Indeed, the left-hand side is:

$$(f_1 \star^2_{1,2} f_2) \star^2_{2,3} f_3 = \tilde{\tau}_2\Big(\varphi_2\tilde{\tau}_2\big(\varphi_1(f_1) \odot \varphi_2(f_2)\big) \odot \varphi_3(f_3)\Big)$$

But by construction in the Homological Perturbation Lemma 3.7.5, $\varphi_2\tilde{\tau}_2$ is homotopic to $\mathrm{id}_{\mathcal{A}(-)[\![\hbar]\!]}$. Therefore $\tilde{\tau}_2\big(\varphi_2\tilde{\tau}_2\big(\varphi_1(f_1) \odot \varphi_2(f_2)\big) \odot \varphi_3(f_3)\big)$ and $\tilde{\tau}_2\big(\big(\varphi_1(f_1) \odot \varphi_2(f_2)\big) \odot \varphi_3(f_3)\big)$ are homotopic, and hence equal. The other side is analogous, and $\odot$ is associative.

Finally, choosing $w = v$, we see that for any $f \in \mathcal{A}[\![\hbar]\!]$,

$$1 \star f = \tilde{\tau}_v\big(\varphi_u(1) \odot \varphi_v(f)\big) = \tilde{\tau}_v\big(\varphi_v(f)\big) = f$$

since $\varphi_u = \mathrm{id} : \mathbb{Q} = X^{\otimes 0} \to (\mathrm{C}_\bullet(U^0) \otimes X^{\otimes 0})/S_0 = \mathbb{Q}$ sends $1 \mapsto 1$. That $f \star 1 = f$ is similar. This completes the proof of Theorem 3.7.2.

3.7.10 Proof of Proposition 3.7.4 We now prove that if the deformation δ to the prefactorization algebra $\mathcal{A}(-)$ satisfies $\delta = \hbar\Delta \mod \hbar^2$, then $f \star g = fg + \frac{\hbar}{2}\{f,g\} \mod \hbar^2$. Since everything is $\mathbb{Q}[\![\hbar]\!]$-linear, we can specialize to working over $\mathbb{Q}[\hbar]/(\hbar^2)$, and thereby work with the prefactorization algebra $\big(\mathcal{A}(-)[\hbar]/(\hbar^2), \partial + \hbar\Delta\big)$ constructed in Corollary 3.4.7.

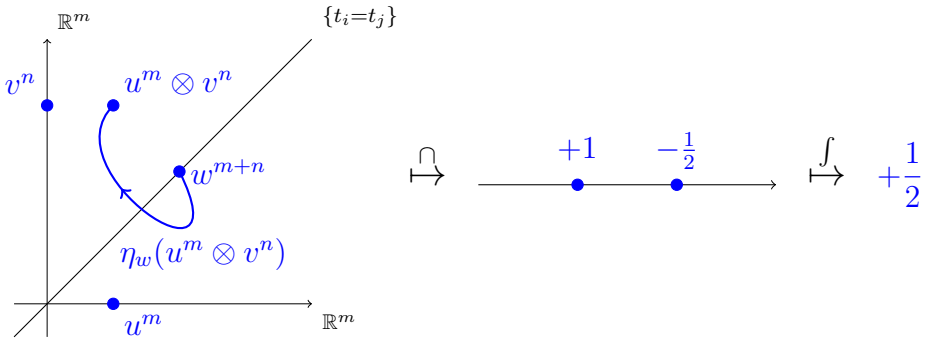
Choose $u, v, w \in \mathbb{Z}[\frac{1}{2}, \frac{1}{4}, \dots]$ with $u < v$. Working modulo terms of order $\hbar^2$ and higher, we expand the definitions from Theorem 3.7.2 and the Homological Perturbation Lemma 3.7.5:

$$\begin{aligned} f \star g &= \tilde{\tau}_w\big(\varphi_u(f) \odot \varphi_v(g)\big) \\ &= \textstyle\int \circ \big(\mathrm{id} + \hbar\Delta \circ \eta_w\big)\big(\varphi_u(f) \odot \varphi_v(g)\big) \mod \hbar^2 \\ &= fg + \hbar \textstyle\int \circ \Delta \circ \eta_w\big(\varphi_u(f) \odot \varphi_v(g)\big) \mod \hbar^2 \end{aligned}$$

We have used that the $\mathrm{GV}\textsc{ect}[X,\pi]$ map $\int : \mathcal{A}(U) \to \mathcal{A}$ is a commutative algebra homomorphism for any open $U \subseteq \mathbb{R}$.

Suppose now that $f \in X^{\otimes m}/S_n$ and $g \in X^{\otimes n}/S_m$. Then $\varphi_u(f) \odot \varphi_v(g) = (u^m \otimes v^n) \otimes (f \otimes g) \in (\mathrm{C}_0(\mathbb{R}^{m+n}) \otimes X^{\otimes(m+n)})/(S_m \times S_n) \to (\mathrm{C}_0(\mathbb{R}^{m+n}) \otimes X^{\otimes(m+n)})/S_{m+n}$. Applying the homotopy η_w replaces $u^m \otimes v^n \in \mathrm{C}_0(\mathbb{R}^{m+n})$ with a chain $\eta_w(u^m \otimes v^n) \in \mathrm{C}_1(\mathbb{R}^{m+n})$ satisfying $\partial\eta_w(u^m \otimes v^n) = u^m \otimes v^n - w^{m+n}$.

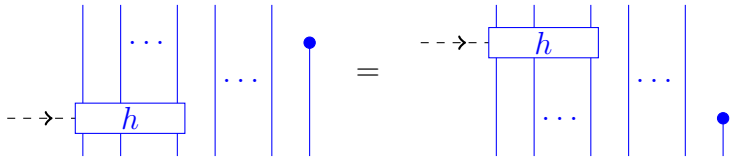
We now apply the operator $\Delta = \mathrm{ave}\sum_{i<j}(\cap_{ij} \otimes \pi_{ij})$, where π_{ij} is the map that applies the bivector π to the ith and jth tensorands of $X^{\otimes(m+n)}$. By looking at the action on $f \otimes g \in X^{\otimes(m+n)}$, we see that applying $\cap \otimes \pi$ to $\eta_w(u^m \otimes v^n) \otimes (f \otimes g)$ returns 0 unless $i \in \{1, \dots, m\}$ and $j \in \{m+1, \dots, m+n\}$. In this case, using that $u < v$ and that w^{m+n} is totally diagonal, we see that $\int \circ \cap_{ij} (\eta_w(u^m \otimes v^n)) = \frac{1}{2}$. This is clear from a cartoon:



The sum over choices of i and j exactly implements the product rule when differentiating $f, g \in \mathcal{A}$ (compare with the multiplicative prefactor ℓm in Definition 3.2.2). Thus $\int \Delta(u^m v^n \otimes fg) = \frac{1}{2}\pi(f \otimes g) = \frac{1}{2}\{f,g\}$. This completes the proof of Proposition 3.7.4.

3.7.11 Proposition ($\star$ acts by bidifferential operators) Our last result concerns the higher-order-in-$\hbar$ terms in the universal star-product $\star$. Suppose that the deformation δ is as constructed in Theorem 3.6.3. Suppose also an additional axiom for the homotopy h, which we would could have easily included in the "dependence only on chosen indices" request of

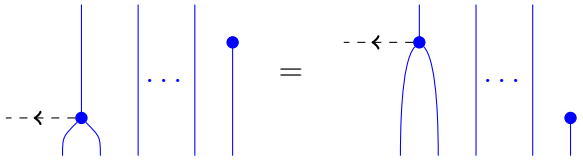
Construction 3.5.6:



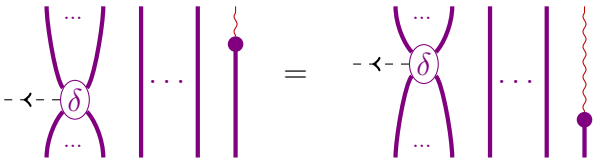
Then we claim that each $\hbar$-coefficient of $\star$ is a bidifferential operator. Put another way, for each n, we claim that $\star : \mathcal{A}^{\otimes 2} \to \mathcal{A}[\hbar]/(\hbar^{n+1})$ is a differential operator in each variable with respect to the commutative algebra structure on $\mathcal{A}$.

3.7.12 Proof of Proposition 3.7.11 The claim essentially follows from the existence of a diagrammatic description of δ. Working modulo $\hbar^n$, we described δ in Theorem 3.6.3 as a finite sum of vertices $\delta^{(1)} = \hbar\Delta$ and $\delta^{(k)}$s for $k < n$, drawn in the |-strands. Then to apply δ to some direct summand  of $\mathcal{A}$, one sums over all ways to attach a vertex from δ to the top of some strands (and then averages the outputs); this is exactly the action of the binomial coefficient in Theorem 3.6.3.

We have:

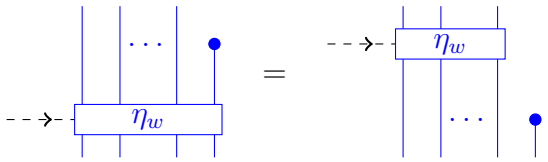


Set  $= \int \otimes \mathrm{id}_X$. Along with the request above for h and the functoriality for diagonal maps, we conclude:



for any $\delta = \delta^{(k)}_{1\ldots j}$

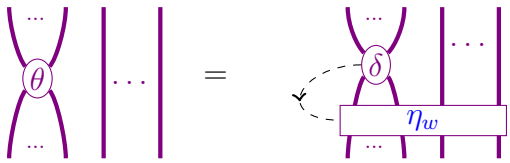
We are free to choose the homotopy η_w however we want, only its action $\mathrm{C}_0(\mathbb{R}^n) \to \mathrm{C}_1(\mathbb{R}^n)$ appears in any formula, and any map $\eta_w : \mathrm{C}_0(\mathbb{R}^n) \to \mathrm{C}_1(\mathbb{R}^n)$ satisfying $\partial \circ \eta_w = \mathrm{id} - w^n \circ \int$ can be extended to a homotopy on all of $\mathrm{C}_\bullet(\mathbb{R}^n)$. We will use some homotopy η_w satisfying:



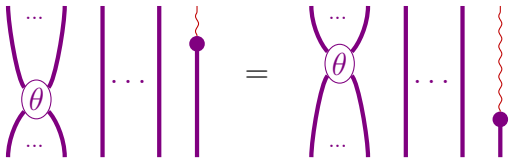
on C_0.

For example, given a zero-cell $c \in \mathbb{Z}[\frac{1}{2}, \frac{1}{4}, \ldots]^n$, we could set $\eta_w(c)$ to be the average of the $n!$ paths connecting w^n to c along the edges the rectangular solid with opposite corners c and w^n.

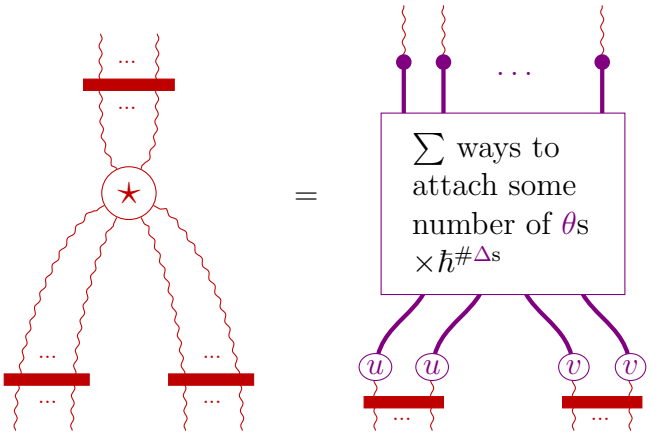
Let us abbreviate $\theta^{(k)}_{1\dots j;n} = \delta^{(k)}_{1\dots j;n} \circ \eta_w$, and $\theta^{(1)} = \Delta \circ \eta_w$:



Then the diagrammatics continue to allow to pull down vertices:



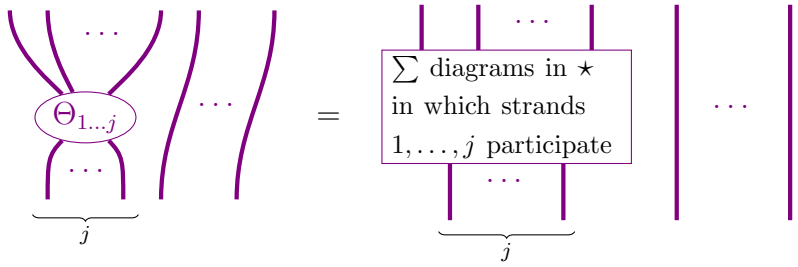
With these abbreviations, we have a nice description of the $\star$-product $\star^w_{u,v} = \tilde{\tau}_w \circ \odot \circ (\varphi_u \otimes \varphi_w) = \int \circ (\mathrm{id} - \theta)^{-1} \circ \odot \circ (\varphi_u \otimes \varphi_w)$. It is implemented by a sum of all diagrams of the following shape:



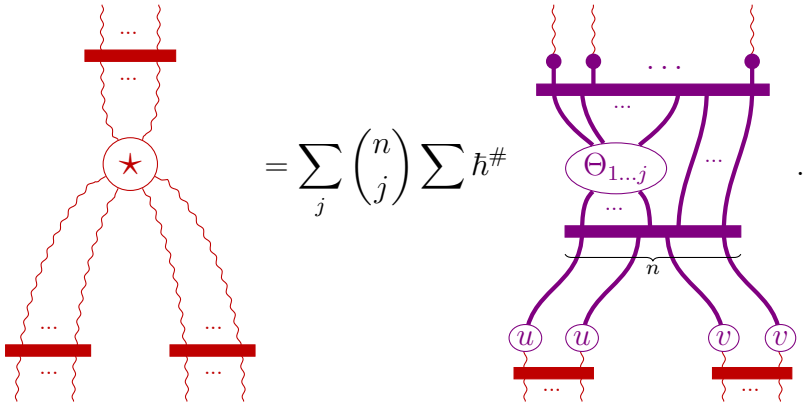
We have written $\textcircled{u}$ for $\varphi_u : X \to \mathrm{C}_0(\mathbb{R}) \otimes X$, and the power on $\hbar$ counts the number of $\textcircled{\Delta}$s in the fully expanded-out diagram. What's important to emphasize is that each $\textcircled{\theta}$ vertex occurs at its own level, and at each level we sum over all ways to attach it to the diagram below it.

We now sort the sum by the degree in $\hbar$, and by the data of which incoming strands participate and which pass through to the top without interacting with some θ vertex. Fixing a power in $\hbar$ and some collection of incoming strands, we can lump into a single vertex Θ the sum of all diagrams that contribute to that power in $\hbar$ and in which the fixed

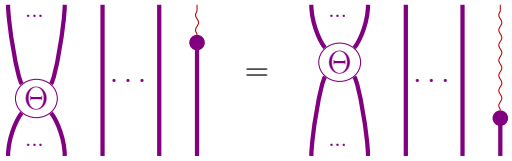
collection of incoming strands are precisely the strands that interact.



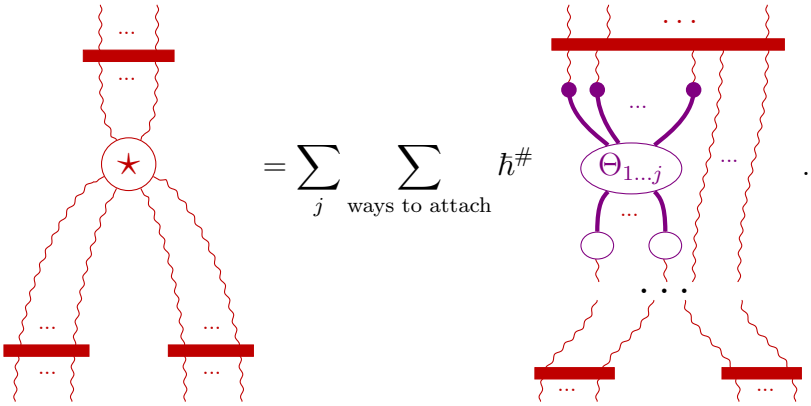
By definition, the corresponding coefficient of $\star$ is precisely a sum over all ways to attached a single Θ vertex:



Finally, we note that we can still pull the integration maps past the Θ vertices:

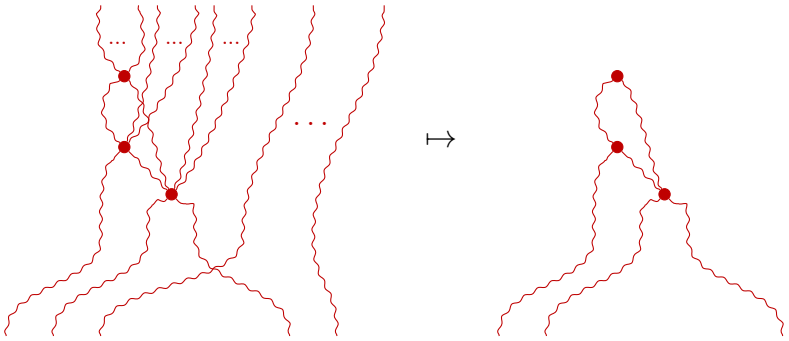


On the strands that do not participate in the Θ vertex, we pull the integration map $\int = \bullet$ all the way down. Since $\int u = 1$, we conclude that $\star : X^{\otimes m}/S_m \otimes X^{\otimes n}/S_n \to \mathcal{A}[\hbar]/(\hbar^n)$ is a sum over ways to attach one of finitely many vertices:



The sum is over ways to attach the Θ vertex to the incoming strands. The  s are either  s or  s depending on whether they attach to the left or the right, and the dots represent the permutation necessary to attach the Θ. The counts of the ways to attach a vertex are exactly the numerical factors arising from the product rule when applying a (bi)differential operator. Thus each coefficient of $\star : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}[\![\hbar]\!]$ is a bidifferential operator, completing the proof.

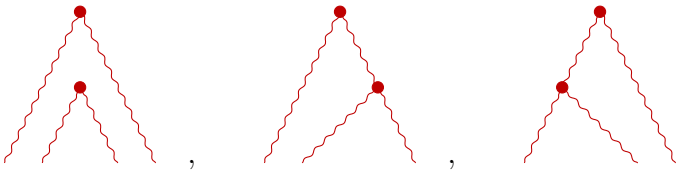
3.7.13 Remark (relation to Kontsevich's Feynman diagrams) Given a diagram drawn in  strands and  s representing a morphism $X^{\otimes m}/S_m \otimes X^{\otimes n}/S_n \to \mathcal{A}$, define its *core* to be the labeled directed graph formed by deleting all strands that connect to the top of the diagram. (We will not mark the directions, which correspond to the vertical direction on the page, nor the labels, which remember which of the bottom-most vertices go to the left and which go to the right, and also for each vertex which of its two lower edges is "left" and which is "right.")



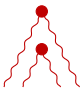
Then the sum over all diagrams with a given core is a bidifferential operator $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, and this gives a spanning set for the space of bidifferential operators on $\mathcal{A} \in \textsc{Vect}[X, \pi]$. One can summarize our proof of Proposition 3.7.11 by saying that in the sum-of-diagrams defining $\star$, all diagrams with the same core appear with the same weight.

In [Kon03], Kontsevich wrote down a sum over certain Feynman diagrams. Kontsevich's Feynman diagrams correspond in an obvious way to the cores of our diagrams, with the restriction that the only Feynman diagrams that appear as cores are those with no cycles. Given a Feynman diagram and a Poisson structure on a finite-dimensional vector space, Kontsevich defined a bidifferential operator; in the case when the diagram has no cycles, the bidifferential operator is precisely the image of the one we constructed from the corresponding core under the canonical map $\textsc{Vect}[X, \pi] \to \textsc{Vect}$ given by the Poisson vector space. Moreover, for each Feynman diagram, Kontsevich constructed a definite integral, and proved that summing over diagrams weighted by the values of these definite integrals gives a universal associative $\star$-product. The values of Kontsevich's integrals are real, but are not expected to be rational [FW10]. On the other hand, the construction in this chapter can be understood as assigning a rational weight to each of Kontsevich's Feynman diagrams in such a way that the corresponding weighted sum of diagrams defines a $\star$-product.

3.7.14 Example ($\star \bmod \hbar^3$) By definition, $f \star g = f \odot g + \frac{\hbar}{2}\pi(f,g) \bmod \hbar^2$. The $\hbar^n$-term in any universal $\star$-product is necessarily homogeneous of degree n in the Poisson structure π. Thus there are three cores contributing to the $\hbar^2$ term:



On a finite-dimensional vector space V with coordinates $\{x_i\}_{i=1}^{\dim V}$ and Poisson structure $\sum_{i,j} \pi_{i,j}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, the three cores correspond to the bidifferential operators:



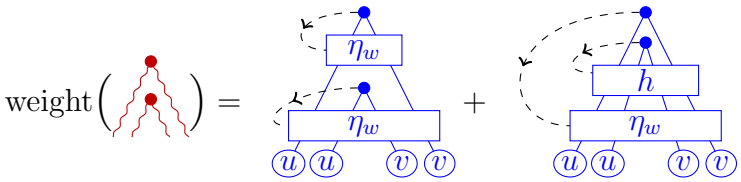
$$: \quad (f,g) \mapsto \sum_{i,j,k,l} \pi_{i,j}(x)\, \pi_{k,l}(x)\, \frac{\partial^2 f}{\partial x_i \partial x_k}\, \frac{\partial^2 g}{\partial x_j \partial x_k}$$

$$: \quad (f,g) \mapsto \sum_{i,j,k,l} \pi_{i,j}(x)\, \frac{\partial \pi_{k,l}}{\partial x_j}\, \frac{\partial^2 f}{\partial x_i \partial x_k}\, \frac{\partial g}{\partial x_l}$$

$$: \quad (f,g) \mapsto \sum_{i,j,k,l} \pi_{i,j}(x)\, \frac{\partial \pi_{k,l}}{\partial x_i}\, \frac{\partial f}{\partial x_k}\, \frac{\partial^2 g}{\partial x_j \partial x_l}$$

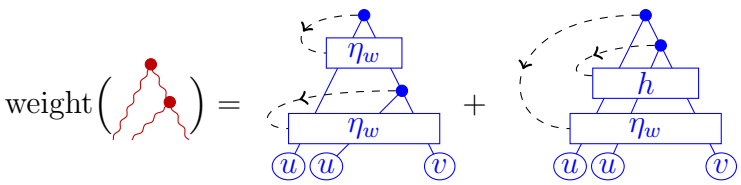
It is now not too hard to compute the weights of each of these cores that our construction assigns.

We choose $u < v$ and w in $\mathbb{Z}[\frac{1}{2}, \frac{1}{4}, \dots]$, and a lift of the homotopy h back to $\mathrm{C}_\bullet$ as in Proposition 3.5.10. The weight of the first core could receive contributions from two terms:

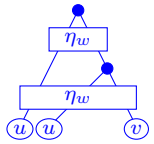
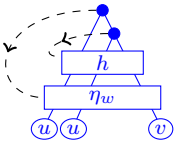


However, the second of these vanishes identically by locality. Evaluating the first gives a factor of $\frac{1}{8}$.

The second and third cores are assigned the same weights, and so for definiteness we will describe the former. Its weight receives contributions from two terms, neither of which is necessarily zero:



We choose η_w as in Proof 3.7.12 to send any point $c \in \mathbb{Z}[\frac{1}{2}, \dots]^n$ to the average over all paths connecting c to w^n along the edges of a rectangular solid. Then, with a little work, one can calculate the values of each diagram. The values of the individual diagrams depend on the value of w, but their sum is independent:

		
$w \neq u, v$	$\frac{1}{12}$	0
$w = u$	0	$\frac{1}{12}$
$w = v$	$\frac{1}{8}$	$-\frac{1}{24}$

In fact, the calculations are slightly different for the three cases $w < u$, $u < w < v$, and $v < w$, but all give the same answer. The sum $\frac{1}{12}$ is not surprising: it is the same $\frac{1}{12}$ that appears in the Baker–Campbell–Hausdorff formula $\log(\exp a \exp b) = a + b + \frac{1}{2}[a, b] + \frac{1}{12}\big([a, [a, b]] + [[a, b], b]\big) + \dots$ in noncommuting variables a and b.

All together, as in many other formal quantizations, our $\star$-product begins:

$$(f \star g)(x) = f(x)\, g(x) + \hbar \left(\frac{1}{2} \sum_{i,j} \pi_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \right) + \hbar^2 \left(\frac{1}{8} \sum_{i,j,k,l} \pi_{i,j}(x)\, \pi_{k,l}(x) \frac{\partial^2 f}{\partial x_i \partial x_k} \frac{\partial^2 g}{\partial x_j \partial x_k} + \frac{1}{12} \sum_{i,j,k,l} \pi_{i,j} \frac{\partial \pi_{k,l}}{\partial x_j} \frac{\partial^2 f}{\partial x_i \partial x_k} \frac{\partial g}{\partial x_l} + \frac{1}{12} \sum_{i,j,k,l} \pi_{i,j} \frac{\partial \pi_{k,l}}{\partial x_i} \frac{\partial f}{\partial x_k} \frac{\partial^2 g}{\partial x_j \partial x_l} \right) + O(\hbar^3)$$

To compute the $\hbar^3$ term requires a sum of roughly 50 diagrams, and so we leave it to the reader.

3.7.15 Example (Moyal product) A formal Poisson structure π is *constant* if $\pi^{(n)} = 0$ for $n \neq 0$. Some straightforward combinatorics verifies that our $\star$-product returns the Moyal $\star$-product in the case of constant Poisson structure.

3.7.16 Remark (universal enveloping algebra) Another situation is when $\pi^{(n)} = 0$ for $n \neq 1$. Then $\pi^{(1)} : X \otimes X \to X$ is a Lie bracket, and a version of Poincaré–Birkoff–Witt theorem implies that our $\star$-product makes $\mathcal{A}[\![\hbar]\!]$ isomorphic to the completion of the universal enveloping algebra of $(X, \hbar\pi^{(1)})$. In characteristic 0, there is a distinguished isomorphism between symmetric and universal enveloping algebras, and it is well-known that Kontsevich's

$\star$-product corresponds to that canonical isomorphism composed with Duflo's isomorphism. One should expect that our $\star$-product agrees with Kontsevich's in this case, but this has not been verified.

3.7.17 Remark (further questions) As a final comment, it should be emphasized that throughout our construction we relied heavily on the "affine structure" on our universal formal Poisson manifold (in which the algebra $\mathcal{A}$ is constructed as a completed symmetric algebra on a "vector space" X). Thus our construction is not functorial in any obvious way for morphisms of formal Poisson manifolds. It would be very interesting to study how our $\star$-product transforms under general formal Poisson maps.

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