The classical master equation in the finite-dimensional case

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Contents

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These lectures are based on a joint work with Giovanni Felder. In particular, the presentation of the theory of BV integration is based on our understanding of the works of Khudaverdian and Severa.

1 Introduction

The kind of a problem one wants to solve, in a simple form, is the following. You have a manifold X — in physics it will be infinite-dimensional, but in our cases it will be finite-dimensional — and you have some sort of "top form" ω, and you want to integrate $\int_X e^{-S/\hbar}\omega$. You have a non-compact group G which acts on X, and S and ω are G-invariant. So this integral, because of the extra G-invariance, is infinite. So what you really want to define is $\int_{X/G} e^{-S/\hbar}\omega$.

So what's the problem? You learn in second year calculus what is a quotient manifold. But the usual situation is that X is something like a vector space V, something you really understand, and the quotient is much more complicated variety.

Usually, the asymptotics as $\hbar \to 0$ depends on $X_{\text{crt}} = \{x \in X \text{s.t. } \mathrm{d}S(x) = 0\}$. Suppose that $X_{\text{crt}}/G = \{\text{pt}\}$, and that this point in the quotient is a nondegenerate critical point. This implies that X_{crt} is a single G-orbit, and that $\mathrm{d}S$ is nondegenerate on the normal bundle $\mathrm{N}_{X_{\text{crt}}}(X)$. So you want to write the asymptotics of the integrals in terms of this normal bundle, and S restricted to it. This is essentially known how to do: it's the usually Faddeev–Popov description.

But there's a bigger problem. In many cases the symmetries are what physicists call "open." What this means is the following. You have a family of vector fields λ_i on X such that $\lambda_i(\omega) = \lambda_i(S) = 0$. Moreover, the restriction of $\langle \lambda_i \rangle$ on X_{crt} is integrable (and, if you want, has one leaf). But the distribution $\gamma = \langle \lambda_i \rangle$ on X is not integral.

So you want to describe the integral, and this is the setup for BV. **Question from the audience:** The initial problem seems to have disappeared. What integral do you want to define? **David:** That's a good question. It's not at all clear that the question makes sense. You want to write an expression which makes sense in the general case, and gives the right answer in the usual case. I

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will give three introductory lectures, and then on Friday I will describe work with Giovanni Felder, where we try to make the problem precise.

Kolya asked me to start from the beginning. And I should apologize from the beginning. The basic language to describe the set-up uses supermanifolds. These are elementary, but kind of clumsy.

2 Superlinear algebra

Objects of the category are $\mathbb{Z}/2$-graded vector spaces $V = V_{\bar{0}} \oplus V_{\bar{1}}$. **Arthur:** Can you write ± 1? **David:** No. For me, $\mathbb{Z}/2$ is additive.

We define tensor products as usual. $V' \otimes V'' = (V_0' \otimes V_0'' \oplus V_1' \otimes V_1'') \oplus (\dots)$, and associativity is as usual. But the *symmetry* or *braiding* is $x \otimes y \mapsto (-1)^{|x|\cdot|y|} y \otimes x$. (Here and throughout, $|v| = 0$ for $v \in V_0$ and $|v| = 1$ for $V \in V_1$, and the formula is defined on homogeneous elements and extended by linearity.) So you change the notion of commutative. The problem is the moment you start doing this, you have infinite amount of signs. It is impossible to keep track. We will go back to this homogeneous-element language in a moment, but the only way to define things like "what is a dual space" is to move slightly more abstract.

Let $\mathcal{C}$ be an abelian category, with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which we assume is strictly associative, with unit $\mathbb{1}$. **Arthur:** What is strictly associative? **David:** That $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and the associativity is the identity. **Arthur:** Are there any examples? **David:** No, but it makes the formulas easier. Pretend it's true for usual vector spaces. **Alan:** There is a theorem that any category can be strictified. **David:** Yes.

A *symmetric braiding* is a natural isomorphism $b_{X,Y} : X \otimes Y \overset{\sim}{\to} Y \otimes X$ s.t. $b_{X,Y} \circ b_{Y,X} = \mathrm{id}$ and $b_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes b_{X,Z}) \circ (b_{X,Y} \otimes \mathrm{id}_Z)$. Also I should say $b_{\mathbb{1},X} = \mathrm{id}$ and so on. Moreover, the group S_n of permutations acts naturally on $X^{\otimes n}$ in such a way that $s_i = (i, i+1)$ acts by $\mathbb{1}^{\otimes i-1} \otimes b_{X,X} \otimes \mathbb{1}^{n-i-1}$. So we can definite subobjects $\mathrm{Sym}^n X$ and $\bigwedge^n X$ of $X^{\otimes n}$.

Given such a category, we can develop a linear algebra. First of all, the dual objects. Given $X \in \mathcal{C}$, we say that a triple (Y, i, ϵ) is *the dual to* X if we have $i : \mathbb{1} \to X \otimes Y$ and $\epsilon : Y \otimes X \to \mathbb{1}$ and the composition $X \overset{\sim}{\to} \mathbb{1} \otimes X \overset{i}{\to} X \otimes Y \otimes X \overset{\epsilon}{\to} X \otimes \mathbb{1} \overset{\sim}{\to} X$ and also $Y \to \cdots \to Y$ are the identity. The dual is unique, if it exists. If X has a dual, we denote it by X^*.

The *inner hom* $\underline{\mathrm{Hom}}(X,Y)$ is defined by $\mathrm{Hom}(Z, \underline{\mathrm{Hom}}(X,Y)) = \mathrm{Hom}(Z \otimes X, Y)$. If X has dual, then $\underline{\mathrm{Hom}}(X,Y) = X^* \otimes Y$.

Given any $f \in \mathrm{End}(X)$, we define $\mathrm{tr}\,\alpha \in K = \mathrm{End}(\mathbb{1})$ as the composition $\mathbb{1} \overset{i}{\to} X \otimes X^\vee \overset{\alpha\otimes\mathrm{id}}{\longrightarrow} X \otimes X^\vee \overset{b_{X,X^\vee}}{\longrightarrow} \overset{\epsilon}{\to} \mathbb{1}$. One can show that $\mathrm{tr}(\alpha\beta) = \mathrm{tr}(\beta\alpha)$. We define $\dim X = \mathrm{tr}(\mathrm{id}_X)$.

For example, when $\mathcal{C} = \mathrm{SV{\small ECT}}$, and $X = X_0 \oplus X_1$, then $\alpha \in \mathrm{End}(X)$ has the form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, and $\mathrm{tr}\,\alpha = \mathrm{tr}\,A - \mathrm{tr}\,D$.

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Vera: α is even or odd? **David:** $\mathrm{Hom}(X,Y) = \underline{\mathrm{Hom}}(X,Y)_0$. So *endomorphism* always means even part.

Let $X \in \mathrm{SV{\scriptstyle ECT}}_f(\mathbb{K})$. Then $\mathrm{Aut}(X)$ is a connected algebraic $\mathbb{K}$-group. **Alan:** What does connected mean when $\mathbb{K} = \mathbb{R}$? **David:** Connected as an algebraic group. We defined a homomorphism $\mathrm{tr} : \mathrm{Lie}(\mathrm{Aut}(X)) \to \mathbb{K}$. It extends uniquely to an algebraic group homomorphism $\mathrm{Ber} : \mathrm{Aut}(X) \to \mathbb{K}^\times$. At least in characteristic 0.

Arthur: How do we make $\mathrm{Aut}(X)$ an algebraic group? **David:** I will explain.

In any symmetric braided category $\mathcal{C}$, we can define a notion of "commutative algebra". It is $A \in \mathcal{C}$ and a morphism $A \otimes A \to A$ and $\mathbb{1} \to A$ which satisfy usual axioms. Analogously, we can define notions of Hopf algebras. Recall that $\mathbb{K}^{m|n} = (\mathbb{K}^m)_0 \oplus (\mathbb{K}^n)_1$. Then consider $\mathbb{K}^{1|1}$, which has an algebra structure. The basis is $\{1, \epsilon\}$, and we define $\epsilon^2 = 0$. Moreover, $\mathbb{K}^{1|1}$ has a Hopf algebra structure, in which $\Delta(1) = 1 \otimes 1$ and $\Delta(\epsilon) = 1 \otimes \epsilon + \epsilon \otimes 1$. Claim: $(\Delta(\epsilon))^2 = 0$. Why? $\Delta(\epsilon)^2 = (1 \otimes \epsilon + \epsilon \otimes 1)(1 \otimes \epsilon + \epsilon \otimes 1) = \epsilon \otimes \epsilon - \epsilon \otimes \epsilon = 0$. Because of the anticommutativity. This is why this is a Hopf algebra in super land, and not in the usual case.

So, let A be an algebra in SVECT. Then we have $\mathrm{tr} : \mathrm{End}_A(X \otimes A) \to A$. Then therefore $\mathrm{Ber} : \mathrm{Aut}(X \otimes A)_0 \to A_0^\times$. **Vera:** This is not a general property of braided tensor category. **David:** No, it's special. A general question is: when do you have a determinant? A necessary condition is that dimensions be integers. But I don't know counterexamples: maybe this is enough.

An *even symplectic structure* on $X \in$ SVECT is an isomorphism $X \xrightarrow{\sim} X^\vee$ s.t. $\alpha^\vee = -\alpha$. For any $X \in$ SVECT, we define ΠX by $(\Pi X)_0 = X_1$ and $(\Pi X)_1 = X_0$. This is bad definition — what means "equals"? More precisely, we set $\Pi X = \mathbb{K}^{0|1} \otimes X$. Here you have to think: I could multiply from the left, or I could multiply from the right. Of course, they are isomorphic, but there is a sign, and all the problems with signs come from forgetting this.

An *odd symplectic structure* on X is an isomorphism $\omega : X \xrightarrow{\sim} \Pi X^\vee$ such that $\omega^\vee = -\omega$.

Consider the functor F : SVECT $\to$ VECT, $(X_0, X_1) \mapsto X_0 \oplus X_1$. This functor is not compatible with the braiding. So when you start describing objects in terms of their images in VECT, it is very easy to make a mistake. And this is implicitly what I did in the first definition.

3 Supermanifolds

Let $U \subseteq \mathbb{R}^n$ be an open set, and $q \geq 0$. We define $C(U)^q = \mathscr{C}^\infty(U) \otimes \bigwedge^\bullet \mathbb{R}^q$. This is a basic superalgebra. A *supermanifold* is a manifold M_0 with a sheaf of rings $\mathcal{O}_M$ such that for any $m \in M_0$ there exists open $U \ni m$ such that $\mathcal{O}_M(U) \cong C(U)^q$. **Alan:** The U on the right is not the same as the U on the left? **David:** By isomorphic, I mean, $U \ni m$ is small enough to be isomorphic to an open in $\mathbb{R}^n$. But fine, there is a diffeomorphism $\phi : U \to \tilde{U} \subseteq \mathbb{R}^n$ open such that $\mathcal{O}_M(U) \cong C(\tilde{U})^q$.

For any supermanifolds M, N, we can define the product $M \times N$ in a natural way.

Here is an example of a supermanifold. Let $E \to M_0$ be a vector bundle. Then I define a supermanifold ΠE such that $\Pi E_0 = M$ and $\mathcal{O}_{\Pi E}(U) = \Gamma_{\text{smooth}}(\bigwedge^\bullet E^\vee)$. If I had not put $\bigwedge^\bullet$ but symmetric, then these would be functions on E that are polynomial in fibers.

Then there is the following theorem, which is correct but very misleading. Any supermanifold is isomorphic to ΠE for some vector bundle $E \to M_0$. Moreover, one can describe $E \to M_0$ explicitly, by $\mathcal{E} = I/I^2$, where $I = \{f \in \mathcal{O}_M \text{s.t. } f|_{M_0} = 0\}$. This isn't really a vector bundle, but a projective module over $\mathscr{C}^\infty(M_0)$, and those are the same as vector bundles. **Alan:** What does it mean to restrict to M_0? **David:** The real point: this ring $\mathcal{O}_M$ has unique maximal nilpotent ideal I.

Why is this theorem misleading? Because $\text{Aut}(\Pi E)$ as a supermanifold is much bigger than $\text{Aut}(E \to M)$ as a vector bundle. The corollary is that the theorem is not true in algebraic geometry. What happens is: your manifold in first approximation looks like a normal bundle. There are obstructions to being really a normal bundle, and these are in cohomology. And this $\mathscr{C}^\infty$ cohomology is 0 by partitions of unity, but the obstructions might not vanish in other categories of manifolds.

Vector fields on M are the same as derivations $A = \Gamma(M_0, \mathcal{O}_M)$. So we can think about vector fields. $\text{Vect}(M)$ is a super Lie algebra, by the same reason as before. (If $\lambda_1, \lambda_2 \in \text{Vect}(M)$, then $\lambda_1\lambda_2 - (-1)^{|\lambda_1||\lambda_2|}\lambda_2\lambda_1 \in \text{Vect}(M)$.)

We can define differential forms $\Omega^1(M) = \text{Hom}_{\mathcal{O}}(\text{Vect}(M), \mathcal{O})$, where $\mathcal{O} = \mathcal{O}_M(M)$. And $\Omega^k(M) = \bigwedge^k_{\mathcal{O}} \Omega^1(M)$. For example, $M = \mathbb{R}^{0|1}$. In this case, $\text{Vect}(M) = \text{Diff}(\mathbb{R}[\epsilon], \epsilon^2 = 0, |\epsilon| = 1)$, and this is generator by $\xi = \frac{\partial}{\partial\epsilon}$, with $|\xi| = 1$. Then $\Omega^1 = \mathrm{d}\epsilon\, \mathcal{O}$, with $|\mathrm{d}\epsilon| = 1$. Then $\Omega^k(M)$ is a free $\mathcal{O}$-module of rank 1 for all $k > 1$.

Now, de Rham differential.

We have seen that $\mathbb{R}^{0|1}$ has a group structure, which we will call $\mathbb{G}_a^-$. This is the same as saying that $\mathcal{O}(\mathbb{R}^{0|1}) = \mathbb{R}^{1|1}$ has a Hopf algebra structure. What is an *action* of $\mathbb{G}_a^-$ on a supermanifold? First of all, it is a map $\mathbb{G}_a^- \times M \to M$. This is the same as an algebra homomorphism $\phi : A \to A \otimes \mathbb{R}[\epsilon]$, where $A = \mathcal{O}_M$. Let's define $\bar\phi$ by demanding that $\phi(a) = a + \bar\phi(a)\,\epsilon$. The condition that ϕ is an algebra homomorphism implies that the map $a \mapsto \bar\phi(a)$ is given by an odd vector field. The condition that we have a group action implies that $\lambda^2 = 0$. In other words, and action of $\mathbb{G}_a^-$ on M is the same as an odd vector field λ on M such that $\lambda^2 = 0$.

Let M be a usual manifold, and $\mathrm{T}_M \to M$ the tangent bundle. Then the differential forms $\Omega(M) = \mathcal{O}(\Pi\mathrm{T}_M)$.

Given a supermanifold M, consider the functor $F_M : \textsc{SuperAlg} \to \textsc{Set}$ given by $F_M(A) = \text{Hom}_{\textsc{SAlg}}(\mathcal{O}(M), A)$. Then F_M completely determines M, and is a way to think about supermanifolds. For example, given two supermanifolds M, N, we can consider the functor $F_{M,N}$: $\textsc{SAlg} \to \textsc{Set}$ given by $F_{M,N}(A) = \text{Hom}_{\textsc{SAlg}}(\mathcal{O}(N), \mathcal{O}(M) \otimes A)$. If $F_{M,N}$ is representable by a supermanifold, we denote said supermanifold by $\underline{\text{Maps}}(M, N)$. Then for any supermanifold X, we have $\text{Maps}(X, \underline{\text{Maps}}(M, N)) = \text{Maps}(X \times M, N)$. Here "Maps" is just hom in $\textsc{SMan}$. The same definition can be given in the usual category of manifolds, but if you consider the category of finite-dimensional manifolds, then $F_{M,N}$ is almost never representable — it's representable only if M_0 is zero-dimensional.

Now we claim: We have a canonical isomorphism $\underline{\mathrm{Maps}}(\mathbb{G}_a^-, M) = \Pi\mathrm{T}_M$. Corollary: We have a natural action of $\mathbb{G}_a^-$ on $\Pi\mathrm{T}_M$. This action is the de Rham differential. Proof: we have to show that for any supermanifold X we have a natural isomorphism $\mathrm{Maps}(X \times \mathbb{G}_a^-, M) \overset{\sim}{\to} \mathrm{Maps}(X, \Pi\mathrm{T}_M)$. So let $A = \mathcal{O}_M$ and $B = \mathcal{O}_X$. Then $\mathrm{Maps}(X \times \mathbb{G}_a^-, M) = \mathrm{Hom}(B, A[\epsilon]) = \{(\alpha, \beta)\text{s.t. } \alpha \in \mathrm{Hom}(B, A), \beta \in \Gamma(X, \alpha^*\mathrm{T}_M)_1$. The maps $X \to \Pi\mathrm{T}_M$ are the parametrized by the same data.

So now we can define the de Rham differential for any supermanifold M by the same formula.

Finally, the Cartan formula. For $\lambda \in \mathrm{Vect}(M)$ and $\alpha \in \Omega^\bullet(M)$ we have $\mathcal{L}_\lambda(\alpha) = \imath_\lambda(\mathrm{d}\alpha) + (-1)^{|\lambda|}\mathrm{d}(\imath_\lambda \alpha)$.

4 Odd symplectic manifolds

An *odd symplectic manifold* is a supermanifold M with $\omega \in \Omega^2(M)$ which is nondegenerate and closed and odd. We have the usual theorems:

1. **Darboux.** For any odd symplectic manifold (M, ω) and $m_0 \in M_0$ there exists an open $U \ni m_0$ and embedding $U \subseteq \mathbb{R}^n$ and isomorphism $\mathcal{O}(U) = \mathscr{C}^\infty(U) \otimes \bigwedge^*(\xi_1, \dots, \xi_n)$ such that $\omega|_U = \sum \mathrm{d}x_i \wedge \mathrm{d}\xi^i$.
2. Any odd symplectic manifold is isomorphic (as a symplectic manifold) to $\Pi(\mathrm{T}^*_{M_0})$ with its canonical odd symplectic structure.

Let (M, ω) be an odd symplectic manifold, $\dim M = (n, n)$. A submanifold $L \subseteq M$ is *Lagrangian* if $\omega|_L \equiv 0$ and $\dim L = (p, q)$ with $p + q = n$. For example, $M = \mathbb{R}^{n|n}$ and $\omega = \sum \mathrm{d}x_i \wedge \mathrm{d}\xi^i$. Take $L = \mathbb{R}^{p|n-p} = \{x_i = 0, i > p,\ \xi^i = 0, i \leq p\}$.

3. For any Lagrangian submanifold $L \subseteq M$, there exists an open neighborhood $\tilde{M}$ of L in M such that $\tilde{M} \cong \Pi\mathrm{T}^*_L$. So it's exactly like Alan's theorem.

If (M, ω) is an odd symplectic manifold then for any $h \in \mathcal{O}(M)$ we can define a vector field λ_h on M such that $\lambda_h \lrcorner \omega = \mathrm{d}h$.

Let $E \to X$ be a usual vector bundle. Then we have a canonical symplectomorphism $\Pi\mathrm{T}^*(\Pi E) \overset{\sim}{\to} \Pi\mathrm{T}^*(E^*)$. The construction locally is as follows. Set $E = V \times U$. Then $\Pi E = \Pi V \times U$ and $\Pi\mathrm{T}^*(\Pi E) = \Pi\mathrm{T}^*(\Pi V \times U) = V^* \times \Pi V \times \Pi\mathrm{T}^* W$. On the other hand, $E^* = V^* \times U$, and $\Pi\mathrm{T}^*(E^*) = \Pi\mathrm{T}^*(V^* \times U) = \Pi\mathrm{T}^*(V^*) \times \Pi\mathrm{T}^* U = \Pi V \times V^* \times \Pi\mathrm{T}^* U$. So we have locally a map $\phi : \Pi\mathrm{T}^*(\Pi(V \times U)) \overset{\sim}{\to} \Pi\mathrm{T}^*(V^* \times U)$. Then one can check that ϕ does not depend on the choice of trivialization, and defines a symplectomorphism $\Pi\mathrm{T}^*(\Pi E) \to \Pi\mathrm{T}^*(E^*)$.

Wednesday, January 11

5 Integration (Berezin)

Let (W,ω) be an odd symplectic vector space. Then $W \cong \mathbb{R}^{n|n}$ with coordinates $(y^1,\dots,y^n,\xi_1,\dots,\xi_n)$ and $\omega = \sum y^i \wedge \xi_i$. Easy exercise: $\omega\wedge\omega = 0$. Now, we claim: we can associate with (W,ω) canonically a line $\mathcal{B}_W$ of degree $|n|$. In particular, we have an action $\mathrm{Symp}(W,\omega) \to \mathrm{Aut}(\mathcal{B}_W)$. And for any supercommutative algebra A we obtain a homomorphism $\mathrm{Aut}_A(W\otimes A,\omega) \to \mathrm{Aut}(\mathcal{B}_W \otimes A) = A_0^\times$. Of course, your first reaction is: this is impossible! The symplectic group is semisimple, and so has no characters! But we are saying that there is a map of superalgebraic groups $\underline{\mathrm{Symp}}(W,\omega) \to \underline{\mathbb{G}_m}$.

Vera: What do you mean by degree? **David:** The line will be odd or even depending on n. But the automorphism group will be the same.

Consider the space $\bigwedge^\bullet W = \bigoplus_{n\geq 0} \bigwedge^n W$. Since $\omega\wedge\omega = 0$, the multiplication by ω, i.e. $\alpha \mapsto \omega\wedge\alpha$, defines a differential on $\bigwedge^\bullet W$. Define $\mathcal{B}_W = \mathrm{H}^\bullet(\bigwedge W, \wedge\omega)$. We claim: $\mathcal{B}_W$ is one-dimensional generated by $y^1\wedge\dots\wedge y^n$. Proof: Consider the grading on $\bigwedge^\bullet W$ such that $|y^i| = 1$ and $|\xi_i| = -1$. Let $\bigwedge^\bullet W[i] \subseteq \bigwedge^\bullet W$ be the subspace of elements of degree i. Then $\bigwedge^\bullet W = \bigoplus_{i=-\infty}^n \bigwedge^\bullet W[i]$, and $\wedge\omega : \bigwedge^\bullet W[i] \to \bigwedge^\bullet W[i]$, because $|\omega| = 0$.

Let $L \in \mathrm{End}(\bigwedge^\bullet W)$ be given by $L = \imath(\frac{\partial}{\partial y^k})\,\imath(\frac{\partial}{\partial \xi_k})$, where $\imath(\partial)$ means "substitute", and $(\frac{\partial}{\partial y^k},\frac{\partial}{\partial \xi_k})$ is basis dual to (y,ξ). Then it is easy to see that

$$(\wedge\omega)\circ L + L\circ(\wedge\omega)\big|_{\bigwedge^\bullet W[i]} = (n-i)\mathrm{id}$$

And this implies that for any supercommutative algebra A, we have $\mathrm{H}^\bullet(\bigwedge_A W\otimes A, \wedge\omega) \cong \bigwedge^\bullet W[n] = A\, y^1\wedge\dots\wedge y^n$.

This way it is obtained $\mathcal{B} : \underline{\mathrm{Symp}}(W,\omega) \to \underline{\mathbb{G}_m}$.

Let $V (\cong \mathbb{R}^{p|q})$ be a linear superspace. Set $W = \Pi \mathrm{T}^* V = V \oplus \Pi V^\vee$. This has natural odd symplectic structure ω. And so we have $\mathcal{B} : \underline{\mathrm{Symp}}(W,\omega) \to \underline{\mathbb{G}_m}$. On the other hand, we have $\underline{\mathrm{Aut}}(V) \hookrightarrow \underline{\mathrm{Symp}}(W,\omega)$, because it's a canonical construction. And so we obtain $\mathrm{Ber} : \underline{\mathrm{Aut}}(V) \to \underline{\mathbb{G}_m}$. This is the best way I know to define Berezinian.

Vectors in $\mathcal{B}(W)$ are called *semidensities*. The same way is when if you want to integrate in the usual way: in a vector space you can define *densities* which are elements of top exterior power. Here we can define semidensities, which are square roots of densities. We can do this because, as opposed to the usual case, the symplectic group is not semisimple.

Now, let (M,ω) be an odd symplectic manifold of dimension $(n|n)$. We define $\mathcal{B}_M$ as the homology of multiplication by ω on $\bigwedge^\bullet(\mathrm{T}^*M) = \Omega^\bullet M$. You do this point by point. Lemma, but it's an immediate consequence of what I said before:

1. $\mathcal{B}_M$ is a line bundle over M of degree $|n|$.

2. If we identify M with $\Pi T^* M_0$ — remember than any odd symplectic manifold is non-canonically isomorphic like this — then any volume form $\kappa \in \Omega^n(M_0)$ gives a trivialization $\mathcal{B}_M \cong \mathcal{O}_M$.

Really we only used the top form $dy^1 \wedge \cdots \wedge dy^n$ to trivialize $\mathcal{B}_W$.

Sections of $\mathcal{B}_M$ over M are called *semidensities*.

6 BV Laplacian

The *BV Laplacian* will be a map $\Delta : \Gamma(M, \mathcal{B}_M) \to \Gamma(M, \mathcal{B}_M)$. I will give a simple-minded definition, and then say some words which you can ignore.

Given $\alpha \in \Gamma(M, \mathcal{B}_M)$, choose $\alpha_0 \in \ker(\wedge\omega) \in \Omega^\bullet M[n]$ which represents α. Consider $d\alpha_0 \in \Omega^\bullet M$. Then $d\alpha_0 \in \Omega^\bullet M[n-1]$. Because we know that in degree $(n-1)$, wedging with ω doesn't have cohomology, there exists $\beta_0 \in \Omega^\bullet M$ such that $d\alpha_0 = \omega \wedge \beta_0$. Then we define $\Delta\alpha$ as the image in $\Gamma(M, \mathcal{B}_M)$ of $d\beta_0$. So this is the definition, and you can easily see that it doesn't depend on the choices.

Ok, a remark, which people who don't know spectral sequences can ignore. Since ω is closed — we haven't so far used it — the two differentials d and $\wedge\omega$ on $\Omega^\bullet M$ commute. So we can consider the spectral sequence to compute $H^\bullet(d + \wedge\omega)$ such that the E_1 term is $H^\bullet(\wedge\omega) = \mathcal{B}_M$. Then the second differential is trivial, and the third is Δ. If you look at the definition of Δ, it is just the standard definition of differential in a spectral sequence. Furthermore, all later differentials vanish. So if you know spectral sequences, this is the explanation for Δ.

Ed: But usually differentials shift degree. **David:** You see, degree here is fictitious. ω doesn't shift degree. Because even things are in degree $+1$, and odd is in degree -1. **Ed:** But the third differential, it should be like a move of a knight on a chess board. It should shift both degrees. **David:** That's completely correct, but we don't have a bicomplex. We have degrees mod two, and we have another degree that d shifts and ω preserves.

There is one formula, which is particularly important.

Let (M, ω) be an odd symplectic manifold, and $h \in \mathcal{O}(M)$. Then h defines a hamiltonian vector field ξ_h such that $dh = \imath(\xi_h)(\omega)$. Then the lemma is: for any semidensity α on M, we have — vector fields act on all sorts of natural objects —

$$\xi_h(\alpha) = \Delta(h\alpha) - (-1)^{|h|} h \Delta(\alpha)$$

Proof: Choose a representative α_0 of α. By the Cartan formula, we have

$$\mathcal{L}_{\xi_h} \alpha_0 = \imath_{\xi_h} d\alpha_0 + (-1)^{|\xi_h|} d(\imath_{\xi_h} \alpha_0)$$

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We use the notation from before. Then:

$$= \imath_{\xi_h}(\omega \wedge \beta_0) + (-1)^{|h|}\mathrm{d} \circ \imath_{\xi_0}(\alpha_0) = \mathrm{d}h \wedge \beta_0 + (-1)^{|h|}\mathrm{d}(\imath_{\xi_0}(\alpha_0)) \mod \omega$$
$$= \mathrm{d}\beta_1 + (-1)^{|h|}h\mathrm{d}\beta_0 \mod \omega$$

where $\beta_1 = h\beta_0 + (-1)^{|h|}\imath_{\xi_h}\alpha_0$. Then I let you finish the proof. The first part is $\Delta(h\alpha)$ and the second is $h\Delta(\alpha)$.

We mention also the following. Let W_0 be a usual manifold with $\dim W_0 = n$, and $W = \Pi \mathrm{T}^* W_0$, and $p : W \to W_0$ the projection. Every odd symplectic manifold can be made to look like this, although not canonically. We claim:

1. For any $\eta \in \Omega^n(W_0)$, $p^*\eta \wedge \omega = 0$. So we obtain a map $\tau = \tau_{W_0} : \Omega^n(W_0) \to \Gamma(W_W, \mathcal{B}_W)$.
2. $\operatorname{Im}\tau \subseteq \ker \Delta$.

7 BV Integration

Let $M = \mathbb{R}^{n|n}$ with $\omega = \mathrm{d}y^i \wedge \mathrm{d}\xi_i$. Let $L = \mathbb{R}^{p|q}$ the Lagrangian subspace in M with coordinates $y^1, \dots, y^p$ and $\xi_{p+1}, \dots, \xi_n$, where $p + q = n$. We denote by $I = I_L \subseteq \mathcal{O}(M)$ the ideal of functions vanishing on L.

Proposition: There exists a linear functional $\int_L : \Gamma_{\mathrm{cpt}}(M, \mathcal{B}_M) \to \mathbb{R}$, unique up to scalar, such that

1. $\int_L \mathcal{L}_{\xi_h}\alpha = 0$ for all $\alpha \in \Gamma_{\mathrm{cpt}}(M, \mathcal{B}_M)$ and $h \in I_L$
2. $\int_L \alpha = 0$ if $\alpha|_L = 0$, i.e. $\alpha \in I_L\Gamma(\mathcal{B}_M)$.

Moreover, $\int_L \Delta\alpha = 0$ for all $\alpha \in \Gamma_{\mathrm{cpt}}(M, \mathcal{B}_M)$.

Proof: Every semidensity has a form $\alpha = f\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n$ for $f \in \mathcal{O}(M)$. Then condition 2. implies that $\int_L \alpha$ depends only on the restriction of f on L. The ideal I is generated by y^i for $i > p$ and ξ_i for $i \leq p$. So we think about f as a function $f(y^1, \dots, y^p, \xi_{p+1}, \dots, \xi_n)$. The condition $\int_L \mathcal{L}_{\xi_i}(f) = 0$ for $i \leq p$ is equivalent that $\int_L \frac{\partial f}{\partial y^i} = 0$. The condition $\int \mathcal{L}_{y^i} f = 0$ is equivalent to $\int \frac{\partial f}{\partial \xi_i} = 0$. Thus we can assume that f has the form $f(y_1, \dots, y_p)\xi_{p+1} \wedge \cdots \wedge \xi_n$, and $\int_L$ defines a functional on $\mathscr{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^p)$ invariant under shifts. So $\int_L f = \int_{\mathbb{R}^p} f_{\mathrm{top}}\, \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_p$, up to some constant, where f_{top} is the top terms of $f|_L$ in terms of the ξ_i for $i > p$. So the integral is uniquely determined. Existence is easy.

To prove that $\int_L \Delta\alpha = 0$, we will use an explicit formula for Δ in coordinates y^i, ξ_i. Namely, you can easily see that $\Delta(f\, \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n) = \frac{\partial^2 f}{\partial y^i \partial \xi_i} \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$. In particular, $\Delta(f\, \mathrm{d}y)$ does not have a top term, and so $\int_L \Delta f = 0$.

This is of course a local computation, but as usual a local computation immediately determines a global result. To define the integration globally we have to normalize the local definition. We can assume locally that our semidensity α is supported on a Darboux coordinate patch U with coordinates $x^i, x_i^\dagger, \theta^j, \theta_j^\dagger$ where: $1 \leq i \leq p$ and $1 \leq j \leq q$; x^i and $\theta_j^\dagger$ are even coordinates; $\omega = \sum_i \mathrm{d}x^i \wedge \mathrm{d}x_i^\dagger + \sum_j \mathrm{d}\theta^j \wedge \mathrm{d}\theta_j^\dagger$; and the Lagrangian L is given by the equations $\theta_j^\dagger = x_i^\dagger = 0$. Then there exists a unique function $f_\alpha \in \mathcal{O}(U)$ such that $\alpha \sim f_\alpha \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p \wedge \mathrm{d}\theta_1^\dagger \wedge \cdots \wedge \mathrm{d}\theta_q^\dagger$. We normalize $\int_L \alpha$ by:

$$\int_L \alpha = \int_{\mathbb{R}^p} f_{\mathrm{top}}(x)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p$$

Here f_{top} is the coefficient of $\theta^1 \wedge \cdots \wedge \theta^q$ in the coordinate expression of f_α.

Then it is easy to check that this definition satisfies conditions 1. and 2. and does not depend on the Darboux coordinate system. For example, let V be a usual n-dimensional vector space, and set $L = \Pi V$. In this case $W = \Pi \mathrm{T}^* L = V^* \times \Pi V$ and we can identify $\Gamma(W, \mathcal{B}_W)$ with the space $\Omega^\bullet(V^*, \bigwedge V^*)$ of $\bigwedge^\bullet V^*$-valued differential forms on V^*. Then $\int_L \omega = \omega_{\mathrm{top}}(0)$, where $\omega_{\mathrm{top}} \in \Omega^n(V^*, \bigwedge^n V^*)$.

Let L be a supermanifold with an orientation on L_0. Let $W = \Pi \mathrm{T}^* L$. Then of course, L lies as a zero section, as Alan suggests. We define $\mathcal{D}_L = \mathcal{B}_W / I_L \mathcal{B}_W$. This is the "line bundle of densities" on L. By 2., $\int_L$ defines a functional on $\int_L : \Gamma_{\mathrm{cpt}}(L, \mathcal{D}_L) \to \mathbb{R}$. It is easy to see that 1. implies $\int_L \mathcal{L}_\xi \kappa = 0$ for all $\kappa \in \Gamma_{\mathrm{cpt}}(L, \mathcal{D}_L)$ and $\xi \in \mathrm{Vect}(L)$. Because for any vector field ξ on L, then we have $\xi = \xi_h$ for the function h on $\Pi \mathrm{T}^* L$ linear in the fibers that's defined by h.

So this is exactly what you want from integration. You want a notion of "densities", with an action of vector fields, and you want an integration which is equivariant for the vector fields, meaning that it vanishes on total derivatives.

Ed: What do you mean by "$\mathcal{B}_W / I_L \mathcal{B}_W$"? It is restriction of $\mathcal{B}_W$ to L? **David:** It depends on what you think a "line bundle" is. If you think geometrically, then this is restriction. If you mean "invertible sheaf", then you mean quotient by vanishing ideal.

There is a thing which is completely not developed. What is a singular super algebraic variety? What is integration there?

In any case, sheaf $\mathcal{B}_W$ is defined everywhere. And any Lagrangian submanifold defines a functional on $\Gamma_{\mathrm{cpt}}(W, \mathcal{B}_W)$.

Now let me explain one serious problem with integration. Let $L = \mathbb{R}^{1|2}$. So we have coordinates x_1, ξ_1, ξ_2. Then $\mathcal{D}_L = \mathcal{O}(L)$, because we have chosen coordinates and so we can trivialize. So $\Gamma(L, \mathcal{D}_L)_0 = \mathcal{O}(L)_0 = \{f(x) + g(x)\xi_1\xi_2\}$. So now you can define $\int_L (f(x) + g(x)\,\xi_1\xi_2) = \int_{\mathbb{R}} g(x)\,\mathrm{d}x$. This is condition $f(x), g(x) \in \mathscr{C}^\infty_{\mathrm{cpt}}(\mathbb{R})$.

Now consider $L = (0,1) \times \mathbb{R}^{0|2}$. We can define $\int_L$ on $\alpha = f(x) + \xi_1\xi_2\, g(x)$, where $f(x), g(x)$ are any smooth function on $[0,1]$. This gives a well-defined functional, but it is NOT invariant under the transformation $(x, \xi_1, \xi_2) \mapsto (x + x\xi_1\xi_2, \xi_1, \xi_2)$ if f, g do not have compact support. So you have well-defined integration for functions with compact support. But the notion of integration for functions without compact support is not well-defined.

See, what is our manifold? Our manifold is $(0,1)\times$supermanifold which you don't see. You should think that there are imaginary walls at 0 and 1 in the super direction. And when you make transformation, you shift the walls, so you get a different integration.

This is why it's hard to define integration on singular manifold. You cannot just remove the singularity and hope that the integral converges. If you have compact support away from the singularity, it's fine. But the moment you hit the boundary, it's a problem.

8 BV equation

Let $L \subseteq M$ be an oriented Lagrangian submanifold. (By which I mean to imply that M is symplectic.) Let $\alpha \in \Gamma_{\text{cpt}}(M, \mathcal{B}_M)$, $h \in \mathcal{O}(L)_1$ an odd function. Then ξ_h is an even vector field on M. So ξ_h defines a flow $g_t : M \to M$ such that $\frac{\partial g_t}{\partial t} = \xi_h$. Let $L_t = g(t)L$.

Lemma: $\frac{\partial}{\partial t} \int_{L_t} \alpha \big|_{t=0} = \int_L h\, \Delta\alpha$.

Corollary: If $\Delta\alpha = 0$, then $\int_{L_t} \alpha$ does not depend on t. Of course, you can take vector fields depending on t — that part doesn't matter.

So this is a kind of basic part of BV story. If you want to integrate over a Lagrangian, you can change the Lagrangian.

Proof: By the definition, $\frac{\partial}{\partial t} \int_{L_t} \alpha \big|_{t=0} = \int_L \mathcal{L}_{\xi_h}\alpha$, but $\mathcal{L}_{\xi_h}\alpha = \Delta(h\alpha) - (-1)^{|h|} h\, \Delta\alpha = \Delta(h\alpha) + h\, \Delta\alpha$, since $h \in \mathcal{O}(M)_1$. But $\int_L \Delta(\cdot) = 0$. So $\frac{\partial}{\partial t} \int_{L_t} \alpha \big|_{t=0} = \int_L h\, \Delta\alpha$.

So this really is the essence of BV method.

Now, in physics, which I don't know what is about, people integrate expressions of the form $\int_L e^{S/\hbar}\alpha$, where $S \in \mathcal{O}(M)$ and $\alpha \in \Gamma(M, \mathcal{B}_M)$. You can also put α depending on $\hbar$. So you want to know: when is $\Delta(e^{S/\hbar}\alpha) = 0$?

Fix an invertible α. Then we can define $\Delta_\alpha : \mathcal{O}(M) \to \mathcal{O}(M)$ by $\Delta_\alpha(f) = \alpha^{-1}\Delta(\alpha f)$. Then, ignoring dependence of h, we want to know when is $\Delta_\alpha(e^{S/\hbar}) = 0$?

Then from formulas that we have, it's easy to see that $\Delta_\alpha(fg) = \Delta_\alpha f\, g + (-1)^{|f|} f\, \Delta_\alpha g + (-1)^{|f|+1}\{f, g\}$.

So the leading term in $\Delta_\alpha(e^{S/\hbar})$ is equal to $\{S, S\}/\hbar^2$. The equation $\Delta_\alpha(e^{S/\hbar}) = 0$ is called the *quantum BV equation*. The equation $\{S, S\} = 0$ is the *classical BV equation*.

Friday, January 13, morning talk

9 How to integrate

Now we'd like to go back and discuss integration, and generalize a little bit. Let $G \curvearrowright X$ be a usual G-manifold, and $S_0 \in \mathcal{O}(X)^G$ and ν an invariant top-form. Suppose also that X is oriented.

We want to study $\int_X e^{S_0}\nu$. Assume that G is split (not compact — e.g. $\mathbb{R}^n$ or $\mathrm{GL}(\mathbb{R})$). Then $\int_X e^{S_0}\nu = \infty$, and we really want to study $\int_{X/G} e^{S_0}\nu$. Faddeev–Popov suggests to study $0^{\dim G}\int_X e^{S_0}\nu$. This is of course not defined — it is $0 \times \infty$ — but this is what we want to compute, and it has a chance. Remark: if G is compact, then $\int_X e^{S_0}\nu < \infty$, and to $0^{\dim G}\int_X e^{S_0}\nu = 0$. So Faddeev–Popov does not work when G is compact.

What does it mean to compute this? Let $\mathfrak{g}$ be the Lie algebra of G and $\tilde{X} = X \times \Pi\mathfrak{g}$. We can pull S_0 and ν back to $\tilde{X}$. Suppose we choose coordinates $\theta_1, \dots, \theta_m$ for $\Pi\mathfrak{g}$. Then if it is defined, the Fubini theorem says:

$$\int_{\tilde{X}} e^{S_0}(\nu \times \mathrm{d}\theta_1 \wedge \cdots \wedge \mathrm{d}\theta_m) = \int_X e^{S_0}\nu = \int_X e^{S_0}\nu \times \int \mathrm{d}\theta_1 \times \cdots \times \int \mathrm{d}\theta_m = \int_X e^{S_0}\nu \times 0^{\dim \mathfrak{g}}$$

because $\int_{\mathbb{R}} \mathrm{d}\theta = 0$.

A little more generally, let's fix a volume form $\mu^* \in \bigwedge^{\top}\mathfrak{g}$ on $\mathfrak{g}^*$. This defines a semidensity $\theta = \tau(\mu^*)$ on $\Pi\mathrm{T}^*(\Pi\mathfrak{g}) = \mathfrak{g}^* \times \Pi\mathfrak{g}$, where τ is the inclusion of volume forms on $\mathfrak{g}^*$ into semidensities on $\mathfrak{g}^* \times \Pi\mathfrak{g} = \Pi\mathrm{T}^*(\mathfrak{g}^*)$. In the above, μ^* is the Lebesgue measure for the choice of coordinates.

So, the strategy for defining the integral (first attempt):

1. Find a function S on $\tilde{W} = \Pi\mathrm{T}^*\tilde{X}$, and a semidensity α on $\tilde{W}$ such that $\Delta(e^S\alpha) = 0$. And $S|_{\tilde{X}} = S_0$ and $\alpha|_{\tilde{X}} = \nu\theta$.
2. Find a hamiltonian deformation $\tilde{Y} \subset \tilde{W}$ of $\tilde{X}$ such that $\int_{\tilde{Y}} e^{\tilde{S}}\alpha$ makes sense.

This will be a solution, because we know that integral does not change under hamiltonian deformation. By definition, $\int_{\tilde{X}} e^{S_0}(\nu\theta) = \int_{\tilde{X}} e^S\alpha$.

Vera: What is deformation? **David:** Deformation of Lagrangian. How will you do this? You will find graph of d of a function on the base. Ours will need to be an odd function.

Alan: Is it correct to think of $0^{\dim G}$ as $1/\operatorname{Vol}(G)$? **David:** What do you mean "volume of G"? What it is $0^{\dim G} = 1/\operatorname{Vol}(\mathfrak{g})$.

Theo: Last time, we said that we only have invariance for compactly-supported integrals. **David:** This is where $S \rightsquigarrow S/\hbar$ will be important. Then as $\hbar \to 0$, the support will be near points.

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10 Step 1: defining S

We write $S = S_0 + S_1$, and $S_1 = S_1^{\mathfrak{g}} + S_1^{\text{act}}$. We want $S_1 \in \mathcal{O}(\Pi\mathrm{T}^*(X \times \Pi\mathfrak{g}))$. We will set $S_1^{\mathfrak{g}} \in \mathcal{O}(\Pi\mathfrak{g}) = [,]$. What does this mean? $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, or in other words $[,] \in \bigwedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$, and $\bigwedge^2 \mathfrak{g}^* \subseteq \mathcal{O}(\Pi\mathfrak{g})$ and $\mathfrak{g} \subseteq \mathcal{O}(\mathfrak{g}^*)$. Because $\mathcal{O}(\Pi\mathrm{T}^*\Pi\mathfrak{g}) = \mathcal{O}(\Pi\mathfrak{g} \times \mathfrak{g}^*) = \mathcal{O}(\Pi\mathfrak{g}) \otimes \mathcal{O}(\mathfrak{g}^*)$, we can consider $[,]$ as a function on $\Pi\mathrm{T}^*\tilde{X}$.

To describe S_1^{act}, we note that we have a map $\mathfrak{g} \to \mathrm{Vect}(X)$. It other words, we have $\phi : \Pi\mathfrak{g} \to \mathcal{O}(\Pi\mathrm{T}^*X)$, because we think of vector fields as linear functions on T^*. So we understand the map $\mathfrak{g} \to \mathrm{Vect}(X)$ as $\phi \in \mathcal{O}(\Pi\mathrm{T}^*X \times \Pi\mathfrak{g})$. But $\mathcal{O}(\Pi\mathrm{T}^*X \times \Pi\mathfrak{g}) \subseteq \mathcal{O}(\Pi\mathrm{T}^*(X \times \Pi\mathfrak{g}))$.

So $S_1 = S_1^{\mathfrak{g}} + \phi$, and $S = S_0 + S_1$. Also set $\alpha = \nu \wedge \theta$. Then:

1. $(S, S) = 0$, where $(,)$ is the Poisson bracket on $\Pi\mathrm{T}^*\tilde{X}$. This is the classical master equation.
2. $\Delta(e^{S/\hbar}\alpha) = 0$ if ν is preserved by action of G and G is unimodular. Actually, one can compensate the other: failure of measure to be preserved by G can be equal to determinant of adjoint action.

Kolya: Is it true that this will be equivalent to $\Delta(S) = 0$? **David:** Most probably. But let me say it slightly differently. $W \cong \Pi\mathrm{T}^*(X \times \mathfrak{g}^*)$, by the lemma from the beginning of today. So we have a map $\Omega^{\text{top}}(X \times \mathfrak{g}^*) \to$ semidensities. So what we actually have is a form $\nu \times \mu$, where μ is our choice of Lebesgue measure on $\mathfrak{g}^*$. So what we really need is that the action of G on $X \times \mathfrak{g}^*$ (coadjoint in the second variable) should preserve the measure. We set $\alpha \in \mathcal{B}(W)$ the image of $\nu \times \mu$. This is the correct way to say it.

11 Step 2: deforming the Lagrangian

Now we make an additional assumption. This is needed to tell the difference between an S^1 action and an $\mathbb{R}$ action. We assume that there exists a map $\kappa : X \to V$ such that for any (generic) $v \in V$, $\kappa^{-1}(v)$ is a section of $g : X \to X/\mathfrak{g}$. Essentially this means that fiber of this map cuts every fiber at a point. So this rules out $G = S^1$, because every fiber over S^1 has at least two points. This encodes analytically the assumption that group acts freely.

Let $\mathrm{d}v$ be a Lebesgue measure on V. Pick $\delta : V \to \mathbb{R}$ a function such that $\int_V \delta \,\mathrm{d}v = 1$. Consider $Y = V^* \times \Pi V^*$ and denote by β the semidensity on $Z = \Pi\mathrm{T}^*Y = \Pi\mathrm{T}^*(V \times V^*)$ corresponding to the top form $e^{2\pi i \langle v, v^* \rangle}\delta(v)\,\mathrm{d}v\,\mathrm{d}v^*$ on $V \times V^* = Z_0$. It is easy to see that $\int_Y \beta = 1$.

So instead of studying $\int_{\tilde{X}} e^{S/\hbar}\alpha$, we can study

$$\int_{\tilde{X} \times \mathfrak{g}^* \subset W = \tilde{W} \times Z} e^{S/\hbar}\alpha \boxtimes \beta.$$

Now consider the (odd) function $\tilde{\kappa}$ on $\tilde{X} \times Y = X \times \Pi\mathfrak{g} \times V^* \times \Pi V^*$, given by $\tilde{\kappa}(x, \xi, v^*, \zeta) = \langle \kappa(x), \zeta \rangle$.

Now the claim is, let $L_\kappa \subseteq W$ be the Lagrangian corresponding to $\tilde{\kappa}$. Then

$$\int_{L_\kappa} e^{S/\hbar} \alpha \boxtimes \beta = \int_X e^{S_0(x)/\hbar} \det(x)\, \delta(\kappa(x))\, \nu$$

What is this determinant? For any point $x \in X$, we have a map $\hat{\kappa}_x : G \to \mathfrak{g}^*$ which is $g \mapsto \kappa(g \cdot x)$. Then the determinant is $\det(x) = \det(\mathrm{d}\hat{\kappa}_x)|_e$, because $\mathrm{d}\hat{\kappa}_x|_e : \mathfrak{g} \to \mathfrak{g}^*$, and we have fixed a volume form on $\mathfrak{g}$. And the right-hand side is the usual change of variables to define $\int_{X/G} e^{S_0(x)/\hbar} \overline{\mathrm{d}x}$.

How did we get the above formula? We have $L_\kappa \to X$, and we integrated out the fiber.

Now, why do you need this formula? In order to study integral, you study asymptotics. And to do this, you look near critical point, and from the RHS point of view you must study very complicated critical subvariety. But from LHS point of view, we just have an isolated critical point on supermanifold L_κ. Of course, any critical point on L_κ lives over a critical point on X, but you have more refined information, because the function on L_κ does not come just from the base.

Alan: I don't understand where this map κ comes from? Suppose that $X = G$ with the left action? **David:** Let's do it. Set $G = \mathrm{GL}(n)$, then $\mathfrak{g}^* = \mathfrak{gl}(n)$, and we have a map from invertible matrices to matrices! Generically it is a birational map. **Ed:** What is this Lagrangian L_κ? **David:** In usual matrices $f : X \to \mathbb{R}$, then $L_f \subseteq \mathrm{T}^*X$ is the Lagrangian $\{x, \mathrm{d}f(x)\}$. **Ed:** What is the status of this formula? It is a defintion? **David:** No. Both sides are separately well-defined, and they are equal.

You see, you take X, and you have two maps. You have map $X \to X/G$ the projection. And you have $\kappa : X \to \mathfrak{g}^*$. Now you take a fiber of second map, say $\kappa^{-1}(0)$ (if 0 is generic point). Then you have $\kappa^{-1}(0) \subseteq X$, and you have measure on X and measure on $\mathfrak{g}$ and so you have a measure on the fiber. And you also have a measure on X/G. And the composition $\kappa^{-1}(0) \to X \to X/G$ is bijection. So you have two measures, and they are not equal. But their ratio is this determinant, which we can describe.

Now you can see what happens in the bad case. Let's consider that $G = S^1$. You ask for some map $\kappa : X \to \mathbb{R} = \mathrm{Lie}(S^1)$. Then $\kappa(\text{orbit})$ is a map from the circle to the real line. So image is an interval, and inverse image of any point is two points. Each of these points contributes to the integral, but they exactly cancel.

Alan: There are two conditions to have a map κ. First is the existence of section of $X \to X/G$. Second is that group can be embedded into Euclidean space of same dimension. This is certainly true for the classical groups. **David:** Yes. And actually you only need birational maps, because we are only interested in computing volumes.

Let's consider simplest possible case. $X = \mathbb{R}$ and $G = \mathbb{R}$. You want to integrate $\int_{\mathbb{R}} \mathrm{d}x$. But this is ∞. So your next guess is $\int_{\mathbb{R}\times\mathbb{R}^{0|1}} \mathrm{d}x \times \mathrm{d}\theta$. Then this is $\infty \times 0 = 0$. So what you do is you embed $\mathbb{R}^{1|1}$ as Lagrangian in symplectic space. And there, the "$0 \times \infty$" is not absolutely convergent. But the integral does converge along deformed Lagrangian.

Ed: And this depends on κ. Because of boundary terms. **David:** I didn't try to make it precise. You can fix boundary data, and consider the deformations that have compact support. There is a good class of deformations, and the answer doesn't depend on the choice because they are related by the arguments from last time. There should be a theorem to this effect — I didn't try to write it down.

Kolya: Do you really need that κ^{-1} of any point is a cross section? **David:** No. You can take your δ to have support one those points for which κ^{-1} is a cross section. But when $G = S^1$, you don't have any such points.

So, what happened. We had $\tilde{W}$. And that had enough room to solve master equation, but not enough room to deform the contour. And then you extend to $W = \tilde{W} \times \Pi\mathrm{T}^*(\text{something})$, and now you have enough room to deform the contour.

Friday, January 13, afternoon talk

12 A generalization

Now we give an example of a generalization. Suppose you have a vector space X and a vector space V, and a map $\phi : V \to \mathrm{Vect}(X)$. Then ϕ defines a distribution $\mathcal{V}$ on X, which we suppose to have constant rank — indeed, we assume that for all $x \in X$, the map $V \to \mathrm{T}_x X$ is injective. Suppose $S_0 : X \to \mathbb{R}$ is a function such that $\phi(v)(S_0) = 0$ for all $v \in V$. This is an analogue of being invariant under the action of a group. Then as before you can define $\tilde{X} = X \times \Pi V$ and a function $S = S_0 + S_1$ on $\tilde{W} = \Pi\mathrm{T}^*\tilde{X}$, as before, then:

- $\{S, S\} = 0$ iff the distribution $\mathcal{V}$ is integrable.

Theo: This is the same formula as before? How do I define S_1? I know how to do this in the integrable case. Is the statement that it is impossible to choose an appropriate S_1 in the nonintegrable case? **David:**

Let's consider an example. Let $p : V \to X$ be a usual vector bundle (different V from above). Pick a metric on fibers of V, and $S_0 : V \to \mathbb{R}$ the square length, and let ∇ be an orthogonal connection on V, which is not flat. Let $E = p^*\mathrm{T}X$ a bundle over V. I want to define a function S on $\Pi\mathrm{T}^*(\Pi E)$. It will be $S = S_0 + S_1 + S_2$. S_0 comes from the projection $E \to V$.

S_1 comes from a vector field on ΠE. Any vector field on the manifold gives you a function on the cotangent bundle. A vector field on ΠE is the same as a differentiation of $\mathcal{O}(\Pi E) = \mathcal{O}(X, \mathrm{Sym}(V^*)) \otimes_{\mathcal{O}(X)} \Omega^\bullet(X) = \Omega^\bullet(X, \mathrm{Sym}\, V^*)$. The connection ∇ defines a derivation $\tilde{\nabla}$ on $\Omega^\bullet(X, V)$ such that $\tilde{\nabla}^2 = \wedge F_\nabla$, and therefore a derivation of $\Omega^\bullet(X, \mathrm{Sym}\, V^*)$. Finally, S_1 is the function on $\Pi\mathrm{T}^*(\Pi E)$ corresponding to this.

Easy to see that $\{S_0 + S_1, S_0 + S_1\} = 0$ iff $F_\nabla = 0$. This is related to above as follows: the connection determines a distribution, which is flat only if curvature F is 0.

Now, $F_\nabla \in \Omega^2(X, \mathrm{End}(V))$. Using the metric S_0, we can consider F_∇ as an element of $\Omega^2(X, \bigwedge^2 V)$. In other words, we can consider F_∇ as a function on $\Pi\mathrm{T}^*(\Pi E)$ quadratic along fibers. Let $S_2 =$ this function.

Claim: Let $S = S_0 + S_1 + S_2$. Then $\{S, S\} = 0$.

So this is a non-integrable situation where you can still construct solution to classical master equation with different terms. This is what happens, for example, in the Poisson sigma model.

Alan: This is the Bianchi identity? **David:** Yes, of course.

We know that $\Pi\mathrm{T}^*(\Pi E) = \Pi\mathrm{T}^*(E^*)$. So we have an isomorphism $\Omega^{\mathrm{top}}(E^*) \hookrightarrow$ BV-harmonic semidensities on $\Pi\mathrm{T}^*(\Pi E)$.

Now, in $\Omega^{\mathrm{top}}(E^*)$ we have canonical element, from the metric on V, because when you look at tangent bundle to E^*, the tangent and cotangent bundles on X each appear and all that's left is some V part. Said another way: cotangent bundle has canonical measure, and in addition you have something along V, but you have a metric there. Call by α the BV-harmonic semidensity corresponding to this element.

Consider $e^{-S/\hbar}\alpha \in \mathcal{B}(\Pi\mathrm{T}^*(\Pi E)) = \Omega^\bullet V$. Call by β the corresponding element of $\Omega^\bullet V$, and let $\bar\beta$ the component of β of degree $= \dim V$. Claim:

1. $\mathrm{d}\bar\beta = 0$
2. $\bar\beta \to 0$ rapidly at infinity in V — assume X is compact. So $\bar\beta$ defines a class $[\bar\beta] \in \mathrm{H}^\bullet_{\mathrm{cpt}}(V)$.
3. The restriction of $[\bar\beta]$ on fibers of $V \to X$ gives the fundamental class. In other words, this is a canonical Thom isomorphism.

Conjecture: $\bar\beta =$ Mathai–Quillen form.

Mathai–Quillen showed how for any vector bundle with metric you can canonically write, using superconnections, a differential form which represents the Euler class.

13 On solution of classical BV equation

Let S_0 be a polynomial function on $X = \mathbb{R}^n$. Let $\tau \subseteq \Gamma(X, \mathrm{T}X)$ be the vector fields λ such that $\lambda(S_0) = 0$. It is a kind of abstract formulation of the problem: you have a function, some vector fields, ... You want to write a solution to classical equation which resolves it.

Set $\mathcal{O} = \mathcal{O}(X)$. We have a natural map $\delta : \Gamma(X, \mathrm{T}X) \to \mathcal{O}(X)$ given by $\delta(\lambda) = \lambda(S_0)$, and $\tau = \ker\delta$. We want to work with $J = \mathcal{O}/\operatorname{Im}\delta$.

We consider a Tate resolution of J. What is this? It is a differential graded commutative ring, nonpositively graded, $R = \bigoplus_{n\leq 0} R_n$ with differential $\delta : R_n \to R_{n+1}$ such that

1. R is a free $\mathcal{O}$-algebra

2. $\mathrm{H}^\bullet(R,\delta) = \mathrm{H}^0(R,\delta) = J$.

It is easy to construct this.

Harold: Do you require that $R_0 = \mathcal{O}$? **David:** You can add that requirement. It's not particularly relevant.

In other words, there exists a non-positively graded vector space $V = \bigoplus_{n\leq 0} V$ such that $R = \mathrm{Sym}(V)$.

We can define $W = V \times V^*[-1]$. So V lives in negative degrees, and $V^*[-1]$ lives in positive degrees. And can we define a ring $\mathcal{O}(W)$ and its completion $\bar{R}$ along V. So W is odd (i.e. shifted) cotangent bundle of V, and $\bar{R}$ is the formal completion of $\mathcal{O}(W)$ along the subspace V.

Harold: I'm confused by the completion? **David:** Consider ideal of elements having higher and higher degree. Take completion for this ideal.

Then one can check that the natural Poisson structure on $\mathcal{O}(W)$ extends to a Poisson structure on $\bar{R}$. The theorem is the following:

1. There exists $S \in \bar{R}$ such that $\{S,S\} = 0$ and $S|_V = S_0$.
2. Any two such S differ by a gauge transformation (i.e. an explicit symplectic transformation of W).

Kolya: What does it mean "explicit"? **David:** The group generated by these has explicit, fairly simple structure.

This is a relatively straightforward theorem: you see the obstruction is zero, ...

Given S, we can define an operator $\check{S}_R : a \mapsto \{S,a\}$. It depends on the resolution R. Since the symplectic structure is odd and $\{S,S\} = 0$, we have $\check{S}^2 = 0$, and we can define $\mathcal{P}_R(S_0) = \mathrm{H}^*(\check{S}_R)$. Then $\mathcal{P}_R(S_0)$ has a natural structure of a Poisson algebra. **Kolya:** From where comes the Poisson structure? **David:** If you have any kind of Poisson algebra, and you take cohomology with respect to a differential that is a derivation of both the bracket and the mutliplication, then you get Poisson algebra.

The nontrivial problem is of well-definedness of this Poisson algebra. The moment you choose a resolution, then $\check{S}_R$ is defined up to gauge, and you can check that gauge transformations don't change the Poisson algebra.

The main theorem: $\mathcal{P}_R(S_0)$ does not depend on R.

What does "does not depend" mean? There can be two meanings. Either: when you change R, you get something isomorphic. But ours is stronger: when you change R, you get a canonical isomorphism. So we have a well-defined $\mathcal{P}(S_0)$.

Now we have a conjecture, which is about how to describe this Poisson algebra. To formulate the conjecture, I need to remind you something about Lie algebroids, also called "Lie-Rinehart algebra". A *Lie algebroid* is a pair $(A,\mathfrak{g})$, where A is a commutative algebra, $\mathfrak{g}$ is an A-module and a Lie algebra acting on A by derivations, such that $[\tau, f\sigma] = \tau(f)\,\sigma + f\,[\tau,\sigma]$ and $(f\tau)(g) = f\,\tau(g)$. For example, $A = \mathscr{C}^\infty(M)$ and $\mathfrak{g} = \mathrm{Vect}(M)$. In this case, we can define the Chevalley–Eilenberg complex $\Omega^\bullet(A,\mathfrak{g}) = \mathrm{Hom}_A(\bigwedge_A^\bullet \mathfrak{g}, A)$ with differential. It's the de Rham complex in the example.

Given S_0, we define $L = \Gamma(X,\tau) =$ vector fields λ such that $\lambda(S_0) = 0$. And we define $L_0 \subseteq L$ by $L_0 = \{\xi(S_0)\,\eta - \eta(S_0)\,\xi\}$. Claim (completely obvious): the pair $(J, L/L_0)$ has the natural structure of a Lie algebroid.

Conjecture: $\mathcal{P}(S_0)$ is the Chevalley–Eilenberg cohomology of the Lie algebroid $(J, L/L_0)$.

We have proven this for $i = 0, 1$.

Now, there are two parts to this conjecture. By the construction, there is a Poisson structure of degree 1 on $\mathcal{P}(S_0)$. So we can define, in principle, a pairing $\mathrm{H}^0_{\mathrm{CE}} \times \mathrm{H}^0_{\mathrm{CE}} \to \mathrm{H}^1_{\mathrm{CE}}$. Now you have two problems. One, proving the isomorphism. But two, if such an isomorphism exists, then it means that CE complex has a Poisson structure. And we only know how to define it in lowest term. There should be a general a priori description.