Lie Groups and Lie Algebras

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List of Theorems

Introduction

In the Fall Semester, 2008, Prof. Mark Haiman taught Math 261A: Lie Groups, at the University of California Berkeley. The course covered the structure of Lie groups, Lie algebras, and their (complex) representations. The textbooks were [2] and [11]. I was one of many students in that class, and typed detailed notes [8], including all the motivation, discussion, questions, errors, and personal confusions and commentary. What you're reading right now is a first attempt to make those notes more presentable. It is also a study aid for my qualifying exam. As such, we present only the definitions, theorems, and proofs, with little motivation. I have made limited rearrangements of the material. Each subsection corresponds to one or two one-hour lectures. Needless to say, the pedagogy (and, since I was taking dictation, many of the words), are due to M. Haiman. In particular, I have quoted almost verbatim the problem sets M. Haiman assigned in the class (the reader may find my answers to some of the exercises in the appendices of [8]). Of course, any and all errors are mine.

In addition to [2, 11], the reader might be interested in getting a sense of previous renditions of UC Berkeley's Math 261. In 2006 a three-professor tag-team taught a one-semester Lie Groups and Lie Algebras course; detailed notes are available [18], and occasionally I have referenced those notes, especially when I was absent or lost, or when my notes are otherwise lacking. They go quickly through the material — about twice as fast as we did — eschewing most proofs. For a very different version of the course, the reader may be interested in the year-long Lie Groups and Lie Algebras course taught in 2001 [12].

These notes are typeset using TEXShop Pro on a MacBook running OS 10.5; the backend is pdfLATEX. Pictures and diagrams are drawn using PGF/Ti*k*Z. For a full list of packages used, you may peruse the LATEX source code for this document, available at `http://math.berkeley.edu/~theojf/LieGroupsBook.tex`.

In addition to Mark Haiman, the people without whom I would not have been able to put together this text were Alex Fink, Dustin Cartwright, Manuel Reyes, and the other participants in the class.

0.1 Notation

We will generally use $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ for categories; named categories are written in small-caps, so that for example A-MOD is the category of (finite-dimensional) A-modules. Objects in a category are generally denoted $A, B, C, \dots$, with the exception that for Lie algebras we use fraktur letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$. The classical Lie groups we refer to with roman letters ($\mathrm{GL}(n, \mathbb{C})$, etc.), and we write $\mathrm{M}(n)$ for the algebra of $n \times n$ matrices. Famous fields and rings are in black-board-bold: $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}$, and we use $\mathbb{K}$ rather than k for a generic field. The *natural numbers* $\mathbb{N}$ is always the set of *non-negative* integers, so that it is a *rig*, or "ring with identities but without negations".

In a category with products, we use $\{\mathrm{pt}\}$ for the terminal object, and $\times$ for the monoidal structure; a general monoidal category is written with $\otimes$ for the product. We do not include associators and other higher-categorical things.

For morphisms in a category we use lower-case Greek and Roman letters $\alpha, \beta, a, b, c, \dots$. An *element* of an object A in monoidal category is a morphism $a : \{\mathrm{pt}\} \to A$, or with $\{\mathrm{pt}\}$ replaced by the monoidal unit; we will write this as "$a \in A$" following the usual convention. When the category is concrete with products, this agrees with the set-theoretic meaning. The identity map on any object $A \in \mathcal{A}$ we write as 1_A. We always write the identity matrix as 1, or 1_n for the $n \times n$ identity matrix when we need to specify its size. Similarly, 0 and 0_n refer to the zero matrix.

If M is a manifold (we will never need more general geometric spaces), we will write $\mathscr{C}(M)$ for the continuous, smooth, analytic, or holomorphic functions on it, depending on what is natural for the given space. Thus if M is a real manifold, we will always use the symbol $\mathscr{C}$ for the sheaf of infinitely-differentiable or analytic functions on it, depending on whether the ambient category is that of infinitely-differentiable manifolds or analytic manifolds. When working over the complex numbers, $\mathscr{C}$ may refer to the sheaf of complex analytic functions or of holomorphic functions. Moreover, the word "smooth" may mean any of "infinitely-differentiable", "analytic", or "holomorphic", depending on the choice of ambient category. When a statement does not hold in this generality, we will specify. We write $\mathrm{T}M$ for the tangent bundle of M, and $\mathrm{T}_p M$ for the fiber over the point p.

Chapter 1

Motivation: Closed Linear Groups

1.1 Definition of a Lie Group

[8, Lecture 1]

1.1.1 Group objects

1.1 Definition *Let $\mathcal{C}$ be a category with finite products; denote the terminal object by* $\{\mathrm{pt}\}$. *A* group object *in $\mathcal{C}$ is an object G along with maps $\mu : G \times G \to G$, $i : G \to G$, and $e : \{\mathrm{pt}\} \to G$, such that the following diagrams commute:*

$$\begin{tikzcd} G \times G \times G \arrow[r, "1_G \times \mu"] \arrow[d, "\mu \times 1_G"] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G \end{tikzcd} \tag{1.1.1}$$

$$\begin{tikzcd} & G \times G \arrow[d, "\mu"] & \\ \{\mathrm{pt}\} \times G \arrow[ur, "e \times 1_G"] \arrow[r, "\sim"'] & G & G \times \{\mathrm{pt}\} \arrow[ul, "1_G \times e"'] \arrow[l, "\sim"] \end{tikzcd} \tag{1.1.2}$$

$$\begin{tikzcd} & G \times G \arrow[r, "1_G \times i"] & G \times G \arrow[dr, "\mu"] & \\ G \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr] & & \{\mathrm{pt}\} \arrow[r, "e"] & G \\ & G \times G \arrow[r, "i \times 1_G"'] & G \times G \arrow[ur, "\mu"'] & \end{tikzcd} \tag{1.1.3}$$

In equation 1.1.2, the isomorphisms are the canonical ones. In equation 1.1.3, the map $G \to \{\mathrm{pt}\}$ *is the unique map to the terminal object, and* $\Delta : G \to G \times G$ *is the canonical diagonal map.*

If (G, μ_G, e_G, i_G) *and* (H, μ_H, e_H, i_H) *are two group objects, a map* $f : G \to H$ *is a* group object homomorphism *if the following commute:*

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "f\times f"] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"] & H \end{tikzcd} \qquad \begin{tikzcd} & G \arrow[dd, "f"] \\ \{\mathrm{pt}\} \arrow[ur, "e_G"] \arrow[dr, "e_H"'] & \\ & H \end{tikzcd} \tag{1.1.4}$$

(That f *intertwines* i_G *with* i_H *is then a corollary.)*

1.2 Definition *A* (left) group action *of a group object* G *in a category* $\mathcal{C}$ *with finite products is a map* $\rho : G \times X \to X$ *such that the following diagrams commute:*

$$\begin{tikzcd} G \times G \times X \arrow[r, "1_G\times\rho"] \arrow[d, "\mu\times 1_X"] & G \times X \arrow[d, "\rho"] \\ G \times X \arrow[r, "\rho"] & X \end{tikzcd} \tag{1.1.5}$$

$$\begin{tikzcd} & G \times X \arrow[d, "\rho"] \\ \{\mathrm{pt}\} \times X \arrow[ur, "e\times 1_X"] \arrow[r, "\sim"'] & X \end{tikzcd} \tag{1.1.6}$$

(The diagram corresponding to equation 1.1.3 is then a corollary.) A right action *is a map* $X \times G \to X$ *with similar diagrams. We denote a left group action* $\rho : G \times X \to X$ *by* $\rho : G \curvearrowright X$*.*

Let $\rho_X : G \times X \to X$ *and* $\rho_Y : G \times Y \to Y$ *be two group actions. A map* $f : X \to Y$ *is* G-equivariant *if the following square commutes:*

$$\begin{tikzcd} G \times X \arrow[r, "\rho_X"] \arrow[d, "1_G\times f"] & X \arrow[d, "f"] \\ G \times Y \arrow[r, "\rho_Y"] & Y \end{tikzcd} \tag{1.1.7}$$

1.1.2 Analytic and Algebraic Groups

1.3 Definition *A* Lie group *is a group object in a category of manifolds. In particular, a Lie group can be infinitely differentiable (in the category* $\mathscr{C}^\infty$-Man*) or* analytic *(in the category* $\mathscr{C}^\omega$-Man*)* *when over* $\mathbb{R}$*, or* complex analytic *or* almost complex *when over* $\mathbb{C}$*. We will take "Lie group" to mean* analytic Lie group *over either* $\mathbb{C}$ *or* $\mathbb{R}$*. In fact, the different notions of real Lie group coincide, a fact that we will not directly prove, as do the different notions of complex Lie group. As always, we will use the word "smooth" for any of "infinitely differentiable", "analytic", or "holomorphic".*

A Lie action *is a group action in the category of manifolds.*

A (linear) algebraic group *over* $\mathbb{K}$ *(algebraically closed) is a group object in the category of (affine) algebraic varieties over* $\mathbb{K}$*.*

1.4 Example The *general linear group* $\mathrm{GL}(n,\mathbb{K})$ of $n\times n$ invertible matrices is a Lie group over $\mathbb{K}$ for $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. When $\mathbb{K}$ is algebraically closed, $\mathrm{GL}(n,\mathbb{K})$ is an algebraic group. It acts algebraically on $\mathbb{K}^n$ and on projective space $\mathbb{P}(\mathbb{K}^n)=\mathbb{P}^{n-1}(\mathbb{K})$.

1.2 Definition of a Closed Linear Group

[8, Lectures 2 and 3]

We write $\mathrm{GL}(n,\mathbb{K})$ for the group of $n\times n$ invertible matrices over $\mathbb{K}$, and $\mathrm{M}(n,\mathbb{K})$ for the algebra of all $n\times n$ matrices. We regularly leave off the $\mathbb{K}$.

1.5 Definition *A* closed linear group *is a subgroup of* $\mathrm{GL}(n)$ *(over* $\mathbb{C}$ *or* $\mathbb{R}$*) that is closed as a topological subspace.*

1.2.1 Lie algebra of a closed linear group

1.6 Lemma / Definition *The following describe the same function* $\exp:\mathrm{M}(n)\to\mathrm{GL}(n)$*, called the* matrix exponential*.*

1. $\exp(a)\overset{\text{def}}{=}\sum_{n\geq 0}\frac{a^n}{n!}$.
2. $\exp(a)\overset{\text{def}}{=}e^{ta}\Big|_{t=1}$*, where for fixed* $a\in\mathrm{M}(n)$ *we define* e^{ta} *as the solution to the initial value problem* $e^{0a}=1$*,* $\frac{d}{dt}e^{ta}=ae^ta$*.*
3. $\exp(a)\overset{\text{def}}{=}\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n$.

If $ab-ba=0$*, then* $\exp(a+b)=\exp(a)+\exp(b)$*.*

The function $\exp:\mathrm{M}(n)\to\mathrm{GL}(n)$ *is a local isomorphism of analytic manifolds. In a neighborhood of* $1\in\mathrm{GL}(n)$*, the function* $\log a\overset{\text{def}}{=}-\sum_{n>0}\frac{(1-a)^n}{n}$ *is an inverse to* $\exp$*.*

1.7 Lemma / Definition *Let* H *be a closed linear group. The* Lie algebra of H *is the set*

$$\mathrm{Lie}(H)=\{x\in\mathrm{M}(n):\exp(\mathbb{R}x)\subseteq H\}\tag{1.2.1}$$

1. $\mathrm{Lie}(H)$ *is a* $\mathbb{R}$*-subspace of* $\mathrm{M}(n)$*.*

2. Lie(H) *is closed under the* bracket $[,] : (a,b) \mapsto ab - ba$.

1.8 Definition *A* Lie algebra *over* $\mathbb{K}$ *is a vector space* $\mathfrak{g}$ *along with an antisymmetric map* $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ *satisfying the* Jacobi identity*:*

$$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0 \tag{1.2.2}$$

A homomorphism of Lie algebras *is a linear map preserving the bracket. A* Lie subalgebra *is a vector subspace closed under the bracket.*

1.9 Example The algebra $\mathfrak{gl}(n) = \mathrm{M}(n)$ of $n \times n$ matrices is a Lie algebra with $[a,b] = ab - ba$. It is $\mathrm{Lie}(\mathrm{GL}(n))$. Lemma/Definition 1.7 says that $\mathrm{Lie}(H)$ is a Lie subalgebra of $\mathrm{M}(n)$.

1.2.2 Some analysis

1.10 Lemma *Let* $\mathrm{M}(n) = V \oplus W$ *as a real vector space. Then there exists an open neighborhood* $U \ni 0$ *in* $\mathrm{M}(n)$ *and an open neighborhood* $U' \ni 1$ *in* $\mathrm{GL}(n)$ *such that* $(v,w) \mapsto \exp(v)\exp(w) : V \oplus W \to \mathrm{GL}(n)$ *is a homeomorphism* $U \to U'$.

1.11 Lemma *Let* H *be a closed subgroup of* $\mathrm{GL}(n)$*, and* $W \subseteq \mathrm{M}(n)$ *be a linear subspace such that* 0 *is a limit point of the set* $\{w \in W \text{ s.t. } \exp(w) \in H\}$*. Then* $W \cap \mathrm{Lie}(H) \neq 0$.

Proof Fix a Euclidian norm on W. Let $w_1, w_2, \dots \to 0$ be a sequence in $\{w \in W \text{ s.t. } \exp(w) \in H\}$, with $w_i \neq 0$. Then $w_i/|w_i|$ are on the unit sphere, which is compact, so passing to a subsequence, we can assume that $w_i/|w_i| \to x$ where x is a unit vector. The norms $|w_i|$ are tending to 0, so $w_i/|w_i|$ is a large multiple of w_i. We approximate this: let $n_i = \lceil 1/|w_i| \rceil$, whence $n_i w_i \approx w_i/|w_i|$, and $n_i w_i \to x$. But $\exp w_i \in H$, so $\exp(n_i w_i) \in H$, and H is a closed subgroup, so $\exp x \in H$.

Repeating the argument with a ball of radius r to conclude that $\exp(rx)$ is in H, we conclude that $x \in \mathrm{Lie}(H)$. □

1.12 Proposition *Let* H *be a closed subgroup of* $\mathrm{GL}(n)$*. There exist neighborhoods* $0 \in U \subseteq \mathrm{M}(n)$ *and* $1 \in U' \subseteq \mathrm{GL}(n)$ *such that* $\exp : U \xrightarrow{\sim} U'$ *takes* $\mathrm{Lie}(H) \cap U \xrightarrow{\sim} H \cap U'$.

Proof We fix a complement $W \subseteq \mathrm{M}(n)$ such that $\mathrm{M}(n) = \mathrm{Lie}(H) \oplus W$. By Lemma 1.11, we can find a neighborhood $V \subseteq W$ of 0 such that $\{v \in V \text{ s.t. } \exp(v) \in H\} = \{0\}$. Then on $\mathrm{Lie}(H) \times V$, the map $(x,w) \mapsto \exp(x)\exp(w)$ lands in H if and only if $w = 0$. By restricting the first component to lie in an open neighborhood, we can approximate $\exp(x+w) \approx \exp(x)\exp(w)$ as well as we need to — there's a change of coordinates that completes the proof. □

1.13 Corollary H *is a submanifold of* $\mathrm{GL}(n)$ *of dimension equal to the dimension of* $\mathrm{Lie}(H)$.

1.14 Corollary $\exp(\mathrm{Lie}(H))$ *generates the identity component* H_0 *of* H.

1.15 Remark In any topological group, the connected component of the identity is normal.

1.16 Corollary $\mathrm{Lie}(H)$ *is the tangent space* $\mathrm{T}_1 H \overset{\text{def}}{=} \{\gamma'(0) \text{ s.t. } \gamma : \mathbb{R} \to H, \gamma(0) = 1\} \subseteq \mathrm{M}(n)$.

1.3 Classical Lie groups

[8, Lectures 4 and 5]

We mention only the classical compact semisimple Lie groups and the classical complex semisimple Lie groups. There are other very interesting classical Lie groups, c.f. [14].

1.3.1 Classical Compact Lie groups

1.17 Lemma / Definition *The* quaternians $\mathbb{H}$ *is the unital* $\mathbb{R}$*-algebra generated by* i, j, k *with the multiplication* $i^2 = j^2 = k^2 = ijk = -1$*; it is a non-commutative division ring. Then* $\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}$*, and* $\mathbb{H}$ *is a subalgebra of* $\mathrm{M}(4, \mathbb{R})$*. We defined the* complex conjugate *linearly by* $\bar{i} = -i$*,* $\bar{j} = -j$*, and* $\bar{k} = -k$*; complex conjugation is an anti-automorphism, and the fixed-point set is* $\mathbb{R}$*. The* Euclidean norm *of* $\zeta \in \mathbb{H}$ *is given by* $\|\zeta\| = \bar{\zeta}\zeta$*.*

The Euclidean norm *of a column vector* $x \in \mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$ *is given by* $\|x\|^2 = \bar{x}^T x$*, where* $\bar{x}$ *is the component-wise complex conjugation of* x*.*

If $x \in \mathrm{M}(n, \mathbb{R}), \mathrm{M}(n, \mathbb{C}), \mathrm{M}(n, \mathbb{H})$ *is a matrix, we define its* Hermetian conjugate *to be the matrix* $x^* = \bar{x}^T$*; Hermetian conjugation is an antiautomorphism of algebras* $\mathrm{M}(n) \to \mathrm{M}(n)$*.* $\mathrm{M}(n, \mathbb{H}) \hookrightarrow \mathrm{M}(2n, \mathbb{C})$ *is a* $*$*-embedding.*

Let $j = \left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right) \in \mathrm{M}(2, \mathrm{M}(n, \mathbb{C})) = \mathrm{M}(2n, \mathbb{C})$ *be a block matrix. We define* $\mathrm{GL}(n, H) \stackrel{\text{def}}{=} \{x \in \mathrm{GL}(2n, \mathbb{C}) \text{ s.t. } jx = \bar{x}j\}$*. It is a closed linear group.*

1.18 Lemma / Definition *The following are closed linear groups, and are compact:*

- *The* (real) special orthogonal group $\mathrm{SO}(n, \mathbb{R}) \stackrel{\text{def}}{=} \{x \in \mathrm{M}(n, \mathbb{R}) \text{ s.t. } x^*x = 1 \text{ and } \det x = 1\}$.
- *The* (real) orthogonal group $\mathrm{O}(n, \mathbb{R}) \stackrel{\text{def}}{=} \{x \in \mathrm{M}(n, \mathbb{R}) \text{ s.t. } x^*x = 1\}$.
- *The* special unitary group $\mathrm{SU}(n) \stackrel{\text{def}}{=} \{x \in \mathrm{M}(n, \mathbb{C}) \text{ s.t. } x^*x = 1 \text{ and } \det x = 1\}$.
- *The* unitary group $\mathrm{U}(n) \stackrel{\text{def}}{=} \{x \in \mathrm{M}(n, \mathbb{C}) \text{ s.t. } x^*x = 1\}$.
- *The* (real) symplectic group $\mathrm{Sp}(n, \mathbb{R}) \stackrel{\text{def}}{=} \{x \in \mathrm{M}(n, \mathbb{C}) \text{ s.t. } x^*x = 1\}$.

There is no natural quaternionic determinant.

1.3.2 Classical Complex Lie groups

The following groups make sense over any field, but it's best to work over an algebraically closed field. We work over $\mathbb{C}$.

1.19 Lemma / Definition *The following are closed linear groups over* $\mathbb{C}$*, and are algebraic:*

- *The* (complex) special linear group $\mathrm{SL}(n, \mathbb{C}) \stackrel{\text{def}}{=} \{x \in \mathrm{GL}(n, \mathbb{C}) \text{ s.t. } \det x = 1\}$.
- *The* (complex) special orthogonal group $\mathrm{SO}(n, \mathbb{C}) \stackrel{\text{def}}{=} \{x \in \mathrm{SL}(n, \mathbb{C}) \text{ s.t. } x^T x = 1\}$.
- *The* (complex) symplectic group $\mathrm{Sp}(n, \mathbb{C}) \stackrel{\text{def}}{=} \{x \in \mathrm{GL}(2n, \mathbb{C}) \text{ s.t. } x^T jx = j\}$.

1.3.3 The Classical groups

In full, we have defined the following "classical" closed linear groups:

	Group Name	Group Description	Algebra Name	Algebra Description	$\dim_{\mathbb{R}}$
Compact	$\mathrm{SO}(n,\mathbb{R})$	$\{x \in \mathrm{M}(n,\mathbb{R}) \text{ s.t. } x^*x = 1, \det x = 1\}$	$\mathfrak{so}(n,\mathbb{R})$	$\{x \in \mathrm{M}(n,\mathbb{R}) \text{ s.t. } x^* + x = 0\}$	$\binom{n}{2}$
	$\mathrm{SU}(n)$	$\{x \in \mathrm{M}(n,\mathbb{C}) \text{ s.t. } x^*x = 1, \det x = 1\}$	$\mathfrak{su}(n)$	$\{x \in \mathrm{M}(n,\mathbb{C}) \text{ s.t. } x^* + x = 0, \operatorname{tr} x = 0\}$	$n^2 - 1$
	$\mathrm{Sp}(n,\mathbb{R})$	$\{x \in \mathrm{M}(n,\mathbb{H}) \text{ s.t. } x^*x = 1\}$	$\mathfrak{sp}(n,\mathbb{R})$	$\{x \in \mathrm{M}(n,\mathbb{H}) \text{ s.t. } x^* + x = 0\}$	$2n^2 + n$
	$\mathrm{GL}(n,\mathbb{H})$	$\{x \in \mathrm{GL}(2n,\mathbb{C}) \text{ s.t. } jx = \bar{x}j\}$	$\mathfrak{gl}(n,\mathbb{H})$	$\{x \in M_{2n}(\mathbb{C}) \text{ s.t. } jx = \bar{x}j\}$	$4n^2$
Complex	$\mathrm{SL}(n,\mathbb{C})$	$\{x \in \mathrm{M}(n,\mathbb{C}) \text{ s.t. } \det x = 1\}$	$\mathfrak{sl}(n,\mathbb{C})$	$\{x \in \mathrm{M}(n,\mathbb{C}) \text{ s.t. } \operatorname{tr} x = 0\}$	$2(n^2 - 1)$
	$\mathrm{SO}(n,\mathbb{C})$	$\{x \in \mathrm{M}(n,\mathbb{C}) \text{ s.t. } x^Tx = 1, \det x = 1\}$	$\mathfrak{so}(n,\mathbb{C})$	$\{x \in \mathrm{M}(n,\mathbb{C}) \text{ s.t. } x^T + x = 0\}$	$n(n-1)$
	$\mathrm{Sp}(n,\mathbb{C})$	$\{x \in \mathrm{M}(n,\mathbb{C}) \text{ s.t. } x^Tjx = j\}$	$\mathfrak{sp}(n,\mathbb{C})$	$\{x \in \mathrm{M}(n,\mathbb{C}) \text{ s.t. } x^Tj + jx = 0\}$	$2\binom{2n+1}{2}$

1.20 Proposition *Via the natural embedding $\mathrm{M}(n,\mathbb{H}) \hookrightarrow \mathrm{M}(2n,\mathbb{C})$, we have:*

$$\mathrm{Sp}(n) = \mathrm{GL}(n,\mathbb{H}) \cap U(2n) \tag{1.3.1}$$
$$= \mathrm{GL}(n,\mathbb{H}) \cap \mathrm{Sp}(n,\mathbb{C}) \tag{1.3.2}$$
$$= U(2n) \cap \mathrm{Sp}(n,\mathbb{C}) \tag{1.3.3}$$

1.4 Homomorphisms of closed linear groups

1.21 Definition *Let H be a closed linear group. The* adjoint action *$H \curvearrowright H$ is given by by ${}^g h \overset{\text{def}}{=} ghg^{-1}$, and this action fixes $1 \in H$. This induces the* adjoint action *$\mathrm{Ad} : H \curvearrowright \mathrm{T}_1 H = \mathrm{Lie}(H)$. It is given by $g \cdot y = gyg^{-1}$, where now $y \in \mathrm{Lie}(H)$.*

1.22 Lemma *Let H and G be closed linear groups and $\phi : H \to G$ a smooth homomorphism. Then $\phi(1) = 1$, so $d\phi : \mathrm{T}_1 H \to \mathrm{T}_1 G$ by $X \mapsto (\phi(1 + tX))'(0)$. The diagram of actions commutes:*

$$\begin{tikzcd} H \arrow[r, phantom, "\curvearrowright"] \arrow[d, "\phi"] & \mathrm{T}_1 H \arrow[d, "d\phi"] \\ G \arrow[r, phantom, "\curvearrowright"] & \mathrm{T}_1 G \end{tikzcd}$$

This is to say:

$$d\phi(\mathrm{Ad}(h)Y) = \mathrm{Ad}(\phi(h))\, d\phi(Y) \tag{1.4.1}$$

Thus $d\phi\,[X,Y] = [d\phi\, X, d\phi\, Y]$, so $d\phi$ is a Lie algebra homomorphism.

If H is connected, the map $d\phi$ determines the map ϕ.

Exercises

1. (a) Show that the orthogonal groups $\mathrm{O}(n, \mathbb{R})$ and $\mathrm{O}(n, \mathbb{C})$ have two connected components, the identity component being the special orthogonal group SO_n, and the other consisting of orthogonal matrices of determinant -1.

   (b) Show that the center of $\mathrm{O}(n)$ is $\{\pm I_n\}$.

   (c) Show that if n is odd, then $\mathrm{SO}(n)$ has trivial center and $\mathrm{O}(n) \cong \mathrm{SO}(n) \times (\mathbb{Z}/2\mathbb{Z})$ as a Lie group.

   (d) Show that if n is even, then the center of $\mathrm{SO}(n)$ has two elements, and $\mathrm{O}(n)$ is a semidirect product $(\mathbb{Z}/2\mathbb{Z}) \ltimes \mathrm{SO}(n)$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{SO}(n)$ by a non-trivial outer automorphism of order 2.

2. Construct a smooth group homomorphism $\Phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ which induces an isomorphism of Lie algebras and identifies $\mathrm{SO}(3)$ with the quotient of $\mathrm{SU}(2)$ by its center $\{\pm I\}$.

3. Construct an isomorphism of $\mathrm{GL}(n, \mathbb{C})$ (as a Lie group and an algebraic group) with a closed subgroup of $\mathrm{SL}(n+1, \mathbb{C})$.

4. Show that the map $\mathbb{C}^* \times \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ given by $(z, g) \mapsto zg$ is a surjective homomorphism of Lie and algebraic groups, find its kernel, and describe the corresponding homomorphism of Lie algebras.

5. Find the Lie algebra of the group $U \subseteq \mathrm{GL}(n, \mathbb{C})$ of upper-triangular matrices with 1 on the diagonal. Show that for this group, the exponential map is a diffeomorphism of the Lie algebra onto the group.

6. A *real form* of a complex Lie algebra $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{g}_\mathbb{R}$ such that that $\mathfrak{g} = \mathfrak{g}_\mathbb{R} \oplus i\mathfrak{g}_\mathbb{R}$, or equivalently, such that the canonical map $\mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \to \mathfrak{g}$ given by scalar multiplication is an isomorphism. A real form of a (connected) complex closed linear group G is a (connected) closed real subgroup $G_\mathbb{R}$ such that $\mathrm{Lie}(G_\mathbb{R})$ is a real form of $\mathrm{Lie}(G)$.

   (a) Show that $\mathrm{U}(n)$ is a compact real form of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SU}(n)$ is a compact real form of $\mathrm{SL}(n, \mathbb{C})$.

   (b) Show that $\mathrm{SO}(n, \mathbb{R})$ is a compact real form of $\mathrm{SO}(n, \mathbb{C})$.

   (c) Show that $\mathrm{Sp}(n, \mathbb{R})$ is a compact real form of $\mathrm{Sp}(n, \mathbb{C})$.

Chapter 2

Mini-course in Differential Geometry

2.1 Manifolds

[8, Lectures 6, 7, and 8]

2.1.1 Classical definition

2.1 Definition *Let X be a (Hausdorff) topological space. A* chart *consists of the data $U \underset{open}{\subseteq} X$ and a homeomorphism $\phi : U \xrightarrow{\sim} V \underset{open}{\subseteq} \mathbb{R}^n$. $\mathbb{R}^n$ has* coordinates *x_i, and $\xi_i \overset{\text{def}}{=} x_i \circ \phi$ are* local coordinates *on the chart. Charts (U, ϕ) and (U', ϕ') are* compatible *if on $U \cap U'$ the ξ'_i are smooth functions of the ξ_i and conversely. I.e.:*

$$
\begin{tikzcd}
U \arrow[d, "\phi"] & & & & & U \cap U' \arrow[dll, "\bar{\phi}"'] \arrow[drr, "\bar{\phi}'"] & & & & & U' \arrow[d, "\phi'"] \\
V & V & \supseteq & W \arrow[rrrr, "\bar{\phi}' \circ \bar{\phi}^{-1}", "\text{smooth with smooth inverse}"'] & & & & W' & \subseteq & V' & V'
\end{tikzcd}
\tag{2.1.1}
$$

An atlas *on X is a covering by pairwise compatible charts.*

2.2 Lemma *If U and U' are compatible with all charts of $\mathcal{A}$, then they are compatible with each other.*

2.3 Corollary *Every atlas has a unique maximal extension.*

2.4 Definition *A* manifold *is a Hausdorff topological space with a maximal atlas. It can be* real, infinitely-differentiable, complex, analytic, *etc., by varying the word "smooth" in the compatibility condition equation 2.1.1.*

2.5 Definition *Let U be an open subset of a manifold X. A function $f : U \to \mathbb{R}$ is* smooth *if it is smooth on local coordinates in all charts.*

2.1.2 Sheafs

2.6 Definition *A* sheaf of functions $\mathscr{S}$ *on a topological space* X *assigns a ring* $\mathscr{S}(U)$ *to each open set* $U \underset{\text{open}}{\subseteq} X$ *such that:*

1. *if* $V \subseteq U$ *and* $f \in \mathscr{S}(U)$, *then* $f|_V \in \mathscr{S}(V)$, *and*
2. *if* $U = \bigcup_\alpha U_\alpha$ *and* $f : U \to \mathbb{R}$ *such that* $f|_{U_\alpha} \in \mathscr{S}(U_\alpha)$ *for each* α, *then* $f \in \mathscr{S}(U)$.

The stalk *of a sheaf at* $x \in X$ *is the space* $\mathcal{S}_x \stackrel{\text{def}}{=} \lim_{U \ni x} \mathscr{S}(U)$.

2.7 Lemma *Let* X *be a manifold, and assign to each* $U \underset{\text{open}}{\subseteq} X$ *the ring* $\mathscr{C}(U)$ *of smooth functions on* U. *Then* $\mathscr{C}$ *is a sheaf. Conversely, a topological space* X *with a sheaf of functions* $\mathscr{S}$ *is a manifold if and only if there exists a covering of* X *by open sets* U *such that* $(U, \mathscr{S}|_U)$ *is isomorphic as a space with a sheaf of functions to* $(V, \mathscr{S}^{\mathbb{R}^n}|_V)$ *for some* $V \subseteq \mathbb{R}^n$ *open.*

2.1.3 Manifold constructions

2.8 Definition *If* X *and* Y *are smooth manifolds, then a* smooth map $f : X \to Y$ *is a continuous map such that for all* $U \subseteq Y$ *and* $g \in \mathscr{C}(U)$, *then* $g \circ f \in \mathscr{C}(f^{-1}(U))$. *Manifolds form a category* Man *with products: a* product of manifolds $X \times Y$ *is a manifold with charts* $U \times V$.

2.9 Definition *Let* M *be a manifold,* $p \in M$ *a point, and* $\gamma_1, \gamma_2 : \mathbb{R} \to M$ *two paths with* $\gamma_1(0) = \gamma_2(0) = p$. *We say that* γ_1 *and* γ_2 *are* tangent at p *if* $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ *for all smooth* f *on a nbhd of* p, *i.e. for all* $f \in \mathscr{C}_p$. *Each equivalence class of tangent curves is called a* tangent vector.

2.10 Definition *Let* M *be a manifold and* $\mathscr{C}$ *its sheaf of smooth functions. A* point derivation *is a linear map* $\delta : \mathscr{C}_p \to \mathbb{R}$ *satisfying the* Leibniz rule*:*

$$\delta(fg) = \delta f \, g(p) + f(p) \, \delta g \tag{2.1.2}$$

2.11 Lemma *Any tangent vector* γ *gives a point derivation* $\delta_\gamma : f \mapsto (f \circ \gamma)'(0)$. *Conversely, every point derivation is of this form.*

2.12 Lemma / Definition *Let* M *and* N *be manifolds, and* $f : M \to N$ *a smooth map sending* $p \mapsto q$. *The following are equivalent, and define* $(df)_p : \mathrm{T}_p M \to \mathrm{T}_q N$, *the* differential *of* f *at* p*:*

1. *If* $[\gamma] \in \mathrm{T}_p M$ *is represented by the curve* γ, *then* $(df)_p(X) \stackrel{\text{def}}{=} [f \circ \gamma]$.
2. *If* $X \in \mathrm{T}_p M$ *is a point-derivation on* $\mathcal{S}_{M,p}$, *then* $(df)_p(X) : \mathcal{S}_{N,q} \to \mathbb{R}$ *(or* $\mathbb{C}$*) is defined by* $\psi \mapsto X[\psi \circ f]$.
3. *In coordinates,* $p \in U \underset{\text{open}}{\subseteq} \mathbb{R}^m$ *and* $q \in W \underset{\text{open}}{\subseteq} \mathbb{R}^n$, *then locally* f *is given by* $f_1, \dots, f_n$ *smooth functions of* $x_1, \dots, x_m$. *The tangent spaces to* $\mathbb{R}^n$ *are in canonical bijection with* $\mathbb{R}^n$, *and a linear map* $\mathbb{R}^m \to \mathbb{R}^n$ *should be presented as a matrix:*

$$\text{Jacobian}(f, x) \stackrel{\text{def}}{=} \frac{\partial f_i}{\partial x_j} \tag{2.1.3}$$

2.13 Lemma *We have the* chain rule*: if $M \xrightarrow{f} N \xrightarrow{g} K$, then $d(g \circ f)_p = (dg)_{f(p)} \circ (df)_p$.*

2.14 Theorem (Inverse Mapping Theorem) *1. Given smooth $f_1, \ldots, f_n : U \to \mathbb{R}$ where $p \in U \underset{open}{\subseteq} \mathbb{R}^n$, then $f : U \to \mathbb{R}^n$ maps some neighborhood $V \ni p$ bijectively to $W \underset{open}{\subseteq} \mathbb{R}^n$ with s/a/h inverse iff $\det \mathrm{Jacobian}(f, x) \neq 0$.*

2. A smooth map $f : M \to N$ of manifold restricts to an isomorphism $p \in U \to W$ for some neighborhood U if and only if $(df)_p$ is a linear isomorphism.

2.1.4 Submanifolds

2.15 Proposition *Let M be a manifold and N a topological subspace with the induced topology such that for each $p \in N$, there is a chart $U \ni p$ in M with coordinates $\{\xi_i\}_{i=1}^m : U \to \mathbb{R}^m$ such that $U \cap N = \{q \in U \text{ s.t. } \xi_{n+1}(q) = \cdots = \xi_m(q) = 0\}$. Then $U \cap N$ is a chart on N with coordinates $\xi_1, \ldots, \xi_n$, and N is a manifold with an atlas given by $\{U \cap N\}$ as U ranges over an atlas of M. The sheaf of smooth functions $\mathscr{C}_N$ is the sheaf of continuous functions on N that are locally restrictions of smooth functions on M. The embedding $N \hookrightarrow M$ is smooth, and satisfies the universal property that any smooth map $f : Z \to M$ such that $f(Z) \subseteq N$ defines a smooth map $Z \to N$.*

2.16 Definition *The map $N \hookrightarrow M$ in Proposition 2.15 is an* immersed submanifold. *A map $Z \to M$ is an* immersion *if it factors as $Z \xrightarrow{\sim} N \hookrightarrow M$ for some immersed submanifold $N \hookrightarrow M$.*

2.17 Proposition *If $N \hookrightarrow M$ is an immersed submanifold, then N is locally closed.*

2.18 Proposition *Any closed linear group $H \subseteq \mathrm{GL}(n)$ is an immersed analytic submanifold. If $\mathrm{Lie}(H)$ is a $\mathbb{C}$-subspace of $\mathrm{M}(n, \mathbb{C})$, then H is a holomorphic submanifold.*

Proof The following diagram defines a chart near $1 \in H$, which can be moved by left-multiplication wherever it is needed:

$$\begin{tikzcd} & 0 \arrow[d, phantom, "\Cap"] & 1 \arrow[d, phantom, "\Cap"] & \\ \mathrm{M}(n) \supseteq & U \arrow[r, shift left, "\exp"] & V \arrow[l, shift left, "\log"] & \subseteq \mathrm{GL}(n) \\ & \mathrm{Lie}(H) \cap U \arrow[u, hook] \arrow[r, shift left] & H \cap V \arrow[u, hook] \arrow[l, shift left] & \end{tikzcd} \tag{2.1.4}$$

□

2.19 Lemma *Given $\mathrm{T}_pM = V_1 \oplus V_2$, there is an open neighborhood $U_1 \times U_2$ of p such that $V_i = \mathrm{T}_pU_i$.*

2.20 Lemma *If $s : N \to M \times N$ is a s/a/h section, then s is a (closed) immersion.*

2.21 Proposition *A smooth map $f : N \to M$ is an immersion on a neighborhood of $p \in N$ if and only if $(df)_p$ is injective.*

2.2 Vector Fields

[8, Lectures 8, 9, and 10]

2.2.1 Definition

2.22 Definition *Let M be a manifold. A* vector field *assigns to each $p \in M$ a vector x_p, i.e. a point derivation:*

$$x_p(fg) = f(p)\, x_p(g) + x_p(f)\, g(p) \tag{2.2.1}$$

We define $(xf)(p) \stackrel{\text{def}}{=} x_p(f)$. Then $x(fg) = f\, x(g) + x(f)\, g$, so x is a derivation*. But it might be discontinuous. A vector field x is* smooth *if $x : \mathscr{C}_M \to \mathscr{C}_M$ is a map of sheaves. Equivalently, in local coordinates the components of x_p must depend smoothly on p. By changing (the conditions on) the sheaf $\mathscr{C}$, we may define analytic or holomorphic vector fields.*

Henceforth, the word "vector field" will always mean "smooth (or analytic or holomorphic) vector field". Similarly, we will use the word "smooth" to mean smooth or analytic or holomorphic, depending on our category.

2.23 Lemma *The commutator $[x, y] \stackrel{\text{def}}{=} xy - yx$ of derivations is a derivation.*

Proof An easy calculation:

$$xy(fg) = xy(f)\, g + x(f)\, y(g) + y(f)\, x(g) + f\, xy(g) \tag{2.2.2}$$

Switch X and Y, and subtract:

$$[x, y](fg) = [x, y](f)\, g + f\, [x, y](g) \tag{2.2.3}$$

□

2.24 Definition *A* Lie algebra *is a vector space $\mathfrak{l}$ with a bilinear map $[,] : \mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$ (i.e. a linear map $[,] : \mathfrak{l} \otimes \mathfrak{l} \to \mathfrak{l}$), satisfying*

1. *Antisymmetry:* $[x, y] + [y, x] = 0$
2. *Jacobi:* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

2.25 Proposition *Let V be a vector space. The bracket $[x, y] \stackrel{\text{def}}{=} xy - yx$ makes $\operatorname{End}(V)$ into a Lie algebra.*

2.26 Lemma / Definition *Let $\mathfrak{l}$ be a Lie algebra. The* adjoint action $\operatorname{ad} : \mathfrak{l} \to \operatorname{End}(\mathfrak{l})$ *given by* $\operatorname{ad} x : y \mapsto [x, y]$ *is a* derivation*:*

$$(\operatorname{ad} x)[y, z] = [(\operatorname{ad} x)y, z] + [y, (\operatorname{ad} x)z] \tag{2.2.4}$$

Moreover, $\operatorname{ad} : \mathfrak{l} \to \operatorname{End}(\mathfrak{l})$ *is a* Lie algebra homomorphism*:*

$$\operatorname{ad}([x, y]) = (\operatorname{ad} x)(\operatorname{ad} y) - (\operatorname{ad} y)(\operatorname{ad} x) \tag{2.2.5}$$

2.2.2 Integral Curves

Let ∂_t be the vector field $f \mapsto \frac{d}{dt} f$ on $\mathbb{R}$.

2.27 Proposition *Given a smooth vector field x on M and a point $p \in M$, there exists an open interval $I \underset{open}{\subseteq} \mathbb{R}$ such that $0 \in I$ and a smooth curve $\gamma : I \to M$ satisfying:*

$$\gamma(0) = p \tag{2.2.6}$$

$$(d\gamma)_t(\partial_t) = x_{\gamma(t)} \, \forall t \in I \tag{2.2.7}$$

When M is a complex manifold and x a holomorphic vector field, we can demand that $I \underset{open}{\subseteq} \mathbb{C}$ is an open domain containing 0, and that $\gamma : I \to M$ be holomorphic.

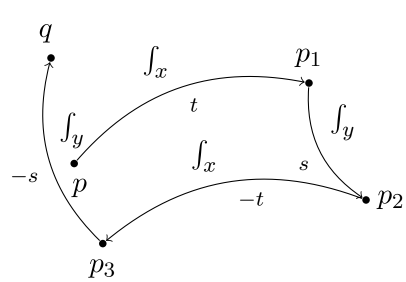
Proof In local coordinates, $\gamma : \mathbb{R} \to \mathbb{R}^n$, and we can use existence and uniqueness theorems for solutions to differential equations; then we need that a smooth (analytic, holomorphic) differential equation has a smooth (analytic, holomorphic) solution.

But there's a subtlety. What if there are two charts, and solutions on each chart, that diverge right where the charts stop overlapping? Well, since M is Hausdorff, if we have two maps $I \to M$, then the locus where they agree is closed, so if they don't agree on all of I, then we can go to the maximal point where they agree and look locally there. $\square$

2.28 Definition *The* integral curve $\int_{x,p}(t)$ *of x at p is the maximal curve satisfying equations 2.2.6 and 2.2.7.*

2.29 Proposition *The integral curve $\int_{x,p}$ depends smoothly on $p \in M$.*

2.30 Proposition *Let x and y be two vector fields on a manifold M. For $p \in M$ and $s, t \in \mathbb{R}$, define q by the following picture:*



Then for any smooth function f, we have $f(q) - f(p) = st[x,y]_p f + O(s,t)^3$.

Proof Let $\alpha(t) = \int_{x,p}(t)$, so that $f(\alpha(t))' = xf(\alpha(t))$. Iterating, we see that $\left(\frac{d}{dt}\right)^n f(\alpha(t)) = x^n f(\alpha(t))$, and by Taylor series expansion,

$$f(\alpha(t)) = \sum \frac{1}{n!} \left(\frac{d}{dt}\right)^n f(\alpha(0)) t^n = \sum \frac{1}{n!} x^n f(p) t^n = e^{tx} f(p). \tag{2.2.8}$$

By varying p, we have:

$$f(q) = \left(e^{-sy} f\right)(p_3) \tag{2.2.9}$$

$$= \left(e^{-tx} e^{-sy} f\right)(p_2) \tag{2.2.10}$$

$$= \left(e^{sy} e^{-tx} e^{-sy} f\right)(p_1) \tag{2.2.11}$$

$$= \left(e^{tx} e^{sy} e^{-tx} e^{-sy} f\right)(p) \tag{2.2.12}$$

We already know that $e^{tx} e^{sy} e^{-tx} e^{-sy} = 1 + st[x, y] + \text{higher terms}$. Therefore $f(q) - f(p) = st[x, y]_p f + O(s,t)^3$. □

2.2.3 Group Actions

2.31 Proposition *Let M be a manifold, G a Lie group, and $G \curvearrowright M$ a Lie group action, i.e. a smooth map $\rho : G \times M \to M$ satisfying equations 1.1.5 and 1.1.6. Let $x \in \mathrm{T}_e G$, where e is the identity element of the group G. The following descriptions of a vector field $\ell x \in \mathrm{Vect}(M)$ are equivalent:*

1. *Let $x = [\gamma]$ be the equivalence class of tangent paths, and let $\gamma : I \to G$ be a representative path. Define $(\ell x)_m = [\tilde{\gamma}]$ where $\tilde{\gamma}(t) \stackrel{\text{def}}{=} \rho\big(\gamma(t)^{-1}, m\big)$. On functions, ℓx acts as:*

$$(\ell x)_m f \stackrel{\text{def}}{=} \left.\frac{d}{dt}\right|_{t=0} f\big(\gamma(t)^{-1} m\big) \tag{2.2.13}$$

2. *Arbitrarily extend x to a vector field $\tilde{x}$ on a neighborhood $U \subseteq G$ of e, and lift this to $\tilde{\tilde{x}}$ on $U \times M$ to point only in the U-direction: $\tilde{\tilde{x}}_{(u,m)} \stackrel{\text{def}}{=} (\tilde{x}_u, 0) \in \mathrm{T}_u U \times \mathrm{T}_m M$. Let ℓx act on functions by:*

$$(\ell x) f \stackrel{\text{def}}{=} -\tilde{\tilde{x}}(f \circ p)\Big|_{\{e\} \times M = M} \tag{2.2.14}$$

3. *$(\ell x)_m \stackrel{\text{def}}{=} -(d\rho)_{(e,m)}(x, 0)$*

2.32 Proposition *Let G be a Lie group, M and N manifolds, and $G \curvearrowright M$, $G \curvearrowright N$ Lie actions, and let $f : M \to N$ be G-equivariant. Given $x \in \mathrm{T}_e G$, define $\ell^M x$ and $\ell^N x$ vector fields on M and N as in Proposition 2.31. Then for each $m \in M$, we have:*

$$(df)_m(\ell^M x) = (\ell^N x)_{f(m)} \tag{2.2.15}$$

2.33 Definition *Let $G \curvearrowright M$ be a Lie action. We define the* adjoint action *of G on* $\mathrm{Vect}(M)$ *by ${}^g y \stackrel{\text{def}}{=} dg(y)_{gm} = (dg)_m(y_m)$. Equivalently, $G \curvearrowright \mathscr{C}_M$ by $g : f \mapsto f \circ g^{-1}$, and given a vector field thought of as a derivation $y : \mathscr{C}_M \to \mathscr{C}_M$, we define ${}^g y \stackrel{\text{def}}{=} g y g^{-1}$.*

2.34 Example Let $G \curvearrowright G$ by right multiplication: $\rho(g, h) \stackrel{\text{def}}{=} h g^{-1}$. Then $G \curvearrowright \mathrm{T}_e G$ by the *adjoint action* $\mathrm{Ad}(g) = d(g - g^{-1})_e$, i.e. if $x = [\gamma]$, then $\mathrm{Ad}(g) x = [g\, \gamma(t)\, g^{-1}]$.

2.35 Definition *Let $\rho : G \curvearrowright M$ be a Lie action. For each $g \in G$, we define ${}^gM$ to be the manifold M with the action ${}^g\rho : (h, m) \mapsto \rho(ghg^{-1}, m)$.*

2.36 Corollary *For each $g \in G$, the map $g : M \to {}^gM$ is G-equivariant. We have:*

$$ {}^g\ell x = dg(\ell x) = \ell^{{}^gM} x = \ell(\mathrm{Ad}(g)x) \tag{2.2.16} $$

2.37 Proposition *Let $\rho : G \curvearrowright G$ by $\rho_g : h \mapsto hg^{-1}$. Then $\ell : \mathrm{T}_eG \to \mathrm{Vect}(G)$ is an isomorphism from T_eG to* left-invariant *vector fields, such that $(\ell x)_e = x$.*

Proof Let $\lambda : G \curvearrowright G$ be the action by left-multiplication: $\lambda_g(h) = gh$. Then for each g, λ_g is ρ-equivariant. Thus $d\lambda_g(\ell x) = {}^{\lambda_g}(\ell x) = \ell x$, so ℓx is left-invariant, and $(\ell x)_e = x$ since $\rho(g, e) = g^{-1}$. Conversely, a left-invariant field is determined by its value at a point:

$$ (\ell x)_g = (d\lambda_g)_e(\ell x_e) = (d\lambda_g)_e(x) \tag{2.2.17} $$

$\square$

2.2.4 Lie algebra of a Lie group

2.38 Lemma / Definition *Let $G \curvearrowright M$ be a Lie action. The subspace of $\mathrm{Vect}(M)$ of G-invariant derivations is a Lie subalgebra of $\mathrm{Vect}(M)$.*

Let G be a Lie group. The Lie algebra of G *is the Lie subalgebra $\mathrm{Lie}(G)$ of $\mathrm{Vect}(G)$ consisting of left-invariant vector fields, i.e. vector fields invariant under the action $\lambda : G \curvearrowright G$ given by $\lambda_g : h \mapsto gh$.*

We identify $\mathrm{Lie}(G) \overset{\text{def}}{=} \mathrm{T}_eG$ as in Proposition 2.37.

2.39 Lemma *Given $G \curvearrowright M$ a Lie action, $x \in \mathrm{Lie}(G)$ represented by $x = [\gamma]$, and $y \in \mathrm{Vect}(M)$, we have:*

$$ \left.\frac{d}{dt}\right|_{t=0} {}^{\gamma(t)}yf = [\ell x, y]f \tag{2.2.18} $$

Proof

$$ \left.\frac{d}{dt}\right|_{t=0} {}^{\gamma(t)}yf\,(p) = \left.\frac{d}{dt}\right|_{t=0} \gamma(t)\,y\,\gamma(t)^{-1}f\,(p) \tag{2.2.19} $$

$$ = \left.\frac{d}{dt}\right|_{t=0} \gamma(t)\,y\,f(\gamma(t)\,p) \tag{2.2.20} $$

$$ = \left.\frac{d}{dt}\right|_{t=0} \gamma(t)\,y\,f(\gamma(0)\,p) + \gamma(0)\ \left.\frac{d}{dt}\right|_{t=0} yf(\gamma(t)p) \tag{2.2.21} $$

$$ = \ell x(yf)(p) + y\ \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)p) \tag{2.2.22} $$

$$ = \ell x(yf)(p) + y(-\ell x\, f)(p) \tag{2.2.23} $$

$$ = [\ell x, y]f\,(p) \tag{2.2.24} $$

$\square$

2.40 Corollary *Let $G \curvearrowright M$ be a Lie action. If $x, y \in \mathrm{Lie}(G)$, where $x = [\gamma]$, then*

$$\ell^M\left(\ell^{\mathrm{Ad}}(-x)\,y\right) = \frac{d}{dt}\bigg|_{t=0} \ell\big(\mathrm{Ad}(\gamma(t))y\big)f = [\ell x, \ell y]\,f \tag{2.2.25}$$

2.41 Lemma *The Lie bracket defined on $\mathrm{Lie}(\mathrm{GL}(n)) = \mathfrak{gl}(n) = \mathrm{T}_e\,\mathrm{GL}(n) = \mathrm{M}(n)$ defined in Lemma/Definition 2.38 is the matrix bracket $[x, y] = xy - yx$.*

Proof We represent $x \in \mathfrak{gl}(n)$ by $[e^{tx}]$. The adjoint action on $\mathrm{GL}(n)$ is given by $\mathrm{Ad}_G(g)\,h = ghg^{-1}$, which is linear in h and fixes e, and so passes immediately to the action $\mathrm{Ad} : \mathrm{GL}(n) \curvearrowright \mathrm{T}_e\,\mathrm{GL}(n)$ given by $\mathrm{Ad}_{\mathfrak{g}}(g)\,y = gyg^{-1}$. Then

$$[x, y] = \frac{d}{dt}\bigg|_{t=0} e^{tx}ye^{-tx} = xy - yx. \tag{2.2.26}$$

□

2.42 Corollary *If H is a closed linear group, then Lemma/Definitions 2.38 and 1.7 agree.*

Exercises

1. (a) Show that the composition of two immersions is an immersion.

   (b) Show that an immersed submanifold $N \subseteq M$ is always a closed submanifold of an open submanifold, but not necessarily an open submanifold of a closed submanifold.

2. Prove that if $f : N \to M$ is a smooth map, then $(df)_p$ is surjective if and only if there are open neighborhoods U of p and V of $f(p)$, and an isomorphism $\psi : V \times W \to U$, such that $f \circ \psi$ is the projection on V.

   In particular, deduce that the fibers of f meet a neighborhood of p in immersed closed submanifolds of that neighborhood.

3. Prove the implicit function theorem: a map (of sets) $f : M \to N$ between manifolds is smooth if and only if its graph is an immersed closed submanifold of $M \times N$.

4. Prove that the curve $y^2 = x^3$ in $\mathbb{R}^2$ is not an immersed submanifold.

5. Let M be a complex holomorphic manifold, p a point of M, X a holomorphic vector field. Show that X has a complex integral curve γ defined on an open neighborhood U of 0 in $\mathbb{C}$, and unique on U if U is connected, which satisfies the usual defining equation but in a complex instead of a real variable t.

   Show that the restriction of γ to $U \cap \mathbb{R}$ is a real integral curve of X, when M is regarded as a real analytic manifold.

6. Let $\mathrm{SL}(2, \mathbb{C})$ act on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations $\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = (az+b)/(cz+d)$. Determine explicitly the vector fields $f(z)\mathfrak{d}_z$ corresponding to the infinitesimal action of the basis elements

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

of $\mathfrak{sl}(2, \mathbb{C})$, and check that you have constructed a Lie algebra homomorphism by computing the commutators of these vector fields.

7. (a) Describe the map $\mathfrak{gl}(n, \mathbb{R}) = \mathrm{Lie}(\mathrm{GL}(n, \mathbb{R})) = \mathrm{M}(n, \mathbb{R}) \to \mathrm{Vect}(\mathbb{R}^n)$ given by the infinitesimal action of $\mathrm{GL}(n, \mathbb{R})$.

   (b) Show that $\mathfrak{so}(n, \mathbb{R})$ is equal to the subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ consisting of elements whose infinitesimal action is a vector field tangential to the unit sphere in $\mathbb{R}^n$.

8. (a) Let X be an analytic vector field on M all of whose integral curves are unbounded (i.e., they are defined on all of $\mathbb{R}$). Show that there exists an analytic action of $R = (\mathbb{R}, +)$ on M such that X is the infinitesimal action of the generator ∂_t of $\mathrm{Lie}(\mathbb{R})$.

   (b) More generally, prove the corresponding result for a family of n commuting vector fields X_i and action of $\mathbb{R}^n$.

9. (a) Show that the matrix $\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$ belongs to the identity component of $\mathrm{GL}(2, \mathbb{R})$ for all positive real numbers a, b.

   (b) Prove that if $a \neq b$, the above matrix is not in the image $\exp(\mathfrak{gl}(2, \mathbb{R}))$ of the exponential map.

Chapter 3

General theory of Lie groups

3.1 From Lie algebra to Lie group

3.1.1 The exponential map

[8, Lecture 10]

We state the following results for Lie groups over $\mathbb{R}$. When working with complex manifolds, we can replace $\mathbb{R}$ by $\mathbb{C}$ throughout, whence the interval $I \underset{\text{open}}{\subseteq} \mathbb{R}$ is replaced by a connected open domain $I \underset{\text{open}}{\subseteq} \mathbb{C}$. As always, the word "smooth" may mean "infinitely differentiable" or "analytic" or ....

3.1 Lemma *Let G be a Lie group and $x \in \mathrm{Lie}(G)$. Then there exists a unique Lie group homomorphism $\gamma_x : \mathbb{R} \to G$ such that $(d\gamma_x)_0(\partial t) = x$. It is given by $\gamma_x(t) = (\int_e \ell x)(t)$.*

Proof Let $\gamma : I \to G$ be the maximal integral curve of ℓx passing through e. Since ℓx is left-invariant, $g\gamma(t)$ is an integral curve through q. Let $g = \gamma(s)$ for $s \in I$; then $\gamma(t)$ and $\gamma(s)\gamma(t)$ are integral curves through $\gamma(s)$, to they must coincide: $\gamma(s+t) = \gamma(s)\gamma(t)$, and $\gamma(-s) = \gamma(s)^{-1}$ for $s \in I \cap (-I)$. So γ is a groupoid homomorphism, and by defining $\gamma(s+t) \overset{\text{def}}{=} \gamma(s)\gamma(t)$ for $s, t \in I$, $s + t \notin I$, we extend γ to $I + I$. Since $\mathbb{R}$ is archimedean, this allows us to extend γ to all of $\mathbb{R}$; it will continue to be an integral curve, so really I must have been $\mathbb{R}$ all along. $\square$

3.2 Corollary *There is a bijection between one-parameter subgroups of G (homomorphisms $\mathbb{R} \to G$) and elements of the Lie algebra of G.*

3.3 Definition *The* exponential map $\exp : \mathrm{Lie}(G) \to G$ *is given by* $\exp x \overset{\text{def}}{=} \gamma_x(1)$, *where γ_x is as in Lemma 3.1.*

3.4 Proposition *Let $x^{(b)}$ be a* smooth family *of vector fields on M parameterized by $b \in B$ a manifold, i.e. the vector field $\tilde{x}$ on $B \times M$ given by $\tilde{x}_{(b,m)} = (0, x_m^{(b)})$ is smooth. Then $(b, p, t) \mapsto \left(\int_p x^{(b)}\right)(t)$ is a smooth map from an open neighborhood of $B \times M \times \{0\}$ in $B \times M \times \mathbb{R}$ to M. When each $x^{(b)}$ has infinite-time solutions, we can take the open neighborhood to be all of $B \times M \times \mathbb{R}$.*

Proof Note that

$$\left(\int_{(b,p)} \tilde{x}\right)(t) = \left(b, \left(\int_p x^{(b)}\right)(t)\right) \tag{3.1.1}$$

So $B \times M \times \mathbb{R} \to B \times M \xrightarrow{\pi} M$ by $(b,p,t) \mapsto \left(\int_{(b,p)} \tilde{x}\right)(t) \mapsto \left(\int_p x^{(b)}\right)(t)$ is a composition of smooth functions, hence is smooth. □

3.5 Theorem (Exponential Map) *For each Lie group G, there is a unique smooth map* $\exp : \mathrm{Lie}(G) \to G$ *such that for $x \in \mathrm{Lie}(G)$, the map $t \mapsto \exp(tx)$ is the integral curve of ℓx through e; $t \mapsto \exp(tx)$ is a Lie group homomorphism $\mathbb{R} \to G$.*

3.6 Example When $G = \mathrm{GL}(n)$, the map $\exp : \mathfrak{gl}(n) \to \mathrm{GL}(n)$ is the matrix exponential.

3.7 Proposition *The differential at the origin $(d\exp)_0$ is the identity map $1_{\mathrm{Lie}(G)}$.*

Proof $d(\exp tx)_0(\partial_t) = x$. □

3.8 Corollary $\exp$ *is a local homeomorphism.*

3.9 Definition *The local inverse of* $\exp : \mathrm{Lie}(G) \to G$ *is called "*$\log$*".*

3.10 Proposition *If G is connected, then* $\exp(\mathrm{Lie}(G))$ *generates G.*

3.11 Proposition *If $\phi : H \to G$ is a group homomorphism, then the following diagram commutes:*

$$\begin{tikzcd} H \arrow[r, "\phi"] & G \\ \mathrm{Lie}(H) \arrow[u, "\exp"] \arrow[r, "(d\phi)_e"] & \mathrm{Lie}(G) \arrow[u, "\exp"'] \end{tikzcd} \tag{3.1.2}$$

If H is connected, then $d\phi$ determines ϕ.

3.1.2 The Fundamental Theorem

[8, Lecture 11]

Like all good algebraists, we assume that Axiom of Choice.

3.12 Theorem (Fundamental Theorem of Lie Groups and Algebras)

1. *The functor $G \mapsto \mathrm{Lie}(G)$ gives an equivalence of categories between the category* scLieGp *of simply-connected Lie groups (over $\mathbb{R}$ or $\mathbb{C}$) and the category* LieAlg *of finite-dimensional Lie algebras (over $\mathbb{R}$ or $\mathbb{C}$).*

2. *"The" inverse functor $\mathfrak{h} \mapsto \mathrm{Grp}(\mathfrak{h})$ is left-adjoint to* $\mathrm{Lie} : $ LieAlg $\to$ LieGp.

We outline the proof. Consider open neighborhood U and V so that the horizontal maps are a homeomorphism:

$$\begin{array}{ccc} \mathrm{Lie}(G) & & G \\ \cup\!\! | & & \cup\!\! | \\ U & \underset{\log}{\overset{\exp}{\rightleftarrows}} & V \\ \Psi & & \Psi \\ 0 & & e \end{array} \tag{3.1.3}$$

Consider the restriction $\mu : G \times G \to G$ to $(V \times V) \cap \mu^{-1}(V) \to V$, and use this to define a "partial group law" $b : \text{open} \to U$, where open $\subseteq U \times U$, via

$$b(x, y) \stackrel{\text{def}}{=} \log(\exp x \exp y) \tag{3.1.4}$$

We will show that the Lie algebra structure of $\mathrm{Lie}(G)$ determines b.

Moreover, given $\mathfrak{h}$ a finite-dimensional Lie algebra, we will need to define b and build $\tilde{H}$ as the group freely generated by U modulo the relations $xy = b(x, y)$ if $x, y, b(x, y) \in U$. We will need to prove that $\tilde{H}$ is a Lie group, with U as an open submanifold.

3.13 Corollary *Every Lie subalgebra $\mathfrak{h}$ of $\mathrm{Lie}(G)$ is $\mathrm{Lie}(H)$ for a unique connected subgroup $H \hookrightarrow G$, up to equivalence.*

The standard proof of Theorem 3.12 is to first prove Corollary 3.13 and then use Theorem 4.99. We will use Theorem 4.89 rather than Theorem 4.99.

3.14 Theorem (Baker-Campbell-Hausdorff Formula (second part only))

1. *Let $\mathcal{T}(x, y)$ be the free tensor algebra generated by x and y, and $\mathcal{T}(x, y)[[s, t]]$ the (non-commutative) ring of formal power series in two commuting variables s and t. Define $b(tx, sy) \stackrel{\text{def}}{=} \log(\exp(tx) \exp(sy)) \in \mathcal{T}(x, y)[[s, t]]$, where $\exp$ and $\log$ are the usual formal power series. Then*

$$b(tx, sy) = tx + sy + st\frac{1}{2}[x, y] + st^2\frac{1}{12}[x, [x, y]] + s^2t\frac{1}{12}[y, [y, x]] + \dots \tag{3.1.5}$$

   has coefficients all coefficients given by Lie bracket polynomials in x and y.

2. *Given a Lie group G, there exists a neighborhood $U' \ni 0$ in $\mathrm{Lie}(G)$ such that $U' \subseteq U \underset{\log}{\overset{\exp}{\rightleftarrows}} V \subseteq G$ and $b(x, y)$ converges on $U' \times U'$ to $\log(\exp x \exp y)$.*

We need more machinery than we have developed so far to prove part 1. We work with analytic manifolds; on $\mathscr{C}$ manifolds, we can make an analogous argument using the language of differential equations.

Proof (of part 2.) For a clearer exposition, we distinguish the maps $\exp : \mathrm{Lie}(G) \to G$ from $e^x \in \mathbb{R}[[x]]$.

We begin with a basic identity. $\exp(tx)$ is an integral curve to ℓx through e, so by left-invariance, $t \mapsto g\exp(tx)$ is the integral curve of ℓx through g. Thus, for f analytic on G,

$$\frac{d}{dt}\big[f(g\exp tx)\big] = \big((\ell x)f\big)(g\exp tx) \tag{3.1.6}$$

We iterate:

$$\left(\frac{d}{dt}\right)^n \big[f(g\exp tx)\big] = \big((\ell x)^n f\big)(g\exp tx) \tag{3.1.7}$$

If f is analytic, then for small t the Taylor series converges:

$$\begin{aligned}
f(g\exp tx) &= \sum_{n=0}^{\infty} \left(\frac{d}{dt}\right)^n \big[f(g\exp tx)\big]\Big|_{t=0} \frac{t^n}{n!} && (3.1.8)\\
&= \sum_{n=0}^{\infty} \big((\ell x)^n f\big)(g\exp tx)\Big|_{t=0} \frac{t^n}{n!} && (3.1.9)\\
&= \sum_{n=0}^{\infty} \big((\ell x)^n f\big)(g)\frac{t^n}{n!} && (3.1.10)\\
&= \sum_{n=0}^{\infty} \left(\frac{(t\,\ell x)^n}{n!} f\right)(g) && (3.1.11)\\
&= \big(e^{t\,\ell x} f\big)(g) && (3.1.12)
\end{aligned}$$

We repeat the trick:

$$f(\exp tx \exp sy) = \big(e^{s\,Ly} f\big)(\exp tx) = \big(e^{t\,\ell x} e^{s\,Ly} f\big)(e) = \big(e^{tx} e^{sy} f\big)(e) \tag{3.1.13}$$

The last equality is because we are evaluating the derivations at e, where $\ell x = x$.

We now let $f = \log : V \to U$, or rather a coordinate of log. Then the left-hand-side is just $\log(\exp tx \exp sy)$, and the right hand side is $\big(e^{tx}e^{sy}\log\big)(e) = \big(e^{b(tx,sy)}\log\big)(e)$, where b is the formal power series from part 1. — we have shown that the right hand side converges. But by interpreting the calculations above as formal power series, and expanding log in Taylor series, we see that the formal power series $\big(e^{b(tx,sy)}\log\big)(e)$ agrees with the formal power series $\log\big(e^{b(tx,sy)}\big) = b(tx, sy)$. This completes the proof of part 2. □

3.2 Universal Enveloping Algebras

3.2.1 The Definition

[8, Lecture 12]

3.15 Definition *A* representation *of a Lie group is a homomorphism* $G \to \mathrm{GL}(n,\mathbb{R})$ *(or* $\mathbb{C}$*). A* representation *of a Lie algebra is a homomorphism* $\mathrm{Lie}(G) \to \mathfrak{gl}(n) = \mathrm{End}(V)$*; the space* $\mathrm{End}(V)$ *is a Lie algebra with the bracket given by* $[x, y] = xy - yx$*.*

3.16 Definition *Let V be a vector space. The* tensor algebra *over V is the free unital non-commuting algebra $\mathcal{T}V$ generated by a basis of V. Equivalently:*

$$\mathcal{T}V \stackrel{\text{def}}{=} \bigoplus_{n\geq 0} V^{\otimes n} \tag{3.2.1}$$

The multiplication is given by $\otimes : V^{\otimes k} \times V^{\otimes l} \to V^{\otimes (k+l)}$. $\mathcal{T}$ is a functor, and is left-adjoint to Forget : $\textsc{Alg} \to \textsc{Vect}$.

3.17 Definition *Let $\mathfrak{g}$ be a Lie algebra. The* universal enveloping algebra *is*

$$\mathcal{U}\mathfrak{g} \stackrel{\text{def}}{=} \mathcal{T}\mathfrak{g}/\big\langle [x,y] - (xy - yx)\big\rangle \tag{3.2.2}$$

$\mathcal{U} : \textsc{LieAlg} \to \textsc{Alg}$ *is a functor, and is left-adjoint to* Forget : $\textsc{Alg} \to \textsc{LieAlg}$.

3.18 Corollary *The category of $\mathfrak{g}$-modules is equal to the category of $\mathcal{U}\mathfrak{g}$-modules.*

3.19 Example A Lie algebra $\mathfrak{g}$ is *abelian* if the bracket is identically 0. If $\mathfrak{g}$ is abelian, then $\mathcal{U}\mathfrak{g} = \mathcal{S}\mathfrak{g}$, where $\mathcal{S}V$ is the symmetric algebra generated by the vector space V (so that $\mathcal{S}$ is left-adjoint to Forget : $\textsc{ComAlg} \to \textsc{Vect}$).

3.20 Example If $\mathfrak{f}$ is the free Lie algebra on generators $x_1, \dots, x_d$, defined in terms of a universal property, then $\mathcal{U}\mathfrak{f} = \mathcal{T}(x_1, \dots, x_d)$.

3.21 Definition *A vector space V is* graded *if it comes with a direct-sum decomposition $V = \bigoplus_{n\geq 0} V_n$. A morphism of graded vector spaces preserves the grading. A* graded algebra *is an algebra object in the category of graded vector spaces. I.e. it is a vector space $V = \bigoplus_{n\geq 0}$ along with a unital associative multiplication $V \otimes V \to V$ such that if $v_n \in V_n$ and $v_m \in V_m$, then $v_n v_m \in V_{n+m}$.*

A vector space V is filtered *if it comes with an increasing sequence of subspaces*

$$\{0\} \subseteq V_{\leq 0} \subseteq V_{\leq 1} \subseteq \cdots \subseteq V \tag{3.2.3}$$

such that $V = \bigcup_{n\geq 0} V_n$. A morphism of graded vector spaces preserves the filtration. A filtered algebra *is an algebra object in the category of filtered vector spaces. I.e. it is a filtered vector space along with a unital associative multiplication $V \otimes V \to V$ such that if $v_n \in V_{\leq n}$ and $v_m \in V_{\leq m}$, then $v_n v_m \in V_{\leq (n+m)}$.*

Given a filtered vector space V, we define $\operatorname{gr} V \stackrel{\text{def}}{=} \bigoplus_{n\geq 0} \operatorname{gr}_n V$, where $\operatorname{gr}_n V \stackrel{\text{def}}{=} V_{\leq n}/V_{\leq (n-1)}$.

3.22 Lemma gr *is a functor. If V is a filtered algebra, then* $\operatorname{gr} V$ *is a graded algebra.*

3.23 Example Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. Then $\mathcal{U}\mathfrak{g}$ has a natural filtration inherited from the filtration of $\mathcal{T}\mathfrak{g}$, since the ideal $\langle xy - yx = [x,y]\rangle$ preserves the filtration. Since $\mathcal{U}\mathfrak{g}$ is generated by $\mathfrak{g}$, so is $\operatorname{gr}\mathcal{U}\mathfrak{g}$; since $xy - yx = [x,y] \in \mathcal{U}_{\leq 1}$, $\operatorname{gr}\mathcal{U}\mathfrak{g}$ is commutative, and so there is a natural projection $\mathcal{S}\mathfrak{g} \twoheadrightarrow \operatorname{gr}\mathcal{U}\mathfrak{g}$.

3.2.2 Poincaré-Birkhoff-Witt Theorem

[8, Lectures 12 and 13]

3.24 Theorem (Poincaré-Birkhoff-Witt) *The map $\mathcal{S}\mathfrak{g} \to \operatorname{gr}\mathcal{U}\mathfrak{g}$ is an isomorphism of algebras.*

3.25 Corollary $\mathfrak{g} \hookrightarrow \mathcal{U}\mathfrak{g}$. *Thus every Lie algebra is isomorphic to a Lie subalgebra of some* $\operatorname{End}(V)$*, namely* $V = \mathcal{U}\mathfrak{g}$.

Proof (of Theorem 3.24) Pick an ordered basis $\{x_\alpha\}$ of $\mathfrak{g}$; then the monomials $x_{\alpha_1}\dots x_{\alpha_n}$ for $\alpha_1 \le \dots \le \alpha_n$ are an ordered basis of $\mathcal{S}\mathfrak{g}$, where we take the "deg-lex" ordering: a monomial of lower degree is immediately smaller than a monomial of high degree, and for monomials of the same degree we alphabetize. Since $\mathcal{S}\mathfrak{g} \twoheadrightarrow \operatorname{gr}\mathcal{U}\mathfrak{g}$ is an algebra homomorphism, the set $\{x_{\alpha_1}\dots x_{\alpha_n} \text{ s.t. } \alpha_1 \le \dots \le \alpha_n\}$ spans $\operatorname{gr}\mathcal{U}\mathfrak{g}$. It suffices to show that they are independent in $\operatorname{gr}\mathcal{U}\mathfrak{g}$. For this it suffices to show that the set $S \overset{\text{def}}{=} \{x_{\alpha_1}\dots x_{\alpha_n} \text{ s.t. } \alpha_1 \le \dots \le \alpha_n\}$ is independent in $\mathcal{U}\mathfrak{g}$.

Let $I = \langle xy - yx - [x,y]\rangle$ be the ideal of $\mathcal{T}\mathfrak{g}$ such that $\mathcal{U}\mathfrak{g} = \mathcal{T}\mathfrak{g}/I$. Define $J \subseteq \mathcal{T}\mathfrak{g}$ to be the span of expressions of the form

$$\xi = x_{\alpha_1}\cdots x_{\alpha_k}\left(x_\beta x_\gamma - x_\gamma x_\beta - [x_\beta, x_\gamma]\right) x_{\nu_1}\cdots x_{\nu_l} \tag{3.2.4}$$

where $\alpha_1 \le \dots \le \alpha_k \le \beta > \gamma$, and there are no conditions on ν_i, so that J is a right ideal. We take the deg-lex ordering in $\mathcal{T}\mathfrak{g}$. The leading monomial in equation 3.2.4 is $x_{\vec\alpha}x_\beta x_\gamma x_{\vec\nu}$. Thus S is an independent set in $\mathcal{T}\mathfrak{g}/J$. We need only show that $J = I$.

The ideal I is generated by expressions of the form $x_\beta x_\gamma - x_\gamma x_\beta - [x_\beta, x_\gamma]$ as a two-sided ideal. If $\beta > \gamma$ then $\big(x_\beta x_\gamma - x_\gamma x_\beta - [x_\beta, x_\gamma]\big) \in J$; by antisymmetry, if $\beta < \gamma$ we switch them and stay in J. If $\beta = \gamma$, then $\big(x_\beta x_\gamma - x_\gamma x_\beta - [x_\beta, x_\gamma]\big) = 0$. Thus J is a right ideal contained in I, and the two-sided ideal generated by J contains I. Thus the two-sided ideal generated by J is I, and it suffices to show that J is a two-sided ideal.

We multiply $x_\delta\xi$. If $k > 0$ and $\delta \le \alpha_1$, then $x_\delta\xi \in J$. If $\delta > \alpha_1$, then $x_\delta\xi \equiv x_{\alpha_1}x_\delta x_{\alpha_2}\dots + [x_\delta, x_{\alpha_1}]x_{\alpha_2}\dots \mod J$. And both $x_\delta x_{\alpha_2}\dots$ and $[x_\delta, x_{\alpha_1}]x_{\alpha_2}\dots$ are in J by induction on degree. Then since $\alpha_1 < \delta$, $x_{\alpha_1}x_\delta x_{\alpha_2}\dots \in J$ by (transfinite) induction on δ.

So suffice to show that if $k = 0$, then we're still in J. I.e. if $\alpha > \beta > \gamma$, then we want to show that $x_\alpha\left(x_\beta x_\gamma - x_\gamma x_\beta - [x_\beta, x_\gamma]\right) \in J$. Well, since $\alpha > \beta$, we see that $x_\alpha x_\beta - x_\beta - [x_\alpha, x_\beta] \in J$, and same with $\beta \leftrightarrow \gamma$. So, working modulo J, we have

$$\begin{aligned}
x_\alpha\left(x_\beta x_\gamma - x_\gamma x_\beta - [x_\beta, x_\gamma]\right) &\equiv \left(x_\beta x_\alpha + [x_\alpha, x_\beta]\right)x_\gamma - \left(x_\gamma x_\alpha + [x_\alpha, x_\gamma]\right)x_\beta - x_\alpha[x_\beta, x_\gamma] \\
&\equiv x_\beta\left(x_\gamma x_\alpha + [x_\alpha, x_\gamma]\right) + [x_\alpha, x_\beta]x_\gamma - x_\gamma\left(x_\beta x_\alpha + [x_\alpha, x_\beta]\right) \\
&\quad - [x_\alpha, x_\gamma]x_\beta - x_\alpha[x_\beta, x_\gamma] \\
&\equiv x_\gamma x_\beta x_\alpha + [x_\beta, x_\gamma]x_\alpha + x_\beta[x_\alpha, x_\gamma] + [x_\alpha, x_\beta]x_\gamma - x_\gamma\left(x_\beta x_\alpha + [x_\alpha, x_\beta]\right) \\
&\quad - [x_\alpha, x_\gamma]x_\beta - x_\alpha[x_\beta, x_\gamma] \\
&= [x_\beta, x_\gamma]x_\alpha + x_\beta[x_\alpha, x_\gamma] + [x_\alpha, x_\beta]x_\gamma - x_\gamma[x_\alpha, x_\beta] - [x_\alpha, x_\gamma]x_\beta - x_\alpha[x_\beta, x_\gamma] \\
&\equiv -[x_\alpha, [x_\beta, x_\gamma]] + [x_\beta, [x_\alpha, x_\gamma]] - [x_\gamma, [x_\alpha, x_\beta]] \\
&= 0 \text{ by Jacobi.}
\end{aligned}$$

□

3.2.3 $\mathcal{U}\mathfrak{g}$ is a bialgebra

[8, Lecture 13]

3.26 Definition *An* algebra *over* $\mathbb{K}$ *is a vector space* U *along with a* $\mathbb{K}$*-linear "multiplication" map* $\mu : U \underset{\mathbb{K}}{\otimes} U \to U$ *which is* associative*, i.e. the following diagram commutes:*

$$\begin{tikzcd} U \otimes U \otimes U \arrow[r, "1_U \otimes \mu"] \arrow[d, "\mu\otimes 1_U"] & U \otimes U \arrow[d, "\mu"] \\ U \otimes U \arrow[r, "\mu"] & U \end{tikzcd} \tag{3.2.5}$$

We demand that all our algebras be unital*, meaning that there is a linear map* $e : \mathbb{K} \to U$ *such that the maps* $U = \mathbb{K} \otimes U \overset{e\otimes 1_U}{\to} U \otimes U \overset{\mu}{\to} U$ *and* $U = U \otimes K \overset{1_U \otimes e}{\to} U \otimes U \overset{\mu}{\to} U$ *are the identity maps. We will call the image of* $1 \in \mathbb{K}$ *under* e *simply* $1 \in U$*.*

A coalgebra *is an algebra in the opposite category. I.e. it is a vector space* U *along with a "comultiplication" map* $\Delta : U \to U \otimes U$ *so that the following commutes:*

$$\begin{tikzcd} U \arrow[r, "\Delta"] \arrow[d, "\Delta"] & U \otimes U \arrow[d, "1_U \otimes \Delta"] \\ U \otimes U \arrow[r, "\Delta\otimes 1_U"] & U \otimes U \otimes U \end{tikzcd} \tag{3.2.6}$$

In elements, if $\Delta x = \sum x_{(1)} \otimes x_{(2)}$*, then we demand that* $\sum x_{(1)} \otimes \Delta(x_{(2)}) = \sum \Delta(x_{(1)}) \otimes x_{(1)}$*. We demand that our coalgebras be* counital*, meaning that there is a linear map* $\epsilon : U \to \mathbb{K}$ *such that the maps* $U \overset{\Delta}{\to} U \otimes U \overset{\epsilon\otimes 1_U}{\mathbb{K}} \otimes U = U$ *and* $U \overset{\Delta}{\to} U \otimes U \overset{1_U\otimes\epsilon}{U} \otimes \mathbb{K} = U$ *are the identity maps.*

A bialgebra *is an algebra in the category of coalgebras, or equivalently a coalgebra in the category of algebras. I.e. it is a vector space* U *with maps* $\mu : U \otimes U \to U$ *and* $\Delta : U \to U \otimes U$ *satisfying equations 3.2.5 and 3.2.6 such that* Δ *and* ϵ *are (unital) algebra homomorphisms or equivalently such that* μ *and* e *are (counital) coalgebra homomorphism. We have defined the multiplication on* $U \otimes U$ *by* $(x \otimes y)(z \otimes w) = (xz) \otimes (yw)$*, and the comultiplication by* $\Delta(x \otimes y) = \sum\sum x_{(1)} \otimes y_{(1)} \otimes x_{(2)} \otimes y_{(2)}$*, where* $\Delta x = \sum x_{(1)} \otimes x_{(2)}$ *and* $\Delta y = \sum y_{(1)} \otimes y_{(2)}$*; the unit and counit are* $e \otimes e$ *and* $\epsilon \otimes \epsilon$*.*

3.27 Definition *Let* U *be a bialgebra, and* $x \in U$*. We say that* x *is* primitive *if* $\Delta x = x \otimes 1 + 1 \otimes x$*, and that* x *is* grouplike *if* $\Delta x = x \otimes x$*. The set of primitive elements of* U *we denote by* $\operatorname{prim} U$*.*

3.28 Proposition $\mathcal{U}\mathfrak{g}$ *is a bialgebra with* $\operatorname{prim}\mathcal{U}\mathfrak{g} = \mathfrak{g}$.

Proof To define the comultiplication, it suffices to show that $\Delta : \mathfrak{g} \to \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$ given by $x \mapsto x \otimes 1 + 1 \otimes x$ is a Lie algebra homomorphism, whence it uniquely extends to an algebra homomorphism by the universal property. We compute:

$$[x\otimes 1+1\otimes x, y\otimes 1+1\otimes y]_{\mathcal{U}\otimes\mathcal{U}} = [x\otimes 1, y\otimes 1]+[1\otimes x, 1\otimes y] \tag{3.2.7}$$
$$= [x,y]\otimes 1+1\otimes [x,y] \tag{3.2.8}$$

To show that Δ thus defined is coassiciative, it suffices to check on the generating set $\mathfrak{g}$, where we see that $\Delta^2(x) = x\otimes 1\otimes 1+1\otimes x\otimes 1+1\otimes 1\otimes x$.

By definition, $\mathfrak{g}\subseteq\mathcal{U}\mathfrak{g}$. To show equality, we use Theorem 3.24. We filter $\mathcal{U}\mathfrak{g}\otimes\mathcal{U}\mathfrak{g}$ in the obvious way, and since Δ is an algebra homomorphism, we see that $\Delta\big(\mathcal{U}\mathfrak{g}_{\leq 1}\big)\subseteq\big(\mathcal{U}\mathfrak{g}\otimes\mathcal{U}\mathfrak{g}\big)_{\leq 1}$, whence $\Delta\big(\mathcal{U}\mathfrak{g}_{\leq n}\big)\subseteq\big(\mathcal{U}\mathfrak{g}\otimes\mathcal{U}\mathfrak{g}\big)_{\leq n}$. Thus Δ induces a map $\bar\Delta$ on $\operatorname{gr}\mathcal{U}\mathfrak{g}=\mathcal{S}\mathfrak{g}$, and $\bar\Delta$ makes $\mathcal{S}\mathfrak{g}$ into a bialgebra.

Let $\xi\in\mathcal{U}\mathfrak{g}_{\leq n}$ be primitive, and define its image to be $\bar\xi\in\operatorname{gr}_n\mathcal{U}\mathfrak{g}$; then $\bar\xi$ must also be primitive. But $\mathcal{S}\mathfrak{g}\otimes\mathcal{S}\mathfrak{g}=\mathbb{K}[y_\alpha,z_\alpha]$, where $\{x_\alpha\}$ is a basis of $\mathfrak{g}$ (whence $\mathcal{S}\mathfrak{g}=\mathbb{K}[x_\alpha]$), and we set $y_\alpha=x_\alpha\otimes 1$ and $z_\alpha=1\otimes x_\alpha]$. We check that $\bar\Delta(x_\alpha)=y_\alpha+z_\alpha$, and so if $f(x)\in\mathcal{S}\mathfrak{g}$, we see that $\Delta f(x)=f(y+z)$. So $f\in\mathcal{S}\mathfrak{g}$ is primitive if and only if $f(y+z)=f(y)+f(z)$, i.e. iff f is homogenoues of degree 1. Therefore $\operatorname{prim}\operatorname{gr}\mathcal{U}\mathfrak{g}=\operatorname{gr}_1\mathcal{U}\mathfrak{g}$, and so if $\xi\in\mathcal{U}\mathfrak{g}$ is primitive, then $\bar\xi\in\operatorname{gr}_1\mathcal{U}\mathfrak{g}$ so $\xi=x+c$ for some $x\in\mathfrak{g}$ and some $c\in\mathbb{K}$. Since x is primitive, c must be also, and the only primitive constant is 0.$\square$

3.2.4 Geometry of the Universal Enveloping Algebra

[8, Lecture 14]

3.29 Definition *Let X be a space and $\mathscr{S}$ a sheaf of functions on X. We define the sheaf $\mathscr{D}$ of* grothendieck differential operators *inductively. Given $U\underset{open}{\subseteq}X$, we define $\mathscr{D}_{\leq 0}(U)=\mathscr{S}(U)$, and $\mathscr{D}_{\leq n}(U)=\{x:\mathscr{S}(U)\to\mathscr{S}(U) \text{ s.t. } [x,f]\in\mathscr{D}_{\leq(n-1)}(U)\,\forall f\in\mathscr{S}(U)\}$, where $\mathscr{S}(U)\curvearrowright\mathscr{S}(U)$ by left-multiplication. Then $\mathscr{D}(U)=\bigcup_{n\geq 0}\mathscr{D}_{\leq n}(U)$ is a filtered sheaf; we say that $x\in\mathscr{D}_{\leq n}(U)$ is an "nth-order differential operator on U".*

3.30 Lemma *$\mathscr{D}$ is a sheaf of filtered algebras, with the multiplication on $\mathscr{D}(U)$ inherited from* $\operatorname{End}(\mathscr{S}(U))$. *For each n, $\mathscr{D}_{\leq n}$ is a sheaf of Lie subalgebras of $\mathscr{D}$.*

3.31 Theorem (Grothendieck Differential Operators) *Let X be a manifold, $\mathscr{C}$ the sheaf of smooth functions on X, and $\mathscr{D}$ the sheaf of differential operators on $\mathscr{C}$ as in Definition 3.29. Then $\mathscr{D}(U)$ is generated as a noncommutative algebra by $\mathscr{C}(U)$ and $\operatorname{Vect}(U)$, and $\mathscr{D}_{\leq 1}=\mathscr{C}(U)\oplus\operatorname{Vect}(U)$.*

3.32 Proposition *Let G be a Lie group, and $\mathscr{D}(G)^G$ the subalgebra of left-invariant differential operators on G. The natural map $\mathcal{U}\mathfrak{g}\to\mathscr{D}(G)^G$ generated by the identification of $\mathfrak{g}$ with left-invariant vector fields is an isomorphism of algebras.*

3.3 The Baker-Campbell-Hausdorff Formula

[8, Lecture 14]

3.33 Lemma *Let U be a bialgebra with comultiplication Δ. Define $\hat{\Delta} : U[[s]] \to (U \otimes U)[[s]]$ by linearity; then $\hat{\Delta}$ is an s-adic-continuous algebra homomorphism, and so commutes with formal power series.*

Let $\psi \in U[[s]]$ with $\psi(0) = 0$. Then ψ is primitive term-by-term — $\hat{\Delta}(\psi) = \psi \otimes 1 + 1 \otimes \psi$, if and only if e^ψ is "group-like" in the sense that $\hat{\Delta}(e^\psi) = e^\psi \otimes e^\psi$, where we have defined $\otimes : U[[s]] \otimes U[[s]] \to (U \otimes U)[[s]]$ by $s^n \otimes s^m \mapsto s^{n+m}$.

Proof $e^\psi \otimes e^\psi = (1 \otimes e^\psi)(e^\psi \otimes 1) = e^{1\otimes\psi} e^{\psi\otimes 1} = e^{1\otimes\psi + \psi\otimes 1}$ □

3.34 Lemma *Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, and identify $\mathcal{U}\mathfrak{g}$ with the left-invariant differential operators on G, as in Proposition 3.32. Let $\mathscr{C}(G)_e$ be the stalk of smooth functions defined in some open set around e (we write $\mathscr{C}$ for the sheaf of functions on G; when G is analytic, we really mean the sheaf of analytic functions on G). Then if $u \in \mathcal{U}\mathfrak{g}$ satisfied $uf(e) = 0$ for each $f \in \mathscr{C}(G)_e$, then $u = 0$.*

Proof For $g \in G$, we have $uf(g) = u(\lambda_{g^{-1}} f)(e) = \lambda_{g^{-1}}(uf)(e) = 0$. □

3.35 Theorem (Baker-Campbell-Hausdorff Formula) *1. Let $\mathfrak{f}$ be the free Lie algebra on two generators x, y; recall that $\mathcal{U}\mathfrak{f} = \mathcal{T}(x, y)$. Define the formal power series $b(tx, sy) \in \mathcal{T}(x, y)[[s, t]]$, where s and t are commuting variables, by*

$$e^{b(tx,sy)} \stackrel{\text{def}}{=} e^{tx} e^{sy} \tag{3.3.1}$$

Then $b(tx, sy) \in \mathfrak{f}[[s, t]]$, i.e. b is a series all of whose coefficients are Lie algebra polynomials in the generators x and y.

2. If G is a Lie group (in the analytic category), then there are open neighborhoods $0 \in U' \underset{open}{\subseteq} U \underset{open}{\subseteq} \mathrm{Lie}(G) = \mathfrak{g}$ and $0 \in V' \underset{open}{\subseteq} V \underset{open}{\subseteq} G$ such that $U \underset{\log}{\overset{\exp}{\rightleftarrows}} V$ and $U' \rightleftarrows V'$ and such that $b(x, y)$ converges on $U' \times U'$ to $\log(\exp x \exp y)$.

Proof 1. Let $\hat{\Delta} : \mathcal{T}(x, y)[[s, t]] = \mathcal{U}\mathfrak{f}[[s, t]] \to (\mathcal{U}\mathfrak{f} \otimes \mathcal{U}\mathfrak{f})[[s, t]]$ as in Lemma 3.33. Since $e^{tx}e^{sy}$ is grouplike —

$$\hat{\Delta}(e^{tx}e^{sy}) = \hat{\Delta}(e^{tx})\,\hat{\Delta}(e^{sy}) = \left(e^{tx} \otimes e^{tx}\right)(e^{sy} \otimes e^{sy}) = e^{tx}e^{sy} \otimes e^{tx}e^{ty} \tag{3.3.2}$$

— we see that $b(tx, sy)$ is primitive term-by-term.

2. Let U, V be open neighborhoods of $\mathrm{Lie}(G)$ and G respectively, and pick V' so that $\mu : G\times G \to G$ restricts to a map $V' \times V' \to V$; let $U' = \log(V')$. Define $\beta(x, y) = \log(\exp x \exp y)$; then β is an analytic function $U' \times U' \to U'$.

Let $x, y \in \mathrm{Lie}(G)$ and $f \in \mathscr{C}(G)_e$. Then $\left(e^{tx}e^{sy}f\right)(e)$ is the Taylor series expansion of $f(\exp tx \exp sy)$, as in the proof of Theorem 3.14. Let $\tilde{\beta}$ be the formal power series that is the Taylor expansion of β; then $e^{\tilde{\beta}(tx,sy)}f(e)$ is also the Taylor series expansion of $f(\exp tx \exp sy)$. This implies that for every $f \in \mathscr{C}(G)_e$, $e^{\tilde{\beta}(tx,sy)}f(e)$ and $e^{tx}e^{sy}f(e)$ have the same coefficients.

But the coefficients are left-invariant differential operators applied to f, so by Lemma 3.34 the series $e^{\tilde{\beta}(tx,sy)}$ and $e^{tx}e^{sy}$ must agree. Upon applying the formal logarithms, we see that $b(tx, sy) = \tilde{\beta}(tx, sy)$.

But $\tilde{\beta}$ is the Taylor series of the analytic function β, so by shrinking U' (and hence V') we can assure that it converges. □

3.4 Lie Subgroups

3.4.1 Relationship between Lie subgroups and Lie subalgebras

[8, Lecture 15]

3.36 Definition *Let G be a Lie group. A* Lie subgroup *of G is a subgroup H of G with its own Lie group structure, so that the inclusion $H \hookrightarrow G$ is a local immersion. We will write "$H \le G$" when H is a Lie subgroup of G.*

3.37 Theorem (Identification of Lie subalgebras and Lie subgroups) *Every Lie subalgebra of* $\mathrm{Lie}(G)$ *is* $\mathrm{Lie}(H)$ *for a unique connected Lie subgroup $H \le G$.*

Proof We first prove uniqueness. If H is a Lie subgroup of G, with $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{g} = \mathrm{Lie}(G)$, then the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, hook] & G \\ \mathfrak{h} \arrow[u, "\exp"] \arrow[r, hook] & \mathfrak{g} \arrow[u, "\exp"'] \end{tikzcd} \tag{3.4.1}$$

This shows that $\exp_G(\mathfrak{h}) \subseteq H$, and so $\exp_G(\mathfrak{h}) = \exp_H(\mathfrak{h})$, and if H is connected, this generates H. So H is uniquely determined by $\mathfrak{h}$ as a group. Its manifold structure is also uniquely determined: we pick U, V so that the vertical arrows are an isomorphism:

$$\begin{tikzcd} e \in V \subseteq G \arrow[d, shift left, "\log"] \\ 0 \in U \subseteq \mathfrak{g} \arrow[u, shift left, "\exp"] \end{tikzcd} \tag{3.4.2}$$

Then $\exp(U \cap \mathfrak{h}) \underset{\log}{\overset{\sim}{\to}} U \cap \mathfrak{h}$ is an immersion into $\mathfrak{g}$, and this defines a chart around $e \in H$, which we can push to any other point $h \in H$ by multiplication by h. This determines the topology and manifold structure of H.

We turn now to the question of existence. We pick U and V as in equation 3.4.2, and then choose $V' \underset{\text{open}}{\subseteq} V$ and $U' \overset{\text{def}}{=} \log V'$ such that:

1. $(V')^2 \subseteq V$ and $(V')^{-1} = V'$
2. $b(x, y)$ converges on $U' \times U'$ to $\log(\exp x \exp y)$

3. $hV'h^{-1} \subseteq V$ for $h \in V'$

4. $e^{\operatorname{ad} x} y$ converges on $U' \times U'$ to $\log\Big((\exp x)(\exp y)(\exp x)^{-1}\Big)$

5. $b(x,y)$ and $e^{\operatorname{ad} x} y$ are elements of $h \cap U$ for $x, y \in \mathfrak{h} \cap U'$

Each condition can be independently achieved on a small enough open set. In condition 4., we consider extend the formal power series e^t to operators, and remark that in a neighborhood of $0 \in \mathfrak{g}$, if $h = \exp x$, then $\operatorname{Ad} h = e^{\operatorname{ad} x}$. Moreover, the following square always commutes:

$$\begin{tikzcd} G \arrow[r, "g \mapsto hgh^{-1}"] & G \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\operatorname{Ad}(h)"] & \mathfrak{g} \arrow[u, "\exp"'] \end{tikzcd} \tag{3.4.3}$$

Thus, we define $W = \exp(\mathfrak{h} \cap U')$, which is certainly an immersed submanifold of G, as $\mathfrak{h} \cap U'$ is an open subset of the immersed submanifold $\mathfrak{h} \hookrightarrow \mathfrak{g}$. We define H to be the subgroup generated by W. Then H and W satisfy the hypotheses of Proposition 3.38. ☐

3.38 Proposition *We use the word "manifold" to mean "object in a particular chosen category of sheaves of functions". We use the word "smooth" to mean "morphism in this category".*

Let H be a group and $U \subseteq H$ such that $e \in U$ and U has the structure of a manifold. Assume further that the maps $U \times U \to H$, ${}^{-1} : U \to H$, and (for each h in a generating set of H) $\operatorname{Ad}(h) : U \to H$ mapping $u \mapsto huh^{-1}$ have the following properties:

1. *The preimage of $U \subseteq H$ under each map is open in the domain.*

2. *The restriction of the map to this preimage is smooth.*

Then H has a unique structure as a group manifold such that U is an open submanifold.

Proof The conditions 1. and 2. are preserved under compositions, so $\operatorname{Ad}(x)$ satisfies both conditions for any $x \in H$. Let $e \in U' \underset{\text{open}}{\subseteq} U$ so that $(U')^3 \subseteq U$ and $(U')^{-1} = U'$.

For $x \in H$, view each coset xU' as a manifold via $U' \xrightarrow{x\cdot} xU'$. For any $U'' \underset{\text{open}}{\subseteq} U'$ and $x, y \in G$, consider $yU'' \cap xU'$; as a subset of xU', it is isomorphic to $x^{-1}yU'' \cap U'$. If this set is empty, then it is open. Otherwise, $x^{-1}yu_2 = u_1$ for some $u_2 \in U''$ and $u_1 \in U'$, so $y^{-1}x = u_2u_1^{-1} \in (U')^2$ and so $y^{-1}xU' \subseteq U$. In particular, the $\{y^{-1}x\} \times U' \subseteq \mu^{-1}(U) \cap (U \times U)$. By the assumptions, $U' \to y^{-1}xU$ is smooth, and so $x^{-1}yU'' \cap U'$, the preimage of U'', is open in U'. Thus the topologies and smooth structures on xU' and yU' agree on their overlap.

In this way, we can put a manifold structure on H by declaring that $S \underset{\text{open}}{\subseteq} H$ if $S \cap xU' \underset{\text{open}}{\subseteq} xU'$ for all $x \in H$ — the topology is locally the topology of $U \ni e$, and so is Hausdorff — , and that a function f on $S \underset{\text{open}}{\subseteq} H$ is smooth if its restriction to each $S \cap xU'$ is smooth.

If we were to repeat this story with right cosets rather than left cosets we would get the a similar structure: all the left cosets xU' are compatible, an all the right cosets $U'x$ are compatible. To show that a right coset is compatible with a left coset, it suffices to show that for each $x \in H$, xU' and $U'x$ have compatible smooth structures. We consider $xU' \cap U'x \subseteq xU'$, which we transport to $U' \cap x^{-1}U'x \subseteq U'$. Since we assumed that conjugation by x was a smooth map, we see that right and left cosets are compatible.

We now need only check that the group structure is by smooth maps. We see that $(xU')^{-1} = (U')^{-1}x^{-1} = U'x^{-1}$, and multiplication is given by $\mu : xU' \times U'y \to xUy$. Left- and right-multiplication maps are smooth with respect to the left- and right-coset structures, which are compatible, and we assumed that $\mu : U' \times U' \to U$ was smooth. $\square$

3.4.2 Review of Algebraic Topology

[8, Lecture 16]

3.39 Definition *A* groupoid *is a category all of whose morphisms are invertible.*

3.40 Definition *A space X is* connected *if the only subsets of X that are both open and closed are $\emptyset$ and X.*

3.41 Definition *Let X be a space and $x, y \in X$. A* path *from x to y, which we write as $x \rightsquigarrow y$, is a continuous function $[0,1] \to X$ such that $0 \mapsto x$ and $1 \mapsto y$. Given $p : x \rightsquigarrow y$ and $q : y \rightsquigarrow z$, we define the* concatenation *$p \cdot q$ by*

$$p \cdot q(t) \overset{\text{def}}{=} \begin{cases} p(2t), & 0 \le t \le \frac{1}{2} \\ q(2t-1), & \frac{1}{2} \le t \le 1 \end{cases} \tag{3.4.4}$$

We write $x \sim y$ if there is a path connecting x to y; $\sim$ is an equivalence relation, and the equivalence classes are path components *of X. If X has only one path component, then it is* path connected.

Let A be a distinguished subset of Y and $f, g : Y \to X$ two functions that agree on A. A homotopy *$f \underset{A}{\sim} g$ relative to A is a continuous map $h : Y \times [0,1] \to X$ such that $h(0,y) = f(y)$, $h(1,y) = g(y)$, and $h(t,a) = f(a) = g(a)$ for $a \in A$. If $f \sim g$ and $g \sim h$, then $f \sim h$ by concatenation. The* fundamental groupoid *$\pi_1(X)$ of X has objects the points of X and arrows $x \to y$ the homotopy classes of paths $x \rightsquigarrow y$. We write $\pi_1(X,x)$ for the set of morphisms $x \to x$ in $\pi_1(X)$. The space X is* simply connected *if $\pi_1(X,x)$ is trivial for each $x \in X$.*

3.42 Example A path connected space is connected, but a connected space is not necessarily path connected. A path is a homotopy of constant maps $\{\text{pt}\} \to X$, where A is empty.

3.43 Definition *Let X be a space. A* covering space *of X is a space E along with a "projection" $\pi : E \to X$ such that there is a non-empty discrete space S and a covering of X by open sets such that for each U in the covering, there exists an isomorphism $\pi^{-1}(U) \xrightarrow{\sim} S \times U$ such that the following diagram commutes:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, phantom, "\cong"] \arrow[dr, "\pi"'] & & S \times U \arrow[dl, "\text{project}"] \\ & U & \end{tikzcd} \tag{3.4.5}$$

3.44 Proposition *Let $\pi_E : E \to X$ be a covering space.*

1. *Given any path $x \rightsquigarrow y$ and a lift $e \in \pi^{-1}(x)$, there is a unique path in E starting at e that projects to $x \rightsquigarrow y$.*

2. *Given a homotopy $\underset{A}{\sim} : Y \rightrightarrows X$ and a choice of a lift of the first arrow, there is a unique lift of the homotopy, provided Y is locally compact.*

Thus E induces a functor $E : \pi_1(X) \to \textsc{Set}$, sending $x \mapsto \pi_E^{-1}(X)$.

3.45 Definition *A space X is* locally path connected *if each $x \in X$ has arbitrarily small path connected neighborhoods. A space X is* locally simply connected *if it has a covering by simply connected open sets.*

3.46 Proposition *Assume that X is path connected, locally path connected, and locally simply connected. Then:*

1. *X has a simply connected covering space $\tilde\pi : \tilde X \to X$.*

2. *$\tilde X$ satisfies the following universal property: Given $f : X \to Y$ and a covering $\pi : E \to Y$, and given a choice of $x \in X$, an element of $\tilde x \in \tilde\pi^{-1}(x)$, and an element $e \in \pi^{-1}(f(x))$, then there exists a unique $\tilde f : \tilde X \to E$ sending $\tilde x \mapsto e$ such that the following diagram commutes:*

$$\begin{tikzcd} \tilde X \arrow[r, dotted, "\tilde f"] \arrow[d, "\tilde\pi"] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{3.4.6}$$

3. *If X is a manifold, so is $\tilde X$. If f is smooth, so is $\tilde f$.*

3.47 Proposition 1. *Let G be a connected Lie group, and $\tilde G$ its simply-connected cover. Pick a point $\tilde e \in \tilde G$ over the identity $e \in G$. Then $\tilde G$ in its given manifold structure is uniquely a Lie group with identity $\tilde e$ such that $\tilde G \to G$ is a homomorphism. This induces an isomorphism of Lie algebras $\mathrm{Lie}(\tilde G) \xrightarrow{\sim} \mathrm{Lie}(G)$.*

2. *$\tilde G$ satisfies the following universal property: Given any Lie algebra homomorphism $\alpha : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$, there is a unique homomorphism $\phi : \tilde G \to H$ inducing α.*

Proof 1. If X and Y are simply-connected, then so is $X \times Y$, and so by the universal property $\tilde G \times \tilde G$ is the universal cover of $G \times G$. We lift the functions $\mu : G \times G \to G$ and $i : G \to G$ to $\tilde G$ by declaring that $\tilde\mu(\tilde e, \tilde e) = \tilde e$ and that $i(\tilde e) = \tilde e$; the group axioms (equations 1.1.1 to 1.1.3) are automatic.

2. Write $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$, and let $\alpha : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then the graph $\mathfrak{f} \subseteq \mathfrak{g} \times \mathfrak{h}$ is a Lie subalgebra. By Theorem 3.37, $\mathfrak{f}$ corresponds to a subgroup $F \leq \tilde{G} \times H$. We check that the map $F \hookrightarrow \tilde{G} \times H \to G$ induces the map $\mathfrak{f} \to \mathfrak{g}$ on Lie algebras.

$$\begin{tikzcd} F \arrow[r, phantom, "\subseteq"] \arrow[dr, dotted, "\mathfrak{f} \xrightarrow{\sim} \mathfrak{g}"'] & \tilde{G} \times H \arrow[d] \\ & G \end{tikzcd} \tag{3.4.7}$$

F is connected and simply connected, and so by the universal property, $F \cong \tilde{G}$. Thus F is the graph of a homomorphism $\phi : \tilde{G} \to H$. □

3.5 A dictionary between algebras and groups

[8, Lecture 17]

We have completed the proof of Theorem 3.12, the equivalence between the category of finite-dimensional Lie algebras and the category of simply-connected Lie groups, subject only to Theorem 4.89. Thus a Lie algebra includes most of the information of a Lie group. We foreshadow a dictionary, most of which we will define and develop later:

Lie Algebra $\mathfrak{g}$	Lie Group G (with $\mathfrak{g} = \mathrm{Lie}(G)$)
Subalgebra $\mathfrak{h} \leq \mathfrak{g}$	Connected Lie subgroup $H \leq G$
Homomorphism $\mathfrak{h} \to \mathfrak{g}$	$\tilde{H} \to G$ provided $\tilde{H}$ simply connected
Module/representation $\mathfrak{g} \to \mathfrak{gl}(V)$	Representation $\tilde{G} \to \mathrm{GL}(V)$ ($\tilde{G}$ simply connected)
Submodule $W \leq V$ with $\mathfrak{g} : W \to W$	Invariant subspace $G : W \to W$
$V^{\mathfrak{g}} \overset{\text{def}}{=} \{v \in V \text{ s.t. } \mathfrak{g}v = 0\}$	$V^{\tilde{G}} = \{v \in V \text{ s.t. } Gv = v\}$
$\mathrm{ad} : \mathfrak{g} \curvearrowright \mathfrak{g}$ via $\mathrm{ad}(x)y = [x, y]$	$\mathrm{Ad} : G \curvearrowright G$ via $\mathrm{Ad}(x)y = xyx^{-1}$
An *ideal* $\mathfrak{a}$, i.e. $[\mathfrak{g}, \mathfrak{a}] \leq \mathfrak{a}$, i.e. sub-$\mathfrak{g}$-module	A is a normal Lie subgroup, provided G is connected
$\mathfrak{g}/\mathfrak{a}$ is a Lie algebra	G/A is a Lie group only if A is closed in G
Center $Z(\mathfrak{g}) = \mathfrak{g}^{\mathfrak{g}}$	$Z_0(G)$ the identity component of center; this is closed
Derived subalgebra $\mathfrak{g}' \overset{\text{def}}{=} [\mathfrak{g}, \mathfrak{g}]$, an ideal	Should be commutator subgroup, but that's not closed: the closure also doesn't work, although if G is compact, then the commutator subgroup is closed.
Semidirect product $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}$ with $\mathfrak{h} \curvearrowright \mathfrak{a}$ and $\mathfrak{a}$ an ideal	If A and H are closed, then $A \cap H$ is discrete, and $\tilde{G} = \tilde{H} \ltimes \tilde{A}$

3.5.1 Basic Examples: one- and two-dimensional Lie algebras

We classify the one- and two-dimensional Lie algebras and describe their corresponding Lie groups. We begin by working over $\mathbb{R}$.

The only one-dimensional Lie algebra is abelian. Its connected Lie groups are the line $\mathbb{R}$ and the circle S^1.

There is a unique abelian two-dimensional Lie algebra, given by a basis $\{x, y\}$ with relation $[x, y] = 0$. This integrates to three possible groups: $\mathbb{R}^2$, $\mathbb{R} \times (\mathbb{R}/\mathbb{Z})$, and $(\mathbb{R}/\mathbb{Z})^2$.

There is a unique nonabelian Lie algebra up to isomorphism, which we call $\mathfrak{b}$. It has a basis $\{x, y\}$ and defining relation $[x, y] = y$:

$$\begin{tikzcd} -x \arrow[r, "\mathrm{ad}\, y"] & y \arrow[loop below, "\mathrm{ad}\, X"] \end{tikzcd} \tag{3.5.1}$$

We can represent $\mathfrak{b}$ as a subalgebra of $\mathfrak{gl}(2$ by $x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\mathfrak{b}$

exponentiates under $\exp : \mathfrak{gl}(2 \to \mathrm{GL}(2$ to the group

$$B = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \text{ s.t. } a \in \mathbb{R}_+, b \in \mathbb{R} \right\} \tag{3.5.2}$$

We check that $B = \mathbb{R}_+ \ltimes \mathbb{R}$, and B is connected and simply connected.

3.48 Lemma *A discrete normal subgroup A of a connected Lie group G is in the center. In particular, any discrete normal subgroup is abelian.*

3.49 Corollary *The group B defined above is the only connected group with Lie algebra $\mathfrak{b}$.*

Proof Any other must be a quotient of B by a discrete normal subgroup, but the center of B is trivial. □

We turn now to the classification of one- and two-dimensional Lie algebras and Lie groups over $\mathbb{C}$. Again, there is only the abelian one-dimensional algebra, and there are two Lie algebras: the abelian one and the nonabelian one.

The simply connected abelian one-(complex-)dimensional Lie group is $\mathbb{C}$ under $+$. Any quotient factors (up to isomorphism) through the cylinder $\mathbb{C} \to \mathbb{C}^\times : z \mapsto e^z$. For any $q \in \mathbb{C}^\times$ with $|q| \neq 1$, we have a discrete subgroup $q^{\mathbb{Z}}$ of $\mathbb{C}^\times$, by which we can quotient out; we get a torus $E(q) = \mathbb{C}^\times / q^{\mathbb{Z}}$. For each q, $E(q)$ is isomorphic to $(\mathbb{R}/\mathbb{Z})^2$ as a real Lie algebra, but the holomorphic structure depends on q. This exhausts the one-dimensional complex Lie groups.

The groups that integrate the abelian two-dimensional complex Lie algebra are combinations of one-dimensional Lie groups: $\mathbb{C}^2, \mathbb{C} \times E, \mathbb{C}^\times \times \mathbb{C}^\times$, etc.

In the non-abelian case, the Lie algebra $\mathfrak{b}_+ \leq \mathfrak{gl}(2$ integrates to $B_{\mathbb{C}} \leq \mathrm{GL}(2$ given by:

$$B_{\mathbb{C}} = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \text{ s.t. } a \in \mathbb{C}^\times, b \in \mathbb{C} \right\} = \mathbb{C}^\times \ltimes \mathbb{C} \tag{3.5.3}$$

This is no longer simply connected. $\mathbb{C} \curvearrowright \mathbb{C}$ by $z \cdot w = e^z w$, and the simply-connected cover of B is

$$\tilde{B}_{\mathbb{C}} = \mathbb{C} \ltimes \mathbb{C} \quad (w, z)(w', z') \overset{\text{def}}{=} (w + e^z w', z + z') \tag{3.5.4}$$

This is an extension:

$$0 \to \mathbb{Z} \to \tilde{B}_{\mathbb{C}} \to B_{\mathbb{C}} \to 0 \tag{3.5.5}$$

with the generator of $\mathbb{Z}$ being $2\pi i$. Other quotients are $\tilde{B}_{\mathbb{C}}/n\mathbb{Z}$.

Exercises

1. (a) Let S be a commutative $\mathbb{K}$-algebra. Show that a linear operator $d : S \to S$ is a derivation if and only if it annihilates 1 and its commutator with the operator of multiplication by every function is the operator of multiplication by another function.

(b) Grothendieck's inductive definition of differential operators on S goes as follows: the differential operators of order zero are the operators of multiplication by functions; the space $D_{\leq n}$ of operators of order at most n is then defined inductively for $n > 0$ by $D_{\leq n} = \{d \text{ s.t. } [d, f] \in D_{\leq n-1} \text{ for all } f \in S\}$. Show that the differential operators of all orders form a filtered algebra D, and that when S is the algebra of smooth functions on an open set in $\mathbb{R}^n$ [or $\mathbb{C}^n$], D is a free left S-module with basis consisting of all monomials in the coordinate derivations $\partial/\partial x^i$.

2. Calculate all terms of degree ≤ 4 in the Baker-Campbell-Hausdorff formula (equation 3.1.5).

3. Let $F(d)$ be the free Lie algebra on generators $x_1, \ldots, x_d$. It has a natural $\mathbb{N}^d$ grading in which $F(d)_{(k_1,\ldots,k_d)}$ is spanned by bracket monomials containing k_i occurences of each generator X_i. Use the PBW theorem to prove the generating function identity

$$\prod_{\mathbf{k}} \frac{1}{(1 - t_1^{k_1} \ldots t_d^{k_d})^{\dim F(d)_{(k_1,\ldots,k_d)}}} = \frac{1}{1 - (t_1 + \cdots + t_d)}$$

4. Words in the symbols $x_1, \ldots, x_d$ form a monoid under concatentation, with identity the empty word. Define a *primitive word* to be a non-empty word that is not a power of a shorter word. A *primitive necklace* is an equivalence class of primitive words under rotation. Use the generating function identity in Problem 3 to prove that the dimension of $F(d)_{k_1,\ldots,k_d}$ is equal to the number of primitive necklaces in which each symbol x_i appears k_i times.

5. A *Lyndon word* is a primitive word that is the lexicographically least representative of its primitive necklace.

(a) Prove that w is a Lyndon word if and only if w is lexicographically less than v for every factorization $w = uv$ such that u and v are non-empty.

(b) Prove that if $w = uv$ is a Lyndon word of length > 1 and v is the longest proper right factor of w which is itself a Lyndon word, then u is also a Lyndon word. This factorization of w is called its *right standard factorization.*

(c) To each Lyndon word w in symbols $x_1, \ldots, x_d$ associate the bracket polynomial $p_w = x_i$ if $w = x_i$ has length 1, or, inductively, $p_w = [p_u, p_v]$, where $w = uv$ is the right standard factorization, if w has length > 1. Prove that the elements p_w form a basis of $F(d)$.

6. Prove that if q is a power of a prime, then the dimension of the subspace of total degree $k_1 + \cdots + k_q = n$ in $F(q)$ is equal to the number of monic irreducible polynomials of degree n over the field with q elements.

7. This problem outlines an alternative proof of the PBW theorem (Theorem 3.24).

(a) Let $L(d)$ denote the Lie subalgebra of $\mathcal{T}(x_1, \ldots, x_d)$ generated by $x_1, \ldots, x_d$. Without using the PBW theorem—in particular, without using $F(d) = L(d)$—show that the value given for $\dim F(d)_{(k_1,\ldots,k_d)}$ by the generating function in Problem 3 is a lower bound for $\dim L(d)_{(k_1,\ldots,k_d)}$.

(b) Show directly that the Lyndon monomials in Problem 5(b) span $F(d)$.

(c) Deduce from (a) and (b) that $F(d) = L(d)$ and that the PBW theorem holds for $F(d)$.

(d) Show that the PBW theorem for a Lie algebra $\mathfrak{g}$ implies the PBW theorem for $\mathfrak{g}/\mathfrak{a}$, where $\mathfrak{a}$ is a Lie ideal, and so deduce PBW for all finitely generate Lie algebras from (c).

(e) Show that the PBW theorem for arbitrarty Lie algebras reduces to the finitely generated case.

8. Let $b(x, y)$ be the Baker-Campbell-Hausdorff series, i.e., $e^{b(x,y)} = e^x e^y$ in noncommuting variables x, y. Let $F(x, y)$ be its linear term in y, that is, $b(x, sy) = x + sF(x, y) + O(s^2)$.

(a) Show that $F(x, y)$ is characterized by the identity

$$\sum_{k,l \geq 0} \frac{x^k\, F(x, y)\, x^l}{(k + l + 1)!} = e^x y. \tag{3.5.6}$$

(b) Let λ, ρ denote the operators of left and right multiplication by x, and let f be the series in two commuting variables such that $F(x, y) = f(\lambda, \rho)(y)$. Show that

$$f(\lambda, \rho) = \frac{\lambda - \rho}{1 - e^{\rho - \lambda}}$$

(c) Deduce that

$$F(x, y) = \frac{\operatorname{ad} x}{1 - e^{-\operatorname{ad} x}}(y).$$

9. Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, $0 \in U' \subseteq U \subseteq \mathfrak{g}$ and $e \in V' \subseteq V \subseteq G$ open neighborhoods such that exp is an isomorphism of U onto V, $\exp(U') = V'$, and $V'V' \subseteq V$. Define $\beta : U' \times U' \to U$ by $\beta(x, y) = \log(\exp(x) \exp(y))$, where $\log : V \to U$ is the inverse of exp.

(a) Show that $\beta(x, (s + t)y) = \beta(\beta(x, ty), sy)$ whenever all arguments are in U'.

(b) Show that the series $(\operatorname{ad} x)/(1 - e^{-\operatorname{ad} x})$, regarded as a formal power series in the coordinates of x with coefficients in the space of linear endomorphisms of $\mathfrak{g}$, converges for all x in a neighborhood of 0 in $\mathfrak{g}$.

(c) Show that on some neighborhood of 0 in $\mathfrak{g}$, $\beta(x, ty)$ is the solution of the initial value problem

$$\beta(x, 0) = x \tag{3.5.7}$$

$$\frac{d}{dt}\beta(x, ty) = F(\beta(x, ty), y), \tag{3.5.8}$$

where $F(x, y) = \big((\operatorname{ad} x)/(1 - e^{-\operatorname{ad} x})\big)(y)$.

(d) Show that the Baker-Campbell-Hausdorff series $b(x, y)$ also satises the identity in part (a), as an identity of formal power series, and deduce that it is the formal power series solution to the IVP in part (c), when $F(x, y)$ is regarded as a formal series.

(e) Deduce from the above an alternative proof that $b(x, y)$ is given as the sum of a series of Lie bracket polynomials in x and y, and that it converges to $\beta(x, y)$ when evaluated on a suitable neighborhood of 0 in $\mathfrak{g}$.

(f) Use part (c) to calculate explicitly the terms of $b(x, y)$ of degree 2 in y.

10. (a) Show that the Lie algebra $\mathfrak{so}(3, \mathbb{C})$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

    (b) Construct a Lie group homomorphism $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(3, \mathbb{C})$ which realizes the isomorphism of Lie algebras in part (a), and calculate its kernel.

11. (a) Show that the Lie algebra $\mathfrak{so}(4, \mathbb{C})$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$.

    (b) Construct a Lie group homomorphism $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(4, \mathbb{C})$ which realizes the isomorphism of Lie algebras in part (a), and calculate its kernel.

12. Show that every closed subgroup H of a Lie group G is a Lie subgroup, so that the inclusion $H \hookrightarrow G$ is a closed immersion.

13. Let G be a Lie group and H a closed subgroup. Show that G/H has a unique manifold structure such that the action of G on it is smooth.

14. Show that the intersection of two Lie subgroups H_1, H_2 of a Lie group G can be given a canonical structure of Lie subgroup so that its Lie algebra is $\mathrm{Lie}(H_1) \cap \mathrm{Lie}(H_2) \subseteq \mathrm{Lie}(G)$.

15. Find the dimension of the closed linear group $\mathrm{SO}(p, q, \mathbb{R}) \subseteq \mathrm{SL}(p+q, \mathbb{R})$ consisting of elements which preserve a non-degenerate symmetric bilinear form on $\mathbb{R}^{p+q}$ of signature (p, q). When is this group connected?

16. Show that the kernel of a Lie group homomorphism $G \to H$ is a closed subgroup of G whose Lie algebra is equal to the kernel of the induced map $\mathrm{Lie}(G) \to \mathrm{Lie}(H)$.

17. Show that if H is a normal Lie subgroup of G, then $\mathrm{Lie}(H)$ is a Lie ideal in $\mathrm{Lie}(G)$.

Chapter 4

General theory of Lie algebras

4.1 $\mathcal{U}\mathfrak{g}$ is a Hopf algebra

[8, Lecture 18]

4.1 Definition *A* Hopf algebra *over $\mathbb{K}$ is a (unital, counital) bialgebra $(U, \mu, e, \Delta, \epsilon)$ along with a bialgebra map $\mathcal{S} : U \to U^{\text{op}}$ called the* antipode*, where U^{op} is U as a vector space, with the opposite multiplication and the opposite comultiplication. I.e. we define $\mu^{\text{op}} : U \otimes U \to U$ by $\mu^{\text{op}}(x \otimes y) = \mu(y \otimes x)$, and $\Delta^{\text{op}} : U \to U \otimes U$ by $\Delta^{\text{op}}(x) = \sum x_{(2)} \otimes x_{(1)}$, where $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$. The antipode $\mathcal{S}$ is required to make the following pentagons commute:*

$$\begin{tikzcd}
 & U \otimes U \arrow[r, "1_G \otimes \mathcal{S}"] & U \otimes U \arrow[dr, "\mu"] & \\
U \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \mathbb{K} \arrow[r, "e"] & U \\
 & U \otimes U \arrow[r, "\mathcal{S} \otimes 1_G"'] & U \otimes U \arrow[ur, "\mu"'] &
\end{tikzcd} \tag{4.1.1}$$

4.2 Definition *An algebra (U, μ, e) is* commutative *if $\mu^{\text{op}} = \mu$. A coalgebra (U, Δ, ϵ) is* cocommutative *if $\Delta^{\text{op}} = \Delta$.*

4.3 Example Let G be a finite group and $\mathscr{C}(G)$ the algebra of functions on it. Then $\mathscr{C}(G)$ is a commutative Hopf algebra with $\Delta(f)(x, y) = f(xy)$, where we have identified $\mathscr{C}(G) \otimes \mathscr{C}(G)$ with $\mathscr{C}(G \times G)$, and $\mathcal{S}(f)(x) = f(x^{-1})$.

Let G be an algebraic group, and $\mathscr{C}(G)$ the algebra of polynomial functions on it. Then Then $\mathscr{C}(G)$ is a commutative Hopf algebra with $\Delta(f)(x, y) = f(xy)$, where we have identified $\mathscr{C}(G) \otimes \mathscr{C}(G)$ with $\mathscr{C}(G \times G)$, and $\mathcal{S}(f)(x) = f(x^{-1})$.

Let G be a group and $\mathbb{K}[G]$ the group algebra of G, with mutliplication defined by $\mu(x \otimes y) = xy$ for $x, y \in G$. Then G is a cocommutative Hopf algebra with $\Delta(x) = x \otimes x$ and $\mathcal{S}(x) = x^{-1}$ for $x \in G$.

Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{U}\mathfrak{g}$ its universal enveloping algebra. We have seen already (Proposition 3.28) that $\mathcal{U}\mathfrak{g}$ is naturally a bialgebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$; we make $\mathcal{U}\mathfrak{g}$ into a Hopf algebra by defining $\mathcal{S}(x) = -x$ for $x \in \mathfrak{g}$.

4.4 Lemma / Definition *Let U be a cocommutative Hopf algebra. Then the antipode is an involution. Moreover, the category of (algebra-) representations of U has naturally the structure of a symmetric monoidal category with duals. In particular, to each pair of representations V, W of U, there are natural ways to make $V \otimes_{\mathbb{K}} W$ and $\mathrm{Hom}_{\mathbb{K}}(V, W)$ into U-modules. Then any (di)natural functorial contruction of vector spaces — for example $V \otimes W \cong W \otimes V$, $\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W)$, and $W : V \mapsto \mathrm{Hom}(\mathrm{Hom}(V, W), W)$ — in fact corresponds to a homomorphism of U-modules.*

Proof Proving the last part would require we go further into category theory than we would like. We describe the U-action on $V \otimes_{\mathbb{K}} W$ and on $\mathrm{Hom}_{\mathbb{K}}(V, W)$ when V and W are U-modules. For $u \in U$, let $\Delta(u) = \sum u_{(1)} \otimes u_{(2)} = \sum u_{(2)} \otimes u_{(1)}$, and write the actions of u on $v \in V$ and on $w \in W$ as $u \cdot v \in V$ and $u \cdot w \in W$. Let $\phi \in \mathrm{Hom}_{\mathbb{K}}(V, W)$. Then we define:

$$u \cdot (v \otimes w) \stackrel{\text{def}}{=} \sum (u_{(1)} \cdot v) \otimes (u_{(2)} \cdot w) \tag{4.1.2}$$

$$u \cdot \phi \stackrel{\text{def}}{=} \sum u_{(1)} \circ \phi \circ \mathcal{S}(u_{(2)}) \tag{4.1.3}$$

Moreover, the counit map $\epsilon : U \to \mathbb{K}$ makes $\mathbb{K}$ into U-module, and it is the unit of the monoidal structure. □

4.5 Remark Equation 4.1.2 makes the category of U-modules into a monoidal category for any bialgebra U. One can define duals via equation 4.1.3, but if U is not cocommutative, then $\mathcal{S}$ may not be an involution, so a choices is required as to which variation of equation 4.1.3 to take. Moreover, when U is not cocommutative, we do not, in general, have an isomorphism $V \otimes W \cong W \otimes V$. See [17] and references therein for more discussion of Hopf algebras.

4.6 Example When $U = \mathcal{U}\mathfrak{g}$ and $x \in \mathfrak{g}$, then x acts on $V \otimes W$ by $v \otimes w \mapsto xv \otimes w + v \otimes w$, and on $\mathrm{Hom}(V, W)$ by $\phi \mapsto x \circ \phi - \phi \circ x$.

4.7 Definition *Let (U, μ, e, ϵ) be a "counital algebra" over $\mathbb{K}$, i.e. an algebra along with an algebra map $\epsilon : U \to \mathbb{K}$; thus ϵ makes $\mathbb{K}$ into a U-module. Let V be a U-module. An element $v \in V$ is* U-invariant *if the linear map $\mathbb{K} \to V$ given by $1 \mapsto v$ is a U-module homomorphism. We write V^U for the vector space of U-invariant elements of V.*

4.8 Lemma *When U is a cocommutative Hopf algebra, the space $\mathrm{Hom}_{\mathbb{K}}(V, W)^U$ of U-invariant linear maps is the same as the space $\mathrm{Hom}_U(V, W)$ of U-module homomorphisms.*

4.9 Example The $\mathcal{U}\mathfrak{g}$-invariant elements of a $\mathfrak{g}$-module V is the set $V^{\mathfrak{g}} = \{v \in V \text{ s.t. } x \cdot v = 0\, \forall x \in \mathfrak{g}\}$. We shorten the word "$\mathcal{U}\mathfrak{g}$-invariant" to "$\mathfrak{g}$-invariant". A linear map $\phi \in \mathrm{Hom}_{\mathbb{K}}(V, W)$ is $\mathfrak{g}$-invariant if and only if $x \circ \phi = \phi \circ x$ for every $x \in \mathfrak{g}$.

4.10 Definition *The* center *of a Lie algebra $\mathfrak{g}$ is the space of $\mathfrak{g}$-invariant elements of $\mathfrak{g}$ under the adjoint action: $Z(\mathfrak{g}) \stackrel{\text{def}}{=} \mathfrak{g}^{\mathfrak{g}} = \{x \in \mathfrak{g} \text{ s.t. } [\mathfrak{g}, x] = 0\}$.*

4.2 Structure Theory of Lie Algebras

4.2.1 Many Definitions

[8, Lectures 17 and 18]

As always, we write "$\mathfrak{g}$-module" for "$\mathcal{U}\mathfrak{g}$-module".

4.11 Definition *A $\mathfrak{g}$-module V is* simple *or* irreducible *if there is no submodule $W \subseteq V$ with $0 \neq W \neq V$. A Lie algebra is* simple *if it is simple as a $\mathfrak{g}$-module under the adjoint action. An* ideal *of $\mathfrak{g}$ is a $\mathfrak{g}$-submodule of $\mathfrak{g}$ under the adjoint action.*

4.12 Proposition *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in $\mathfrak{g}$, then so is $[\mathfrak{a}, \mathfrak{b}]$.*

4.13 Definition *The* upper central series *of a Lie algebra $\mathfrak{g}$ is the series $\mathfrak{g} \geq \mathfrak{g}_1 \geq \mathfrak{g}_2 \geq \dots$ where $\mathfrak{g}_0 \stackrel{\text{def}}{=} \mathfrak{g}$ and $\mathfrak{g}_{n+1} \stackrel{\text{def}}{=} [\mathfrak{g}, \mathfrak{g}_n]$. The Lie algebra $\mathfrak{g}$ is* nilpotent *if $\mathfrak{g}_n = 0$ for some n.*

4.14 Definition *The* derived subalgebra *of a Lie algebra $\mathfrak{g}$ is the algebra $\mathfrak{g}' \stackrel{\text{def}}{=} [\mathfrak{g}, \mathfrak{g}]$. The* derived series *of $\mathfrak{g}$ is the series $\mathfrak{g} \geq \mathfrak{g}' \geq \mathfrak{g}'' \geq \dots$ given by $\mathfrak{g}^{(0)} \stackrel{\text{def}}{=} \mathfrak{g}$ and $\mathfrak{g}^{(n+1)} \stackrel{\text{def}}{=} [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$. The Lie algebra $\mathfrak{g}$ is* solvable *if $\mathfrak{g}^{(n)} = 0$ for some n. An ideal $\mathfrak{r}$ in $\mathfrak{g}$ is solvable if it is solvable as a subalgebra. By Proposition 4.12, if $\mathfrak{r}$ is an ideal of $\mathfrak{g}$, then so is $\mathfrak{r}^{(n)}$.*

4.15 Example The Lie algebra of upper-triangular matrices in $\mathfrak{gl}(n)$ is solvable. A converse to this statement is Corollary 4.38. The Lie algebra of strictly upper triangular matrices is nilpotent.

4.16 Definition *A Lie algebra $\mathfrak{g}$ is* semisimple *if its only solvable ideal is* 0.

4.17 Remark If $\mathfrak{r}$ is a solvable ideal of $\mathfrak{g}$ with $\mathfrak{r}^{(n)} = 0$, then $\mathfrak{r}^{(n-1)}$ is abelian. Conversely, any abelian ideal of $\mathfrak{g}$ is solvable. Thus it is equivalent to replace the word "solvable" in Definition 4.16 with the word "abelian".

4.18 Proposition *Any nilpotent Lie algebra is solvable. A non-zero nilpotent Lie algebra has non-zero center.*

4.19 Proposition *A subquotient of a solvable Lie algebra is solvable. A subquotient of a nilpotent algebra is nilpotent. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ and if $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ are both solvable, then $\mathfrak{g}$ is solvable. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{a}$ is nilpotent and if $\mathfrak{g} \curvearrowright \mathfrak{a}$ nilpotently, then $\mathfrak{g}$ is nilpotent. Thus a central extension of a nilpotent Lie algebra is nilpotent.*

Proof The derived and upper central series of subquotients are subquotients of the derived and upper central series. For the second statement, we start taking the derived series of $\mathfrak{g}$, eventually landing in $\mathfrak{a}$ (since $\mathfrak{g}/\mathfrak{a} \to 0$), which is solvable. The nilpotent claim is similar.

4.20 Example Let $\mathfrak{g} = \langle x, y : [x, y] = y \rangle$ be the two-dimensional nonabelian Lie algebra. Then $\mathfrak{g}^{(1)} = \langle y \rangle$ and $\mathfrak{g}^{(2)} = 0$, but $\mathfrak{g}_2 = [\mathfrak{g}, \langle y \rangle] = \langle y \rangle$ so $\mathfrak{g}$ is solvable but not nilpotent.

4.21 Definition *The* lower central series *of a Lie algebra* $\mathfrak{g}$ *is the series* $0 \leq Z(\mathfrak{g}) \leq \mathfrak{z}_2 \leq \dots$ *defined by* $\mathfrak{z}_0 = 0$ *and* $\mathfrak{z}_{k+1} = \{x \in \mathfrak{g} \text{ s.t. } [\mathfrak{g}, x] \subseteq \mathfrak{z}_k\}$.

4.22 Proposition *For any of the derived series, the upper central series, and the lower central series, quotients of consecutive terms are abelian.*

4.23 Proposition *Let* $\mathfrak{g}$ *be a Lie algebra and* $\{\mathfrak{z}_k\}$ *its lower central series. Then* $\mathfrak{z}_n = \mathfrak{g}$ *for some* n *if and only if* $\mathfrak{g}$ *is nilpotent.*

4.2.2 Nilpotency: Engel's Theorem and Corollaries

[8, Lecture 19]

4.24 Lemma / Definition *A matrix* $x \in \mathrm{End}(V)$ *is* nilpotent *if* $x^n = 0$ *for some* n. *A Lie algebra* $\mathfrak{g}$ acts by nilpotents *on a vector space* V *if for each* $x \in \mathfrak{g}$, *its image under* $\mathfrak{g} \to \mathrm{End}(V)$ *is nilpotent. If* $\mathfrak{g} \curvearrowright V, W$ *by nilpotents, then* $\mathfrak{g} \curvearrowright V \otimes W$ *and* $g \curvearrowright \mathrm{Hom}(V, W)$ *by nilpotents. If* $v \in V$ *and* $\mathfrak{g} \curvearrowright V$, *define the* annihilator *of* v *to be* $\mathrm{ann}_{\mathfrak{g}}(v) = \{x \in \mathfrak{g} \text{ s.t. } xv = 0\}$. *For any* $v \in V$, $\mathrm{ann}_(v)$ *is a Lie subalgebra of* $\mathfrak{g}$.

4.25 Theorem (Engel's Theorem) *If* $\mathfrak{g}$ *is a finite-dimensional Lie algebra acting on* V *(possibly infinite-dimensional) by nilpotent endomorphisms, and* $V \neq 0$, *then there exists a non-zero vector* $v \in V$ *such that* $\mathfrak{g}v = 0$.

Proof It suffices to look at the image of $\mathfrak{g}$ in $\mathfrak{gl}(V) = \mathrm{Hom}(V, V)$. Then $\mathrm{ad} : \mathfrak{g} \curvearrowright \mathfrak{g}$ is by nilpotents.

Pick $v_0 \in V$ so that $\mathrm{ann}_{\mathfrak{g}}(v_0)$ has maximal dimension and let $\mathfrak{h} = \mathrm{ann}_{\mathfrak{g}}(v_0)$. It suffices to show that $\mathfrak{h} = \mathfrak{g}$; suppose to the contrary that $\mathfrak{h} \subsetneq \mathfrak{g}$. By induction on dimension, the theorem holds for $\mathfrak{h}$. Consider the vector space $\mathfrak{g}/\mathfrak{h}$; then $\mathfrak{h} \curvearrowright \mathfrak{g}/\mathfrak{h}$ by nilpotents, so we can find $x \in \mathfrak{g}/\mathfrak{h}$ nonzero with $\mathfrak{h}x = 0$. Let $\hat{x}$ be a preimage of x in $\mathfrak{g}$. Then $\hat{x} \in \mathfrak{g} \smallsetminus \mathfrak{h}$ and $[\mathfrak{h}, \hat{x}] \subseteq \mathfrak{h}$. Then $\mathfrak{h}_1 \stackrel{\text{def}}{=} \langle \hat{x} \rangle + \mathfrak{h}$ is a subalgebra of $\mathfrak{g}$.

The space $U \stackrel{\text{def}}{=} \{u \in V \text{ s.t. } \mathfrak{h}u = 0\}$ is non-zero, since $v_0 \in U$. We see that U is an $\mathfrak{h}_1$-submodule of $\mathfrak{h}_1 \curvearrowright V$: $hu = 0 \in U$ for $h \in \mathfrak{h}$, and $h(\hat{x}u) = [h, \hat{x}]u + \hat{x}hu = 0u + \hat{x}0 = 0$ so $\hat{x}u \in U$. All of $\mathfrak{g}$ acts on all of V by nilpotents, so in particular $x|_U$ is nilpotent, and so there is some vector $v_1 \in U$ with $xv_1 = 0$. But then $\mathfrak{h}_1 v_1 = 0$, contradicting the maximality of $\mathfrak{h} = \mathrm{ann}(v_0)$. □

4.26 Corollary

1. *If* $\mathfrak{g} \curvearrowright V$ *by nilpotents and* V *is finite dimension, then* V *has a basis in which* $\mathfrak{g}$ *is strictly upper triangular.*

2. *If* $\mathrm{ad}\, x$ *is nilpotent for all* $x \in \mathfrak{g}$ *finite-dimensional, then* $\mathfrak{g}$ *is a nilpotent Lie algebra.*

3. *Let* V *be a simple* $\mathfrak{g}$*-module. If an ideal* $\mathfrak{a} \leq \mathfrak{g}$ *acts nilpotently on* V *then* $\mathfrak{a}$ *acts as* 0 *on* V.

4.27 Lemma / Definition *If* V *is a finite-dimensional* $\mathfrak{g}$ *module, then there exists a* Jordan-Holder series $0 = M_0 < M_1 < M_2 < \dots < \mathrm{M}(n) = V$ *such that each* M_i *is a* $\mathfrak{g}$*-submodule and each* M_{i+1}/M_i *is simple.*

4.28 Corollary *Let V be a finite-dimensional $\mathfrak{g}$-module and $0 = M_0 < M_1 < M_2 < \cdots < \mathrm{M}(n) = V$ a Jordan-Holder series for V. An ideal $\mathfrak{a} \leq \mathfrak{g}$ acts by nilpotents on V if and only if $\mathfrak{a}$ acts by 0 on each M_{i+1}/M_i. Thus there is a largest ideal of $\mathfrak{g}$ that acts by nilpotents on V.*

4.29 Definition *The largest ideal of $\mathfrak{g}$ that acts by nilpotents on V is the* nilpotency ideal *of the action $\mathfrak{g} \curvearrowright V$.*

4.30 Proposition *Any nilpotent ideal $\mathfrak{a} \leq \mathfrak{g}$ acts nilpotently on $\mathfrak{g}$.*

4.31 Corollary *Any finite-dimensional Lie algebra has a largest nilpotent ideal: the nilpotency ideal of* ad*.*

4.32 Remark Not every ad-nilpotent element of a Lie algebra is necessarily in the nilpotency ideal of ad.

4.33 Definition *Let $\mathfrak{g}$ be a Lie algebra and V a finite-dimensional $\mathfrak{g}$-module. Then V defines a* trace form *β_V: a symmetric bilinear form on $\mathfrak{g}$ given by $\beta_V(x, y) \stackrel{\text{def}}{=} \mathrm{tr}_V(x, y)$. The* radical *or* kernel *of β_V is the set $\ker \beta_V \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \text{ s.t. } \beta_V(x, \mathfrak{g}) = 0\}$.*

4.34 Remark The more standard notation seems to be $\mathrm{rad}\,\beta$ for what we call $\ker \beta$, c.f. [8]. We prefer the term "kernel" largely to avoid the conflict of notation with Lemma/Definition 4.36. Any bilinear form β on V defines two linear maps $V \to V^*$, where V^* is the dual vector space to V, given by $x \mapsto \beta(x, -)$ and $x \mapsto \beta(-, x)$. Of course, when β is symmetric, these are the same map, and we can unambiguously call the map $\beta : V \to V^*$. Then $\ker \beta_V$ defined above is precisely the kernel of the map $\beta_V : \mathfrak{g} \to \mathfrak{g}^*$.

The following proposition follows from considering Jordan-Holder series:

4.35 Proposition *If and ideal $\mathfrak{a} \leq \mathfrak{g}$ of a finite-dimensional Lie algebra acts nilpotently on a finite-dimensional vector space V, then $\mathfrak{a} \leq \ker \beta_V$.*

4.2.3 Solvability: Lie's Theorem and Corollaries

[8, Lectures 19 and 20]

4.36 Lemma / Definition *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ has a largest solvable ideal, the* radical $\mathrm{rad}\,\mathfrak{g}$*.*

Proof If ideals $\mathfrak{a}, \mathfrak{b} \leq \mathfrak{g}$ are solvable, then $\mathfrak{a} + \mathfrak{b}$ is solvable, since we have an exact sequence of $\mathfrak{g}$-modules

$$0 \to \mathfrak{a} \to \mathfrak{a} + \mathfrak{b} \to (\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \to 0 \tag{4.2.1}$$

which is also an extension of a solvable algebra (a quotient of $\mathfrak{b}$) by a solvable ideal. □

4.37 Theorem (Lie's Theorem) *Let $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra over $\mathbb{K}$ of characteristic 0, and V a non-zero $\mathfrak{g}$-module. Assume that $\mathbb{K}$ contains eigenvalues of the actions of all $x \in \mathfrak{g}$. Then V has a one-dimensional $\mathfrak{g}$-submodule.*

Proof Without loss of generality $\mathfrak{g} \neq 0$; then $\mathfrak{g}' \neq \mathfrak{g}$ by solvability. Pick any $\mathfrak{g} \geq \mathfrak{h} \geq \mathfrak{g}'$ a codimension-1 subspace. Since $\mathfrak{h} \geq \mathfrak{g}'$, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Pick $x \in \mathfrak{g} \smallsetminus \mathfrak{h}$, whence $\mathfrak{g} = \langle x \rangle + \mathfrak{h}$.

Being a subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ is solvable, and by induction on dimension $\mathfrak{h} \curvearrowright V$ has a one-dimensional $\mathfrak{h}$-submodule $\langle w \rangle$. Thus there is some linear map $\lambda : \mathfrak{h} \to \mathbb{K}$ so that $h \cdot w = \lambda(h) w$ for each $h \in \mathfrak{h}$. Let $W = \mathbb{K}[x]w$ for $x \in \mathfrak{g} \smallsetminus \mathfrak{h}$ as above. Then $W = \mathcal{U}(\mathfrak{g})w$, as $\mathfrak{g} = \mathfrak{h} + \langle x \rangle$ and $\mathfrak{h}w \subseteq \mathbb{K}w$.

By induction on m, each $\langle 1, x, \dots, x^m \rangle w$ is an $\mathfrak{h}$-submodule of W:

$$h(x^m w) = x^m h w + \sum_{k+l=m-1} x^k [h, x] x^l w \tag{4.2.2}$$

$$= \lambda(h) x^m w + x^k h' x^l w \tag{4.2.3}$$

where $h' = [h, x] \in \mathfrak{h}$. Thus $h' x^l w \in \langle 1, \dots, x^l \rangle w$ by induction, and so $x^k h' x^l w \in \langle 1, \dots, x^{k+l} \rangle w = \langle 1, \dots, x^{m-1} \rangle w$.

Moreover, we see that W is a generalized eigenspace with eigenvalue $\lambda(h)$ for all $h \in \mathfrak{h}$, and so $\operatorname{tr}_W h = (\dim W)\lambda(h)$, by working in a basis where h is upper triangular. But for any a, b, $\operatorname{tr}[a, b] = 0$; thus $\operatorname{tr}_W [h, x] = 0$ so $\lambda([h, x]) = 0$. Then equations 4.2.2 to 4.2.3 and induction on m show that W is an actual eigenspace.

Thus we can pick $v \in W$ an eigenvector of x, and then v generates a one-dimensional eigenspace of $x + \mathfrak{h} = \mathfrak{g}$, i.e. a one-dimensional $\mathfrak{g}$-submodule. □

4.38 Corollary *Let $\mathfrak{g}$ and V satisfy the conditions of Theorem 4.37. Then V has a basis in which $\mathfrak{g}$ is upper-diagonal.*

4.39 Corollary *Let $\mathfrak{g}$ be a solvable finite-dimensional Lie algebra over an algebraically closed field of characteristic* 0*. Then every simple finite-dimensional $\mathfrak{g}$-module is one-dimensional.*

4.40 Corollary *Let $\mathfrak{g}$ be a solvable finite-dimensional Lie algebra over a field of characteristic* 0*. Then $\mathfrak{g}'$ acts nilpotently on any finite-dimensional $\mathfrak{g}$-module.*

4.41 Remark In spite of the condition on the ground field in Theorem 4.37, Corollary 4.40 is true over any field of characteristic 0. Indeed, let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and $\mathbb{K} \leq \mathbb{L}$ a field extension. The upper central, lower central, and derived series are all preserved under $\mathbb{L} \otimes_{\mathbb{K}}$, so $\mathbb{L} \otimes_{\mathbb{K}} \mathfrak{g}$ is solvable if and only if $\mathfrak{g}$ is. Moreover, $\mathfrak{g} \curvearrowright V$ nilpotently if and only if $\mathbb{L} \otimes_{\mathbb{K}} \mathfrak{g} \curvearrowright \mathbb{L} \otimes_{\mathbb{K}} V$ nilpotently. Thus we may as well "extend by scalars" to an algebraically closed field.

4.42 Corollary *Corollary 4.31 asserts that any Lie algebra $\mathfrak{g}$ has a largest ideal that acts nilpotently on $\mathfrak{g}$. When $\mathfrak{g}$ is solvable, then any element of $\mathfrak{g}'$ is* ad*-nilpotent. Hence the set of* ad*-nilpotent elements of $\mathfrak{g}$ is an ideal.*

4.2.4 The Killing Form

[8, Lecture 20]

We recall Definition 4.33.

4.43 Proposition *Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. The trace form $\beta_V : (x, y) \mapsto \operatorname{tr}_V(xy)$ on $\mathfrak{g}$ is invariant under the $\mathfrak{g}$-action:*

$$\beta_V\Big([z, x], y\Big) + \beta_V\Big(x, [z, y]\Big) = 0 \tag{4.2.4}$$

4.44 Definition *Let $\mathfrak{g}$ be a Lie algebra. The* Killing form $\beta \stackrel{\text{def}}{=} \beta_{(\mathfrak{g},\mathrm{ad})}$ *on $\mathfrak{g}$ is the trace form of the adjoint representation $\mathfrak{g} \curvearrowright \mathfrak{g}$.*

4.45 Proposition *Let $\mathfrak{g}$ be a Lie algebra, V a $\mathfrak{g}$-module, and $W \subseteq V$ a $\mathfrak{g}$-submodule. Then $\beta_V = \beta_W + \beta_{V/W}$.*

4.46 Corollary *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a} \leq \mathfrak{g}$ an ideal. Then $\beta_{(\mathfrak{g}/\mathfrak{a},\mathrm{ad})}|_{\mathfrak{a}\times\mathfrak{g}} = 0$, so $\beta|_{\mathfrak{a}\times\mathfrak{g}} = \beta_{\mathfrak{a}}|_{\mathfrak{a}\times\mathfrak{g}}$. In particular, the Killing form of $\mathfrak{a}$ is $\beta|_{\mathfrak{a}\times\mathfrak{a}}$.*

4.47 Proposition *Let V be a $\mathfrak{g}$-module of a Lie algebra $\mathfrak{g}$. Then $\ker \beta_V$ is an ideal of $\mathfrak{g}$.*

Proof The invariance of β_V implies that the map $\beta_V : \mathfrak{g} \to \mathfrak{g}^*$ given by $x \mapsto \beta_V(x, -)$ is a $\mathfrak{g}$-module homomorphism, whence $\ker \beta_V$ is a submodule a.k.a. and ideal of $\mathfrak{g}$. □

The following is a corollary to Theorem 4.25, using the Jordan-form decomposition of matrices:

4.48 Proposition *Let $\mathfrak{g}$ be a Lie algebra, V a $\mathfrak{g}$-module, and $\mathfrak{a}$ an ideal of $\mathfrak{g}$ that acts nilpotently on V. Then $\mathfrak{a} \subseteq \ker \beta_V$.*

4.49 Corollary *If the Killing form β of a Lie algebra $\mathfrak{g}$ is* nondegenerate *(i.e. if $\ker \beta = 0$), then $\mathfrak{g}$ is semisimple.*

4.2.5 Jordan Form

[8, Lecture 20], [18, page 57][1]

4.50 Theorem (Jordan decomposition) *Let V be a finite-dimensional vector space over an algebraically closed field $\mathbb{K}$. Then:*

[1] My notes from MH's class [8] on the proof Theorem 4.50 are sadly incomplete, and so I have quoted from [18]. The statement of the Jordan decomposition in [8, Proposition 20.3] is stronger than in [18, Lemma 12.4]: in the former, not only do we assert that $s, n \in \mathbb{K}[x]$, but that they are in $x\mathbb{K}[x]$. This is probably equivalent, but I haven't thought about it enough to be sure.

The two statements in this section are leading towards the statement and proof of Proposition 4.52. It seems that this theorem requires an unmotivated piece of linear algebra; for us, this is Lemma 4.51. [18, Lemma 12.3] states this result differently; only by inspecting the proofs are they obviously equivalent:

Lemma *Let s be a diagonalizable linear operator on a vector space V over $\mathbb{K}$ algebraically closed of characteristic 0. If $s = a \operatorname{diag}(\lambda_1, \dots, \lambda_n) a^{-1}$ for a an invertible matrix over V, and given $f : \mathbb{K} \to \mathbb{K}$ an arbitrary function, we define $f(x)$ as $a \operatorname{diag}(f(\lambda_1), \dots, f(\lambda_n)) a^{-1}$. Suppose that $\operatorname{tr}(xf(x)) = 0$ for any $\mathbb{Q}$-linear map $f : \mathbb{K} \to \mathbb{K}$ that restricts to the identity on $\mathbb{Q} \hookrightarrow \mathbb{K}$. Then $s = 0$.*

1. *Every* $a \in \mathfrak{gl}(V)$ *has a unique* Jordan decomposition $a = s + n$, *where* s *is diagonalizable,* n *is nilpotent, and they commute.*

2. $s, n \in \mathbb{K}[a]$, *in the sense that they are linear combinations of powers of* a*; as* a *varies,* s *and* n *need not depend polynomially on* a.

Proof We write a in Jordan form; since strictly-upper-triangular matrices are nilpotent, existence of a Jordan decomposition of a is guaranteed. In particular, the diagonal part s clearly commutes with a, and hence with $n = a - s$. We say this again more specifically, showing that s, n constructed this way are polynomials in x:

Let the characteristic polynomial of a be $\prod_i (x - \lambda_i)^{n_i}$. In particular, $(x - \lambda_i)$ are relatively prime, so by the Chinese Remainder Theorem, there is a polynomial f such that $f(x) = \lambda_i \mod (x - \lambda_i)^{n_i}$. Choose a basis of V in which a is in Jordan form; since restricting to a Jordan block b of a is an algebra homomorphism $\mathbb{K}[a] \twoheadrightarrow \mathbb{K}[b]$, we can compute $f(a)$ block-by-block. Let b be a block of a with eigenvalue λ_i. Then $(b - \lambda_i)^{n_i} = 0$, so $f(b) = \lambda_i$. Thus $s = f(a)$ is diagonal in this basis, and $n = a - f(a)$ is nilpotent.

For uniqueness in part 1., let $x = n' + s'$ be any other Jordan decomposition of a. Then n' and s' commut with a and hence with any polynomial in a, and in particular n' commutes with n and s' commutes with s. But $n' + s' = a = n + s$, so $n' - n = s' - s$. Since everything commutes, $n' - n$ is nilpotent and $s' - s$ is diagonalizable, but the only nilpotent diagonal is 0. $\square$

We now move to an entirely unmotivated piece of linear algebra:

4.51 Lemma *Let* V *be a finite-dimensional vector space over an algebraically closed field* $\mathbb{K}$ *of characteristic* 0. *Let* $B \subseteq A \subseteq \mathfrak{gl}(V)$ *be any subspaces, and define* $T = \{x \in \mathfrak{gl}(V) : [x, A] \subseteq B\}$. *Then if* $t \in T$ *satisfies* $\mathrm{tr}_v(tu) = 0 \ \forall u \in T$, *then* t *is nilpotent.*

We can express this as follows: Let β_V be the trace form on $\mathfrak{gl}(V) \curvearrowright V$. Then $\ker \beta_V|_{T \times T}$ consists of nilpotents.

Proof Let $t = s + n$ be the Jordan decomposition; we wish to show that $s = 0$. We fix a basis $\{e_i\}$ in which s is diagonal: $se_i = \lambda_i e_i$. Let $\{e_{ij}\}$ be the corresponding basis of matrix units for $\mathfrak{gl}(V)$. Then $(\mathrm{ad}\, s)e_{ij} = (\lambda_i - \lambda_j)e_{ij}$.

Now let $\Lambda = \mathbb{Q}\{\lambda_i\}$ be the finite-dimensional $\mathbb{Q}$-vector-subspace of $\mathbb{K}$. We consider an arbitrary $\mathbb{Q}$-linear functional $f : \Lambda \to \mathbb{Q}$; we will show that $f = 0$, and hence that $\Lambda = 0$.

By $\mathbb{Q}$-linearity, $f(\lambda_i) - f(\lambda_j) = f(\lambda_i - \lambda_j)$, and we chose a polynomial $p(x) \in \mathbb{K}[x]$ so that $p(\lambda_i - \lambda_j) = f(\lambda_i) - f(\lambda_j)$; in particular, $p(0) = 0$.

Now we define $u \in \mathfrak{gl}(V)$ by $ue_i = f(\lambda_i)e_i$, and then $(\mathrm{ad}\, u)e_{ij} = (f(\lambda_i) - f(\lambda_j))e_{ij} = p(\mathrm{ad}\, s)e_{ij}$. So $\mathrm{ad}\, u = p(\mathrm{ad}\, s)$.

Since $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism, $\mathrm{ad}\, t = \mathrm{ad}\, s + \mathrm{ad}\, n$, and $\mathrm{ad}\, s, \mathrm{ad}\, n$ commute, and $\mathrm{ad}\, s$ is diagonalizable and $\mathrm{ad}\, n$ is nilpotent. So $\mathrm{ad}\, s + \mathrm{ad}\, n$ is the Jordan decomposition of $\mathrm{ad}\, t$, and hence $\mathrm{ad}\, s = q(\mathrm{ad}\, t)$ for some polynomial $q \in \mathbb{K}[x]$. Then $\mathrm{ad}\, u = (p \circ q)(\mathrm{ad}\, t)$, and since every power of t takes A into B, we have $(\mathrm{ad}\, u)A \subseteq B$, so $u \in T$.

But by construction u is diagonal in the $\{e_i\}$ basis and t is upper-triangular, so tu is upper-triangular with diagonal $\mathrm{diag}(\lambda_i f(\lambda_i))$. Thus $0 = \mathrm{tr}(tu) = \sum \lambda_i f(\lambda_i)$. We apply f to this: $0 = \sum (f(\lambda_i))^2 \in \mathbb{Q}$, so $f(\lambda_i) = 0$ for each i. Thus $f = 0$. $\square$

4.2.6 Cartan's Criteria

[8, Lecture 21]

4.52 Proposition *Let V be a finite-dimensional vector space over a field $\mathbb{K}$ of characteristic 0. Then a subalgebra $\mathfrak{g} \leq \mathfrak{gl}(V)$ is solvable if and only if $\beta_V(\mathfrak{g}, \mathfrak{g}') = 0$, i.e. $\mathfrak{g}' \leq \ker \beta_V$.*

Proof We can extend scalars and assume that $\mathbb{K}$ is algebraically closed, thus we can use Lemma 4.51.

The forward direction follows by Lie's theorem (Theorem 4.37): we can find a basis of V in which $\mathfrak{g}$ acts by upper-triangular matrices, and hence $\mathfrak{g}'$ acts by strictly upper-triangular matrices.

For the reverse, we'll show that $\mathfrak{g}'$ acts nilpotently, and hence is nilpotent by Engel's theorem (Theorem 4.25). We use Lemma 4.51, taking $V = V$, $A = \mathfrak{g}$, and $B = \mathfrak{g}'$. Then $T = \{t \in \mathfrak{gl}(V) \text{ s.t. } [t, \mathfrak{g}] \leq \mathfrak{g}'\}$, and in particular $\mathfrak{g} \leq T$, and so $\mathfrak{g}' \leq T$.

So if $[x, y] = t \in \mathfrak{g}'$, then $\mathrm{tr}_V(tu) = \mathrm{tr}_V([x, y]u) = \mathrm{tr}_V(y[x, u])$ by invariance, and $y \in \mathfrak{g}$ and $[x, u] \in \mathfrak{g}'$ so $\mathrm{tr}_V(y[x, u]) = 0$. Hence t is nilpotent. $\square$

The following is a straightforward corollary:

4.53 Theorem (Cartan's First Criterion) *Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic 0. Then $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}' \leq \ker \beta$.*

Proof We have not yet proven Theorem 4.99, so we cannot assume that $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ for some V. Rather, we let $V = \mathfrak{g}$ and $\tilde{\mathfrak{g}} = \mathfrak{g}/Z(\mathfrak{g})$, whence $\tilde{\mathfrak{g}} \hookrightarrow \mathfrak{gl}(V)$ by the adjoint action. Then $\mathfrak{g}$ is a central extension of $\tilde{\mathfrak{g}}$, so by Proposition 4.19 $\mathfrak{g}$ is solvable if and only if $\tilde{\mathfrak{g}}$ is. By Proposition 4.52, $\tilde{\mathfrak{g}}$ is solvable if and only if $\beta_{\tilde{\mathfrak{g}}}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}') = 0$. But $\beta_{\mathfrak{g}}$ facrots through $\beta_{\tilde{\mathfrak{g}}}$:

$$\beta_{\mathfrak{g}} = \{\mathfrak{g} \times \mathfrak{g} \xrightarrow{/Z(\mathfrak{g})} \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \xrightarrow{\beta_{\tilde{\mathfrak{g}}}} \mathbb{K}\} \tag{4.2.5}$$

Moreover, $\mathfrak{g}' \overset{/Z(\mathfrak{g})}{\twoheadrightarrow} \tilde{\mathfrak{g}}'$, and so $\beta_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}') = \beta_{\tilde{\mathfrak{g}}}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}')$. $\square$

4.54 Corollary *For any Lie algebra $\mathfrak{g}$ in characteristic zero with Killing form β, we have that $\ker \beta$ is solvable, i.e. $\ker \beta \leq \mathrm{rad}\, \mathfrak{g}$.*

The reverse direction of the following is true in any characteristic (Corollary 4.49). The forward direction is an immediate corollary of Corollary 4.54.

4.55 Theorem (Cartan's Second Criterian) *Let $\mathfrak{g}$ be a Lie algebra over characteristic 0, and β its Killing form. Then $\mathfrak{g}$ is semisimple if and only if $\ker \beta = 0$.*

4.56 Corollary *Let $\mathfrak{g}$ be a Lie algebra over characteristic 0. The $\mathfrak{g}$ is semisimple if and only if any extension by scalars of $\mathfrak{g}$ is semisimple.*

4.57 Remark For any Lie algebra $\mathfrak{g}$, $\mathfrak{g}/\,\mathrm{rad}\, \mathfrak{g}$ is semisimple. We will see latter (Theorem 4.86) that in characteristic 0, $\mathrm{rad}\, \mathfrak{g}$ is a direct summand and $\mathfrak{g}$.

4.3 Examples: three-dimensional Lie algebras

[8, Lecture 21]

The classification of three-dimensional Lie algebras over $\mathbb{R}$ or $\mathbb{C}$ is long but can be done by hand, c.f. `http://en.wikipedia.org/wiki/Bianchi_classification`. The classification of four-dimensional Lie algebras has been completed, but beyond this it is hopeless: there are too many extensions of one algebra by another. In Chapter 5 we will classify all semisimple Lie algebras. For now we list two important Lie algebras:

4.58 Lemma / Definition *The* Heisenberg algebra *is a three-dimensional Lie algebra with a basis* x, y, z*, in which* z *is central and* $[x, y] = z$*. The Heisenberg algebra is nilpotent.*

4.59 Lemma / Definition *We define* $\mathfrak{sl}(2)$ *to be the three-dimensional Lie algebra with a basis* e, h, f *and relations* $[h, e] = 2e$*,* $[h, f] = -2f$*, and* $[e, f] = h$*. So long as we are not working over characteristic* 2*,* $\mathfrak{sl}(2)$ *is semisimple; simplicity follows from Corollary 4.62.*

Proof Just compute the Killing form $\beta_{\mathfrak{sl}(2)}$. $\square$

We conclude this section with two propositions and two corollaries; these will play an important role in Chapter 5.

4.60 Proposition *Let* $\mathfrak{g}$ *be a Lie algebra such that every ideal* $\mathfrak{a}$ *of* $\mathfrak{g}$ *and every quotient* $\mathfrak{g}/\mathfrak{a}$ *of* $\mathfrak{g}$ *is semisimple. Then* $\mathfrak{g}$ *is semisimple. Conversely, let* $\mathfrak{g}$ *be a semisimple Lie algebra over characteristic* 0*. Then all ideals and all quotients of* $\mathfrak{g}$ *are semisimple.*

Proof We prove only the converse direction. Let $\mathfrak{g}$ be semisimple, so that β is nondegenerate. Let $\alpha^\perp$ be the orthogonal subspace to $\mathfrak{a}$ with respet to β. Then $\mathfrak{a}^\perp = \ker\{x \mapsto \beta(-, x) : \mathfrak{g} \to \mathrm{Hom}(\mathfrak{a}, \mathfrak{g})\}$, so $\mathfrak{a}^\perp$ is an ideal. Then $\mathfrak{a} \cap \mathfrak{a}^\perp = \ker \beta|_{\mathfrak{a}} \leq \mathrm{rad}\, \mathfrak{a}$, and hence it's solvable and hence is 0. So $\mathfrak{a}$ is semisimple, and also $\mathfrak{a}^\perp$ is. In particular, the projection $\mathfrak{a}^\perp \xrightarrow{\sim} \mathfrak{g}/\mathfrak{a}$ is an isomorphism of Lie algebras, so $\mathfrak{g}/\mathfrak{a}$ is semisimple. $\square$

4.61 Corollary *Every finite-dimensional semisimple Lie algebra* $\mathfrak{g}$ *over characteristic* 0 *is a direct product* $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_m$ *of simple nonabelian algebras.*

Proof Let $\mathfrak{a}$ be a minimal and hence simple idea. Then $[\mathfrak{a}, \mathfrak{a}^\perp] \subseteq \mathfrak{a} \cap \mathfrak{a}^\perp = 0$. Rinse and repeat. $\square$

4.62 Corollary $\mathfrak{sl}(2)$ *is simple.*

4.4 Some Homological Algebra

We will not need too much homological algebra; any standard textbook on the subject, e.g. [3, 6, 19], will contain fancier versions of many of these constructions.

4.4.1 The Casimir

[8, Lectures 21 and 22]

The following piece of linear algebra is a trivial exercise in definition-chasing, and is best checked in either the physicists' index notation or Penrose's graphical language:

4.63 Proposition *Let $\langle,\rangle$ be a nondegenerate not-necessarily-symmetric bilinear form on finite-dimensional V. Let (x_i) and (y_i) be dual bases, so $\langle x_i, y_j\rangle = \delta_{ij}$. Then $\theta = \sum x_i \otimes y_i \in V \otimes V$ depends only on the form $\langle,\rangle$. If $z \in \mathfrak{gl}(V)$ leaves $\langle,\rangle$ invariant, then θ is also invariant.*

4.64 Corollary *Let β be a nondegenerate invariant (symmetric) form on a finite-dimensional Lie algebra $\mathfrak{g}$, and define $c_\beta = \sum x_i y_i$ to be the image of θ in Proposition 4.63 under the multiplication map $\mathfrak{g} \otimes \mathfrak{g} \to \mathcal{U}\mathfrak{g}$. Then c_β is a central element of $\mathcal{U}\mathfrak{g}$.*

4.65 Lemma / Definition *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and V a $\mathfrak{g}$-module so that the trace form β_V is nondegenerate. Define the* Casimir operator *$c_V = c_{\beta_V}$ as in Corollary 4.64. Then c_V has the following properties:*

1. *c_V only depends on β_V.*
2. *$c_V \in Z(\mathcal{U}(\mathfrak{g}))$*
3. *$c_V \in \mathcal{U}(\mathfrak{g})\mathfrak{g}$, i.e. it acts as 0 on $\mathbb{K}$.*
4. *$\operatorname{tr}_V(c_V) = \sum \operatorname{tr}_V(x_i y_i) = \dim \mathfrak{g}$.*

In particular, c_V distinguishes V from the trivial representation.

4.4.2 Review of Ext

[8, Lectures 22 and 23]

4.66 Definition *Let $\mathcal{C}$ be an abelian category. A* complex *(with* homological indexing*) in $\mathcal{C}$ is a sequence $A_\bullet = \dots A_k \xrightarrow{d_k} A_{k-1} \to \dots$ of maps in $\mathcal{C}$ such that $d_k \circ d_{k+1} = 0$ for every k. The* homology *of $A_\bullet$ are the objects $H_k(A_\bullet) \overset{\text{def}}{=} \ker d_k / \operatorname{Im} d_{k+1}$. For each k, $\ker d_k$ is the object of k-*cycles*, and $\operatorname{Im} d_{k+1}$ is the object of k-*boundaries*.*

We can write the same complex with cohomological indexing *by writing $A^k \overset{\text{def}}{=} A_{-k}$, whence the arrows go $\dots \to A^{k-1} \xrightarrow{\delta^k} A^k \to \dots$. The* cohomology *of a complex is $H^k(A^\bullet) \overset{\text{def}}{=} H_{-k}(A_\bullet) = \ker \delta^{k+1} / \operatorname{Im} \delta^k$. The k-*cocycles *are $\ker \delta^{k+1}$ and the k-*coboundaries *are $\operatorname{Im} \delta^k$.*

A complex is exact *at k if $H_k = 0$. A* long exact sequence *is a complex, usually infinite, that is exact everywhere. A* short exact sequence *is a three-term exact complex of the form $0 \to A \to B \to C \to 0$. In particular, $A = \ker(B \to C)$ and $C = A/B$.*

4.67 Definition *Let U be an associative algebra and U-MOD its category of left modules. A* free module *is a module $U \curvearrowright F$ that is isomorphic to a possibly-infinite direct sum of copies of $U \curvearrowright U$. Let M be a U-module. A* free resolution *of M is a complex $F_\bullet = \dots \to F_1 \to F_0 \to 0$ that is exact everywhere except at $k = 0$, where $H_k(F_\bullet) = M$. Equivalently, the* augmented complex *$F_\bullet \to M \to 0$ is exact.*

4.68 Lemma *Given any module M, a free resolution $F_\bullet$ of M exists.*

Proof Let $F_{-1} \stackrel{\text{def}}{=} M_0 \stackrel{\text{def}}{=} M$ and $M_{k+1} \stackrel{\text{def}}{=} \ker(F_k \to F_{k-1})$. Define F_k to be the free module on a generating set of M_k. □

4.69 Lemma / Definition *Let U be an associative algebra and M, N two left U modules. Let $F_\bullet$ be a free resolution of M, and construct the complex*

$$\mathrm{Hom}_U(F_\bullet, N) = \mathrm{Hom}_U(F_0, N) \xrightarrow{\delta^1} \mathrm{Hom}_U(F_1, N) \xrightarrow{\delta^2} \dots \tag{4.4.1}$$

by applying the contravariant functor $\mathrm{Hom}_U(-, N)$ to the complex $F_\bullet$. Define $\mathrm{Ext}^i_U(M, N) \stackrel{\text{def}}{=} H^i(\mathrm{Hom}_U(F_\bullet, N))$. Then $\mathrm{Ext}^0_U(M, N) = \mathrm{Hom}(M, N)$. Moreover, $\mathrm{Ext}^i_U(M, N)$ does not depend on the choice of free resolution $F_\bullet$, and is functorial in M and N.

Proof It's clear that for each choice of a free-resolution of M, we get a functor $\mathrm{Ext}^\bullet(M, -)$.

Let $M \to M'$ be a U-morphism, and $F'_\bullet$ a free resolution of M'. By freeness we can extend the morphism $M \to M'$ to a chain morphism, unique up to chain homotopy:

$$\begin{tikzcd} \cdots \arrow[r] & F_1 \arrow[r] \arrow[d, dotted] & F_0 \arrow[r] \arrow[d, dotted] & M \arrow[d] \\ \cdots \arrow[r] & F'_1 \arrow[r] & F'_0 \arrow[r] & M' \end{tikzcd} \tag{4.4.2}$$

Chain homotopies induce isomorphisms on Hom, so $\mathrm{Ext}^\bullet(M, N)$ is functorial in M; in particular, letting $M' = M$ with a different free resolution shows that $\mathrm{Ext}^\bullet(M, N)$ is well-defined. □

4.70 Lemma / Definition *The functor* $\mathrm{Hom}(-, N)$ *is* left-exact *but not* right-exact*, i.e. if $0 \to A \to B \to C \to 0$ is a short exact sequence then $\mathrm{Hom}(A, N) \leftarrow \mathrm{Hom}(B, N) \leftarrow \mathrm{Hom}(C, N) \leftarrow 0$ is exact, but $0 \leftarrow \mathrm{Hom}(A, N) \leftarrow \mathrm{Hom}(B, N)$ is not necessarily exact. Rather, we get a* long exact sequence in Ext*:*

$$\begin{tikzcd} & \mathrm{Ext}^0(A,N) & \mathrm{Ext}^0(B,N) \arrow[l] & \mathrm{Ext}^0(C,N) \arrow[l] & 0 \arrow[l] \\ & \mathrm{Ext}^1(A,N) & \mathrm{Ext}^1(B,N) \arrow[l] & \mathrm{Ext}^1(C,N) \arrow[l] \arrow[from=1-2, crossing over] & \\ \cdots & \mathrm{Ext}^2(A,N) \arrow[l] & \mathrm{Ext}^2(B,N) \arrow[l] & \mathrm{Ext}^2(C,N) \arrow[l] \arrow[from=2-2, crossing over] & \end{tikzcd} \tag{4.4.3}$$

When $N = A$, the image of $1_A \in \mathrm{Hom}(A, A) = \mathrm{Ext}^0(A, A)$ in $\mathrm{Ext}^1(C, A)$ is the characteristic class *of the extension $0 \to A \to B \to C \to 0$. The characteristic class determines B up to equivalence; in particular, when $1_A \mapsto 0$, then $B \cong A \oplus C$.*

4.4.3 Complete Reducibility

4.71 Lemma *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, $\mathcal{U}\mathfrak{g}$ its universal enveloping algebra, N a $\mathfrak{g}$-module, and F a free $\mathfrak{g}$-module. Then $F \otimes_{\mathbb{K}} N$ is free.*

Proof Let $F = \bigoplus \mathcal{U}\mathfrak{g}$; then $F \otimes N = \left(\bigoplus \mathcal{U}\mathfrak{g}\right) \otimes_{\mathbb{K}} N = \bigoplus (\mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} N)$, so it suffices to show that $G \overset{\text{def}}{=} \mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} N$ is free.

We understand the $\mathcal{U}\mathfrak{g}$-action on G: let $x \in \mathfrak{g}$ and $u \otimes n \in G$, then $x \cdot (u \otimes n) = (xu) \otimes n + u \otimes (x \cdot n)$ as $\Delta x = x \otimes 1 + 1 \otimes x$. Here xu is the product in $\mathcal{U}\mathfrak{g}$ and $x \cdot n$ is the action $\mathfrak{g} \curvearrowright N$.

We can put a filtration on G by $G_{\leq n} = \mathcal{U}\mathfrak{g}_{\leq n} \otimes_{\mathbb{K}} N$. This makes G into a *filtered module*:

$$\mathcal{U}(\mathfrak{g})_{\leq k} G_{\leq l} \subseteq G_{\leq k+l} \tag{4.4.4}$$

Thus $\operatorname{gr} G$ is a $\operatorname{gr}\mathcal{U}\mathfrak{g}$-module, but $\operatorname{gr}\mathcal{U}\mathfrak{g} = \mathcal{S}\mathfrak{g}$, and $S(\mathfrak{g})$ acts through the first term, so $S(\mathfrak{g}) \otimes N$ is a free $S(\mathfrak{g})$-module, by picking any basis of N.

let $\{n_\beta\}$ be a basis of N and $\{x_\alpha\}$ a basis of $\mathfrak{g}$. Then $\{x_{\vec{\alpha}} n_\beta\}$ is a basis of $\operatorname{gr} G = S(\mathfrak{g}) \otimes N$, hence also a basis of $\mathcal{U}(\mathfrak{g}) \otimes N$. Thus $\mathcal{U}(\mathfrak{g}) \otimes N$ is free. We have used Theorem 3.24 implicitly multiple times. □

4.72 Corollary *If M and N are finite-dimensional $\mathfrak{g}$-modules, then:*

$$\operatorname{Ext}^i(M, N) \cong \operatorname{Ext}^i(\mathbb{K}, \operatorname{Hom}(M, N)) \cong \operatorname{Ext}^i(\operatorname{Hom}(N, M), \mathbb{K}) \tag{4.4.5}$$

Proof It suffices to prove the first equality.

Let $F_\bullet \to M$ be a free resolution. A $\mathcal{U}(\mathfrak{g})$-module homomorphism is exactly a $\mathfrak{g}$-invariant linear map:

$$\operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(F, N)^\bullet = \operatorname{Hom}_{\mathbb{K}}(F_\bullet, N)^{\mathfrak{g}} \tag{4.4.6}$$

$$= \operatorname{Hom}_{\mathbb{K}}(F_\bullet \otimes_{\mathbb{K}} N^*, \mathbb{K})^{\mathfrak{g}} \tag{4.4.7}$$

$$= \operatorname{Ext}^\bullet(M \otimes N^*, \mathbb{K}) \tag{4.4.8}$$

Using the finite-dimensionality of N and the lemma that $F^\bullet \otimes N^*$ is a free resolution of $M \otimes N^*$. □

4.73 Lemma *If M, N are finite-dimensional $\mathfrak{g}$-modules and $c \in Z(\mathcal{U}\mathfrak{g})$ such that the characteristic polynomials f and g of c on M and N are relatively prime, then $\operatorname{Ext}^i(M, N) = 0$ for all i.*

Proof By functoriality, c acts $\operatorname{Ext}^i(M, N)$. By centrality, the action of c on $\operatorname{Ext}^i(M, N)$ must satisfy both the characteristic polynomials: $f(c), g(c)$ annihilate $\operatorname{Ext}^i(M, N)$. If f and g are relatively prime, then $1 = af + bg$ for some polynomials a, b; thus 1 annihilates $\operatorname{Ext}^i(M, N)$, which must therefore be 0. □

4.74 Theorem (Schur's Lemma) *Let U be an algebra and N a simple non-zero U-module, and let $\alpha : N \to N$ a U-homomorphism; then $\alpha = 0$ or α is an isomorphism.*

Proof The image of α is a submodule of N, hence either 0 or N. If $\operatorname{Im}\alpha = 0$, then we're done. If $\operatorname{Im}\alpha = N$, then $\ker\alpha \neq 0$, so $\ker\alpha = N$ by simplicity, and α is an isom. □

4.75 Corollary *Let M, N be finite-dimensional simple U-modules such that $c \in Z(U)$ annihilates M but not N; then $\operatorname{Ext}^i(M, N) = 0$ for every i.*

Proof By Theorem 4.74, c acts invertibly on N, so all its eigenvalues (over the algebraic closure) are non-zero. But the eigenvalues of c on M are all 0, so the characteristic polynomials are relatively prime. □

4.76 Theorem (Ext^1 vanishes over a semisimple Lie algebra) *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field $\mathbb{K}$ of characteristic 0, and let M and N be finite-dimensional $\mathfrak{g}$-modules. Then $\operatorname{Ext}^1(M, N) = 0$.*

Proof Using Corollary 4.72 we may assume that $M = \mathbb{K}$. Assume that N is not a trivial module. Then $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ by Corollary 4.61 for $\mathfrak{g}_i$ simple, and some $\mathfrak{g}_i$ acts non-trivially on N. Then β_N does not vanish on $\mathfrak{g}_i$ by Theorem 4.55, and so $\ker_{\mathfrak{g}_i} \beta_N = 0$ by simplicity. Thus we can find a Casimir $c \in Z(\mathcal{U}\mathfrak{g}_i) \subseteq Z(\mathcal{U}\mathfrak{g})$. In particular, $\operatorname{tr}_N(c) = \dim \mathfrak{g}_i \neq 0$, but c annihilates $\mathbb{K}$, and so by Corollary 4.75 $\operatorname{Ext}^1(\mathbb{K}, N) = 0$.

If N is a trivial module, then we use the fact that $\operatorname{Ext}^1(\mathbb{K}, N)$ classifies extensions $0 \to N \to L \to \mathbb{K} \to 0$, which we will classify directly. (See Example 4.82 for a direct verification that Ext^1 classifies extensions in the case of $\mathfrak{g}$-modules.) Writing L in block form (as a vector space, $L = N \oplus \mathbb{K}$), we see that $\mathfrak{g}$ acts on L like $\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$. Then $\mathfrak{g}$ acts by nilpotent matrices, but $\mathfrak{g}$ is semisimple, so $\mathfrak{g}$ annihilates L. Thus the only extension is the trivial one, and $\operatorname{Ext}^1(\mathbb{K}, N) = 0$. □

We list two corollaries, which are important enough to call theorems. We recall the following definition:

4.77 Definition *An object in an abelian category is* simple *if it has no non-zero proper subobjects. An object is* completely reducible *if it is a direct sum of simple objects.*

4.78 Theorem (Weyl's Complete Reducibility Theorem) *Every finite-dimensional representation of a semisimple Lie algebra over characteristic zero is completely reducible.*

4.79 Theorem (Whitehead's Theorem) *If $\mathfrak{g}$ is a semisimple Lie algebra over characteristic zero, and M and N are finite-dimensional non-isomorphic simple $\mathfrak{g}$-modules, then $\operatorname{Ext}^i(M, N)$ vanishes for all i.*

4.4.4 Computing $\operatorname{Ext}^i(\mathbb{K}, M)$

[8, Lecture 23 and 24]

4.80 Proposition *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, and $\mathbb{K}$ the trivial representation. Then $\mathbb{K}$ has a free $\mathcal{U}\mathfrak{g}$ resolution given by:*

$$\cdots \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{K}} \textstyle\bigwedge^2 \mathfrak{g} \xrightarrow{d_2} \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g} \xrightarrow{d_1} \mathcal{U}(\mathfrak{g}) \xrightarrow{\epsilon} \mathbb{K} \to 0 \tag{4.4.9}$$

The maps $d_k : \mathcal{U}(\mathfrak{g}) \otimes \bigwedge^k \mathfrak{g} \to \mathcal{U}(\mathfrak{g}) \otimes \bigwedge^{k-1} \mathfrak{g}$ *for* $k \leq 1$ *are given by:*

$$\begin{aligned} d_k(x_1 \wedge \cdots \wedge x_k) &= \sum_i (-1)^{i-1} x_i \otimes (x_1 \wedge \cdots \hat{x_i} \cdots \wedge x_k) \\ &- \sum_{i<j} (-1)^{i-j+1} 1 \otimes ([x_i, x_j] \wedge x_1 \wedge \cdots \hat{x_i} \cdots \hat{x_j} \cdots \wedge x_k) \end{aligned} \tag{4.4.10}$$

Proof That d_k is well-defined requires only checking that it is antisymmetric. That $d_{k-1} \circ d_k = 0$ is more or less obvious: terms cancel either by being sufficiently separated to appear twice with opposite signs (like in the free resolution of the symmetric polynomial ring), or by syzygy, or by Jacobi.

For exactness, we quote a general principle: Let $F_\bullet(t)$ be a t-varying complex of vector spaces, and choose a basis for each one. Assume that the vector spaces do not change with t, but that the matrix coefficients of the differentials d_k depend algebraically on t. Then the dimension of H^i can jump for special values of t, but does not fall at special values of t. In particular, exactness is a Zariski open condition.

Thus consider the complex with the vector spaces given by equation 4.4.9, but with the differential given by

$$\begin{aligned} d_k(x_1 \wedge \cdots \wedge x_k) &= \sum_i (-1)^{i-1} x_i \otimes (x_1 \wedge \cdots \hat{x_i} \cdots \wedge x_k) \\ &- t \sum_{i<j} (-1)^{i-j+1} 1 \otimes ([x_i, x_j] \wedge x_1 \wedge \cdots \hat{x_i} \cdots \hat{x_j} \cdots \wedge x_k) \end{aligned} \tag{4.4.11}$$

This corresponds to the Lie algebra $\mathfrak{g}_t = (\mathfrak{g}, [x, y]_t \overset{\text{def}}{=} t[x, y])$. When $t \neq 0$, $\mathfrak{g}_t \cong \mathfrak{g}$, by $x \mapsto tx$, but when $t = 0$, $\mathfrak{g}_0$ is abelian, and the complex consists of polynomial rings and is obviously exact.

Thus the t-varying complex is exact at $t = 0$ and hence in an open neighborhood of 0. If $\mathbb{K}$ is not finite, then an open neighborhood of 0 contains non-zero terms, and so the complex is exact for some $t \neq 0$ and hence for all t. If $\mathbb{K}$ is finite, we replace it by its algebraic closure. □

4.81 Corollary $\mathrm{Ext}^\bullet(\mathbb{K}, M)$ *is the cohomology of the* Chevalley complex *with coefficients in* M*:*

$$0 \to M \xrightarrow{\delta^1} \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, M) \xrightarrow{\delta^2} \mathrm{Hom}(\textstyle\bigwedge^2 \mathfrak{g}, M) \to \dots \tag{4.4.12}$$

If $g \in \mathrm{Hom}_{\mathbb{K}}(\bigwedge^{k-1} \mathfrak{g}, M)$*, then the differential* $\delta^k g$ *is given by:*

$$\begin{aligned} \delta^k g(x_1 \wedge \cdots \wedge x_k) &= \sum_i (-1)^{i-1} x_i g(x_1 \wedge \cdots \hat{x_i} \cdots \wedge x_k) \\ &- \sum_{i<j} (-1)^{i-j+1} g([x_i, x_j] \wedge x_1 \wedge \cdots \hat{x_i} \cdots \hat{x_j} \cdots \wedge x_k) \end{aligned} \tag{4.4.13}$$

4.82 Example Let M and N be finite-dimensional $\mathfrak{g}$-modules. Then $\mathrm{Ext}^i(M, N)$ is the cohomology of $\cdots \xrightarrow{\delta^k} \mathrm{Hom}_{\mathbb{K}}(\bigwedge^l \mathfrak{g}, M^* \otimes N) \xrightarrow{\delta^{k+1}} \cdots$. We compute $\mathrm{Ext}^1(M, N)$.

If $\phi \in M^* \otimes N$ and $x \in \mathfrak{g}$, then the action of x on ϕ is given by $x \cdot \phi = x_N \circ \phi - \phi \circ x_M =$ "$[x, \phi]$". A 1-cocycle is a map $f : \mathfrak{g} \to M^* \otimes N$ such that $0 = \delta^1 f(x \wedge y) = f([x, y]) - \big((x \cdot f)(y) - (y \cdot f)(x)\big) =$ "$[x, f(y)] - [y, f(x)]$".

Let $0 \to N \to V \to M \to 0$ be a $\mathbb{K}$-vector space, and choose a splitting $\sigma : M \to V$ as vector spaces. Then $\mathfrak{g}$ acts on $M \oplus N$ by $x \mapsto \begin{bmatrix} x_N & f(x) \\ 0 & x_M \end{bmatrix}$, and the cocycles f exactly classify the possible ways to put something in the upper right corner.

The ways to change the splitting $\sigma \mapsto \sigma' = \sigma + h$ correspond to $\mathbb{K}$-linear maps $h : M \to N$. This changes $f(x)$ by $x_N \circ h - h \circ x_M = \delta^1(h)$.

We have seen that the 1-cocycles classify the splitting, and changing the 1-cocycle by a 1-coboundary changes the splitting but not the extension. So $\mathrm{Ext}^1(M, N)$ classifies extensions up to isomorphism.

4.83 Example Consider abelian extensions of Lie algebras $0 \to \mathfrak{m} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$ where $\mathfrak{m}$ is an abelian ideal of $\hat{\mathfrak{g}}$. Since $\mathfrak{m}$ is abelian, the action $\hat{\mathfrak{g}} \curvearrowright \mathfrak{m}$ factors through $\mathfrak{g} = \hat{\mathfrak{g}}/\mathfrak{m}$. Conversely, we can classify abelian extensions $0 \to \mathfrak{m} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$ given $\mathfrak{g}$ and a $\mathfrak{g}$-module $\mathfrak{m}$.

We pick a $\mathbb{K}$-linear splitting $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$; then $\hat{\mathfrak{g}} = \{\sigma(x) + m\}$ as x ranges over $\mathfrak{g}$ and m over $\mathfrak{m}$, and the bracket is

$$[\sigma(x) + m, \sigma(y) + n] = \sigma([x, y]) + [\sigma(x), n] - [\sigma(y), m] + g(x, y) \tag{4.4.14}$$

where g is the error term measuring how far off σ is from being a splitting of $\mathfrak{g}$-modules. There is no $[m, n]$ term, because $\mathfrak{m}$ is assumed to be an abelian ideal of $\hat{\mathfrak{g}}$.

Then g is antisymmetric. The Jacobi identity on $\hat{\mathfrak{g}}$ is equivalent to g satisfying:

$$\begin{aligned} 0 &= x\,g(y \wedge z) - y\,g(x \wedge z) + z\,g(x \wedge y) - g([x, y] \wedge z) + g([x, z] \wedge y) - g([y, z] \wedge x) && (4.4.15) \\ &= x\,g(y \wedge z) - g([x, y] \wedge z) + \text{cycle permutations} && (4.4.16) \end{aligned}$$

I.e. g is a 2-cocycle in $\mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{m})$. In particular, the 2-cocycles classify extensions of $\mathfrak{g}$ by $\mathfrak{m}$ along with a splitting. If we change the splitting by $f : \mathfrak{g} \to \mathfrak{m}$, then $\mathfrak{g}$ changes by $(x \cdot f)(y) - (y \cdot f)(x) - f([x, y]) = \delta^2(f)$. We have proved:

4.84 Proposition $\mathrm{Ext}^2_{\mathcal{U}\mathfrak{g}}(\mathbb{K}, \mathfrak{m})$ *classifies abelian extensions* $0 \to \mathfrak{m} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$ *up to isomorphism. The element* $0 \in \mathrm{Ext}^2$ *corresponds to the semidirect product* $\hat{\mathfrak{g}} = \mathfrak{g} \ltimes \mathfrak{m}$.

4.85 Corollary *Abelian extensions of semisimple Lie groups are semidirect products.*

4.86 Theorem (Levi's Theorem) *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra over characteristic* 0, *and let* $\mathfrak{r} = \mathrm{rad}(\mathfrak{g})$. *Then* $\mathfrak{g}$ *has a* Levi decomposition*: semisimple* Levi subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$.

Proof Without loss of generality, $\mathfrak{r} \neq 0$, as otherwise $\mathfrak{g}$ is already semisimple.

Assume first that $\mathfrak{r}$ is not a minimal non-zero ideal. In particular, let $\mathfrak{m} \neq 0$ be an ideal of $\mathfrak{g}$ with $\mathfrak{m} \subsetneq \mathfrak{r}$. Then $\mathfrak{r}/\mathfrak{m} = \mathrm{rad}(\mathfrak{g}/\mathfrak{m}) \neq 0$, and by induction on dimension $\mathfrak{g}/\mathfrak{m}$ has a Levi subalgebra. Let $\tilde{\mathfrak{s}}$ be the preimage of this subalgebra in $\mathfrak{g}/\mathfrak{m}$. Then $\tilde{\mathfrak{s}} \cap \mathfrak{r} = \mathfrak{m}$ and $\tilde{\mathfrak{s}}/\mathfrak{m} \xrightarrow{\sim} (\mathfrak{g}/\mathfrak{m})/(\mathfrak{r}/\mathfrak{m}) = \mathfrak{g}/\mathfrak{r}$. Hence $\mathfrak{m} = \mathrm{rad}(\tilde{\mathfrak{s}})$. Again by induction on dimension, $\tilde{\mathfrak{s}}$ has a Levi subalgebra $\mathfrak{s}$; then $\tilde{\mathfrak{s}} = \mathfrak{s} \oplus \mathfrak{m}$ and $\mathfrak{s} \cap \mathfrak{r} = 0$, so $\mathfrak{s} \xrightarrow{\sim} \mathfrak{g}/\mathfrak{r}$. Thus $\mathfrak{s}$ is a Levi subalgebra of $\mathfrak{g}$.

We turn now to the case when $\mathfrak{r}$ is minimal. Being a radical, $\mathfrak{r}$ is solvable, so $\mathfrak{r}' \neq \mathfrak{r}$, and by minimality $\mathfrak{r}' = 0$. So $\mathfrak{r}$ is abelian. In particular, the action $\mathfrak{g} \curvearrowright \mathfrak{r}$ factors through $\mathfrak{g}/\mathfrak{r}$, and so $0 \to \mathfrak{r} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{r} \to 0$ is an abelian extension of $\mathfrak{g}/\mathfrak{r}$, and thus must be semidirect by Corollary 4.85. $\square$

4.87 Remark We always have $Z(\mathfrak{g}) \leq \mathfrak{r}$, and when $\mathfrak{r}$ is minimal, $Z(\mathfrak{g})$ is either 0 or $\mathfrak{r}$. When $Z(\mathfrak{g}) = \mathfrak{r}$, then $0 \to \mathfrak{r} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{r} \to 0$ is in fact an extension of $\mathfrak{g}/\mathfrak{r}$-modules, and so is a direct product by Example 4.82.

We do not prove the following, given as an exercise in [8]:

4.88 Theorem (Malcev-Harish-Chandra Theorem) *All Levi subalgebras of a given Lie algebra are conjugate under the action of the group* $\exp \operatorname{ad} \mathfrak{n} \subseteq \mathrm{GL}(V)$*, where* $\mathfrak{n}$ *is the largest nilpotent ideal of* $\mathfrak{g}$*. (In particular,* $\operatorname{ad} : \mathfrak{n} \curvearrowright \mathfrak{g}$ *is nilpotent, so the power series for* $\exp$ *terminates.)*

We are now ready to complete the proof of Theorem 3.12, with a theorem of Cartan:

4.89 Theorem (Lie's Third Theorem) *Let* $\mathfrak{g}$ *be a Lie algebra over* $\mathbb{R}$*. Then* $\mathfrak{g} = \mathrm{Lie}(G)$ *for some analytic Lie group* G*.*

Proof Find a Levi decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$. If $\mathfrak{s} = \mathrm{Lie}(S)$ and $\mathfrak{r} = \mathrm{Lie}(R)$ where S and R are connected and simply connected, then the action $\mathfrak{s} \curvearrowright \mathfrak{r}$ lifts to an action $S \curvearrowright R$. Thus we can construct $G = S \ltimes R$, and it is a direct computation that $\mathfrak{g} = \mathrm{Lie}(G)$ in this case.

So it suffices to find groups S and R with the desired Lie algebras. We need not even assure that the groups we find are simply-connected; we can always take universal covers. In any case, $\mathfrak{s}$ is simply connected, so the action $\mathfrak{s} \to \mathfrak{gl}(\mathfrak{s})$ is faithful, and thus we can find $S \subseteq \mathrm{GL}(\mathfrak{s})$ with $\mathrm{Lie}(S) = \mathfrak{s}$.

On the other hand, $\mathfrak{r}$ is solvable: the chain $\mathfrak{r} \geq \mathfrak{r}' \geq \mathfrak{r}'' \geq \dots$ eventually gets to 0. We can interpolate between $\mathfrak{r}$ and $\mathfrak{r}'$ by one-co-dimensional vector spaces, which are all necessarily ideals of some $\mathfrak{r}^{(k)}$, and the quotients are one-dimensional and hence abelian. Thus any solvable Lie algebra is an extension by one-dimensional algebras, and this extension also lifts to the level of groups. So $\mathfrak{r} = \mathrm{Lie}(R)$ for some Lie group R. $\square$

4.5 From Zassenhaus to Ado

[8, Lectures 25 and 26]

Ado's Theorem (Theorem 4.99) normally is not proven in a course in Lie Theory. For example, [18, page 8] mentions it only in a footnote, referring the reader to [5, Appendix E]. [11] also relegates Ado's Theorem to an appendix (B.3). In fact, we will see that Ado's Theorem is a direct consequence of Theorem 4.86, although we will need to develop some preliminary facts.

4.90 Lemma / Definition *A* Lie derivation *of a Lie algebra* $\mathfrak{a}$ *is a linear map* $f : \mathfrak{a} \to \mathfrak{a}$ *such that* $f([x, y]) = [f(x), y] + [x, f(y)]$*. Equivalently, a derivation is a one-cocycle in the Chevalley complex with coefficients in* $\mathfrak{a}$*.*

A derivation *of an associative algebra* A *is a linear map* $f : A \to A$ *so that* $f(xy) = f(x)\, y + x\, f(x)$*.*

The product (composition) of (Lie) derivations is not necessarily a (Lie) derivation, but the commutator of derivations is a derivation. We write $\operatorname{Der} \mathfrak{a}$ *for the Lie algebra of Lie derivations of* $\mathfrak{a}$*, and* $\operatorname{Der} A$ *for the algebra of associative derivations of* A*. Henceforth, we drop the adjective "Lie", talking about simply derivations of a Lie algebra.*

We say that $\mathfrak{h} \curvearrowright \mathfrak{a}$ by derivations *if the map* $\mathfrak{h} \to \mathfrak{gl}(\mathfrak{a})$ *in fact lands in* $\operatorname{Der} \mathfrak{a}$*.*

In very general language, let A and B be vector spaces, and $a : A^{\otimes n} \to A$ and $b : B^{\otimes n} \to B$ n-linear maps. Then a *homomorphism* from (A, a) to (B, b) is a linear map $\phi : A \to B$ so that $\phi \circ a = b \circ \phi^{\otimes n}$, and a *derivation* from (A, a) to (B, b) is a linear map $\phi : A \to B$ such that $\phi \circ a = b \circ (\sum_{i=1}^{n} \phi_i)$, where $\phi_i \overset{\text{def}}{=} 1 \otimes \cdots \otimes \phi \otimes \cdots \otimes 1$, with the ϕ in the ith spot. Then the space $\mathrm{Hom}(A, B)$ of homomorphisms is not generally a vector space, but the space $\mathrm{Der}(A, B)$ of derivations is. If $(A, a) = (B, b)$, then $\mathrm{Hom}(A, A)$ is closed under composition and hence a monoid, whereas $\mathrm{Der}(A, A)$ is closed under the commutator and hence a Lie algebra. The notions of "derivation" and "homomorphism" agree for $n = 1$, whence the map ϕ must intertwine a with b. The difference between derivations and homomorphisms is the difference between grouplike and primitive elements of a bialgebra.

4.91 Proposition *Let $\mathfrak{a}$ be a Lie algebra.*

1. *Every derivation of $\mathfrak{a}$ extends uniquely to a derivation of $\mathcal{U}(\mathfrak{a})$.*

2. $\mathrm{Der}\,\mathfrak{a} \to \mathrm{Der}\,\mathcal{U}(\mathfrak{a})$ *is a Lie algebra homomorphism.*

3. *If $\mathfrak{h} \curvearrowright \mathfrak{a}$ by derivations, then $\mathfrak{h}(\mathcal{U}(\mathfrak{a})) \subseteq \mathcal{U}(\mathfrak{a}) \cdot \mathfrak{h}(\mathfrak{a}) \cdot \mathcal{U}(\mathfrak{a})$ the two-sided ideal of $\mathcal{U}(\mathfrak{a})$ generated by the image of the $\mathfrak{h}$ action in $\mathfrak{a}$.*

4. *If $N \leq \mathcal{U}(\mathfrak{a})$ is an $\mathfrak{h}$-stable two-sided ideal, so is N^n.*

Proof
1. Let $d \in \mathrm{Der}\,\mathfrak{a}$, and define $\hat{\mathfrak{a}} \overset{\text{def}}{=} \mathbb{K}d \oplus \mathfrak{a}$; then $\mathcal{U}(\mathfrak{a}) \subseteq \mathcal{U}(\hat{\mathfrak{a}})$. The commutative $[d, -]$ preserves $\mathcal{U}(\mathfrak{a})$ and is the required derivation. Uniqueness is immediate: once you've said how something acts on the generators, you've defined it on the whole algebra.

2. This is an automatic consequence of the uniqueness: the commutator of two derivations is a derivation, so if it's unique, it must be the correct derivation.

3. Let $a_1, \dots, a_k \in \mathfrak{a}$ and $h \in \mathfrak{h}$. Then $h(a_1 \cdots a_k) = \sum_{i=1}^{k} a_1 \cdots h(a_i) \cdots a_k \in \mathcal{U}(\mathfrak{a})\, \mathfrak{h}(\mathfrak{a})\, \mathcal{U}(\mathfrak{a})$.

4. N^n is spanned by monomials $a_1 \cdots a_n$ where all $a_i \in N$. Assuming that $h(a_i) \in N$ for each $h \in \mathfrak{h}$, we see that $h(a_1 \cdots a_k) = \sum_{i=1}^{k} a_1 \cdots h(a_i) \cdots a_k \in N^n$. □

4.92 Lemma / Definition *Let $\mathfrak{h}$ and $\mathfrak{a}$ be Lie algebras and let $\mathfrak{h} \curvearrowright \mathfrak{a}$ by derivations. The* semidirect product $\mathfrak{h} \ltimes \mathfrak{a}$ *is the vector space $\mathfrak{h} \oplus \mathfrak{a}$ with the bracket given by $[h_1 + a_1, h_2 + a_2] = [h_1, h_2]_{\mathfrak{h}} + [a_1, a_2]_{\mathfrak{a}} + h_1 \cdot a_2 - h_2 \cdot a_1$, where be $h \cdot a$ we mean the action of h on a. Then $\mathfrak{h} \ltimes \mathfrak{a}$ is a Lie algebra, and $0 \to \mathfrak{a} \to \mathfrak{h} \ltimes \mathfrak{a} \to \mathfrak{h} \to 0$ is a split short exact sequence in* LieAlg.

4.93 Proposition *Let $\mathfrak{h} \curvearrowright \mathfrak{a}$ by derivations, and let $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}$ be the semidirect product. Then the actions $\mathfrak{h} \curvearrowright \mathcal{U}\mathfrak{a}$ by derivations and $\mathfrak{a} \curvearrowright \mathcal{U}\mathfrak{a}$ by left-multiplication together make a $\mathfrak{g}$-action on $\mathcal{U}\mathfrak{a}$.*

Proof We need only check the commutator of $\mathfrak{h}$ with $\mathfrak{a}$. Let $u \in \mathcal{U}(\mathfrak{a})$, $h \in \mathfrak{h}$, and $a \in \mathfrak{a}$. Then $(h \circ a)u = h(au) = h(a)\, u + a\, h(u) = [h, a]u + a\, h(u)$. Thus $[h, a] \in \mathfrak{g}$ acts as the commutator of operators h and a on $\mathcal{U}(\mathfrak{a})$. □

4.94 Definition *An algebra U is* left-noetherian *if left ideals of U satisfy the* ascending chain condition*. I.e. if any chain of left ideals $I_1 \leq I_2 \leq \dots$ of U stabilizes.*

We refer the reader to any standard algebra textbook for a discussion of noetherian rings. For noncommutative ring theory see [13, 16, 7].

4.95 Proposition *Let U be a filtered algebra. If $\operatorname{gr} U$ is left-noetherian, then so is U.*

In particular, $\mathcal{U}(\mathfrak{a})$ is left-noetherian, since $\operatorname{gr}\mathcal{U}(\mathfrak{a})$ is a polynomial ring on $\dim \mathfrak{a}$ generators.

Proof Let $I \leq U$ be a left ideal. We define $I_{\leq n} = I \cap U_{\leq n}$, and hence $I = \bigcup I_{\leq n}$. We define $\operatorname{gr} I = \bigoplus I_{\leq n}/I_{\leq n-1}$, and this is a left ideal in $\operatorname{gr} U$. If $I \leq J$, then $\operatorname{gr} I \leq \operatorname{gr} J$, using the fact that U injects into $\operatorname{gr} U$ as vector spaces.

So if we have an ascending chain $I_1 \leq I_2 \leq \dots$, then the corresponding chain $\operatorname{gr} I_1 \leq \operatorname{gr} I_2 \leq \dots$ eventually terminate by assumption: $\operatorname{gr} I_n = \operatorname{gr} I_{n_0}$ for $n \geq n_0$. But if $\operatorname{gr} I = \operatorname{gr} J$, then by induction on n, $I_{\leq n} = J_{\leq n}$, and so $I = J$. Hence the original sequence terminates. □

4.96 Lemma *Let $\mathfrak{j} = \mathfrak{h} + \mathfrak{n}$ be a finite-dimensional Lie algebra, where $\mathfrak{h}$ is a subalgebra of $\mathfrak{j}$ and $\mathfrak{n}$ an ideal. Assume that $\mathfrak{g} \curvearrowright W$ is a finite-dimensional representation such that $\mathfrak{h}, \mathfrak{n} \curvearrowright W$ nilpotently. Then $\mathfrak{g} \curvearrowright W$ nilpotently.*

Proof If $W = 0$ there is nothing to prove. Otherwise, by Theorem 4.25 there is some $w \in W^{\mathfrak{n}}$, where $W^{\mathfrak{n}}$ is the subspace of W annihilated by $\mathfrak{n}$. Let $h \in \mathfrak{h}$ and $x \in \mathfrak{n}$. Then:

$$xhw = \underbrace{[x, h]}_{\in \mathfrak{n}} w + h \underbrace{xw}_{=0} = 0 \tag{4.5.1}$$

Thus $hw \in V^{\mathfrak{n}}$, and so $w \in V^{\mathfrak{g}}$. By modding out and iterating, we see that $\mathfrak{g} \curvearrowright V$ nilpotently. □

4.97 Theorem (Zassenhaus's Extension Lemma) *Let $\mathfrak{h}$ and $\mathfrak{a}$ be finite-dimensional Lie algebras so that $\mathfrak{h} \curvearrowright \mathfrak{a}$ by derivations, and let $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}$. Moreover, let V be a finite-dimensional $\mathfrak{a}$-module, and let $\mathfrak{n}$ be the nilpotency ideal of $\mathfrak{a} \curvearrowright V$. If $[\mathfrak{h}, \mathfrak{a}] \leq \mathfrak{n}$, then there exists a finite-dimensional $\mathfrak{g}$-module W and a surjective $\mathfrak{a}$-module map $W \twoheadrightarrow V$, and so that the nilpotency ideal $\mathfrak{m}$ of $\mathfrak{g} \curvearrowright W$ contains $\mathfrak{n}$. If $\mathfrak{h} \curvearrowright \mathfrak{a}$ by nilpotents, then we can arrange for $\mathfrak{m} \subseteq \mathfrak{h}$ as well.*

Proof Consider a Jordan-Holder series $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq \mathrm{M}(n) = V$. Then $\mathfrak{n} = \bigcap \ker(M_i/M_{i-1})$ by Corollary 4.28. We define $N = \bigcap \ker\big(\mathcal{U}\mathfrak{a} \to \operatorname{End}(M_i/M_{i-1})\big)$, an ideal of $\mathcal{U}\mathfrak{a}$. Then $N \supseteq \mathfrak{n} \supseteq [\mathfrak{h}, \mathfrak{a}]$, and so N is an $\mathfrak{h}$-stable ideal of $\mathcal{U}\mathfrak{a}$ by the third part of Proposition 4.91, and N^k is $\mathfrak{h}$-stable by the fourth part.

Since $\mathcal{U}\mathfrak{a}$ is left-noetherian (Proposition 4.95), N^k is finitely generated for each k, and hence N^k/N^{k+1} is a finitely generated $\mathcal{U}\mathfrak{a}$ module. But the action $\mathcal{U}\mathfrak{a} \curvearrowright (N^k/N^{k+1})$ factors through $\mathcal{U}\mathfrak{a}/N$, so in fact N^k/N^{k+1} is a finitely generated $(\mathcal{U}\mathfrak{a}/N)$-module. But $\mathcal{U}\mathfrak{a}/N \cong \bigoplus \operatorname{Im}\big(\mathcal{U}\mathfrak{a} \to \operatorname{End}(M_i/M_{i-1})\big) \subseteq \bigoplus \operatorname{End}(M_i/M_{i-1})$, which is finite-dimensional. So $\mathcal{U}\mathfrak{a}/N$ is finite-dimensional, N^k/N^{k+1} a finitely-generated $(\mathcal{U}\mathfrak{a}/N)$-module, and hence N^k/N^{k+1} is finite-dimensional.

By construction, $N(M_k) \subseteq M_{k-1}$, so N^n annihilates V, where n is the length of the Jordan-Holder series $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq \mathrm{M}(n) = V$. Let $\dim V = d$, and define

$$W \stackrel{\text{def}}{=} \bigoplus_{i=1}^{d} \mathcal{U}\mathfrak{a}/N^n \tag{4.5.2}$$

Then W is finite-dimensional since $\mathcal{U}\mathfrak{a}/N^n \cong \mathcal{U}\mathfrak{a}/N \oplus N/N^2 \oplus \cdots \oplus N^{n-1}/N^n$ as a vector space, and each summand is finite-dimensional. To construct the map $W \twoheadrightarrow V$, we pick a basis $\{v_i\}_{i=1}^d$ of V, and send $(0,\dots,1,\dots,0) \mapsto v_i$, where 1 is the image of $1 \in \mathcal{U}\mathfrak{a}$ in $\mathcal{U}\mathfrak{a}/N^n$, and it is in the ith spot. By construction $\mathcal{U}\mathfrak{a}_{\leq 0}$ acts as scalars, and so N does not contain $\mathcal{U}\mathfrak{a}_{\leq 0}$; thus the map is well-defined. Moreover, $\mathfrak{g} \curvearrowright \mathcal{U}\mathfrak{a}$ by Proposition 4.91, and N is $\mathfrak{h}$-stable and hence $\mathfrak{g}$-stable. Thus $\mathfrak{g} \curvearrowright W$ naturally, and the action $\mathcal{U}\mathfrak{a} \curvearrowright V$ factors through N^n, and so $W \twoheadrightarrow V$ is a map of $\mathfrak{g}$-modules.

By construction, N and hence $\mathfrak{n}$ acts nilpotently on W. But $\mathfrak{n}$ is an ideal of $\mathfrak{g}$: a general element of $\mathfrak{g}$ is of the form $h+a$ for $h \in \mathfrak{h}$ and $a \in \mathfrak{a}$, and $[h+a,\mathfrak{n}] = [h,\mathfrak{n}]+[a,\mathfrak{n}] \subseteq [h,\mathfrak{a}]+[a,\mathfrak{n}] \subseteq \mathfrak{n}+\mathfrak{n} = \mathfrak{n}$. So $\mathfrak{m} \supseteq \mathfrak{n}$, as $\mathfrak{m}$ is the largest nilpotency ideal of $\mathfrak{g} \curvearrowright W$.

We finish by considering the case when $\mathfrak{h} \curvearrowright \mathfrak{a}$ nilpotently. Then $\mathfrak{h} \curvearrowright W$ nilpotently, and since $[\mathfrak{h},\mathfrak{a}] \subseteq \mathfrak{n}$, $\mathfrak{h}+\mathfrak{n}$ is an ideal of $\mathfrak{g}$. By Lemma 4.96, $\mathfrak{h}+\mathfrak{n}$ acts nilpotently on W, and so is a subideal of $\mathfrak{m}$. □

4.98 Corollary *Let $\mathfrak{r}$ be a solvable Lie algebra over characteristic 0, and let $\mathfrak{n}$ be its largest nilpotent ideal. Then every derivation of $\mathfrak{r}$ has image in $\mathfrak{n}$. In particular, if $\mathfrak{r}$ is an ideal of some larger Lie algebra $\mathfrak{g}$, then $[\mathfrak{g},\mathfrak{r}] \subseteq \mathfrak{n}$.*

Proof Let d be a derivation of $\mathfrak{r}$ and $\mathfrak{h} = \mathbb{K}d \oplus \mathfrak{r}$. Then $\mathfrak{h}$ is solvable by Proposition 4.19, and $\mathfrak{h}' \curvearrowright \mathfrak{h}$ nilpotently by Corollary 4.40. But $d(\mathfrak{r}) \subseteq \mathfrak{h}'$ and $\mathfrak{r}$ is an ideal of $\mathfrak{h}$, and so $d(\mathfrak{r})$ acts nilpotently on $\mathfrak{r}$, and is thus a subideal of $\mathfrak{n}$.

The second statement follows from the fact that $[g,-]$ is a derivation; this follows ultimately from the Jacobi identity. □

4.99 Theorem (Ado's Theorem) *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over characteristic 0. Then $\mathfrak{g}$ has a faithful representation $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, and this representation can be chosen so that the largest nilpotent ideal $\mathfrak{n} \leq \mathfrak{g}$ acts nilpotently on V.*[2]

Proof We induct on $\dim \mathfrak{g}$. The $\mathfrak{g} = 0$ case is trivial, and we break the induction step into cases:

Case I: $\mathfrak{g} = \mathfrak{n}$ is nilpotent. Then $\mathfrak{g} \neq \mathfrak{g}'$, and so we choose a subspace $\mathfrak{a} \supseteq \mathfrak{g}'$ of codimensional 1 in $\mathfrak{g}$. This is automatically an ideal, and we pick $x \notin \mathfrak{a}$, and $\mathfrak{h} = \langle x \rangle$. Any one-dimensional subspace is a subalgebra, and $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}$. By induction, $\mathfrak{a}$ has a faithful module V' and acts nilpotently.

The hypotheses of Theorem 4.97 are satisfied, and we get an $\mathfrak{a}$-module homomorphism $W \twoheadrightarrow V'$ with $\mathfrak{g} \curvearrowright W$ nilpotently. As yet, this might not be a faithful representation of $\mathfrak{g}$: certainly

[2] It seems that Ado originally proved a weaker version of Theorem 4.99 over $\mathbb{R}$. The version we present is due to [9]. The dependence on characteristic is removed in [10].

$\mathfrak{a}$ acts faithfully on W because it does so on V', but x might kill W. We pick a faithful nilpotent $\mathfrak{g}/\mathfrak{a} = \mathbb{K}$-module W_1, e.g. $x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathfrak{gl}(2)$. Then $V = W \oplus W_1$ is a faithful nilpotent $\mathfrak{g}$ representation.

Case II: $\mathfrak{g}$ is solvable but not nilpotent. Then $\mathfrak{g}' \leq \mathfrak{n} \lneq \mathfrak{g}$. We pick an ideal $\mathfrak{a}$ of codimension 1 in $\mathfrak{g}$ such that $\mathfrak{n} \subseteq \mathfrak{a}$, and x and $\mathfrak{h} = \mathbb{K}x$ as before, so that $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}$. Then $\mathfrak{n}(\mathfrak{a}) \supseteq \mathfrak{n}$ — if a matrix acts nilpotently on $\mathfrak{g}$, then certainly it does so on $\mathfrak{a}$, and by construction $\mathfrak{n} \subseteq \mathfrak{a}$ — and we have a faithful module $\mathfrak{a} \curvearrowright V'$ by induction, with $\mathfrak{n}(\mathfrak{a}) \curvearrowright V'$ nilpotently. Then $[\mathfrak{h}, \mathfrak{a}] \curvearrowright V'$ nilpotently, since $[\mathfrak{h}, \mathfrak{a}] \subseteq \mathfrak{n}(\mathfrak{a})$ by Corollary 4.98, so we use Theorem 4.97 to get $\mathfrak{g} \curvearrowright W$ and an $\mathfrak{a}$-module map $W \twoheadrightarrow V'$, such that $\mathfrak{n} \curvearrowright W$ nilpotently. We form $V = W \oplus W_1$ as before so that $\mathfrak{g} \curvearrowright V$ is faithful. Since $\mathfrak{n}$ is contained in $\mathfrak{a}$ and $\mathfrak{a}$ acts as 0 on W_1, $\mathfrak{n}$ acts nilpotently on W.

Case III: general. By Theorem 4.86, there is a splitting $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ with $\mathfrak{s}$ semisimple and $\mathfrak{r}$ solvable. By Case II, we have a faithful $\mathfrak{r}$-representation V' with $\mathfrak{n}(\mathfrak{r}) \curvearrowright V'$ nilpotently. By Corollary 4.98 the conditions of Theorem 4.97 apply, so we have $\mathfrak{g} \curvearrowright W$ and an $\mathfrak{r}$-module map $W \twoheadrightarrow V'$, and since $\mathfrak{n} \leq \mathfrak{r}$ we have $\mathfrak{n} \leq \mathfrak{n}(\mathfrak{r})$ so $\mathfrak{n} \curvearrowright W$ nilpotently. We want to get a faithful representation, and we need to make sure it is faithful on $\mathfrak{s}$. But $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ is semisimple, so has no center, so $\operatorname{ad} : \mathfrak{s} \curvearrowright \mathfrak{s}$ is faithful. So we let $W_1 = \mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ as $\mathfrak{g}$-modules, and $\mathfrak{g} \curvearrowright V = W \oplus W_1$ is faithful with $\mathfrak{n}$ acting as 0 on W_1 and nilpotently on W. $\square$

Exercises

1. Classify the 3-dimensional Lie algebras $\mathfrak{g}$ over an algebraically closed field $\mathbb{K}$ of characteristic zero by showing that if $\mathfrak{g}$ is not a direct product of smaller Lie algebras, then either

   - $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{K})$,
   - $\mathfrak{g}$ is isomorphic to the nilpotent Heisenberg Lie algebra $\mathfrak{h}$ with basis X, Y, Z such that Z is central and $[X, Y] = Z$, or
   - $\mathfrak{g}$ is isomorphic to a solvable algebra $\mathfrak{s}$ which is the semidirect product of the abelian algebra $\mathbb{K}^2$ by an invertible derivation. In particular $\mathfrak{s}$ has basis X, Y, Z such that $[Y, Z] = 0$, and $\operatorname{ad} X$ acts on $\mathbb{K}Y + \mathbb{K}Z$ by an invertible matrix, which, up to change of basis in $\mathbb{K}Y + \mathbb{K}Z$ and rescaling X, can be taken to be either $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$ for some nonzero $\lambda \in \mathbb{K}$.

2. (a) Show that the Heisenberg Lie algebra $\mathfrak{h}$ in Problem 1 has the property that Z acts nilpotently in every finite-dimensional module, and as zero in every simple finite-dimensional module.

   (b) Construct a simple infinite-dimensional $\mathfrak{h}$-module in which Z acts as a non-zero scalar. [Hint: take X and Y to be the operators d/dt and t on $\mathbb{K}[t]$.]

3. Construct a simple 2-dimensional module for the Heisenberg algebra $\mathfrak{h}$ over any field $\mathbb{K}$ of characteristic 2. In particular, if $\mathbb{K} = \bar{\mathbb{K}}$, this gives a counterexample to Lie's theorem in non-zero characteristic.

4. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{K}$.

   (a) Show that the intersection $\mathfrak{n}$ of the kernels of all finite-dimensional simple $\mathfrak{g}$-modules can be characterized as the largest ideal of $\mathfrak{g}$ which acts nilpotently in every finite-dimensional $\mathfrak{g}$-module. It is called the *nilradical* of $\mathfrak{g}$.

   (b) Show that the nilradical of $\mathfrak{g}$ is contained in $\mathfrak{g}' \cap \operatorname{rad}(\mathfrak{g})$.

   (c) Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a subalgebra and V a $\mathfrak{g}$-module. Given a linear functional $\lambda : \mathfrak{h} \to \mathbb{K}$, define the associated weight space to be $V_\lambda = \{v \in V : Hv = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}$. Assuming $\operatorname{char}(\mathbb{K}) = 0$, adapt the proof of Lie's theorem to show that if $\mathfrak{h}$ is an ideal and V is finite-dimensional, then V_λ is a $\mathfrak{g}$-submodule of V.

   (d) Show that if $\operatorname{char}(\mathbb{K}) = 0$ then the nilradical of $\mathfrak{g}$ is equal to $\mathfrak{g}' \cap \operatorname{rad}(\mathfrak{g})$. [Hint: assume without loss of generality that $\mathbb{K} = \bar{\mathbb{K}}$ and obtain from Lie's theorem that any finite-dimensional simple $\mathfrak{g}$-module V has a non-zero weight space for some weight λ on $\mathfrak{g}' \cap \operatorname{rad}(\mathfrak{g})$. Then use (c) to deduce that $\lambda = 0$ if V is simple.]

5. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{K}$, $\operatorname{char}(\mathbb{K}) = 0$. Prove that the largest nilpotent ideal of $\mathfrak{g}$ is equal to the set of elements of $\mathfrak{r} = \operatorname{rad}\mathfrak{g}$ which act nilpotently in the adjoint action on $\mathfrak{g}$, or equivalently on $\mathfrak{r}$. In particular, it is equal to the largest nilpotent ideal of $\mathfrak{r}$.

6. Prove that the Lie algebra $\mathfrak{sl}(2, \mathbb{K})$ of 2×2 matrices with trace zero is simple, over a field $\mathbb{K}$ of any characteristic $\neq 2$. In characteristic 2, show that it is nilpotent.

7. In this exercise, well deduce from the standard functorial properties of Ext groups and their associated long exact sequences that $\operatorname{Ext}^1(N, M)$ bijectively classifies extensions $0 \to M \to V \to N \to 0$ up to isomorphism, for modules over any associative ring with unity.

   (a) Let F be a free module with a surjective homomorphism onto N, so we have an exact sequence $0 \to K \to F \to N \to 0$. Use the long exact sequence to produce an isomorphism of $\operatorname{Ext}^1(N, M)$ with the cokernel of $\operatorname{Hom}(F, M) \to \operatorname{Hom}(K, M)$.

   (b) Given $\phi \in \operatorname{Hom}(K, M)$, construct V as the quotient of $F \oplus M$ by the graph of $-\phi$ (note that this graph is a submodule of $K \oplus M \subseteq F \oplus M$).

   (c) Use the functoriality of Ext and the long exact sequences to show that the characteristic class in $\operatorname{Ext}^1(N, M)$ of the extension constructed in (b) is the element represented by the chosen ϕ, and conversely, that if ϕ represents the characteristic class of a given extension, then the extension constructed in (b) is isomorphic to the given one.

8. Calculate $\operatorname{Ext}^i(\mathbb{K}, \mathbb{K})$ for all i for the trivial representation $\mathbb{K}$ of $\mathfrak{sl}(2, \mathbb{K})$, where $\operatorname{char}(\mathbb{K}) = 0$. Conclude that the theorem that $\operatorname{Ext}^i(M, N) = 0$ for $i = 1, 2$ and all finite-dimensional representations M, N of a semi-simple Lie algebra $\mathfrak{g}$ does not extend to $i > 2$.

9. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Show that $\mathrm{Ext}^1(\mathbb{K}, \mathbb{K})$ can be canonically identified with the dual space of $\mathfrak{g}/\mathfrak{g}'$, and therefore also with the set of 1-dimensional $\mathfrak{g}$-modules, up to isomorphism.

10. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Show that $\mathrm{Ext}^1(\mathbb{K}, \mathfrak{g})$ can be canonically identified with the quotient $\mathrm{Der}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$, where $\mathrm{Der}(\mathfrak{g})$ is the space of derivations of $\mathfrak{g}$, and $\mathrm{Inn}(\mathfrak{g})$ is the subspace of inner derivations, that is, those of the form $d(x) = [y, x]$ for some $y \in \mathfrak{g}$. Show that this also classifies Lie algebra extensions $\hat{\mathfrak{g}}$ containing $\mathfrak{g}$ as an ideal of codimension 1.

11. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{K}$, $\mathrm{char}(\mathbb{K}) = 0$. The Malcev-Harish-Chandra theorem says that all Levi subalgebras $\mathfrak{s} \subseteq \mathfrak{g}$ are conjugate under the action of the group $\exp \mathrm{ad}\, \mathfrak{n}$, where $\mathfrak{n}$ is the largest nilpotent ideal of $\mathfrak{g}$ (note that $\mathfrak{n}$ acts nilpotently on $\mathfrak{g}$, so the power series expression for $\exp \mathrm{ad}\, X$ reduces to a finite sum when $X \in \mathfrak{n}$).

    (a) Show that the reduction we used to prove Levi's theorem by induction in the case where the radical $\mathfrak{r} = \mathrm{rad}\, \mathfrak{g}$ is not a minimal ideal also works for the Malcev-Harish-Chandra theorem. More precisely, show that if $\mathfrak{r}$ is nilpotent, the reduction can be done using any nonzero ideal $\mathfrak{m}$ properly contained in $\mathfrak{r}$. If $\mathfrak{r}$ is not nilpotent, use Problem 4 to show that $[\mathfrak{g}, \mathfrak{r}] = \mathfrak{r}$, then make the reduction by taking $\mathfrak{m}$ to contain $[\mathfrak{g}, \mathfrak{r}]$.

    (b) In general, given a semidirect product $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$, where $\mathfrak{m}$ is an abelian ideal, show that $\mathrm{Ext}^1_{\mathcal{U}(\mathfrak{h})}(\mathbb{K}, \mathfrak{m})$ classifies subalgebras complementary to $\mathfrak{m}$, up to conjugacy by the action of $\exp \mathrm{ad}\, \mathfrak{m}$. Then use the vanishing of $\mathrm{Ext}^1(M, N)$ for finite-dimensional modules over a semi-simple Lie algebra to complete the proof of the Malcev-Harish-Chandra theorem.

Chapter 5

Classification of Semisimple Lie Algebras

5.1 Classical Lie algebras over $\mathbb{C}$

5.1.1 Reductive Lie algebras

[8, Lecture 26]

Henceforth every Lie algebra, except when otherwise marked, is finite-dimensional over a field of characteristic 0.

5.1 Lemma / Definition *A Lie algebra $\mathfrak{g}$ is* reductive *if $(\mathfrak{g}, \mathrm{ad})$ is completely reducible.*

A Lie algebra is reductive if and only if it is of the form $\mathfrak{g} = \mathfrak{s} \times \mathfrak{a}$ where $\mathfrak{s}$ is semisimple and $\mathfrak{a}$ is abelian. Moreover, $\mathfrak{a} = Z(\mathfrak{g})$ and $\mathfrak{s} = \mathfrak{g}'$.

Proof Let $\mathfrak{g}$ be a reductive Lie algebra; then $\mathfrak{g} = \bigoplus \mathfrak{a}_i$ as $\mathfrak{g}$-modules, where each $\mathfrak{a}_i$ is an ideal of $\mathfrak{g}$ and $[\mathfrak{a}_i, \mathfrak{a}_j] \subseteq \mathfrak{a}_i \cap \mathfrak{a}_j = 0$ for $i \neq j$. Thus $\mathfrak{g} = \prod \mathfrak{a}_i$ as Lie algebras, and each $\mathfrak{a}_i$ is either simple non-abelian or one-dimensional. □

5.2 Proposition *Any $*$-closed subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ is reductive.*

Proof We define a symmetric real-valued bilinear form $(,)$ on $\mathfrak{gl}(n, \mathbb{C})$ by $(x, y) = \big(\mathrm{tr}(xy^*)\big)$. Then $(x, x) = \sum |x_{ij}|^2$, so $(,)$ is positive-definite. Moreover:

$$([z, x], y) = -(x, [z^*, y]) \tag{5.1.1}$$

so $[z^*, -]$ is adjoint to $[-, z]$.

Let $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{C})$ be any subalgebra and $\mathfrak{a} \leq \mathfrak{g}$ an ideal. Then $\mathfrak{a}^\perp \subseteq \mathfrak{g}^*$ by invariance, where $\mathfrak{g}^*$ is the Lie algebra of Hermitian conjugates of elements of $\mathfrak{g}$. If $\mathfrak{g}$ is $*$-closed, then $\mathfrak{g}^* = \mathfrak{g}$ and $\mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$. By positive-definiteness, $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, and we rinse and repeat to write $\mathfrak{g}$ is a sum of irreducibles. □

5.3 Example The classical Lie algebras $\mathfrak{sl}(n,\mathbb{C}) \overset{\text{def}}{=} \{x \in \mathfrak{gl}(n) \text{ s.t. } \operatorname{tr} x = 0\}$, $\mathfrak{so}(n,\mathbb{C}) \overset{\text{def}}{=} \{x \in \mathfrak{gl}(n) \text{ s.t. } x + x^T = 0\}$, and $\mathfrak{sp}(n,\mathbb{C}) \overset{\text{def}}{=} \{x \in \mathfrak{gl}(2n) \text{ s.t. } jx + x^T j = 0\}$ are reductive. Indeed, since they have no center except in very low dimensions, they are all semisimple. We will see later that they are all simple, except in a few low dimensions.

Since a real Lie algebra $\mathfrak{g}$ is semisimple if $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is, the real Lie algebras $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{so}(n,\mathbb{R})$, and $\mathfrak{sp}(n,\mathbb{R})$ also are semisimple.

5.1.2 Guiding examples: $\mathfrak{sl}(n)$ and $\mathfrak{sp}(n)$ over $\mathbb{C}$

[8, Lectures 27 and 28]

Let $\mathfrak{g} = \mathfrak{sl}(n)$ or $\mathfrak{sp}(n)$. We extract an abelian subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. For $\mathfrak{sl}_n$ we use the diagonal traceless matrices:

$$\mathfrak{h} \overset{\text{def}}{=} \left\{ \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} \text{ s.t. } \sum z_i = 0 \right\} \tag{5.1.2}$$

For $\mathfrak{sp}(n) \overset{\text{def}}{=} \{x \in \mathfrak{gl}(2n) \text{ s.t. } jx + x^T j = 0\}$, it will be helpful to redefine j. We can use any j which defines a non-degenerate antisymmetric bilinear form, and we take:

$$j = \left[\begin{array}{ccc|ccc} & & & & & 1 \\ & 0 & & & \iddots & \\ & & & 1 & & \\ \hline & & -1 & & & \\ & \iddots & & & 0 & \\ -1 & & & & & \end{array}\right] \tag{5.1.3}$$

Let a^R be the matrix a reflected across the antidiagonal. Then we can define $\mathfrak{sp}(n)$ in block diagonal form:

$$\mathfrak{sp}(n) = \left\{ \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \in \mathrm{M}(2n) = \mathrm{M}(2, \mathrm{M}(n)) \text{ s.t. } d = -a^R, b = b^R, c = c^R \right\} \tag{5.1.4}$$

In this basis, we take as our abelian subalgebra

$$\mathfrak{h} \overset{\text{def}}{=} \left\{ \left[\begin{array}{ccc|ccc} z_1 & & & & & \\ & \ddots & & & 0 & \\ & & z_n & & & \\ \hline & & & -z_n & & \\ & 0 & & & \ddots & \\ & & & & & -z_1 \end{array}\right] \right\} \tag{5.1.5}$$

5.4 Proposition *Let $\mathfrak{g} = \mathfrak{sl}(n)$ or $\mathfrak{sp}(n)$. For $\mathfrak{h} \leq \mathfrak{g}$ defined above, the adjoint action* $\operatorname{ad} : \mathfrak{h} \curvearrowright \mathfrak{g}$ *is diagonal.*

Proof We make explicit the basis of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{sl}(n)$, the natural basis is $\{e_{ij}\}_{i\neq j} \cup \{e_{ii} - e_{i+1,i+1}\}_{i=1}^{n-1}$, where e_{ij} is the matrix with a 1 in the (ij) spot and 0s elsewhere. In particular, $\{e_{ii} - e_{i+1,i+1}\}_{i=1}^{n-1}$ is a basis of $\mathfrak{h}$. Let $h \in \mathfrak{h}$ be

$$h = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} \tag{5.1.6}$$

Then $[h, e_{ij}] = (z_i - z_j)e_{ij}$, and $[h, h'] = 0$ when $h' \in \mathfrak{h}$.

For $\mathfrak{g} = \mathfrak{sp}(n)$, the natural basis suggested by equation 5.1.4 is

$$\begin{aligned} &\left\{a_{ij} \overset{\text{def}}{=} \left[\begin{array}{c|c} e_{ij} & 0 \\ \hline 0 & -e_{n+1-j,n+1-i} \end{array}\right]\right\} \cup \left\{b_{ij} \overset{\text{def}}{=} \left[\begin{array}{c|c} 0 & e_{ij} + e_{n+1-j,n+1-i} \\ \hline 0 & 0 \end{array}\right] \text{ s.t. } i+j \leq n+1\right\} \\ &\qquad \cup \left\{c_{ij} \overset{\text{def}}{=} \left[\begin{array}{c|c} 0 & 0 \\ \hline e_{ij} + e_{n+1-j,n+1-i} & 0 \end{array}\right] \text{ s.t. } i+j \leq n+1\right\} \end{aligned} \tag{5.1.7}$$

Of course, when $i = j$, then $\left\{\left[\begin{array}{c|c} e_{ii} & 0 \\ \hline 0 & -e_{n+1-i,n+1-i} \end{array}\right]\right\}$ is a basis of $\mathfrak{h}$. Let $h \in \mathfrak{h}$ be given by

$$h = \left[\begin{array}{ccc|ccc} z_1 & & & & & \\ & \ddots & & & 0 & \\ & & z_n & & & \\ \hline & & & -z_n & & \\ & 0 & & & \ddots & \\ & & & & & -z_1 \end{array}\right] \tag{5.1.8}$$

Then $[h, a_{ij}] = (z_i - z_j)a_{ij}$, $[h, b_{ij}] = (z_i + z_j)b_{ij}$, and $[h, c_{ij}] = (-z_i - z_j)c_{ij}$. $\square$

5.5 Definition *Let $\mathfrak{h}$ be a maximal abelian subalgebra of a finite-dimensional Lie algebra $\mathfrak{g}$ so that* $\mathrm{ad} : \mathfrak{h} \curvearrowright \mathfrak{g}$ *is diagonalizable, so diagonal in an eigenbasis. Write $\mathfrak{h}^*$ for the vector space dual to $\mathfrak{h}$. Each eigenbasis element of $\mathfrak{g}$ defines an eigenvalue to each $h \in \mathfrak{h}$, and this assignment is linear in $\mathfrak{h}$; thus, the eigenbasis of $\mathfrak{g}$ picks out a vector $\alpha \in \mathfrak{h}^*$. The set of such vectors are the* roots *of the pair $(\mathfrak{g}, \mathfrak{h})$.*

We will refine this definition in Definition 5.41, and we will prove that the set of roots of a semisimple Lie algebra $\mathfrak{g}$ is determined up to isomorphism by $\mathfrak{g}$ (in particular, it does not depend on the subalgebra $\mathfrak{h}$).

5.6 Example When $\mathfrak{g} = \mathfrak{sl}(n)$ and $\mathfrak{h}$ is as above, the roots are $\{0\} \cup \{z_i - z_j\}_{i\neq j}$, where $\{z_i\}_{i=1}^n$ are the natural linear functionals $\mathfrak{h} \to \mathbb{C}$. When $\mathfrak{g} = \mathfrak{sp}(n)$ and $\mathfrak{h}$ is as above, the roots are $\{0\} \cup \{\pm 2z_i\} \cup \{\pm z_i \pm z_j\}_{i\neq j}$.

5.7 Lemma / Definition *Let $\mathfrak{g}$ and $\mathfrak{h} \leq \mathfrak{g}$ as in Definition 5.5. Then the roots break $\mathfrak{g}$ into eigenspaces:*

$$\mathfrak{g} = \bigoplus_{\alpha \text{ a root}} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{\alpha\neq 0 \text{ a root}} \mathfrak{g}_\alpha \tag{5.1.9}$$

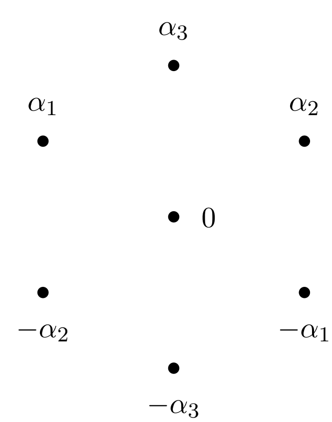
In particular, since $\mathfrak{h}$ is a maximal abelian subalgebra, the 0-eigenspace of $\mathfrak{h} \curvearrowright \mathfrak{g}$ is precisely $\mathfrak{g}_0 = \mathfrak{h}$. Then the spaces $\mathfrak{g}_\alpha$ are the root spaces *of the pair $(\mathfrak{g}, \mathfrak{h})$. By the Jacobi identity, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.*

5.8 Lemma *When $\mathfrak{g} = \mathfrak{sl}(n)$ or $\mathfrak{sp}(n)$ and $\mathfrak{h}$ is as above, then for $\alpha \neq 0$ the root space $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$ is one-dimensional. Let $\mathfrak{h}_\alpha \overset{\text{def}}{=} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Then $\mathfrak{h}_\alpha = \mathfrak{h}_{-\alpha}$ is one-dimensional, and $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_\alpha$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2)$.*

Proof For each root α, pick a basis element $g_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ (in particular, we can use the eigenbasis of $\mathfrak{h} \curvearrowright \mathfrak{g}$ given above), and define $h_\alpha \overset{\text{def}}{=} [g_\alpha, g_{-\alpha}]$. Define $\alpha(h_\alpha) = a$ so that $[h_\alpha, g_{\pm\alpha}] = \pm a g_\alpha$; one can check directly that $a \neq 0$. For the isomorphism, we use the fact that $\mathbb{C}$ is algebraically closed.$\square$

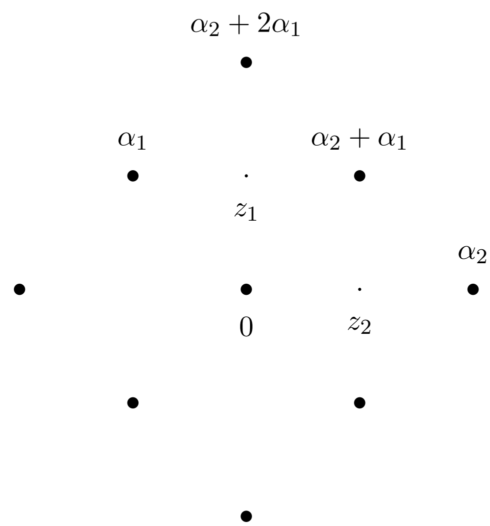
5.9 Definition *Let $\mathfrak{g}$ and $\mathfrak{h} \leq \mathfrak{g}$ as in Definition 5.5. The* rank *of $\mathfrak{g}$ is the dimension of $\mathfrak{h}$, or equivalently the dimension of the dual space $\mathfrak{h}^*$.*

5.10 Example The Lie algebras $\mathfrak{sl}(3)$ and $\mathfrak{sp}(2)$ are rank-two. For $\mathfrak{g} = \mathfrak{sl}(3)$, the dual space $\mathfrak{h}^*$ to $\mathfrak{h}$ spanned by the vectors $z_1 - z_2$ and $z_2 - z_3$ naturally embeds in a three-dimensional vector space spanned by $\{z_1, z_2, z_3\}$, and we choose an inner product on this space in which $\{z_i\}$ is an orthonormal basis. Let $\alpha_1 = z_1 - z_2$, $\alpha_2 = z_2 - z_3$, and $\alpha_3 = z_1 - z_3$. Then the roots $\{0, \pm\alpha_i\}$ form a perfect hexagon:



(5.1.10)

For $\mathfrak{g} = \mathfrak{sp}(2)$, we have $\mathfrak{h}^*$ spanned by $\{z_1, z_2\}$, and we choose an inner product in which this is an orthonormal basis. Let $\alpha_1 = z_1 - z_2$ and $\alpha_2 = 2z_2$. The roots form a diamond:

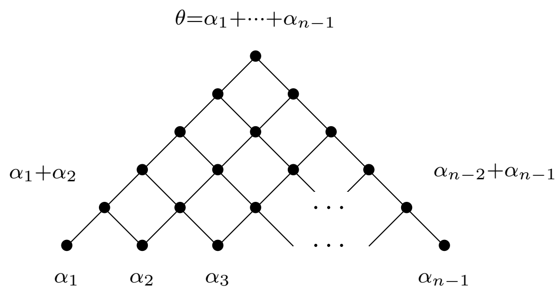


(5.1.11)

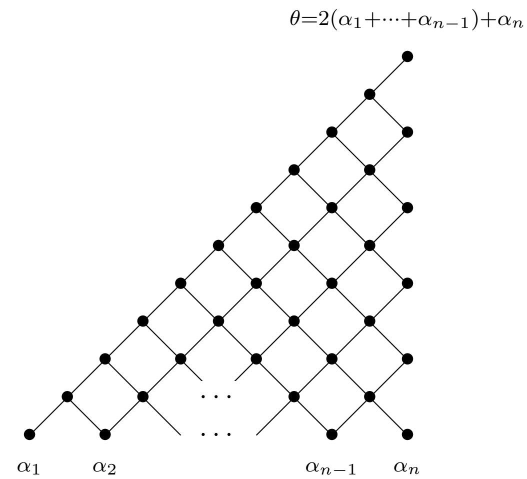
5.11 Lemma *Let $\mathfrak{g}, \mathfrak{h} \leq \mathfrak{g}$ be as in Definition 5.5. Let $v \in \mathfrak{h}$ be chosen so that $\alpha(v) \neq 0$ for every root α. Then v divides the roots into* positive roots *and* negative roots *according to the sign of $\alpha(v)$. A* simple root *is any positive root that is not expressible as a sum of positive roots.*

5.12 Example Let $\mathfrak{g} = \mathfrak{sl}(n)$ or $\mathfrak{sp}(n)$, and choose $v \in \mathfrak{h}$ so that $z_1(v) > z_2(v) > \cdots > z_n(v) > 0$. The positive roots of $\mathfrak{sl}(n)$ are $\{z_i - z_j\}_{i<j}$, and the positive roots of $\mathfrak{sp}(n)$ are $\{z_i - z_j\}_{i<j} \cup \{z_i + z_i\} \cup \{2z_i\}$. The simple roots of $\mathfrak{sl}(n)$ are $\{\alpha_i = z_i - z_{i+1}\}_{i=1}^{n-1}$, and the simple roots of $\mathfrak{sp}(n)$ are $\{\alpha_i = z_i - z_{i+1}\}_{i=1}^{n-1} \cup \{2z_n\}$. In each case, the simple roots are a basis of $\mathfrak{h}^*$. Moreover, the roots are in the $\mathbb{Z}$-span of the simple roots, i.e. the lattice generated by the simple roots, and the positive roots are in the intersection of this lattice with the positive cone, so that the positive roots are in the $\mathbb{N}$-span of the simple roots.

We partially order the positive roots by saying that $\alpha < \beta$ if $\beta - \alpha$ is a positive root. Under this partial order there is a unique maximal positive root θ, the *highest root*; for $\mathfrak{sl}(n)$ we have $\theta = z_1 - z_n = \alpha_1 + \cdots + \alpha_{n-1}$, and for $\mathfrak{sp}(n)$ we have $\theta = 2z_1 = 2(\alpha_1 + \cdots + \alpha_{n-1}) + \alpha_n$. We draw these partial orders:

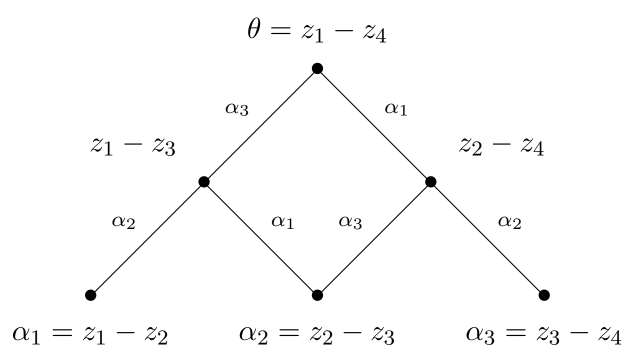


$\mathfrak{sl}(n)$

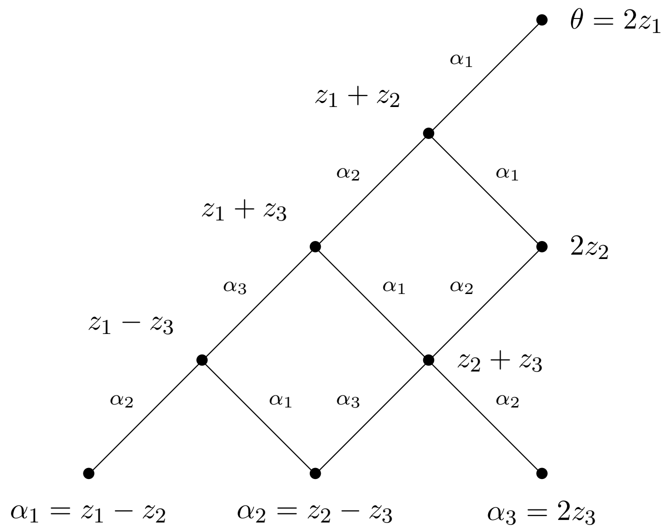


$\mathfrak{sp}(n)$

To make this very clear, we draw the rank-three pictures fully labeled (edges by the difference between consecutive nodes):



$\mathfrak{sl}(4)$



$\mathfrak{sp}(3)$

Using these pictures of $\mathfrak{sl}(n)$ and $\mathfrak{sp}(n)$, we can directly prove the following:

5.13 Proposition *The Lie algebras $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{sp}(n,\mathbb{C})$ are simple.*

Proof Let $\mathfrak{g} = \mathfrak{sl}(n)$ or $\mathfrak{sp}(n)$, and $\mathfrak{h}$, the systems of positive and simple roots, and θ the highest root as above. Recall that for each root $\alpha \neq 0$, the root space $\mathfrak{g}_\alpha$ is one-dimensional, and we pick an eigenbasis $\{g_\alpha\}_{\alpha\neq 0} \cup \{h_i\}_{i=1}^{\text{rank}}$ for the action $\mathfrak{h} \curvearrowright \mathfrak{g}$.

Let $x \in \mathfrak{g}$. It is a standard exercise from linear algebra that $\mathfrak{h}x$ is the span of the eigenvectros g_α, $\alpha \neq 0$, for which the coefficient of x in the eigenbasis is non-zero. In particular, if $x \in \mathfrak{g} \smallsetminus \mathfrak{h}$, then $[\mathfrak{h}, x]$ includes some g_α. By switching the roles of positive and negative roots if necessary, we can assure that α is positive; thus $[\mathfrak{h}, x] \supseteq \mathfrak{g}_\alpha$ for some positive α.

One can check directly that if $\alpha, \beta, \alpha+\beta$ are all nonzero roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. In particular, for any positive root α, $\theta - \alpha$ is a positive root, and so $[\mathfrak{g}, \mathfrak{g}_\alpha] \supseteq [\mathfrak{g}_{\theta-\alpha}, \mathfrak{g}_\alpha] = \mathfrak{g}_\theta$. In particular, $g_\theta \in [\mathfrak{g}, x]$.

But $[\mathfrak{g}_{\theta-\alpha}, \mathfrak{g}_\theta] = \mathfrak{g}_\alpha$, and so $[\mathfrak{g}, g_\theta]$ generates all g_α for α a positive root. We saw already (Lemma 5.8) that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathfrak{h}_\alpha$ is non-zero, and that $[\mathfrak{g}_{\pm\alpha}, \mathfrak{h}_\alpha] = \mathfrak{g}_{\pm\alpha}$. Thus $[\mathfrak{g}, \mathfrak{g}_\alpha] \supseteq \mathfrak{g}_{-\alpha}$, and in particular g_θ generates every g_α for $\alpha \neq 0$, and every h_α. Then g_θ generates all of $\mathfrak{g}$.

Thus x generates all of $\mathfrak{g}$ for any $x \in \mathfrak{g} \smallsetminus \mathfrak{h}$. If $x \in \mathfrak{h}$, then $\alpha(x) \neq 0$ for some α, and then $[\mathfrak{g}_\alpha, x] = \mathfrak{g}_\alpha$, and we repeat the proof with some nonzero element of $\mathfrak{g}_\alpha$. Hence $\mathfrak{g}$ is simple. □

When $\mathfrak{g} = \mathfrak{sl}(n)$, let ϵ_i refer to the matrix e_{ii}, and when $\mathfrak{g} = \mathfrak{sp}(n)$, let ϵ_i refer to the matrix $\left[\begin{array}{c:c} e_{ii} & 0 \\ \hdashline 0 & -e_{n+1-i,n+1-i} \end{array}\right]$. We construct a linear isomorphism $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ by assigning an element $\alpha_i^\vee$ of $\mathfrak{h}$ to each simple root α_i as follows: to $\alpha_i = z_i - z_{i+1}$ for $1 \le i \le n-1$ we assign $\alpha_i^\vee = \epsilon_i - \epsilon_{i+1}$, and to $\alpha_n = 2z_n$ a root of $\mathfrak{sp}(n)$ we assign $\alpha_n^\vee = \epsilon_n$. In particular, $\alpha_i(h_i) = 2$ for each simple root. We define the *Cartan matrix* a by $a_{ij} \stackrel{\text{def}}{=} \alpha_i(h_j)$.

5.14 Example For $\mathfrak{sl}(n)$, we have the following $(n-1)\times(n-1)$ matrix:

$$a = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix} \tag{5.1.12}$$

For $\mathfrak{sp}(n)$, we have the following $n \times n$ matrix:

$$a = \left[\begin{array}{ccccc|c} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & & \vdots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & 2 & -1 & 0 \\ 0 & \dots & 0 & -1 & 2 & -1 \\ \hline 0 & \dots & \dots & 0 & -2 & 2 \end{array}\right] \tag{5.1.13}$$

To each of the above matrices we associate a *Dynkin diagram*. This is a graph with a node for each simple root, and edges assigned by: i and j are not connected if $a_{ij} = 0$; they are singly connected if a_{ij} is a block $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$; we put a double arrow from j to i when the (i,j)-block is $\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$. So for $\mathfrak{sl}(n)$ we get the graph , and for $\mathfrak{sp}(n)$ we get .

5.15 Lemma / Definition *The identification $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ lets us construct reflections of $\mathfrak{h}^*$ by $s_i : \alpha \mapsto \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha$, where $\langle , \rangle$ is the pairing $\mathfrak{h}^* \otimes \mathfrak{h} \to \mathbb{C}$ that we had earlier written as $\langle \alpha, \beta \rangle = \alpha(\beta)$. These reflections generated the* Weyl group *W.*

For each of $\mathfrak{sl}(n)$ and $\mathfrak{sp}(n)$, let $R \subseteq \mathfrak{h}^*$ be the set of roots and W the Weyl group. Then $W \curvearrowright R \smallsetminus \{0\}$. In particular, for $\mathfrak{sl}(n)$, we have $W = S_n$ the symmetric group on n letters, where the reflection $(i, i+1)$ acts as s_i; $W \curvearrowright R \smallsetminus \{0\}$ is transitive. For $\mathfrak{sp}(n)$, we have $W = S_n \ltimes (\mathbb{Z}/2)^n$, the hyperoctahedral group*, generated by the reflections $s_i = (i, i+1) \in S_n$ and s_n the sign change, and the action $W \curvearrowright R \smallsetminus \{0\}$ has two orbits.*

We will spend the rest of this chapter showing that the picture of $\mathfrak{sl}(n)$ and $\mathfrak{sp}(n)$ in this section is typical of simple Lie algebras over $\mathbb{C}$.

5.2 Representation Theory of $\mathfrak{sl}(2)$

[8, Lectures 28 and 29]

Our hero for this section is the Lie algebra $\mathfrak{sl}(2,\mathbb{C}) \stackrel{\text{def}}{=} \langle e, h, f : [e,f] = h,\ [h,e] = 2e,\ [h,f] = -2f \rangle = \{x \in \mathrm{M}(2,\mathbb{C}) \text{ s.t. } \operatorname{tr} x = 0\}$.

5.16 Example As a subalgebra of $\mathrm{M}(2,\mathbb{C})$, $\mathfrak{sl}(2)$ has a tautological representation on $\mathbb{C}^2$, given by $E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $H \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Let v_0 and v_1 be the basis vectors of $\mathbb{C}^2$. Then the representation $\mathfrak{sl}(2) \curvearrowright \mathbb{C}^2$ has the following picture:

$$\begin{tikzcd} v_0 \ \bullet \arrow[loop right, "H"] \arrow[d, bend left, "F"] \\ v_1 \ \bullet \arrow[loop right, "H"] \arrow[u, bend left, "E"] \end{tikzcd} \tag{5.2.1}$$

This is the infinitesimal verion of the action $\mathrm{SL}(2) \curvearrowright \mathbb{C}^2$ given by

$$\begin{aligned} (\exp(-te)) \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x - ty \\ y \end{bmatrix} \end{aligned} \tag{5.2.2}$$

$$\left.\frac{d}{dt}\right|_{t=0} \exp(-te) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ 0 \end{bmatrix} \right\} \tag{5.2.3}$$

$$\left.\frac{d}{dt}\right|_{t=0} \exp(-tf) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ -x \end{bmatrix} \right\} \tag{5.2.4}$$

5.17 Example Since $\mathrm{SL}(2) \curvearrowright \mathbb{C}^2$, it acts also on the space of functions on $\mathbb{C}^2$; by the previous calculations, we see that the action is:

$$e = -y\partial_x, \quad f = -x\partial_y, \quad h = -x\partial_x + y\partial_y \tag{5.2.5}$$

These operations are homogenous — they preserve the total degree of any polynomial — and so the symmetric tensor product $\mathcal{S}^n(\mathbb{C}^2) = \{$homogeneous polynomials of degree n in x and $y\}$ is a submodule of $\mathrm{SL}(2) \curvearrowright \{$functions$\}$. Let $v_i \overset{\text{def}}{=} \binom{n}{i} x^i y^{n-i}$ be a basis vector in $\mathcal{S}^n(\mathbb{C}x \oplus \mathbb{C}y)$. Then the action $\mathrm{SL}(2) \curvearrowright \mathcal{S}^2(\mathbb{C}^2)$ has the following picture:

$$\begin{tikzcd} y^n = v_0 \ \bullet \arrow[loop right, "h=n"] \arrow[d, bend left, "f=1"] \\ nxy^{n-1} = v_1 \ \bullet \arrow[loop right, "h=n-2"] \arrow[u, bend left, "n=e"] \arrow[d, bend left, "f=2"] \\ \binom{n}{2} x^2 y^{n-2} = v_2 \ \bullet \arrow[loop right, "h=n-4"] \arrow[u, bend left, "n-1=e"] \arrow[d, bend left] \\ \vdots \arrow[loop right] \arrow[u, bend left] \arrow[d, bend left] \\ \binom{n}{n-1} x^{n-1} y = v_{n-1} \ \bullet \arrow[loop right, "h=2-n"] \arrow[u, bend left] \arrow[d, bend left, "f=n"] \\ x^n = v_n \ \bullet \arrow[loop right, "h=-n"] \arrow[u, bend left, "1=e"] \end{tikzcd} \tag{5.2.6}$$

Let us call this module V_n. Then V_n is irreducible, because applying e enough times to any non-zero element results in a multiple of v_0, and v_0 generated the module.

5.18 Proposition *Let V be any $(n+1)$-dimensional irreducible module over $\mathfrak{sl}(2)$. Then $V \cong V_n$.*

Proof Suppose that $v \in V$ is an eigenvector of h, so that $hv = \lambda v$. Then $hev = [h,e]v + ehv = 2ev + \lambda ev$, so ev is an h-eigenvector with eigenvalue $\lambda + 2$. Similarly, fv is an h-eigenvector with eigenvalue $\lambda - 2$. So the space spanned by h-eigenvectors of V is a submodule of V; by the irreducibility of V, and using the fact that h has at least one eigenvector, this submodule must be the whole of V, and so h acts diagonally.

By finite-dimensionality, there is an eigenvector v_0 of h with the highest eigenvalue, and so $ev_0 = 0$. By Theorem 3.24, $\{f^k e^l h^m\}$ spans $\mathcal{U}\mathfrak{sl}(2)$, and so $\{v_i \overset{\text{def}}{=} f^i v_0/i!\}$ is a basis of V (by irreducibility, V is generated by v_0). In particular, $v_n = f^n v_0/n!$, the $(n+1)$st member of the basis, has $fv_n = 0$, since V is $(n+1)$-dimensional.

We compute the action of e by induction, using the fact that $hv_k = (\lambda_0 - 2k)v_k$:

$$
\begin{aligned}
ev_0 &= 0 && (5.2.7)\\
ev_1 &= efv_0 = [e,f]v_0 + fev_0 = hv_0 = \lambda_0 v_0 && (5.2.8)\\
ev_2 &= efv_1/2 = [e,f]v_1/2 + fev_1/2 = hv_1/2 + f\lambda_0 v_0/2 \\
&= (\lambda_0 - 2)v_1/2 + \lambda_0 v_1/2 = (\lambda_0 - 1)v_1 && (5.2.9)\\
&\dots \\
ev_k &= efv_{k-1}/k = hv_{k-1}/k + fev_{k-1}/k = (\lambda_0 - 2k + 2)v_{k-1}/k + (\lambda_0 - k + 2)fv_{k-2}/k \\
&= \big((\lambda_0 - 2k + 2)/k + (k-1)(\lambda_0 - k + 2)/k\big)v_{k-1} = (\lambda_0 - k + 1)v_{k-1} && (5.2.10)
\end{aligned}
$$

But $fv_n = 0$, and so:

$$
\begin{aligned}
0 &= efv_n = [e,f]v_n + fev_n = hv_n + (\lambda_0 - n + 1)fv_{n-1} \\
&= (\lambda_0 - 2n)v_n + (\lambda_0 - n + 1)nv_n = \big((n+1)\lambda_0 - (n+1)n\big)v_n && (5.2.11)
\end{aligned}
$$

Thus $\lambda_0 = n$ and V is isomorphic to V_n defined in equation 5.2.6. □

5.3 Cartan subalgebras

5.3.1 Definition and Existence

[8, Lectures 29 and 30]

5.19 Lemma *Let $\mathfrak{h}$ be a nilpotent Lie algebra over a field $\mathbb{K}$, and $\mathfrak{h} \curvearrowright V$ a finite-dimensional representation. For $h \in \mathfrak{h}$ and $\lambda \in \mathbb{C}$, define $V_{\lambda,h} = \{v \in V \text{ s.t. } \exists n \text{ s.t. } (h-\lambda)^n v = 0\}$. Then $V_{\lambda,h}$ is an $\mathfrak{h}$-submodule of V.*

Proof Let $\mathrm{ad} : \mathfrak{h} \curvearrowright \mathfrak{h}$ be the adjoint action; since $\mathfrak{h}$ is nilpotent, $\mathrm{ad}\, h \in \mathrm{End}(\mathfrak{h})$ is a nilpotent endomorphism. Define $\mathfrak{h}_{(m)} \overset{\text{def}}{=} \ker\big((\mathrm{ad}\, h)^m\big)$; then $\mathfrak{h}_{(m)} = \mathfrak{h}$ for m large enough. We will show that $\mathfrak{h}_{(m)} V_{\lambda,h} \subseteq V_{\lambda,h}$ by induction on m; when $m = 0$, $\mathfrak{h}_{(0)} = 0$ and the statement is trivial.

Let $y \in \mathfrak{h}_{(m)}$, whence $[h, y] \in \mathfrak{h}_{(m-1)}$, and let $v \in V_{\lambda,h}$. Then $(h-\lambda)^n v = 0$ for n large enough, and so

$$\begin{aligned}(h-\lambda)^n yv &= y(h-\lambda)^n v + [(h-\lambda)^n, y]v && (5.3.1)\\ &= 0 + [(h-\lambda)^n, y]v && (5.3.2)\\ &= \sum_{k+1=n-1} (h-\lambda)^k [h,y](h-\lambda)^l v && (5.3.3)\end{aligned}$$

since $[\lambda, y] = 0$. By increasing n, we can assure that for each term in the sum at least one of the following happens: l is large enough that $(h-\lambda)^l v = 0$, or k is large enough that $(h-\lambda)^k V_{\lambda,h} = 0$. The large-$l$ terms vanish immediately; the large-k terms vanish upon realizing that $(h-\lambda)V_{h,\lambda} \subseteq V_{h,\lambda}$ by definition and $[h, y]V_{h,\lambda} \subseteq \mathfrak{h}_{(m-1)}V_{h,\lambda} \subseteq V_{h,\lambda}$ by induction on m. □

5.20 Corollary *Let $\mathfrak{h}$ be a nilpotent Lie algebra over $\mathbb{K}$, $\mathfrak{h} \curvearrowright V$ a finite-dimensionalrepresentation, and $\lambda : \mathfrak{h} \to \mathbb{K}$ a linear map. Then $V_\lambda \overset{\text{def}}{=} \bigcap_{h\in\mathfrak{h}} V_{\lambda(h),h}$ is an $\mathfrak{h}$-submodule of V.*

5.21 Proposition *Let $\mathfrak{h}$ be a finite-dimensional nilpotent Lie algebra over an algebraically closed field $\mathbb{K}$ of characteristic 0, and V a finite-dimensional $\mathfrak{h}$-module. For each $\lambda \in \mathfrak{h}^*$, define V_λ as in Corollary 5.20. Then $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$.*

Proof Let $h_1, \dots, h_k \in \mathfrak{h}$, and let $H_k \subseteq \mathfrak{h}$ be the linear span of the h_i. Let $W \overset{\text{def}}{=} \bigcap_{i=1}^k V_{\lambda(h_i),h_i}$. It follows from Theorem 4.37 that $W = \bigcap_{h\in H} V_{\lambda(h),h}$, since we can choose a basis of V in which $\mathfrak{h} \curvearrowright V$ by upper-triangular matrices.

We have seen already that W is a submodule of V. Let $h_{k+1} \notin H_k$; then we can decompose W into generalized eigenspaces of h_{k+1}. We proceed by induction on k until we have a basis of $\mathfrak{h}$. □

5.22 Definition *For $\lambda \in \mathfrak{h}^*$, the space V_λ in Corollary 5.20 is a* weight space *of V, and $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$ the* weight space decomposition.

5.23 Lemma *Let $\mathfrak{h}$ be a finite-dimensional nilpotent Lie algebra over an algebraically closed field of characteristic 0, and let V and W be two finite-dimensional $\mathfrak{h}$ modules. Then the weight spaces of $V \otimes W$ are given by $(V \otimes W)_\lambda = \bigoplus_{\alpha+\beta=\lambda} V_\alpha \otimes W_\beta$.*

Proof $h(v \otimes w) = hv \otimes w + v \otimes hw$ □

5.24 Corollary *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field of characteristic 0, and $\mathfrak{h} \subseteq \mathfrak{g}$ a nilpotent subalgebra. Then the weight spaces of* $\mathrm{ad} : \mathfrak{h} \curvearrowright \mathfrak{g}$ *satisfy $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.*

5.25 Proposition *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field of characteristic 0, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a nilpotent subalgebra. The following are equivalent:*

1. *$\mathfrak{h} = N(\mathfrak{h}) \overset{\text{def}}{=} \{x \in \mathfrak{g} \text{ s.t. } [x, \mathfrak{h}] \subseteq \mathfrak{h}\}$, the* normalizer *of $\mathfrak{h}$ in $\mathfrak{g}$.*

2. *$\mathfrak{h} = \mathfrak{g}_0$ is the 0-weight space of* $\mathrm{ad} : \mathfrak{h} \curvearrowright \mathfrak{g}$.

Proof Define $N^{(i)} \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \text{ s.t. } (\operatorname{ad}\mathfrak{h})^i x \subseteq \mathfrak{h}\}$. Then $N^{(0)} = \mathfrak{h}$ and $N^{(1)} = N(\mathfrak{h})$, and $N^{(i)} \subseteq N^{(i+1)}$. By finite-dimensionality, the sequence $N^{(0)} \subseteq N^{(1)} \subseteq \dots$ must eventually stabilize. By definition $\bigcup N^{(i)} = \mathfrak{g}_0$, so 2. implies 1. But $N^{(i+1)} = N(N^{(i)})$, and so 1. implies 2. $\square$

5.26 Definition *A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ satisfying the equivalent conditions of Proposition 5.25 is a* Cartan subalgebra *of $\mathfrak{g}$.*

5.27 Theorem (Existence of a Cartan Subalgebra) *Every finite-dimensional Lie algebra over an algebraically closed field of characteristic 0 has a Cartan subalgebra.*

Before we prove this theorem, we will need some definitions and lemmas.

5.28 Definition *Let $\mathbb{K}$ be a field; we say that $X \subseteq \mathbb{K}^n$ is* Zariski closed *if $X = \{x \in \mathbb{K}^n \text{ s.t. } p_i(x) = 0 \,\forall i\}$ for some possibly infinite set $\{p_i\}$ of polynomials in $\mathbb{K}[x_1, \dots, x_n]$. A subset $X \subseteq \mathbb{K}^n$ is* Zariski open *if $\mathbb{K}^n \smallsetminus X$ is Zariski closed.*

5.29 Lemma *If $\mathbb{K}$ is infinite and $U, V \subseteq \mathbb{K}^n$ are two non-empty Zariski open subsets, then $U \cap V$ is non-empty.*

Proof Let $\bar{U} \stackrel{\text{def}}{=} \mathbb{K}^n \smallsetminus U$ and similarly for $\bar{V}$. Let $u \in U$ and $v \in V$. If $u = v$ we're done, and otherwise consider the line $L \subseteq \mathbb{K}^n$ passing through u and v, parameterized $\mathbb{K} \xrightarrow{\sim} \mathbb{L}$ by $t \mapsto tu + (1-t)v$. Then $L \cap \bar{U}$ and $L \cap \bar{V}$ are finite, as their preimages under $\mathbb{K} \to \mathbb{L}$ are loci of polynomials. Since $\mathbb{K}$ is infinite, L contains infinitely many points in $U \cap V$. $\square$

5.30 Lemma / Definition *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field of characteristic 0. An element $x \in \mathfrak{g}$ is* regular *if $\mathfrak{g}_{0,x}$ has minimal dimension. If x is regular, then $\mathfrak{g}_{0,x}$ is a nilpotent subalgebra of $\mathfrak{g}$.*

Proof We will write $\mathfrak{h}$ for $\mathfrak{g}_{0,x}$. That $\mathfrak{h}$ is a subalgebra follows from Corollary 5.24. Suppose that $\mathfrak{h}$ is not nilpotent, and let $U \stackrel{\text{def}}{=} \{h \in \mathfrak{h} \text{ s.t. } \operatorname{ad} h|_{\mathfrak{h}} \text{ if not nilpotent}\} \neq 0$. Then $U = \{h \in \mathfrak{h} \text{ s.t. } (\operatorname{ad} h|_{\mathfrak{h}})^d \neq 0\}$ is a Zariski-open subset of $\mathfrak{h}$. Moreover, $V \stackrel{\text{def}}{=} \{h \in \mathfrak{h} \text{ s.t. } h \text{ acts invertibly on } \mathfrak{g}/\mathfrak{h}\}$ is also a non-empty Zariski-open subset of $\mathfrak{h}$, where V is the quotient of $\mathfrak{h}$-modules; it is non-empty because $x \in V$. By Lemma 5.29 (recall that any algebraically closed field is infinite), there exists $y \in U \cap V$. Then $\operatorname{ad} y$ preserves $\mathfrak{g}_{\alpha,x}$ for every α, as $y \in \mathfrak{h} = \mathfrak{g}_{0,x}$, and y acts invertibly on every $\mathfrak{g}_{\alpha,x}$ for $\alpha \neq 0$. Then$\mathfrak{g}_{0,y} \subseteq \mathfrak{g}_{0,x} = \mathfrak{h}$, but $y \in U$ and so $\mathfrak{g}_{0,y} \neq \mathfrak{h}$. This contradicts the minimality of $\mathfrak{h}$. $\square$

Proof (of Theorem 5.27) We let $\mathfrak{g}$, $x \in \mathfrak{g}$, and $\mathfrak{h} = \mathfrak{g}_{0,x}$ be as in Lemma/Definition 5.30. Then $\mathfrak{h} \subseteq \mathfrak{g}_{0,\mathfrak{h}}$ because $\mathfrak{h}$ is nilpotent, and $\mathfrak{g}_{0,\mathfrak{h}} \subseteq \mathfrak{g}_{0,x} = \mathfrak{h}$ because $x \in \mathfrak{h}$. Thus $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. $\square$

We mention one more fact about the Zariski topology:

5.31 Lemma *Let U by a Zariski open set over $\mathbb{C}$. Then U is path connected.*

Proof Let $u, v \in U$ and construct the line L as in the proof of Lemma 5.29. Then $L \cap U$ is isomorphic to $\mathbb{C} \smallsetminus \{\text{finite}\}$, and therefore is path connected. $\square$

5.32 Proposition *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$. Then all Cartan subalgebras of $\mathfrak{g}$ are conjugate by automophisms of $\mathfrak{g}$.*

Proof Consider $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Then $\mathrm{ad}\,\mathfrak{g} \subseteq \mathfrak{gl}(\mathfrak{g})$ is a Lie subalgebra, and so corresponds to a connected Lie subgroup $\mathrm{Int}\,\mathfrak{g} \subseteq \mathrm{GL}(\mathfrak{g})$ generated by $\exp(\mathrm{ad}\,\mathfrak{g})$. Since $\mathfrak{g} \curvearrowright \mathfrak{g}$ be derivations, $\exp(\mathrm{ad}\,\mathfrak{g}) \curvearrowright \mathfrak{g}$ by automorphisms, and so $\mathrm{Int}\,\mathfrak{g} \subseteq \mathrm{Aut}\,\mathfrak{g}$.

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra, and $\mathfrak{g} = \bigoplus \mathfrak{g}_{\alpha,\mathfrak{h}}$ the corresponding weight-space decomposition. Since $\mathfrak{g}$ is finite-dimensional, the set

$$R_{\mathfrak{h}} \overset{\text{def}}{=} \{h \in \mathfrak{h} \text{ s.t. } \alpha(h) \neq 0 \text{ if } \alpha \neq 0 \text{ and } \mathfrak{g}_{\alpha,\mathfrak{h}} \neq 0\} = \{h \in \mathfrak{h} \text{ s.t. } \mathfrak{g}_{0,h} = \mathfrak{h}\} \tag{5.3.4}$$

is non-empty and open, since we can take α to range over a finite set (by finite-dimensionality).

Let $\sigma : \mathrm{Int}\,\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be the canonical action, and consider the restriction to $\sigma : \mathrm{Int}\,\mathfrak{g} \times R_{\mathfrak{h}} \to \mathfrak{g}$. Pick $y \in R_{\mathfrak{h}}$ and let $e \in \mathrm{Int}\,\mathfrak{g}$ be the identity element. We compute the image of the infinitesimal action $d\sigma\left(\mathrm{T}_{(e,y)}(\mathrm{Int}\,\mathfrak{g} \times R_{\mathfrak{h}})\right) \subseteq \mathrm{T}_y\mathfrak{g} \cong \mathfrak{g}$. By construction, varying the first component yields an action by conjugation: $x \mapsto [x, y]$. Thus the image of $\mathrm{T}_e\,\mathrm{Int}\,\mathfrak{g} \times \{0 \in \mathrm{T}_y R_{\mathfrak{h}}\}$ is $(\mathrm{ad}\,y)(\mathfrak{g})$. Since y acts invertibly, $(\mathrm{ad}\,y)(\mathfrak{g}) \supseteq \bigoplus_{\alpha\neq 0} \mathfrak{g}_{\alpha,\mathfrak{h}}$. By varying the second coordinate (recall that $R_{\mathfrak{h}}$ is open), we see that $d\sigma\left(\mathrm{T}_{(e,y)}(\mathrm{Int}\,\mathfrak{g} \times R_{\mathfrak{h}})\right) \supseteq \mathfrak{h} = \mathfrak{g}_{0,\mathfrak{h}}$ also. Thus $d\sigma\left(\mathrm{T}_{(e,y)}(\mathrm{Int}\,\mathfrak{g} \times R_{\mathfrak{h}})\right) = \mathfrak{g} = \mathrm{T}_y\mathfrak{g}$, and so the image $(\mathrm{Int}\,\mathfrak{g})(R_{\mathfrak{h}})$ contains a neighborhood of y and therefore is open.

For each $y \in \mathfrak{g}$, consider the generalized nullspace $\mathfrak{g}_{0,y}$; the dimension of $\mathfrak{g}_{0,y}$ depends on the characteristic polynomial of y, and the coefficients of the characteristic polynomial depend polynomially on the matrix entries of $\mathrm{ad}\,y$. In particular, $\dim \mathfrak{g}_{0,y} \geq r$ if and only if the last r coefficients of the characteristic polynomial of $\mathrm{ad}\,y$ are 0, and so $\{y \in \mathfrak{g} \text{ s.t. } \dim \mathfrak{g}_{0,y} \geq r\}$ is Zariski closed. Therefore $y \mapsto \dim \mathfrak{g}_{0,y}$ is upper semi-continuous in the Zariski topology. In particular, let r be the minimum value of $\dim \mathfrak{g}_{0,y}$, which exists since $\dim \mathfrak{g}_{0,y}$ takes values in integers. Then $\mathrm{Reg} \overset{\text{def}}{=} \{y \in \mathfrak{g} \text{ s.t. } \dim \mathfrak{g}_{0,y} = r\}$, the set of regular elements, is Zariski open and therefore dense. In particular, Reg intersects $(\mathrm{Int}\,\mathfrak{g})(R_{\mathfrak{h}})$.

But if $y \in (\mathrm{Int}\,\mathfrak{g})(R_{\mathfrak{h}})$ then $\dim \mathfrak{g}_{0,y} = \dim \mathfrak{h}$. Therefore $\dim \mathfrak{h}$ is the minimal value of $\dim \mathfrak{g}_{0,y}$ and in particular $R_{\mathfrak{h}} \subseteq \mathrm{Reg}$. Conversely, $\mathrm{Reg} = \bigcup_{\mathfrak{h}' \text{ a Cartan}} R_{\mathfrak{h}'} = \bigcup_{\mathfrak{h}' \text{ a Cartan}} (\mathrm{Int}\,\mathfrak{g}) R_{\mathfrak{h}'}$.

However, $\mathrm{Int}\,\mathfrak{g}$ is a connected group, $R_{\mathfrak{h}}$ is connected being $\mathbb{C}^n$ minus some hyperplanes, and Reg is connected on account of being Zariski open. But the orbits of $(\mathrm{Int}\,\mathfrak{g})R_{\mathfrak{h}}$ are disjoint, and their union is all of Reg, so Reg must consist of a single orbit.

To review: $\mathfrak{h}$ is Cartan and so contains regular elements of $\mathfrak{g}$, and any other regular element of $\mathfrak{g}$ is in the image under $\mathrm{Int}\,\mathfrak{g}$ of a regular element of $\mathfrak{h}$. Thus every Cartan subalgebra is in $(\mathrm{Int}\,\mathfrak{g})\mathfrak{h}$. $\square$

5.3.2 More on the Jordan Decomposition and Schur's Lemma

[8, Lectures 30 and 31]

Recall Theorem 4.50 that every $x \in \mathrm{End}(V)$, where V is a finite-dimensional vector spce over an algebraically closed field, has a unique decomposition $x = x_s + x_n$ where x_s is diagonalizable and x_n is nilpotent. We will strengthen this result in the case when $x \in \mathfrak{g} \to \mathrm{End}(V)$ and $\mathfrak{g}$ is semisimple.

5.33 Lemma *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field, and* $\operatorname{Der}\mathfrak{g} \subseteq \operatorname{End}\mathfrak{g}$ *the algebra of derivations of* $\mathfrak{g}$*. If* $x \in \operatorname{Der}\mathfrak{g}$*, then* $x_s, x_n \in \operatorname{Der}\mathfrak{g}$*.*

Proof For $x \in \operatorname{Der}\mathfrak{g}$, construct the weight-space decomposition $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_{\lambda,x}$ of generalized eigenspaces of x. Since x is a derivation, the weight spaces add: $[\mathfrak{g}_{\mu,x}, \mathfrak{g}_{\nu,x}] \subseteq \mathfrak{g}_{\mu+\nu,x}$. Let $y \in \operatorname{End}\mathfrak{g}$ act as λ on $\mathfrak{g}_{\lambda,x}$; then y is a derivation by the additive property. But y is diagonalizable and commutes with x, and $x - y$ is nilpotent because all its eigenvalues are 0 so $y = x_s$. □

We have an immediate corollary:

5.34 Lemma / Definition *If $\mathfrak{g}$ is a semisimple finite-dimensional Lie algebra over an algebraically closed field $\mathbb{K}$, then every $x \in \mathfrak{g}$ has a unique* Jordan decomposition $x = x_s + x_n$ *such that* $[x_s, x_n] = 0$*,* $\operatorname{ad} x_s$ *is diagonalizable, and* $\operatorname{ad} x_n$ *is nilpotent.*

Proof If $\mathfrak{g}$ is semisimple then $\operatorname{ad} : \mathfrak{g} \to \operatorname{Der}\mathfrak{g}$ is injective as $Z(\mathfrak{g}) = 0$ and surjective because $\operatorname{Der}\mathfrak{g}/\operatorname{ad}\mathfrak{g} = \operatorname{Ext}^1(\mathfrak{g}, \mathbb{K}) = 0$. □

5.35 Theorem (Schur's Lemma over an algebraically closed field) *Let U be an algebra over $\mathbb{K}$ and algebraically closed field, and let V be an irreducible U-module. Then* $\operatorname{End}_U(V) = \mathbb{K}$*.*

Proof Let $\phi \in \operatorname{End}_U(V)$ and $\lambda \in \mathbb{K}$ an eigenvalue of ϕ. Then $\phi - \lambda$ is singular and hence 0 by Theorem 4.74. □

5.36 Proposition *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic* 0*, and let $\sigma : \mathfrak{g} \curvearrowright V$ be a finite-dimensional $\mathfrak{g}$ module. For $x \in \mathfrak{g}$, write x_s and x_n as in Lemma/Definition 5.34, and write $\sigma(x)_s$ and $\sigma(x)_n$ for the diagonalizable and nilpotent parts of $\sigma(x) \in \mathfrak{gl}(\mathfrak{g})$ as given by Theorem 4.50. Then $\sigma(x)_s = \sigma(x_s)$ and $\sigma(x)_n = \sigma(x_n)$.*

Proof We reduce to the case when V is an irreducible $\mathfrak{g}$-module using Theorem 4.78, and we write $\mathfrak{g} = \prod \mathfrak{g}_i$ a product of simples using Corollary 4.61. Then $\mathfrak{g}_i \curvearrowright V$ as 0 for every i except one, for which the action $\mathfrak{g}_i \curvearrowright V$ is faithful. We replace $\mathfrak{g}$ by that $\mathfrak{g}_i$, whence $\sigma : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ with $\mathfrak{g}$ simple.

It suffices to show that $\sigma(x)_s \in \sigma(\mathfrak{g})$, since then $\sigma(x_s) = \sigma(s)$ for some $s \in \mathfrak{g}$, $\sigma(x)_n = \sigma(x) - \sigma(s) = \sigma(x - s)$, and s and $n = x - s$ commute, sum to x, and act diagonalizably and nilpotently since the adjoint action $\operatorname{ad} : \mathfrak{g} \curvearrowright \mathfrak{g}$ is a submodule of $\mathfrak{g} \curvearrowright \mathfrak{gl}(V)$, so $s = x_s$ and $n = x_n$.

By semisimplicity, $\mathfrak{g} = \mathfrak{g}' \subseteq \mathfrak{sl}(V)$. By Theorem 5.35, the centralizer of $\mathfrak{g}$ in $\mathfrak{gl}(V)$ consists of scalars. In characteristic 0, the only scalar in $\mathfrak{sl}(V$ is 0, so the centralizer of $\mathfrak{g}$ in $\mathfrak{sl}(V)$ is 0. Define the normalizer $N(\mathfrak{g}) = \{x \in \mathfrak{sl}(V) \text{ s.t. } [x, \mathfrak{g}] \subseteq \mathfrak{g}\}$; then $N(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{sl}(V)$ containing $\mathfrak{g}$, and $N(\mathfrak{g})$ acts faithfully on $\mathfrak{g}$ since the centralizer of $\mathfrak{g}$ in $\mathfrak{sl}(V)$ is 0, and this action is by derivations. But all derivations are inner, as in the proof of Lemma/Definition 5.34, and so $N(\mathfrak{g}) \curvearrowright \mathfrak{g}$ factors through $\mathfrak{g} \curvearrowright \mathfrak{g}$, and hence $N(\mathfrak{g}) = \mathfrak{g}$.

So it suffices to show that $\sigma(x)_s \in N(\mathfrak{g})$ for $x \in \mathfrak{g}$. Since $\sigma(x)_n$ is nilpotent, it's traceless, and hence in $\mathfrak{sl}(V)$; then $\sigma(x)_s \in \mathfrak{sl}(V)$ as well. We construct a generalized eigenspace decomposition of V with respect to $\sigma(x) : V = \bigoplus V_{\lambda,x}$. Then $\sigma(x)_s$ acts on $V_{\lambda,x}$ by the scalar λ. We also construct a generalized eigenspace decomposition $\mathfrak{g} = \bigoplus \mathfrak{g}_{\alpha,x}$ with respect to the adjoint action $\operatorname{ad} : \mathfrak{g} \curvearrowright \mathfrak{g}$. Since $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, we have $\mathfrak{g}_{\alpha,x} = \mathfrak{g} \cap \operatorname{End}_{\mathbb{K}}(V)_\alpha = \bigoplus \operatorname{Hom}_{\mathbb{K}}(V_{\lambda,x}, V_{\lambda+\alpha,x})$, by tracking the eigenvalues of the right and left actions of $\mathfrak{g}$ on V.

Moreover, $\mathrm{ad}(\sigma(x)_s) = \mathrm{ad}(\sigma(x_s))$ because both act by α on $\mathrm{Hom}_{\mathbb{K}}(V_{\lambda,x}, V_{\lambda+\alpha,x})$ and hence on $\mathfrak{g}_\alpha$. Thus $\sigma(x)_s$ fixes $\mathfrak{g}$ since $\sigma(x_s)$ does. Therefore $\sigma(x)_s \in N(\mathfrak{g})$. $\square$

5.3.3 Precise description of Cartan subalgebras

[8, Lecture 31]

5.37 Lemma *Let $\mathfrak{g}$ be a semisimple Lie algebra over characteristic 0, $\mathfrak{h} \subseteq \mathfrak{g}$ a nilpotent subalgebra, and $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$ the root space decomposition with respect to $\mathfrak{h}$. Then the Killing form β pairs $\mathfrak{g}_\alpha$ with $\mathfrak{g}_{-\alpha}$ nondegenerately, and $\beta(\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}) = 0$ if $\alpha + \alpha' \neq 0$.*

Proof Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{\alpha'}$. For any $h \in \mathfrak{h}$, $(\mathrm{ad}\, h - \alpha(h))^n x = 0$ for some n. So

$$0 = \beta\Big((\mathrm{ad}\, h - \alpha(h))^n\, x, y\Big) = \beta\Big(x, (-\,\mathrm{ad}\, h - \alpha(h))^n\, y\Big) \tag{5.3.5}$$

but $(-\,\mathrm{ad}\, h - \alpha(h))^n$ is invertible on $\mathfrak{g}_{\alpha'}$ unless $\alpha' = -\alpha$. Nondegeneracy follows from nondegeneracy of β on all of $\mathfrak{g}$. $\square$

5.38 Corollary *If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra over characteristic 0, and let $\mathfrak{h} \subseteq \mathfrak{g}$ a nilpotent subalgebra, then the largest nilpotency ideal in $\mathfrak{g}_0$ of the action $\mathrm{ad} : \mathfrak{g}_0 \curvearrowright \mathfrak{g}$ is the 0 ideal.*

Proof The Killing form β pairs $\mathfrak{g}_0$ with itself nondegenerately. As β is the trace form of $\mathrm{ad} : \mathfrak{g}_0 \curvearrowright \mathfrak{g}$, and $\mathrm{ad}(\mathfrak{g})$-nilpotent ideal of $\mathfrak{g}_0$ must be in $\ker \beta = 0$. $\square$

5.39 Proposition *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field $\mathbb{K}$ of characteristic 0, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Then $\mathfrak{h}$ is abelian and $\mathrm{ad} : \mathfrak{h} \curvearrowright \mathfrak{g}$ is diagonalizable.*

Proof By definition, $\mathfrak{h}$ is nilpotent and hence solvable, and by Theorem 4.37 we can find a basis of $\mathfrak{g}$ in which $\mathfrak{h} \curvearrowright \mathfrak{g}$ by upper triangular matrices. Thus $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ acts by strictly upper triangular matrices and hence nilpotently on $\mathfrak{g}$. But $\mathfrak{h} = \mathfrak{g}_0$, and so $\mathfrak{h}' = 0$ by Corollary 5.38. This proves that $\mathfrak{h}$ is abelian.

Let $x \in \mathfrak{h}$. Then $\mathrm{ad}\, x_s = (\mathrm{ad}\, x)_s$ acts as $\alpha(x)$ on $\mathfrak{g}_\alpha$, and in particular x_s centralizes $\mathfrak{h}$. So $x_s \in \mathfrak{g}_0 = \mathfrak{h}$ and so $x_n = x - x_s \in \mathfrak{h}$. But if $n \in \mathfrak{h}$ acts nilpotently on $\mathfrak{g}$, then $\mathbb{K}n$ is an ideal of $\mathfrak{h}$, since $\mathfrak{h}$ is abelian, and acts nilpotently on $\mathfrak{g}$, so $\mathbb{K}n = 0$ by Corollary 5.38. Thus $x_n = 0$ and $x = x_s$. In particular, x acts diagonalizably on $\mathfrak{g}$. To show that $\mathfrak{h}$ acts diagonalizably, we use finite-dimensionality and the classical fact that if n diagonalizable matrices commute, then they can by simultaneously diagonalized. $\square$

5.40 Corollary *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0. Then a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is Cartan if and only if $\mathfrak{h}$ is a maximal diagonalizable abelian subalgebra.*

Proof We first show the maximality of a Cartan subalgebra. Let $\mathfrak{h}$ be a Cartan subalgebra and $\mathfrak{h}_1 \supseteq \mathfrak{h}$ abelian. Then $\mathfrak{h}_1 \subseteq \mathfrak{g}_0 = \mathfrak{h}$ because it normalizes $\mathfrak{h}$.

Conversely, let $\mathfrak{h}$ be a maximal diagonalizable abelian subalgebra of $\mathfrak{g}$, and write $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$ the weight space decomposition of $\mathfrak{h} \curvearrowright \mathfrak{g}$. We want to show that $\mathfrak{h} = \mathfrak{g}_0$, the centralizer of $\mathfrak{h}$. Pick $x \in \mathfrak{g}_0$; then $x_s, x_n \in \mathfrak{g}_0$, and so $x_s \in \mathfrak{h}$ by maximality. In particular, $\mathfrak{g}_0$ is spanned by $\mathfrak{h}$ and ad-nilpotent elements. Thus $\mathfrak{g}_0$ is nilpotent by Theorem 4.25 and therefore solvable, so $\mathfrak{g}_0'$ acts nilpotently on $\mathfrak{g}$. But $\mathfrak{g}_0'$ is an ideal of $\mathfrak{g}_0$ that acts nilpotently, so $\mathfrak{g}_0' = 0$, so $\mathfrak{g}_0$ is abelian. Then any one-dimensional subspace of $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}_0$, and a subspace spanned by a nilpotent acts nilpotently, so $\mathfrak{g}_0$ doesn't have any nilpotents. Therefore $\mathfrak{g}_0 = \mathfrak{h}$. □

5.4 Root systems

5.4.1 Motivation and a Quick Computation

[8, Lectures 31 and 32]

In any semisimple Lie algebra over $\mathbb{C}$ we can choose a Cartan subalgebra, to which we assign combinatorial data. Since all Cartan subalgebras are conjugate, this data, called a *root system*, will not depend on our choice. Conversely, this data will uniquely describe the Lie algebra, based on the representation theory of $\mathfrak{sl}(2)$.

5.41 Definition *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and $\mathfrak{h}$ a Cartan subalgebra. The* root space decomposition *of $\mathfrak{g}$ is the weight decomposition $\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$ of $\mathrm{ad} : \mathfrak{h} \curvearrowright \mathfrak{g}$; each $\mathfrak{g}_\alpha$ is a* root space*, and the set of weights $\alpha \in \mathfrak{h}^*$ that appear in the root space decomposition comprise the* roots *of $\mathfrak{g}$. By Proposition 5.32 the structure of the set of roots depends up to isomorphism only on $\mathfrak{g}$.*

5.42 Lemma / Definition *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with Killing form β, $\mathfrak{h}$ a Cartan subalgebra, and $x_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \neq 0$. To x_α we can associate $y_\alpha \in \mathfrak{g}_{-\alpha}$ with $\beta(x_\alpha, y_\alpha) = -1$ and to the root α we associate a* coroot *h_α with $\beta(h_\alpha, -) = \alpha$. Then $\{x_\alpha, y_\alpha, h_\alpha\}$ span a subalgebra $\mathfrak{sl}(2)_\alpha$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2)$.*

Proof That h_α and y_α are well-defined follows from the nondegeneracy of β. For any $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha$, and $y \in \mathfrak{g}_{-\alpha}$, we have

$$\beta(h, [x, y]) = \beta([x, h], y) \tag{5.4.1}$$
$$= -\alpha(h)\,\beta(x, y) \tag{5.4.2}$$

Thus $[x, y] = -\beta(x, y)h_\alpha$. Moreover, since $x_\alpha \in \mathfrak{g}_\alpha$, $[h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha$, and since $y_\alpha \in \mathfrak{g}_{-\alpha}$, $[h_\alpha, y_\alpha] = -\alpha(h_\alpha)y_\alpha$.

Thus $x_\alpha, y_\alpha, h_\alpha$ span a three-dimensional Lie subalgebra of $\mathfrak{g}$, which is isomorphic to either $\mathfrak{sl}(2)$ or or the Heisenberg algebra. But in every finite-dimensional representation the Heisenberg algebra acts nilpotently, whereas $\mathrm{ad}(h_\alpha) \in \mathrm{End}(\mathfrak{g})$ is diagonalizable. Therefore this subalgebra is isomorphic to $\mathfrak{sl}(2)$, and $\alpha(h_\alpha) \neq 0$. □

5.43 Corollary *Let α be a root of $\mathfrak{g}$. Then $\pm\alpha$ are the only non-zero roots of $\mathfrak{g}$ in $\mathbb{C}\alpha$, and $\dim \mathfrak{g}_\alpha = 1$. In particular, $\mathfrak{sl}(2)_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathbb{C}h_\alpha$.*

Proof We consider $\mathfrak{j}_\alpha \stackrel{\text{def}}{=} \bigoplus_{\alpha' \in \mathbb{C}\alpha \smallsetminus \{0\}} \mathfrak{g}_{\alpha'} \oplus \mathbb{C}h_\alpha$; it is a subalgebra of $\mathfrak{g}$ and an $\mathfrak{sl}(2)_\alpha$-submodule, since $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}] \subseteq \mathfrak{g}_{\alpha+\alpha'}$, and $h_{c\alpha} = ch_\alpha$ for $c \in \mathbb{C}$. Let $\alpha' \in \mathbb{C}\alpha \smallsetminus \{0\}$ be a root; as a weight of the $\mathfrak{sl}(2)_\alpha$ representation, we see that $\alpha' \in \mathbb{Z}\alpha/2$. If any half-integer multiple of α actually appears, then $\alpha/2$ appears, and by switching α to $\alpha/2$ if necessary we can assure that $\mathfrak{j}_\alpha$ contains only representations V_{2m}. But each V_{2m} has contributes a basis vector in weight 0, and the only part of $\mathfrak{j}_\alpha$ in weight 0 is $\mathbb{C}h_\alpha$. Therefore $\mathfrak{j}_\alpha$ is irreducible as an $\mathfrak{sl}(2)_\alpha$ module, contains $\mathfrak{sl}(2)_\alpha$, and so equals $\mathfrak{sl}(2)_\alpha$. □

5.44 Corollary *The roots α span $\mathfrak{h}^*$, and the coroots h_α span $\mathfrak{h}$.*

Proof We let α range over the non-zero roots. Then

$$\bigcap_{\alpha \neq 0} \ker \alpha = Z(\mathfrak{g}) = 0 \tag{5.4.3}$$

$$\sum_{\alpha \neq 0} \mathbb{C}h_\alpha = \mathfrak{g}' \cap \mathfrak{h} = \mathfrak{h} \tag{5.4.4}$$

That $\beta^{-1} : \alpha \mapsto h_\alpha$ is a linear isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ completes the proof. □

5.45 Proposition *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$, and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. Let $R \subseteq \mathfrak{h}^*$ be the set of nonzero roots and $R^\vee \subseteq \mathfrak{h}$ the set of nonzero coroots. Then $\alpha \mapsto \alpha^\vee \stackrel{\text{def}}{=} \frac{2h_\alpha}{\alpha(h_\alpha)}$ defines a bijection $\vee : R \to R^\vee$, and the triple $(R, R^\vee, \vee)$ comprise a* root system *in $\mathfrak{h}$.*

We will define the words “root system” in the next section to generalize the data already computed.

5.4.2 The Definition

[8, Lecture 33]

5.46 Definition *A* root system *is a complex vector space $\mathfrak{h}$, a finite subset $R \subseteq \mathfrak{h}^*$, a subset $R^\vee \subseteq \mathfrak{h}$, a bijection $\vee : R \to R^\vee$, subject to*

RS1 $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$

RS2 *$R = -R$ and $R^\vee = -R^\vee$, with $(-\alpha)^\vee = -(\alpha^\vee)$*

RS3 $\langle \alpha, \alpha^\vee \rangle = 2$

RS4 *If $\alpha, \beta \in R$ are not proportional, then $(\beta + \mathbb{C}\alpha) \cap R$ consists of a “string”:*

$$\left\langle (\beta + \mathbb{C}\alpha) \cap R, \alpha^\vee \right\rangle = \{m, m-2, \ldots, -m+2, -m\} \tag{5.4.5}$$

Nondeg *R spans $\mathfrak{h}^*$ and $R^\vee$ spans $\mathfrak{h}$*

Reduced *$\mathbb{C}\alpha \cap R = \{\pm\alpha\}$ for $\alpha \in R$.*

Two root systems are isomorphic *if there is a linear isomorphism of the underlying vector spaces, inducing an isomorphism on dual spaces, that carries each root system to the other. The* rank *of a root system is the dimension of* $\mathfrak{h}$*.*

5.47 Definition *Given a root system* $(R, R^\vee)$ *on a vector space* $\mathfrak{h}$*, the* Weyl group $W \subseteq \mathrm{GL}(\mathfrak{h}^*)$ *is the group generated by the reflections* $s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ *as* α *ranges over* R*.*

5.48 Proposition

1. *It follows from* **RS3** *that* $s_\alpha^2 = e \in W$ *for each root* α*.*
2. *It follows from* **RS4** *that* $WR = R$*. Thus* W *is finite. Moreover,* W *preserves* $\mathfrak{h}^*_{\mathbb{R}}$*, the* $\mathbb{R}$*-span of* R*.*
3. *The* W*-average of any positive-definite inner product on* $\mathfrak{h}^*_{\mathbb{R}}$ *is a* W*-invariant positive-definite inner product. Let* $(,)$ *be a* W*-invariant positive-definite inner product. Then* s_α *is orthogonal with respect to* $(,)$*, and so* $s_\alpha : \lambda \mapsto \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$*. This inner product establishes an isomorphism* $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$*, under which* $\alpha^\vee \mapsto 2\alpha/(\alpha, \alpha)$*.*
4. *Therefore* **Reduced** *holds with* R *replaced by* $R^\vee$ *if it holds at all.*
5. *Let* W *act on* $\mathfrak{h}$ *dual to its action on* $\mathfrak{h}^*$*. Then* $w(\alpha^\vee) = (w\alpha)^\vee$ *for* $w \in W$ *and* $\alpha \in R$*. Thus* $s_{w\alpha} = w s_\alpha w^{-1}$*.*
6. *If* $V \subseteq \mathfrak{h}^*$ *is spanned by any subset of* R*, then* $R \cap V$ *and its image under* $\vee$ *form another nondegenerate root system.*
7. *Two root systems with the same Weyl group and lattices are related by an isomorphism.*

5.49 Definition *Let* R *be a root system in* $\mathfrak{h}^*$*. Define the* weight lattice *to be* $P \stackrel{\text{def}}{=} \{\lambda \in \mathfrak{h}^* \text{ s.t. } \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\, \forall \alpha^\vee \in R^\vee\}$ *and the* root lattice Q *to be the* $\mathbb{Z}$*-span of* R*. Then* **RS1** *implies that* $R \subseteq Q \subseteq P \subseteq \mathfrak{h}^*$*; by* **Nondeg***, both* P *and* Q *are of full rank and so the index* $P : Q$ *is finite. We define the* coweight lattice *to be* $P^\vee$ *and the* coroot lattice *to be* $Q^\vee$*.*

5.4.3 Classification of rank-two root systems

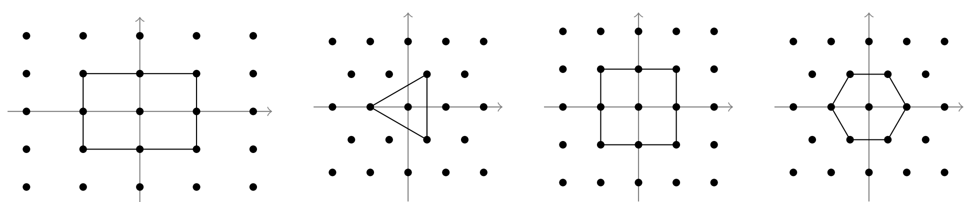
By **Reduced**, there is a unique rank-one root system up to isomorphism, the root system of $\mathfrak{sl}(2)$.

Let R be a rank-two root system; then its Weyl group W is a finite subgroup $W \subseteq \mathrm{GL}(2, \mathbb{R})$ generated by reflections. The only finite subgroups of $\mathrm{GL}(2, \mathbb{R})$ are the cyclic and dihedral groups; only the dihedral groups are generated by reflections, and so $W \cong D_{2m}$ for some m. Moreover, W preserves the root lattice Q.

5.50 Lemma *The only dihedral groups that preserve a lattice are* D_4, D_6, D_8*, and* D_{12}*.*

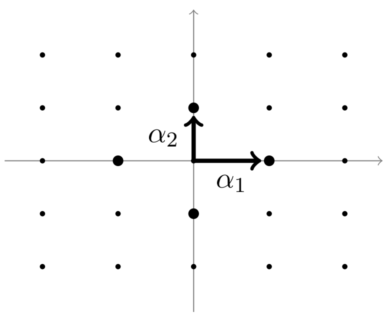
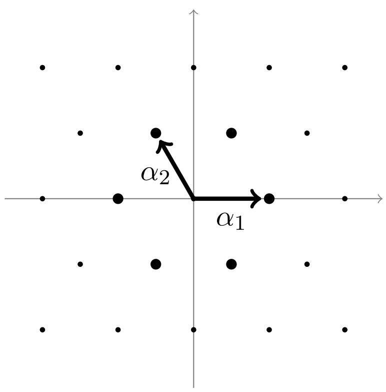
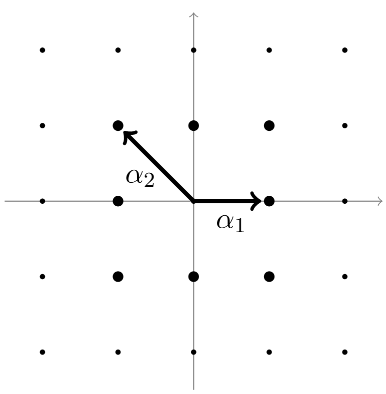
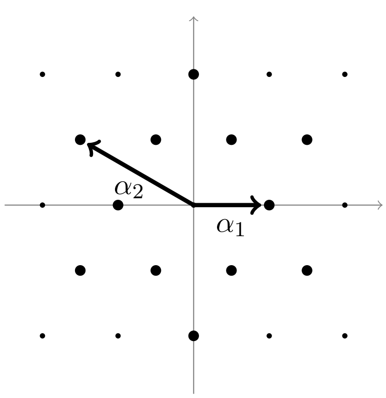
Proof Let r_θ be a rotation by θ. Its eigenvalues are $e^{\pm i\theta}$, and so $\mathrm{tr}(r_\theta) = 2\cos\theta$. If r_θ preserves a lattice, its trace must be an integer, and so $2\cos\theta \in \{1, 0, -1, -2\}$, as $2\cos\theta = 2$ corresponds to the identity rotation, and $|\cos\theta| \leq 1$. Therefore $\theta \in \{\pi, 2\pi/3, \pi/2, \pi/3\}$, i.e. $\theta = 2\pi/m$ for $m \in \{2, 3, 4, 6\}$, and the only valid dihedral groups are D_{2m} for these values of m. $\square$

5.51 Corollary *There are four rank-two root systems, corresponding to the rectangular lattice, the square lattice, and the hexagonal lattice twice:*



$\theta = \pi$ $\theta = 2\pi/3$ $\theta = \pi/2$ $\theta = \pi/3$

For each dihedral group, we can pick two reflections α_1, α_2 with a maximally obtuse angle; these generate W and the lattice. On the next page we list the four rank-two root systems with comments on their corresponding Lie groups:

m	Picture	Name	notes
2		$A_1 \times A_1 = D_2$	corresponding to the Lie algbera $\mathfrak{sl}(2) \times \mathfrak{sl}(2) = \mathfrak{so}(4)$
3		A_2	corresponding to $\mathfrak{sl}(3)$ acting on the traceless diagonals
4		$B_2 = C_2$	$\mathfrak{so}(5) = \mathfrak{sp}(4)$. (When we get higher up, the Bs and Cs will separate, and we will have a new sequence of Ds.)
6		G_2	a new simple algebra of dimension 14 = number of roots plus dimension of root space. We will see later that its smallest representation has dimension 7. There are many descriptions of this representation and the corresponding Lie algebra; the seven-dimensional representation comes from the Octonians, a non-associative, non-commutative "field", and G_2 is the automorphism group of the pure-imaginary part of the Octonians.

5.52 Lemma / Definition *The axioms of a finite root system are symmetric under the interchange $R \leftrightarrow R^\vee$. This interchange assigns a* dual *to each root system.*

Proof Only **RS4** is not obviously symmetric. We did not use **RS4** to classify the two-dimensional root systems; we needed only a corollary:

RS4' $W(R) = R$,

which is obviously symmetric. But **RS4** describes only the two-dimensional subspaces of a root system, and every rank-two root system with **RS4'** replacing **RS4** in fact satisfies **RS4**. This suffices to show that **RS4'** implies **RS4** for finite root systems. □

We remark that the statement is false for infinite root systems, and we presented the definition we did to accommodate the infinite case. We will not discuss infinite root systems further.

5.4.4 Positive roots

[8, Lecture 34][1]

5.53 Definition *A* positive root system *consists of a (finite) root system $R \subseteq \mathfrak{h}_{\mathbb{R}}^*$ and a vector $v \in \mathfrak{h}_{\mathbb{R}}$ so that $\alpha(v) \neq 0$ for every root $\alpha \in R$. A root $\alpha \in R$ is* positive *if $\alpha(v) > 0$, and* negative *otherwise. Let R_+ be the set of positive roots and R_- the set of negative ones; then $R = R_+ \sqcup R_-$, and by* **RS2**, *$R_+ = -R_-$.*

The $\mathbb{R}_{\geq 0}$-span of R_+ is a cone in $\mathfrak{h}_{\mathbb{R}}^*$, and we let Δ be the set of extremal rays in this cone. Since the root system is finite, extremal rays are generated by roots, and we use Reduced *to identify extremal rays with positive roots. Then $\Delta \subseteq R$ is the set of* simple roots*.*

5.54 Lemma *If α and β are two simple roots, then $\alpha - \beta$ is not a root. Moreover, $(\alpha, \beta) \leq 0$ for $\alpha \neq \beta$.*

Proof If $\alpha - \beta$ is a positive root, then $\alpha = \beta + (\alpha - \beta)$ is not simple; if $\alpha - \beta$ is negative then β is not simple.

For the second statement, assume that α and β are any two roots with $(\alpha, \beta) > 0$. If $\alpha \neq \beta$, then they cannot be proportional, and we assume without loss of generality that $(\alpha, \alpha) \leq (\beta, \beta)$. Then $s_\beta(\alpha) = \alpha - \frac{2(\alpha,\beta)}{(\beta,\beta)}\beta = \alpha - \beta$, because $2(\alpha, \beta)/(\beta, \beta) = \langle \alpha, \beta^\vee \rangle$ is a positive integer strictly less than 2. Thus $\alpha - \beta$ is a root if $(\alpha, \beta) > 0$. □

5.55 Lemma *Let $\mathbb{R}^n$ have a positive definite inner product $(,)$, and suppose that $v_1, \dots, v_n \in \mathbb{R}^n$ satisfy $(v_i, v_j) \leq 0$ if $i \neq j$, and such that there exists v_0 with $(v_0, v_i) > 0$ for every i. Then $\{v_1, \dots, v_n\}$ is an independent set.*

Proof Suppose that $0 = c_1 v_1 + \cdots + c_n v_n$. Renumbering as necessary, we assume that $c_1, \dots, c_k \geq 0$, and $c_{k+1}, \dots, c_n \leq 0$. Let $v = c_1 v_1 + \cdots + c_k v_k = |c_{k+1}| v_{k+1} + \cdots + |c_n| v_n$. Then $0 \leq (v, v) =$

[1] I missed a few proofs from class. In particular, the proofs of Lemma 5.54 and Lemma 5.55 are reproduced from [11, page 156].

$(\sum_{i=1}^k c_i v_i, \sum_{j=k+1}^n -c_k v_k) = \sum_{j,k} |c_i c_j|(v_i, v_j) \leq 0$, which can happen only if $v = 0$. But then $0 = (v, v_0) = \sum_{i=1}^k c_i(v_i, v_0) > 0$ unless all c_i are 0 for $i \leq k$. Similarly we must have $c_j = 0$ for $j \geq k+1$, and so $\{v_i\}$ is independent. $\square$

5.56 Corollary *In any positive root system, the set Δ of simple roots is a basis of $\mathfrak{h}^*$.*

Proof By Lemma 5.54, Δ satisfies the conditions of Lemma 5.55 and so is independent. But Δ generates R_+ and hence R, and therefore spans $\mathfrak{h}^*$. $\square$

5.57 Lemma *Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a set of vectors in $\mathbb{R}^m$ with inner product $(,)$, and assume that α_i are all on one side of a hyperplane: there exists v such that $(\alpha_i, v) > 0\ \forall i$. Let W be the group generated by reflections S_{α_i}. Let R_+ be any subset of $\mathbb{R}_{\geq 0}\Delta \smallsetminus \{0\}$ such that $s_i(R_+ \smallsetminus \{\alpha_i\}) \subseteq R_+$ for each i, and such that the set of heights $\{(\alpha, v)\}_{\alpha \in R_+} \subseteq \mathbb{R}_{\geq 0}$ is well-ordered. Then $R_+ \subseteq W(\Delta)$.*

Proof Let $\beta \in R_+$. We proceed by induction on its height.

There exists i such that $(\alpha_i, \beta) > 0$, because if $(\beta, \alpha_i) \leq 0\ \forall i$, then $(\beta, \beta) = 0$ since β is a positive combination of the α_is. Thus $s_i(\beta) = \beta - (\text{positive})\alpha_i$; in particular, $\big(v, s_i(\beta)\big) < (v, \beta)$.

If $\beta \neq \alpha_i$, then $s_i(\beta) \in R_+$ by hypothesis, so by induction $s_i(\beta) \in W(\Delta)$, and hence $\beta = s_i(s_i(\beta)) \in W(\Delta)$. If $\beta = \alpha_i$, it's already in $W(\Delta)$. $\square$

5.58 Corollary *Let R be a finite root system, R_+ a choice of positive roots, and Δ the corresponding set of simple roots. Then $R = W(\Delta)$, and the set $\{s_{\alpha_i}\}_{\alpha_i \in \Delta}$ generates W.*

5.59 Corollary *Let R be a finite root system, R_+ a choice of positive roots, and Δ the corresponding set of simple roots. Then $R \subseteq \mathbb{Z}\Delta$ and $R_+ \subseteq \mathbb{Z}_{\geq 0}\Delta$.*

5.60 Proposition *Let R be a finite root system, and R_+ and R'_+ two choices of positive roots. Then R_+ and R'_+ are W-conjugate.*

Proof Let Δ be the set of simple roots corresponding to R_+. If $\Delta \subseteq R'_+$, then $R_+ \subseteq R'_+$. Then $R_- \subseteq R'_-$ by negating, and $R_+ \supseteq R'_+$ by taking complements, so $R_+ = R'_+$.

Suppose $\alpha_i \in \Delta$ but $\alpha_i \notin R'_+$, and consider the new system of positive roots $s_i(R'_+)$, where $s_i = s_{\alpha_i}$ is the reflection corresponding to α_i. Then $s_i(R'_+) \cap R_+ \supseteq s_i(R'_+ \cap R_+)$, because a system of roots that does not contain α_i does not lose anything under s_i. But $\alpha_i \in R'_-$, so $-\alpha_i \in R'_+$, and so $\alpha_i \in s_i(R'_+)$ and hence in $s_i(R'_+) \cap R_+$. Therefore $\Big|s_i(R'_+) \cap R_+\Big| > \Big|R'_+ \cap R_+\Big|$.

If $s_i(R'_+) \neq R_+$, then we can find $\alpha_j \in \Delta \smallsetminus s_i(R'_+)$. We repeat the argument, at each step making the set $\Big|w(R'_+) \cap R_+\Big|$ strictly bigger, where $w = \cdots s_j s_i \in W$. Since R_+ is a finite set, eventually we cannot get any bigger; this can only happen when $\Delta \subseteq w(R'_+)$, and so $R_+ = w(R'_+)$. $\square$

5.5 Cartan Matrices and Dynkin Diagrams

5.5.1 Definitions

[8, Lecture 34]

5.61 Definition *A* finite-type Cartan matrix *of rank n is an $n \times n$ matrix a_{ij} satisfying the following:*

- $a_{ii} = 2$ *and* $a_{ij} \in \mathbb{Z}_{\leq 0}$ *for* $i \neq j$.
- a *is* symmetrizable*: there exists an invertible diagonal matrix d with da symmetric.*
- a *is* positive*: all principle minors of a are positive.*

An isomorphism *between Cartan matrices a_{ij} and b_{ij} is a permutation $\sigma \in S_n$ such that $a_{ij} = b_{\sigma i, \sigma j}$.*

5.62 Lemma / Definition *Let R be a finite root system, R_+ a system of positive roots, and $\Delta = \{\alpha_1, \dots, \alpha_n\}$ the corresponding simple roots. The* Cartan matrix *of R is the matrix $a_{ij} \overset{\text{def}}{=} \langle \alpha_j, \alpha_i^\vee \rangle = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$.*

The Cartan matrix of a root system is a Cartan matrix. It depends (up to isomorphism) only on the root system. Conversely, a root system is determined up to isomorphism by its Cartan matrix.

Proof That the Cartan matrix depends only on the root system follows from Proposition 5.60. That the Cartan matrix determines the root system follows from Corollary 5.58.

Given a choice of root system and simple roots, let $d_i \overset{\text{def}}{=} (\alpha_i, \alpha_i)/2$, and let $d_{ij} \overset{\text{def}}{=} d_i \delta_{ij}$ be the diagonal matrix with the d_is on the diagonal. Then d is invertible because $d_i > 0$, and $da = (\alpha_i, \alpha_j)$ is obviously symmetric. Let $I \subseteq \{1, \dots, n\}$; then the $I \times I$ principle minor of da is just $\prod_{i \in I} d_i$ times the corresponding principle minor of a. Since $d_i > 0$ for each i and da is the matrix of a positive-definite symmetric bilinear form, we see that a is positive. $\square$

5.5.2 Classification of finite-type Cartan matrices

[8, Lectures 34 and 35]

We classify (finite-type) Cartan matrices by encoding their information in graph-theoretic form ("Dynkin diagrams") and then classifying (indecomposable) Dynkin diagrams.

5.63 Definition *Let a be an integer matrix so that every principle 2×2 sub-matrix has the form $\begin{bmatrix} 2 & -k \\ -l & 2 \end{bmatrix}$ with $k, l \in \mathbb{Z}_{\geq 0}$ and either both k and l are 0 or one of them is 1. Let us call such a matrix* generalized-Cartan.

5.64 Lemma *A Cartan matrix is generalized-Cartan. A generalized-Cartan matrix is not Cartan if any entry is -4 or less.*

Proof Consider a 2×2 sub-matrix $\begin{bmatrix} 2 & -k \\ -l & 2 \end{bmatrix}$. Then if one of k and l is non-zero, the other must also be non-zero by symmetrizability. Moreover, $kl < 4$ by positivity, and so one of k and l must be 1. $\square$

5.65 Definition *Let a be a rank-n generalized-Cartan matrix. Its* diagram *is a graph on n vertices with (labeled, directed) edges determined as follows:*

Let $1 \leq i, j \leq n$, and consider the $\{i,j\} \times \{i,j\}$ submatrix of a. By definition, either k and l are both 0, or one of them is 1 and the other is a positive integer. We do not draw an edge between vertices i and j if $k = l = 0$. We connect i and j with a single undirected edge if $k = l = 1$. For $k = 2, 3$, we draw an arrow with k edges from vertex i to vertex j if the $\{i,j\}$ block is $\begin{bmatrix} 2 & -1 \\ -k & 2 \end{bmatrix}$.

5.66 Definition *A diagram is* Dynkin *if its corresponding generalized-Cartan matrix is in fact Cartan.*

5.67 Lemma / Definition *The diagram of a generalized-Cartan matrix a is disconnected if and only if a is block diagonal, and connected components of the diagram correspond to the blocks of a. A block diagonal matrix a is Cartan if and only if each block is. A connected diagram is* indecomposable. *We write "$\times$" for the disjoint union of Dynkian diagrams.*

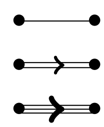
5.68 Example There is a unique indecomposable rank-1 diagram, and it is Dynkin: $A_1 = \bullet$.

The indecomposable rank-2 Dynkin diagrams are:

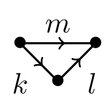
$A_2 =$

$B_2 = C_2 =$

$G_2 =$



5.69 Lemma / Definition *A* subdiagram *of a diagram is a subset of the vertices, with edges induced from the parent diagram. Subdiagrams of a Dynkin diagram correspond to principle submatrices of the corresponding Cartan matrix. Any subdiagram of a Dynkin diagram is Dynkin.*

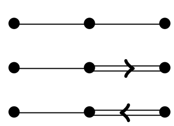
By symmetrizability, if we have a triangle  , then the multiplicities must be related: $m = kl$. So k or l is 1, and you can check that the three possibilities all have determinant ≤ 0. Moreover, a triple edge cannot attach to an edge, and two double edges cannot attach, again by positivity. As such, we will never need to discuss the triple-edge again.

5.70 Example The are three indecomposable rank-3 Dynkin diagrams:

$A_3 =$

$B_3 =$

$C_3 =$



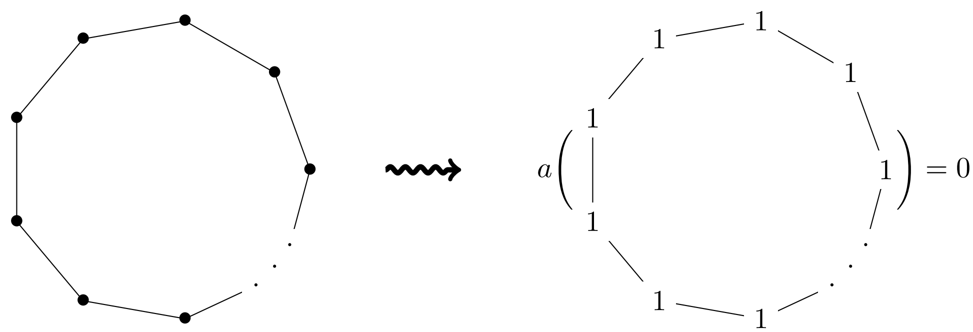
5.71 Definition *Let a be a generalized-Cartan rank-n matrix. We can specify a vector in $\mathbb{R}^n$ by assigning a "weight" to each vertex of the corresponding diagram. The* neighbors *of a vertex are counted with multiplicity: an arrow leaving a vertex contributes only one neighbor to that vertex, but an arrow arriving contributes as many neighbors as the arrow has edges. Naturally, each vertex of a weighted diagram has some number of "weighted neighbors": each neighbor is counted with multiplicity and multiplied by its weight, and these numbers are summed.*

5.72 Lemma *Let a be a generalized-Cartan matrix, and think of a vector $\vec{x}$ as a weighting of the corresponding diagram. With the weighted-neighbor conventions in Definition 5.71, the multiplication $a\vec{x}$ can be achieved by subtracting the number of weighted neighbors of each vertex from twice the weight of that vertex.*

Thus, a generalized-Cartan matrix is singular if its corresponding diagram has a weighting such that each vertex has twice as many (weighted) neighbors as its own weight.

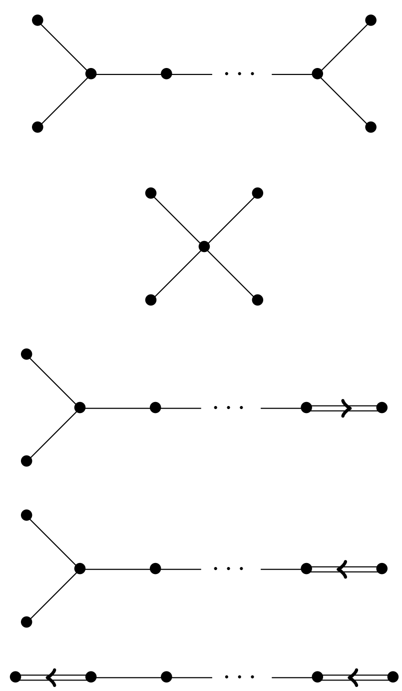
5.73 Corollary *A ring of single edges, and hence any diagram with a ring as a subdiagram, is not Dynkin.*

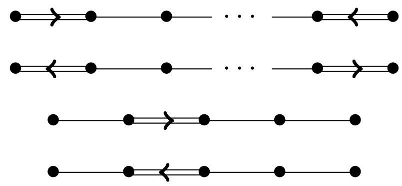
Proof We assign weight 1 to each vertex; this shows that the determinant of the ring is 0:



$\square$

5.74 Corollary *The following diagrams correspond to singular matrices and hence are not Dynkin:*





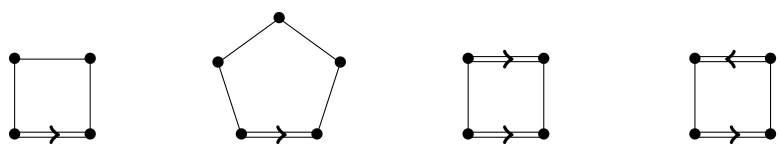
Proof For example, we can show the last two as singular with the following weightings:

$$1 — 2 \Rightarrow 3 — 2 — 1$$

$$2 — 4 \Leftarrow 3 — 2 — 1$$

□

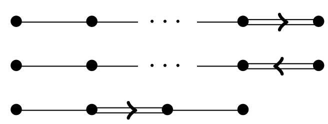
5.75 Lemma *The following diagrams are not Dynkin:*



det $= -4$ det $= -4$ det $= -8$ *not symmetrizable*

5.76 Corollary *The indecomposable Dynkin diagrams with double edges are the following:*

$B_n =$
$C_n =$
$F_4 =$



Proof Any indecomposable Dynkin diagram with a double edge is a chain. The double edge must come at the end of the chain, unless the diagram has rank 4. □

5.77 Lemma *Consider a Y-shaped indecomposable diagram. Let the lengths of the three arms, including the middle vertex, be k, l, m. Then the diagram is Dynkin if and only if $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} > 1$.*

Proof One can show directly that the determinant of such a diagram is $klm(\frac{1}{k} + \frac{1}{l} + \frac{1}{m} - 1)$. We present null-vectors for the three "Egyptian fraction" decompositions of 1 — triples k, l, m such that $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} = 1$:

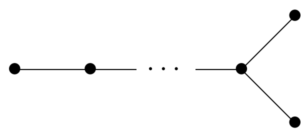
$$\begin{array}{ccccccccc} & & & & 2 & — & 1 & & \\ & & & & | & & & & \\ 1 & — & 2 & — & 3 & — & 2 & — & 1 \end{array} \qquad \begin{array}{ccccccccc} & & 3 & — & 2 & — & 1 & & \\ & & | & & & & & & \\ 2 & — & 4 & — & 3 & — & 2 & — & 1 \end{array}$$

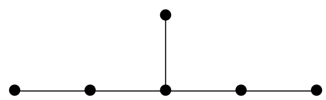
$$\begin{array}{ccccccccccccccc} & & & & 3 & & & & & & & & & & \\ & & & & | & & & & & & & & & & \\ 2 & — & 4 & — & 6 & — & 5 & — & 4 & — & 3 & — & 2 & — & 1 \end{array}$$

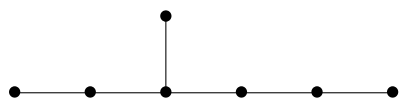
□

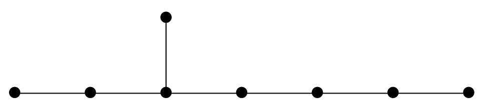
5.78 Corollary *The indecomposable Dynkin diagrams made entirely of single edges are:*

$A_n =$

$D_n =$ 

$E_6 =$ 

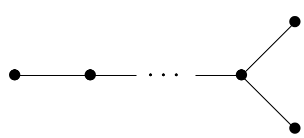
$E_7 =$ 

$E_8 =$ 

All together, we have proven:

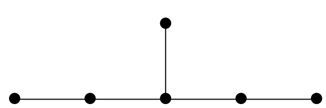
5.79 Theorem (Classification of indecomposable Dynkin diagrams) *A diagram is Dynkin if and only if it is a disjoint union of indecomposable Dynkin diagrams. The indecomposable Dynkin diagrams comprise four infinite families and five "sporadic" cases:*

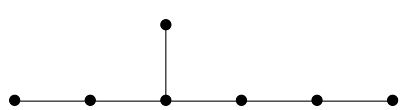
$A_n =$

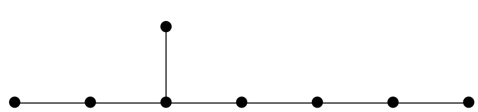
$D_n =$ 

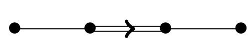
$B_n =$ 

$C_n =$ 

$E_6 =$ 

$E_7 =$ 

$E_8 =$ 

$F_4 =$ 

$G_2 =$ 

5.80 Example We mention the small-rank coindidences. We can continue the E series for smaller n: $E_4 = A_4$, $E_5 = D_5$. E_3 is not defined. The B, C, and D series make sense for $n \geq 2$, whence $B_2 = C_2$ and $D_2 = A_1 \times A_1$ and $D_3 = A_3$. Some diagrams have nontrivial symmetries: for $n \geq 1$, the symmetry group of A_n has order 2, and similarly for D_n for $n \neq 4$. The diagram D_4 has an unexpected symmetry: its symmetry group is S_3, with order 6. The symmetry group of E_6 is order-2.

5.6 From Cartan Matrix to Lie Algebra

[8, Lectures 36 and 37]

In Theorem 5.79, we classified indecomposable finite-type Cartan matrices, and therefore all finite-type Cartan matrices. We can present generators and relations showing that each indecomposable Cartan matrix is the Cartan matrix of some simple Lie algebra — indeed, the infinite families A_n, B_n, C_n, and D_n correspond respectively to the classical Lie algebras $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(2n+1, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$, and $\mathfrak{so}(2n, \mathbb{C})$ — and it is straightforward to show that a disjoint union of Cartan matrices corresponds to a direct product of Lie algebras.

In this section, we explain how to construct a semisimple Lie algebra for any finite-type Cartan matrix, and we show that a semisimple Lie algebra is determined by its Cartan matrix. This will complete the proof of the classification of semisimple Lie algebras. Most, but not all, of the construction applies to generalized-Cartan matrices; the corresponding Lie algebras are *Kac-Moody*, which are infinite-dimensional versions of semisimple Lie algebras. We will not discuss Kac-Moody algebras here.

5.81 Lemma / Definition *Let Δ be a rank-n Dynkin diagram with vertices labeled a basis $\{\alpha_1, \dots, \alpha_n\}$ of a vector space $\mathfrak{h}^*$, and let a_{ij} be the corresponding Cartan matrix. Since a_{ij} is nondegenerate, it defines a map $\vee : \mathfrak{h}^* \to \mathfrak{h}$ by $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$. We define $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_\Delta$ to be the Lie algebra generated by $\{e_i, f_i, h_i\}_{i=1}^n$ subject to the relations*

$$[h_i, e_j] = a_{ij} e_j \tag{5.6.1}$$
$$[h_i, f_j] = -a_{ij} f_j \tag{5.6.2}$$
$$[e_i, f_j] = \delta_{ij} h_i \tag{5.6.3}$$
$$[h_i, h_j] = 0 \tag{5.6.4}$$

For each i, we write $\mathfrak{sl}(2)_i$ for the subalgebra spanned by $\{e_i, f_i, h_i\}$; clearly $\mathfrak{sl}(2)_i \cong \mathfrak{sl}(2)$.

Let $Q = \mathbb{Z}\Delta$ be the root lattice of Δ. Then the free Lie algebra generated by $\{e_i, f_i, h_i\}_{i=1}^n$ has a natural Q-grading, by $\deg e_i = \alpha_i$, $\deg f_i = -\alpha_i$, and $\deg h_i = 0$; under this grading, the relations are homogeneous, so the grading passes to the quotient $\tilde{\mathfrak{g}}_\Delta$.

Let $\tilde{\mathfrak{h}} \subseteq \tilde{\mathfrak{g}}$ be the subalgebra generated by $\{h_i\}_{i=1}^n$; then it is abelian and spanned by $\{h_i\}_{i=1}^n$. The adjoint action $\mathrm{ad} : \mathfrak{h} \curvearrowright \tilde{\mathfrak{g}}$ is diagonalized by the grading: h_i acts on anything of degree $q \in Q$ by $\langle q, \alpha_i^\vee \rangle$.

Let $\tilde{\mathfrak{n}}_+$ be the subalgebra of $\tilde{\mathfrak{g}}$ generated by $\{e_i\}_{i=1}^n$ and let $\tilde{\mathfrak{n}}_-$ be the subalgebra of $\tilde{\mathfrak{g}}$ generated by $\{f_i\}_{i=1}^n$; the algebras $\tilde{\mathfrak{n}}_\pm$ are called the "upper-" and "lower-triangular" subalgebras.

5.82 Proposition *Let $\Delta, \tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, \tilde{\mathfrak{n}}_\pm$ be as in Lemma/Definition 5.81. Then $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+$ as vector spaces; this is the "triangular decomposition" of $\tilde{\mathfrak{g}}$.*

Proof That $\tilde{\mathfrak{n}}_-, \tilde{\mathfrak{h}}, \tilde{\mathfrak{n}}_+$ intersect trivially follows form the grading, so it suffices to show that $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- + \tilde{\mathfrak{h}} + \tilde{\mathfrak{n}}_+$. By inspecting the relations, we see that $(\operatorname{ad} f_i)\tilde{\mathfrak{n}}_- \subseteq \tilde{\mathfrak{n}}_-$, $(\operatorname{ad} f_i)\tilde{\mathfrak{h}} \subseteq \langle f_i \rangle \subseteq \tilde{\mathfrak{n}}_-$, and $(\operatorname{ad} f_i)\tilde{\mathfrak{n}}_+ \subseteq \tilde{\mathfrak{h}} + \tilde{\mathfrak{n}}_+$. Therefore $\operatorname{ad} f_i$ preserves $\tilde{\mathfrak{n}}_- + \tilde{\mathfrak{h}} + \tilde{\mathfrak{n}}_+$, $\tilde{\mathfrak{h}}$ does so obviously, and $\operatorname{ad} e_i$ does so by the obvious symmetry $f_i \leftrightarrow e_i$. Therefore $\tilde{\mathfrak{n}}_- + \tilde{\mathfrak{h}} + \tilde{\mathfrak{n}}_+$ is an ideal of $\tilde{\mathfrak{g}}$ and therefore a subalgebra, but it contains all the generators of $\tilde{\mathfrak{g}}$. □

5.83 Proposition *Let $\Delta, \tilde{\mathfrak{g}}$ be as in Lemma/Definition 5.81, and let $\lambda \in \mathfrak{h}^*$. Write $\mathbb{C}\langle f_1, \dots, f_n \rangle$ for the free algebra generated by noncommuting symbols $f_1, \dots, f_n$ and $M_\lambda \stackrel{\text{def}}{=} \mathbb{C}\langle f_1, \dots, f_n \rangle v_\lambda$ for its free module generated by the symbol v_λ. Then there exists an action of $\tilde{\mathfrak{g}}$ on M_λ such that:*

$$f_i \left(\prod f_{j_k} v_\lambda \right) = \left(f_i \prod f_{j_k} \right) v_\lambda \tag{5.6.5}$$

$$h_i \left(\prod f_{j_k} v_\lambda \right) = \left(\lambda(h_i) - \sum_k a_{i,j_k} \right) \left(\prod f_{j_k} v_\lambda \right) \tag{5.6.6}$$

$$e_i \left(\prod f_{j_k} v_\lambda \right) = \sum_{k \text{ s.t. } j_k = i} f_{j_1} \cdots f_{j_{k-1}} h_i f_{j_{k+1}} \cdots f_{j_l} \, v_\lambda \tag{5.6.7}$$

Proof We have only to check that the action satisfies the relations equations 5.6.1 to 5.6.4. The Q-grading verifies equations 5.6.1, 5.6.2, and 5.6.4; we need only to check equation 5.6.3. When $i \neq j$, the action by e_i ignores any action by f_j, and so we need only check that $[e_i, f_i]$ acts by h_i. Write $\underline{f}$ for some monomial $f_{j_1} \cdots f_{j_n}$. Then $e_i f_i(\underline{f} v_\lambda) = e_i(f_i \underline{f} v_\lambda) = h_i \underline{f} v_\lambda + f_i e_i(\underline{f} v_\lambda)$, clear by the construction. □

5.84 Definition *The $\tilde{\mathfrak{g}}$-module M_λ defined in Proposition 5.83 is the* Verma module *of $\tilde{\mathfrak{g}}$ with weight λ.*

5.85 Corollary *The map $\mathfrak{h} \to \tilde{\mathfrak{h}}$ is an isomorphism, so $\mathfrak{h} \hookrightarrow \tilde{\mathfrak{h}}$. The upper- and lower-triangular algebras $\tilde{\mathfrak{n}}_-$ and $\tilde{\mathfrak{n}}_+$ are free on $\{f_i\}$ and $\{e_i\}$ respectively.*

5.86 Proposition *Assume that Δ is an indecomposable system of simple roots, in the sense that the Dynkin diagram of the Cartan matrix of Δ is connected. Construct $\tilde{\mathfrak{g}}$ as in Lemma/Definition 5.81. Then any proper ideal of $\tilde{\mathfrak{g}}$ is graded, contained in $\tilde{\mathfrak{n}}_- + \tilde{\mathfrak{n}}_+$, and does not contain any e_i or f_i.*

Proof The grading on $\tilde{\mathfrak{g}}$ is determined by the adjoint action of $\mathfrak{h} = \tilde{\mathfrak{h}}$. Let $\mathfrak{a}$ be an ideal of $\tilde{\mathfrak{g}}$ and $a \in \mathfrak{a}$. Let $a = \sum a_q g_q$ where g_q are homogeneous of degree $q \in Q$. Then $[h_i, a] = \sum \langle q, \alpha_i^\vee \rangle a_q g_q$, and so $[\mathfrak{h}, a]$ has the same dimension as the number of non-zero coefficients a_q; in particular, $g_q \in [\mathfrak{h}, a]$. Thus $\mathfrak{a}$ is graded.

Suppose that $\mathfrak{a}$ has a degree-0 part, i.e. suppose that there is some $h \in \mathfrak{h} \cap \mathfrak{a}$. Since the Cartan matrix a is nonsingular, there exists $\alpha_i \in \Delta$ with $\alpha_i(h) \neq 0$. Then $[f_i, h] = \alpha_i(h) f_i \neq 0$, and so $f_i \in \mathfrak{a}$.

Now let $\mathfrak{a}$ be any ideal with $f_i \in \mathfrak{a}$ for some i. Then $h_i = [e_i, f_i] \in \mathfrak{a}$ and $e_i = -\frac{1}{2}[e_i, h_i] \in \mathfrak{a}$. But let α_j be any neighbor of α_i in the Dynkin diagram. Then $a_{ij} \neq 0$, and so $[f_j, h_i] = a_{ij} f_j \neq 0$; then $f_j \in \mathfrak{a}$. Therefore, if the Dynkin diagram is connected, then any ideal of $\tilde{\mathfrak{g}}$ that contains some f_i (or some e_i by symmetry) contains every generator of $\tilde{\mathfrak{g}}$. □

5.87 Corollary *Under the conditions of Proposition 5.86, $\tilde{\mathfrak{g}}$ has a unique maximal proper ideal.*

Proof Let $\mathfrak{a}$ and $\mathfrak{b}$ be any two proper ideals of $\tilde{\mathfrak{g}}$. Then the ideal $\mathfrak{a}+\mathfrak{b}$ does not contain $\mathfrak{h}$ or any e_i or f_i, and so is a proper ideal. □

5.88 Definition *Let Δ be a system of simple roots with connected Dynkin diagram, and let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_\Delta$ be defined as in Lemma/Definition 5.81. We define $\mathfrak{g} = \mathfrak{g}_\Delta$ as the quotient of $\tilde{\mathfrak{g}}$ by its unique maximal proper ideal. Then $\langle h_i, e_i, f_i\rangle \hookrightarrow \mathfrak{g}$, where by $\langle h_i, e_i, f_i\rangle$ we mean the linear span of the generators of $\tilde{\mathfrak{g}}$. Since we quotiented by a maximal ideal, $\mathfrak{g}$ is simple.*

5.89 Theorem (Serre Relations) *Let $\mathfrak{g}$ be as in Definition 5.88, and e_i, f_i the images of the corresponding generators of $\tilde{\mathfrak{g}}$. Then:*

$$(\operatorname{ad} e_j)^{1-a_{ji}} e_i = 0 \tag{5.6.8}$$

$$(\operatorname{ad} f_j)^{1-a_{ji}} f_i = 0 \tag{5.6.9}$$

Proof We will check equation 5.6.9; equation 5.6.8 is exactly analogous. Let s be the left-hand-side of equation 5.6.9, interpreted as an element of $\tilde{\mathfrak{g}}$. We will show that the ideal generated by s is proper.

When $i = j$, $s = 0$, and when $i \neq j$, $a_{ji} \leq 0$, and so the degree of s is $-\alpha_i - (\geq 1)\alpha_j$. In particular, bracketing with f_k and h_k only moves the degree further from 0. Therefore, the claim follows from the following equation:

$$[e_k, s]_{\tilde{\mathfrak{g}}} = 0 \text{ for any } k \tag{5.6.10}$$

When $k \neq i, j$, $[e_k, f_i] = [e_k, f_j] = 0$. So it suffices to check equation 5.6.10 when $k = i, j$. Let $m = -a_{ji}$. When $k = j$, we compute:

$$\begin{aligned}
(\operatorname{ad} e_j)(\operatorname{ad} f_j)^{1+m} f_i &= \left[\operatorname{ad} e_j, (\operatorname{ad} f_j)^{1+m}\right] f_i + (\operatorname{ad} f_j)^{1+m}(\operatorname{ad} e_j) f_i && (5.6.11)\\
&= \left[\operatorname{ad} e_j, (\operatorname{ad} f_j)^{1+m}\right] f_i + 0 && (5.6.12)\\
&= \sum_{l=0}^{m} (\operatorname{ad} f_j)^{m-l}\left(\operatorname{ad}[e_j, f_j]\right)(\operatorname{ad} f_j)^l f_i && (5.6.13)\\
&= \sum_{l=0}^{m} (\operatorname{ad} f_j)^{m-l}(\operatorname{ad} h_j)(\operatorname{ad} f_j)^l f_i && (5.6.14)\\
&= \sum_{l=0}^{m} \left(l(-\langle \alpha_j, \alpha_j^\vee\rangle) - \langle \alpha_i, \alpha_j^\vee\rangle\right)(\operatorname{ad} f_j)^m f_i && (5.6.15)\\
&= \left(\sum_{l=0}^{m} \left(-2l + m\right)\right) (\operatorname{ad} f_j)^m f_i && (5.6.16)\\
&= \left(-2\frac{m(m+1)}{2} + (m+1)m\right) (\operatorname{ad} f_j)^m f_i = 0 && (5.6.17)
\end{aligned}$$

where equation 5.6.12 follows by $[e_i, f_j] = 0$, equation 5.6.13 by the fact that ad is a Lie algebra homomorphism, and the rest is equations 5.6.2 and 5.6.3, that $m = -a_{ji}$, and arithmetic.

When $k = i$, e_i and f_j commute, and we have:

$$(\operatorname{ad} e_i)(\operatorname{ad} f_j)^{1+m} f_i = \left[\operatorname{ad} e_j, (\operatorname{ad} f_j)^{1+m}\right] f_i + (\operatorname{ad} f_j)^{1+m} (\operatorname{ad} e_i) f_i \tag{5.6.18}$$

$$= 0 + (\operatorname{ad} f_j)^{1+m} (\operatorname{ad} e_i) f_i \tag{5.6.19}$$

$$= (\operatorname{ad} f_j)^{1+m} h_i = 0 \tag{5.6.20}$$

provided that $m \geq 1$. When $m = 0$, we use the symmetrizability of the Cartan matrix: if $a_{ji} = 0$ then $a_{ij} = 0$. Therefore

$$(\operatorname{ad} e_i)(\operatorname{ad} f_j)^{1-a_{ji}} f_i = (\operatorname{ad} e_i)[f_j, f_i] = -(\operatorname{ad} e_i)(\operatorname{ad} f_i)^{1-a_{ij}} f_j \tag{5.6.21}$$

which vanishes by the first computation. □

We have defined for each indecomposable Dynkin diagram Δ a simple Lie algebra $\mathfrak{g}_\Delta$. If $\Delta = \Delta_1 \times \Delta_2$ is a disjoint union of Dynkin diagrams, we define $\mathfrak{g}_\Delta \stackrel{\text{def}}{=} \mathfrak{g}_{\Delta_1} \times \mathfrak{g}_{\Delta_2}$.

5.90 Definition *Let V be a (possibly-infinite-dimensional) $\mathfrak{g}$-module. An element $v \in V$ is* integrable *if for each i, the $\mathfrak{sl}(2)_i$-submodule of V generated by v is finite-dimensional. We write $I(V)$ for the set of integrable elements of V.*

5.91 Lemma *Let V be a $\mathfrak{g}$-module. Then $I(V)$ is a $\mathfrak{g}$-submodule.*

Proof Let $N \subseteq V$ be an $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}(2)_i$; then it is isomorphic to V_n defined in Example 5.17. It suffices to show that $e_j N$ is contained within some finite-dimensional $\mathfrak{sl}(2)_i$ submodule of V for $i \neq j$; the rest follows by switching $e \leftrightarrow f$ and permuting the indices, using the fact that $\{e_j, f_j\}$ generate $\mathfrak{g}$.

Then N is spanned by $\{{f_i}^k v_0\}_{k=0}^n$ where $v_0 \in N$ is the vector annihilated by e_i; in particular, ${f_i}^{n+1} v_0 = 0$. Since e_j and f_i commute, $e_j N$ is spanned by $\{{f_i}^k e_j v_0\}_{k=0}^n$. It suffices to compute the $\mathfrak{sl}(2)_i$ module generated by $e_j v_0$, or at least to show that it is finite-dimensional. The action of h_i on $e_j v_0$ is $h_i e_j v_0 = ([h_i, e_j] + e_j h_i) v_0 = (a_{ij} + n) e_j v_0$. For $k \neq n+1$, $f_i^k e_j v_0 = e_j f_i^k v_0 = 0$. Moreover, by Theorem 5.89, $e_i^k e_j v_0 = [e_i^k, e_j] v_0 + e_j e_i^k v_0 = (\operatorname{ad} e_i)^k (e_j) v_0 + 0$, which vanishes for large enough k. Then the result follows by Theorem 3.24 and the fact that $[e_i, f_i] = h_i$. □

5.92 Corollary *Let Δ be a Dynkin diagram and define $\mathfrak{g}$ as above. Then $\mathfrak{g}$ is* ad*-integrable.*

Proof Since $\{e_k, f_k\}$ generate $\mathfrak{g}$, it suffices to show that e_k and f_k are ad-integrable for each k. But the $\mathfrak{sl}(2)_i$-module generated by f_k has f_k as its highest-weight vector, since $[e_i, f_k] = 0$, and is finite-dimensional, since $(\operatorname{ad} f_i)^n f_k = 0$ for large enough n by Theorem 5.89. □

5.93 Corollary *The non-zero weights R of* ad $: \mathfrak{g} \curvearrowright \mathfrak{g}$ *form a root system.*

Proof Axioms **RS1**, **RS2**, **RS3**, **RS4**, and **Nondeg** of Definition 5.46 follow from the ad-integrability. Axiom **Reduced** and that R is finite follow from Lemma 5.57. □

5.94 Theorem (Classification of finite-dimensional simple Lie algebras) *The list given in Theorem 5.79 classifies the finite-dimensional simple Lie algebras over $\mathbb{C}$.*

Proof A Lie algebra with an indecomposable root system is simple, because any such system has a highest root, and linear combination of roots generates the highest root, and the highest root generates the entire algebra. So it suffices to show that two simple Lie algebras with isomorphic root systems are isomorphic.

Let Δ be an indecomposable root system, and define $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ as above. Let $\mathfrak{g}_1$ be a Lie algebra with root system Δ. Then the relations defining $\tilde{\mathfrak{g}}$ hold in $\mathfrak{g}_1$, and so there is a surjection $\tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}_1$; if $\mathfrak{g}_1$ is simple, then the kernel of this surjection is a maximal ideal of $\tilde{\mathfrak{g}}$. But $\tilde{\mathfrak{g}}$ has a unique maximal ideal, and $\mathfrak{g}$ is the quotient by this ideal; thus $\mathfrak{g}_1 \cong \mathfrak{g}$. $\square$

5.95 Example The families $ABCD$ correspond to the classical Lie algebras: $A_n \leftrightarrow \mathfrak{sl}(n+1)$, $B_n \leftrightarrow \mathfrak{so}(2n+1)$, $C_n \leftrightarrow \mathfrak{sp}(n)$, and $D_n \leftrightarrow \mathfrak{so}(2n)$. We recall that we have defined $\mathfrak{sp}(n)$ as the Lie algebra that fixes the nondegenerate antisymmetric $2n \times 2n$ bilinear form: $\mathfrak{sp}(n) \subseteq \mathfrak{gl}(2n)$. The EFG Lie algebras are new.

The coincidences in Example 5.80 correspond to coincidences of classical Lie algebras: $\mathfrak{so}(6) \cong \mathfrak{sl}(4)$, $\mathfrak{so}(5) \cong \mathfrak{sp}(2)$, and $\mathfrak{so}(4) \cong \mathfrak{sl}(2) \times \mathfrak{sl}(2)$. The identity $\mathfrak{so}(3) \cong \mathfrak{sl}(2)$ suggests that we define $B_1 = A_1 = \bullet$, but $\mathfrak{sl}(2)$ is not congruent to $\mathfrak{sp}(1)$ or to $\mathfrak{so}(2)$, so we do not assign meaning to C_1 or D_1, and justifying the name B_1 for $\bullet$ but not C_1 is ad hoc.

Exercises

1. (a) Show that $\mathrm{SL}(2,\mathbb{R})$ is topologically the product of a circle and two copies of $\mathbb{R}$, hence it is not simply connected.

   (b) Let S be the simply connected cover of $\mathrm{SL}(2,\mathbb{R})$. Show that its finite-dimensional complex representations, i.e., real Lie group homomorphisms $S \to \mathrm{GL}(n,\mathbb{C})$, are determined by corresponding complex representations of the Lie algebra $\mathrm{Lie}(S)^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$, and hence factor through $\mathrm{SL}(2,\mathbb{R})$. Thus S is a simply connected real Lie group with no faithful finite-dimensional representation.

2. (a) Let U be the group of 3×3 upper-unitriangular complex matrices. Let $\Gamma \subseteq U$ be the cyclic subgroup of matrices
$$\begin{bmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
   where $m \in \mathbb{Z}$. Show that $G = U/\Gamma$ is a (non-simply-connected) complex Lie group that has no faithful finite-dimensional representation.

   (b) Adapt the solution to Set 4, Problem 2(b) to construct a faithful, irreducible infinite-dimensional linear representation V of G.

3. Following the outline below, prove that if $\mathfrak{h} \subseteq \mathfrak{gl}(n,\mathbb{C})$ is a real Lie subalgebra with the property that every $X \in \mathfrak{h}$ is diagonalizable and has purely imaginary eigenvalues, then the corresponding connected Lie subgroup $H \subseteq GL(n,\mathbb{C})$ has compact closure (this completes the solution to Set 1, Problem 7).

   (a) Show that $\operatorname{ad} X$ is diagonalizable with imaginary eigenvalues for every $X \in \mathfrak{h}$.

(b) Show that the Killing form of $\mathfrak{h}$ is negative semidefinite and its radical is the center of $\mathfrak{h}$. Deduce that $\mathfrak{h}$ is reductive and the Killing form of its semi-simple part is negative definite. Hence the Lie subgroup corresponding to the semi-simple part is compact.

(c) Show that the Lie subgroup corresponding to the center of $\mathfrak{h}$ is a dense subgroup of a compact torus. Deduce that the closure of H is compact.

(d) Show that H is compact — that is, closed — if and only if it further holds that the center of $\mathfrak{h}$ is spanned by matrices whose eigenvalues are rational multiples of i.

4. Let $V_n = \mathcal{S}^n(\mathbb{C}^2)$ be the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$.

(a) Show that for $m \leq n$, $V_m \otimes V_n \cong V_{n-m} \oplus V_{n-m+2} \oplus \cdots \oplus V_{n+m}$, and deduce that the decomposition into irreducibles is unique.

(b) Show that in any decomposition of $V_1^{\otimes n}$ into irreducibles, the multiplicity of V_n is equal to 1, the multiplicity of V_{n-2k} is equal to $\binom{n}{k} - \binom{n}{k-1}$ for $k = 1, \ldots, \lfloor n/2 \rfloor$, and all other irreducibles V_m have multiplicity zero.

5. Let a be a symmetric Cartan matrix, i.e. a is symmetric with diagonal entries 2 and off-diagonal entries 0 or -1. Let Γ be a subgroup of the automorphism group of the Dynkin diagram D of a, such that every edge of D has its endpoints in distinct Γ orbits. Define the *folding* D' of D to be the diagram with a node for every Γ orbit I of nodes in D, with edge weight k from I to J if each node of I is adjacent in D to k nodes of J. Denote by a' the Cartan matrix with diagram D'.

(a) Show that a' is symmetrizable and that every symmetrizable generalized Cartan matrix (not assumed to be of finite type) can be obtained by folding from a symmetric one.

(b) Show that every folding of a finite type symmetric Cartan matrix is of finite type.

(c) Verify that every non-symmetric finite type Cartan matrix is obtained by folding from a unique symmetric finite type Cartan matrix.

6. An indecomposable symmetrizable generalized Cartan matrix a is said to be of *affine type* if $\det(a) = 0$ and all the proper principal minors of a are positive.

(a) Classify the affine Cartan matrices.

(b) Show that every non-symmetric affine Cartan matrix is a folding, as in the previous problem, of a symmetric one.

(c) Let $\mathfrak{h}$ be a vector space, $\alpha_i \in \mathfrak{h}^*$ and $\alpha_i^\vee \in \mathfrak{h}$ vectors such that a is the matrix $\langle \alpha_j, \alpha_i^\vee \rangle$. Assume that this realization is non-degenerate in the sense that the vectors α_i are linearly independent. Define the *affine Weyl group* W to be generated by the reflections s_{α_i}, as usual. Show that W is isomorphic to the semidirect product $W_0 \ltimes Q$ where Q and W_0 are the root lattice and Weyl group of a unique finite root system, and that every such $W_0 \ltimes Q$ occurs as an affine Weyl group.

(d) Show that the affine and finite root systems related as in (c) have the property that the affine Dynkin diagram is obtained by adding a node to the finite one, in a unique way if the finite Cartan matrix is symmetric.

7. Work out the root systems of the orthogonal Lie algebras $\mathfrak{so}(m, \mathbb{C})$ explicitly, thereby verifying that they correspond to the Dynkin diagrams B_n if $m = 2n+1$, or D_n if $m = 2n$. Deduce the isomorphisms $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{so}(5, \mathbb{C}) \cong \mathfrak{sp}(4, \mathbb{C})$, and $\mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{sl}(4, \mathbb{C})$.

8. Show that the Weyl group of type B_n or C_n (they are the same because these two root systems are dual to each other) is the group $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ of signed permutations, and that the Weyl group of type D_n is its subgroup of index two consisting of signed permutations with an even number of sign changes, i.e., the semidirect factor $(\mathbb{Z}/2\mathbb{Z})^n$ is replaced by the kernel of S_n-invariant summation homomorphism $(\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}/2\mathbb{Z}$

9. Let $(\mathfrak{h}, R, R^\vee)$ be a finite root system, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots with respect to a choice of positive roots R_+, $s_i = s_{\alpha_i}$ the corresponding generators of the Weyl group W. Given $w \in W$, let $l(w)$ denote the minimum length of an expression for w as a product of the generators s_i.

   (a) If $w = s_{i_1} \ldots s_{i_r}$ and $w(\alpha_j) \in R_-$, show that for some k we have $\alpha_{i_k} = s_{i_{k+1}} \ldots s_{i_r}(\alpha_j)$, and hence $s_{i_k} s_{i_{k+1}} \ldots s_{i_r} = s_{i_{k+1}} \ldots s_{i_r} s_j$. Deduce that $l(ws_j) = l(w) - 1$ if $w(\alpha_j) \in R_-$.

   (b) Using the fact that the conclusion of (a) also holds for $v = ws_j$, deduce that $l(ws_j) = l(w) + 1$ if $w(\alpha_j) \notin R_-$.

   (c) Conclude that $l(w) = |w(R_+) \cup R_-|$ for all $w \in W$. Characterize $l(w)$ in more explicit terms in the case of the Weyl groups of type A and B/C.

   (d) Assuming that $\mathfrak{h}$ is over $\mathbb{R}$, show that the dominant cone $X = \{\lambda \in \mathfrak{h} : \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i\}$ is a fundamental domain for W, i.e., every vector in $\mathfrak{h}$ has a unique element of X in its W orbit.

   (e) Deduce that $|W|$ is equal to the number of connected regions into which $\mathfrak{h}$ is separated by the removal of all the root hyperplanes $\langle \lambda, \alpha^\vee \rangle$, $\alpha^\vee \in R^\vee$.

10. Let $h_1, \ldots, h_r$ be linear forms in variables $x_1, \ldots, x_n$ with integer coefficients. Let $\mathbb{F}_q$ denote the finite field with $q = p^e$ elements. Prove that except in a finite number of "bad" characteristics p, the number of vectors $v \in \mathbb{F}_q^n$ such that $h_i(v) = 0$ for all i is given for all q by a polynomial $\chi(q)$ in q with integer coefficients, and that $(-1)^n \chi(-1)$ is equal to the number of connected regions into which $\mathbb{R}^n$ is separated by the removal of all the hyperplanes $h_i = 0$.

    Pick your favorite finite root system and verify that in the case where the h_i are the root hyperplanes, the polynomial $\chi(q)$ factors as $(q - e_1) \ldots (q - e_n)$ for some positive integers e_i called the *exponents* of the root system. In particular, verify that the sum of the exponents is the number of positive roots, and that (by Problem 9(e)) the order of the Weyl group is $\prod_i (1 + e_i)$

11. The *height* of a positive root α is the sum of the coefficients c_i in its expansion $\alpha = \sum_i c_i \alpha_i$ on the basis of simple roots.

    Pick your favorite root system and verify that for each $k \geq 1$, the number of roots of height k is equal to the number of the exponents e_i in Problem 10 for which $e_i \geq k$.

12. Pick your favorite root system and verify that if h denotes the height of the highest root plus one, then the number of roots is equal to h times the rank. This number h is called the *Coxeter number*. Verify that, moreover, the multiset of exponents (see Problem 10) is invariant with respect to the symmetry $e_i \mapsto h - e_i$.

13. A *Coxeter element* in the Weyl group W is the product of all the simple reflections, once each, in any order. Prove that a Coxeter element is unique up to conjugacy. Pick your favorite root system and verify that the order of a Coxeter element is equal to the Coxeter number (see Problem 12).

14. The *fundamental weights* λ_i are defined to be the basis of the weight lattice P dual to the basis of simple coroots in $Q^\vee$, i.e., $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$.

    (a) Prove that the stabilizer in W of λ_i is the Weyl group of the root system whose Dynkin diagram is obtained by deleting node i of the original Dynkin diagram.

    (b) Show that each of the root systems E_6, E_7, and E_8 has the property that its highest root is a fundamental weight. Deduce that the order of the Weyl group $W(E_k)$ in each case is equal to the number of roots times the order of the Weyl group $W(E_{k-1})$, or $W(D_5)$ for $k = 6$. Use this to calculate the orders of these Weyl groups.

15. Let $e_1, \dots, e_8$ be the usual orthonormal basis of coordinate vectors in Euclidean space $\mathbb{R}^8$. The root system of type E_8 can be realized in $\mathbb{R}^8$ with simple roots $\alpha_i = e_i - e_{i+1}$ for $i = 1, \dots, 7$ and
$$\alpha_8 = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$
Show that the root lattice Q is equal to the weight lattice P, and that in this realization, Q consists of all vectors $\beta \in \mathbb{Z}^8$ such that $\sum_i \beta_i$ is even and all vectors $\beta \in \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \mathbb{Z}^8$ such that $\sum_i \beta_i$ is odd. Show that the root system consists of all vectors of squared length 2 in Q, namely, the vectors $\pm e_i \pm e_j$ for $i < j$, and all vectors with coordinates $\pm\frac{1}{2}$ and an odd number of coordinates with each sign.

16. Show that the root system of type F_4 has 24 long roots and 24 short roots, and that the roots of each length form a root system of type D_4. Show that the highest root and the highest short root are the fundamental weights at the end nodes of the diagram. Then use Problem 14(a) to calculate the order of the Weyl group $W(F_4)$. Show that $W(F_4)$ acts on the set of short (resp. long roots) as the semidirect product $S_3 \ltimes W(D_4)$, where the symmetric group S_3 on three letters acts on $W(D_4)$ as the automorphism group of its Dynkin diagram.

17. Pick your favorite root system and verify that the generating function $W(t) = \sum_{w \in W} t^{l(w)}$ is equal to $\prod_i (1 + t + \cdots + t^{e_i})$, where e_i are the exponents as in Problem 10.

18. Let S be the subring of W-invariant elements in the ring of polynomial functions on $\mathfrak{h}$. Pick your favorite root system and verify that S is a polynomial ring generated by homogeneous generators of degrees $e_i + 1$, where e_i are the exponents as in Problem 10.

Chapter 6

Representation Theory of Semisimple Lie Groups

6.1 Irreducible Lie-algebra representations

[8, Lectures 38 and 39] [18, Lectures 18 and 19][1]

Any representation of a Lie group induces a representation of its Lie algebra, so we start our story there. We recall Theorem 4.78: any finite-dimensional representation of a semisimple Lie algebra is the direct sum of simple representations. In section 5.2 we computed the finite-dimensional simple representations of $\mathfrak{sl}(2)$; we now generalize that theory to arbitrary finite-dimensional semisimple Lie algebras.

6.1 Lemma / Definition *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system R, and choose a system of positive roots R_+. Let $\mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$ be the upper- and lower-triangular subalgebras; then $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ as a a vector space. We define the* Borel subalgebra *$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$, and $\mathfrak{n}_+$ is an ideal of $\mathfrak{b}$ with $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}_+$.*

Pick $\lambda \in \mathfrak{h}^*$; then $\mathfrak{b}$ has a one-dimensional module $\mathbb{C}v_\lambda$, where $hv_\lambda = \lambda(h)\, v_\lambda$ for $h \in \mathfrak{h}$ and $\mathfrak{n}_+ v_\lambda = 0$.

As a subalgebra, $\mathfrak{b}$ acts on $\mathfrak{g}$ from the right, and so we define the Verma module *of $\mathfrak{g}$ with weight λ by:*

$$M_\lambda \overset{\text{def}}{=} \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}v_\lambda \tag{6.1.1}$$

As a vector space, $M_\lambda \cong \mathcal{U}\mathfrak{n}_- \otimes_{\mathbb{C}} \mathbb{C}v_\lambda$. It is generated as a $\mathfrak{g}$-module by v_λ with the relations $hv_\lambda = \lambda(h)v_\lambda$, $\mathfrak{n}_+ v_\lambda = 0$, and no relations on the action of $\mathfrak{n}_-$ except those from $\mathfrak{g}$.

Proof The explicit description of M_λ follows from Theorem 3.24: $\mathcal{U}\mathfrak{g} = \mathcal{U}\mathfrak{n}_- \otimes \mathcal{U}\mathfrak{h} \otimes \mathcal{U}\mathfrak{n}_+$ as vector spaces. □

[1] I was absent for Lecture 39 in Mark Haiman's class, and so those notes are missing from [8]. To prepare this document, I thus turned to [18] for the proof of the Weyl Character Formula (Theorem 6.15). The presentation of this material there is quite good, although differs from the presentation in [8]; in particular, [18] includes more motivation and examples, and discusses ch (Definition 6.10) in terms of group characters of the corresponding simply-connected Lie group. See also Footnote 1 in [18, page 106].

6.2 Corollary *Any module with highest weight λ is a quotient of M_λ.*

6.3 Lemma *Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the simple roots of $\mathfrak{g}$, and let $Q \stackrel{\text{def}}{=} \mathbb{Z}\Delta$ be the root lattice and $Q_+ \stackrel{\text{def}}{=} \mathbb{Z}_{\geq 0}\Delta$. Then the* weight grading *given by the action of $\mathfrak{h}$ on the Verma module M_λ is:*

$$M_\lambda = \bigoplus_{\beta \in Q_+} (M_\lambda)_{\lambda - \beta} \tag{6.1.2}$$

Moreover, let $N \subseteq M_\lambda$ be a proper submodule. Then $N \subseteq \bigoplus_{\beta \in Q_+ \smallsetminus \{0\}} (M_\lambda)_{\lambda-\beta}$.

Proof The description of the weight grading follows directly from the description of M_λ given in Lemma/Definition 6.1. Any submodule is graded by the action of $\mathfrak{h}$. Since $(M_\lambda)_\lambda = \mathbb{C}v_\lambda$ is one-dimensional and generates M_λ, a proper submodule cannot intersect $(M_\lambda)_\lambda$. □

6.4 Corollary *For any $\lambda \in \mathfrak{h}^*$, the Verma module M_λ has a unique maximal proper submodule. The quotient $M_\lambda \twoheadrightarrow L_\lambda$ is an irreducible $\mathfrak{g}$-module. Conversely, any irreducible $\mathfrak{g}$-module with highest weight λ is isomorphic to L_λ, since it must be a quotient of M_λ by a maximal ideal.*

6.5 Definition *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\Delta = \{\alpha_1, \dots, \alpha_n\}$. We recall the* root lattice *$Q \stackrel{\text{def}}{=} \mathbb{Z}\Delta$ and the* weight lattice *$P \stackrel{\text{def}}{=} \{\lambda \in \mathfrak{h}^* \text{ s.t. } \langle \lambda, Q^\vee \rangle \subseteq \mathbb{Z}\}$. A* dominant integral weight *is an element of $P_+ \stackrel{\text{def}}{=} \{\lambda \in P \text{ s.t. } \langle \lambda, \alpha_i^\vee \rangle \geq 0 \, \forall i\}$.*

We recall Definition 5.90.

6.6 Proposition *If $\lambda \in P_+$, then L_λ consists of integrable elements.*

Proof Since L_λ is irreducible, its submodule of integrable elements is either 0 or the whole module. So it suffices to show that if $\lambda \in P_+$, then v_λ is integrable. Pick a simple root α_i. By construction, $e_i v_\lambda = 0$ and $h_i v_\lambda = \langle \lambda, \alpha_i^\vee \rangle v_\lambda$. Since $\lambda \in P_+$, $\langle \lambda, \alpha_i^\vee \rangle = m \geq 0$ is an integer. Consider the $\mathfrak{sl}(2)_i$-submodule of M_λ generated by v_λ; if m is a nonnegative integer, from the representation theory of $\mathfrak{sl}(2)$ we know that $e_i f_i^{m+1} v_\lambda = 0$. But if $j \neq i$, then $e_j f_i^{m+1} v_\lambda = f_i^{m+1} e_j v_\lambda = 0$. Recalling the grading, we see then that $f_i^{m+1} v_\lambda$ generates a submodule of M_λ, and so $f_i^{m+1} v_\lambda \mapsto 0$ in L_λ. Hence the $\mathfrak{sl}(2)_i$-submodule of L_λ generated by v_λ is finite, and so v_λ is integrable. □

6.7 Definition *Let $\mathfrak{g}$ be a semisimple Lie algebra. We define the category $\hat{\mathcal{O}}$ to be a full subcategory of the category $\mathfrak{g}$-MOD of (possibly-infinite-dimensional) $\mathfrak{g}$ modules. The objects $X \in \hat{\mathcal{O}}$ are required to satisfy the following conditions:*

- *The action $\mathfrak{h} \curvearrowright X$ is diagonalizable.*
- *For each $\lambda \in \mathfrak{h}^*$, the weight space X_λ is finite-dimensional.*
- *There exists a finite set $S \subseteq \mathfrak{h}^*$ such that the weights of X lie in $S + (-Q_+)$.*

6.8 Lemma *The category $\hat{\mathcal{O}}$ is closed under submodules, quotients, extensions, and tensor products.*

Proof The $\mathfrak{h}$-action grades subquotients of any graded module, and acts diagonally. An extension of graded modules is graded, with graded components extensions of the corresponding graded components. Since $\mathfrak{g}$ is semisimple, any extension of finite-dimensional modules is a direct sum, and so the $\mathfrak{h}$-action is diagonal on any extension of objects in $\mathcal{O}$. Finally, tensor products are handled by Lemma 5.23. $\square$

6.9 Definition *Write the additive group $\mathfrak{h}^*$ multiplicatively: $\lambda \mapsto x^\lambda$. The* group algebra $\mathbb{Z}[\mathfrak{h}^*]$ *is the algebra of "polynomials" $\sum c_i x^{\lambda_i}$, with the obvious addition and multiplication. I.e. $\mathbb{Z}[\mathfrak{h}^*]$ is the free abelian group $\bigoplus_{\lambda\in\mathfrak{h}^*} \mathbb{Z}x^\lambda$, with multiplication given on a basis by $x^\lambda x^\mu = x^{\lambda+\mu}$.*

Let $\mathbb{Z}[-Q_+]$ be the subalgebra of $\mathbb{Z}[\mathfrak{h}^*]$ generated by $\{x^\lambda \text{ s.t. } -\lambda \in Q_+\}$. This has a natural topology given by setting $\|x^{-\alpha_i}\| = c^{-\alpha_i}$ for α_i a simple root and c some real constant with $c > 1$. We let $\mathbb{Z}[[-Q_+]]$ be the completion of $\mathbb{Z}[-Q_+]$ with respect to this topology. Equivalently, $\mathbb{Z}[[-Q_+]]$ is the algebra of formal power series *in the variables $x^{-\alpha_1}, \dots, x^{-\alpha_n}$ with integer coefficients.*

Then $\mathbb{Z}[-Q_+]$ is a subalgebra of both $\mathbb{Z}[\mathfrak{h}^*]$ and $\mathbb{Z}[[-Q_+]]$. We will write $\mathbb{Z}[h^*, -Q_+]]$ for the algebra $\mathbb{Z}[\mathfrak{h}^*] \otimes_{\mathbb{Z}[-Q_+]} \mathbb{Z}[[-Q_+]]$.

The algebra $\mathbb{Z}[h^*, -Q_+]]$ is a formal gadget, consisting of formal fractional Laurant series. We use it as a space of generating functions.

6.10 Definition *Let $X \in \hat{\mathcal{O}}$. We define* $\mathrm{ch}(X) \in \mathbb{Z}[h^*, -Q_+]]$ *by:*

$$\mathrm{ch}(X) \stackrel{\text{def}}{=} \sum_{\lambda \text{ a weight of } X} \dim(X_\lambda) x^\lambda \tag{6.1.3}$$

We remark that every coefficient of $\mathrm{ch}(X)$ is a nonnegative integer, and if Y is a subquotient

6.11 Example Let M_λ be the Verma module with weight λ, and let R_+ be the set of positive roots of $\mathfrak{g}$. Then

$$\mathrm{ch}(M_\lambda) = \frac{x^\lambda}{\prod_{\alpha\in R_+}(1 - x^{-\alpha})} \stackrel{\text{def}}{=} x^\lambda \prod_{\alpha\in R_+} \sum_{l=0}^{\infty} x^{-l\alpha} \tag{6.1.4}$$

This follows from Theorem 3.24, the explicit description of $M_\lambda \cong \mathcal{U}\mathfrak{n}_- \otimes \mathbb{C}v_\lambda$, and some elementary combinatorics.

6.12 Proposition *Let $\mathfrak{g}$ be simple Lie algebra, P_+ the set of dominant integral weights, and W the Weyl group. Let $\lambda \in P_+$, and L_λ the irreducible quotient of M_λ given in Corollary 6.4. Then:*

1. $\mathrm{ch}(L_\lambda)$ *is W-invariant.*
2. *If μ is a weight of L_λ, then $\mu \in W(\nu)$ for some $\nu \in P_+ \cap (\lambda - Q_+)$.*
3. *L_λ is finite-dimensional.*

Conversely, every finite-dimensional irreducible $\mathfrak{g}$-module is L_λ for a unique $\lambda \in P_+$.

Proof 1. We use Proposition 6.6: L_λ consists of integrable elements. Let α_i be a root of $\mathfrak{g}$; then L_λ splits as an $\mathfrak{sl}(2)_i$ module: $L_\lambda = \bigoplus V_a$, where each V_a is an irreducible $\mathfrak{sl}(2)_i$

submodule. In particular, $V_a = \mathbb{C}v_{a,m} \oplus \mathbb{C}v_{a,m-2} \cdots \oplus \mathbb{C}v_{a,-m}$ for some m depending on a, where h_i acts on $\mathbb{C}v_{a,l}$ by l. But $\mathrm{ch}(L_\lambda) = \sum_a \mathrm{ch}(V_a) = \sum_a \sum_{j=-m,-m+2,\dots,m} \mathrm{ch}(\mathbb{C}v_{a,m})$. Let $\mathrm{ch}\big(\mathbb{C}(v_a)_l\big) = x^{\mu_{a,l}}$; then $\langle \mu_{a,l}, \alpha_i^\vee \rangle = l$ by definition, and $v_{a,l-2} \in f_i \mathbb{C}v_{a,l}$, and so $s_i\mu_{a,l} = \mu_{a,-l}$. This shows that $\mathrm{ch}(V_a)$ is fixed under the action of s_i, and so $\mathrm{ch}(L_\lambda)$ is also s_i-invariant. But the reflections s_i generate W, and so $\mathrm{ch}(L_\lambda)$ is W-invariant.

2. We partially order P: $\nu \leq \mu$ if $\mu - \nu \in Q_+$. In particular, the weights of L_λ are all less than or equal to λ.

   Let $\lambda \in P$. Then $s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$, and so $W(\lambda) \subseteq \lambda + Q$. If $\lambda \in P_+$ then $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for every i and so $s_i \leq \lambda$; if $\lambda \in P \smallsetminus P_+$ then there is some i with $\langle \lambda, \alpha_i^\vee \rangle < 0$, i.e. some i with $s_i(\lambda) > \lambda$. But W is finite, so for any $\lambda \in P$, $W(\lambda)$ has a maximal element, which must be in P_+. This proves that $P = W(P_+)$.

   Thus, if μ is a weight of L_λ, then $\mu \in W(\nu)$ for some $\nu \in P_+$. But by 1., ν is a weight of L_λ, and so $\nu \leq \lambda$. This proves statements 2.

   Moreover, the W-invariance of $\mathrm{ch}(L_\lambda)$ shows that if $\lambda \in P_+$, then $W(\lambda) \subseteq \lambda - Q_+$, and moreover that P_+ is a fundamental domain of W.

3. The Weyl group W is finite. Consider the two cones $\mathbb{R}_{\geq 0}P_+$ and $-\mathbb{R}_{\geq 0}Q_+$. Since the inner product (the symmetrization of the Cartan matrix) is positive definite and by construction the inner product of anything in $\mathbb{R}_{\geq 0}P_+$ with anything in $-\mathbb{R}_{\geq 0}Q_+$ is negative, the two cones intersect only at 0. Thus there is a hyperplane separating the cones: i.e. there exists a linear functional $\eta : \mathfrak{h}_\mathbb{R}^* \to \mathbb{R}$ such that its value is positive on P_+ but negative on $-Q_+$. Then $\lambda - Q_+$ is below the $\eta = \eta(\lambda)$ hyperplane. But $-Q_+$ is generated by $-\alpha_i$, each of which has a negative value under η, and so $\lambda - Q_+$ contains only finitely many points μ with $\eta(\mu) \geq 0$. Thus $P_+ \cap (\lambda - Q_+)$ is finite, and hence so is its image under W.

For the converse statement, let L be a finite-dimensional irreducible $\mathfrak{g}$-module, and let $v \in L$ be any vector. Then consider $\mathfrak{n}_+ v$, the image of v under repeated application of various e_is. By finite-dimensionality, $\mathfrak{n}_+ v$ must contain a vector $l \in \mathfrak{n}_+ v$ so that $e_i l = 0$ for every i. By the $\mathfrak{sl}(2)$ representation theory, l must be homogeneous, and indeed a top-weight vector of L, and by the irreducibility l generates L. Let the weight of l be λ; then the map $v_\lambda \to l$ generates a map $M_\lambda \twoheadrightarrow L$. But M_λ has a unique maximal submodule, and since L is irreducible, this maximal submodule must be the kernel of the map $M_\lambda \twoheadrightarrow L$. Thus $L \cong L_\lambda$. □

6.13 Lemma / Definition *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra, and $\Delta = \{\alpha_1, \dots, \alpha_n\}$ its simple root system. For each $i = 1, \dots, n$, we define a* fundamental weight $\Lambda_i \in \mathfrak{h}^*$ *by $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$. Then $P_+ = \mathbb{Z}_{\geq 0}\{\Lambda_1, \dots, \Lambda_n\}$.*

The following are equivalent, and define the Weyl vector *ρ:*

1. *$\rho = \sum_{i=1}^n \Lambda_i$. I.e. $\langle \rho, \alpha_j^\vee \rangle = 1$ for every j.*

2. *$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$.*

Proof Let $\rho_2 = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$. Since $s_i(R_+ \smallsetminus \{\alpha_i\}) = R_+$ but $s_i(\alpha_i) = -\alpha_i$, we see that $s_i(\tilde{\rho}) = \tilde{\rho} - \alpha_i$, and so $\langle \tilde{\rho}, \alpha_i^\vee \rangle = 1$ for every i. The rest is elementary linear algebra. □

6.14 Definition *Let $\lambda \in P_+$. We define the* character χ^λ *of λ to be* $\mathrm{ch}(L_\lambda)$.

6.15 Theorem (Weyl Character Formula) *Let $\epsilon : W \to \{\pm 1\}$ be given by $\epsilon(w) = \det_{\mathfrak{h}} w$; i.e. ϵ is the group homomorphism generated by $s_i \mapsto -1$ for each i. Let $\lambda \in P_+$. Then χ^λ can be computed as follows:*

$$\chi^\lambda = \frac{\displaystyle\sum_{w \in W} \epsilon(w)\, x^{w(\lambda+\rho)-\rho}}{\displaystyle\prod_{\alpha \in R_+} (1 - x^{-\alpha})} = \frac{\displaystyle\sum_{w \in W} \epsilon(w)\, x^{w(\lambda+\rho)}}{\displaystyle\prod_{\alpha \in R_+} (x^{\alpha/2} - x^{-\alpha/2})} \tag{6.1.5}$$

The equality of fractions follows simply from the description $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$.

6.16 Remark The sum in equation 6.1.5 is finite.

Indeed, the numerator and denominator on the right-hand-side fraction are obiously antisymmetric in W, and so the whole expression is W-invariant. The numerator on the left-hand-side fraction is a polynomial, and each $(1 - x^{-\alpha})$ is invertible as a power series: $(1 - x^{-\alpha})^{-1} = \sum_{n=0}^{\infty} x^{-n\alpha}$. So the fraction is a W-invariant power series, and hence a polynomial.

To prove Theorem 6.15 we will need a number of lemmas. In Example 6.11 we computed the character of the Verma module M_λ. Then Theorem 6.15 asserts that:

$$\mathrm{ch}(L_\lambda) = \sum_{w \in W} \epsilon(w)\ \mathrm{ch}\Big(M_{w(\lambda+\rho)-\rho}\Big) \tag{6.1.6}$$

As such, we will begin by understanding M_λ better. We recall Lemma/Definition 4.65: Let $(,)$ be the Killing form on $\mathfrak{g}$, and $\{x_i\}$ any basis of $\mathfrak{g}$ with dual basis $\{y_j\}$, i.e. $(x_i, y_j) = \delta_{ij}$ for every i, j; then $c = \sum x_i y_i \in \mathcal{U}\mathfrak{g}$ is central, and does not depend on the choice of basis.

6.17 Lemma *Let $\lambda \in \mathfrak{h}^*$ and M_λ the Verma module with weight λ. Let $c \in \mathcal{U}\mathfrak{g}$ be the Casimir, corresponding to the Killing form on $\mathfrak{g}$. Then c acts on M_λ by multiplication by $(\lambda, \lambda + 2\rho)$.*

Proof Let $\mathfrak{g}$ have rank n. Write R for the set of roots of $\mathfrak{g}$, R_+ for the positive roots, and Δ for the simple roots, as we have previously.

Recall Lemma 5.37. We construct a basis of $\mathfrak{g}$ as follows: we pick an orthonormal basis $\{u_i\}_{i=1}^n$ of $\mathfrak{h}$. For each α a non-zero root of $\mathfrak{g}$, the space $\mathfrak{g}_\alpha$ is one-dimensional; let x_α be a basis vector in $\mathfrak{g}_\alpha$. Then the dual basis to $\{u_i\}_{i=1}^n \cup \{x_\alpha\}_{\alpha \in R \smallsetminus \{0\}}$ is $\{u_i\}_{i=1}^n \cup \{y_\alpha\}_{\alpha \in R \smallsetminus \{0\}}$, where $y_\alpha = \frac{x_{-\alpha}}{(x_\alpha, x_{-\alpha})}$. Then:

$$c = \sum_{i=1}^n u_i^2 + \sum_{\alpha \in R \smallsetminus \{0\}} x_\alpha y_\alpha = \sum_{i=1}^n u_i^2 + \sum_{\alpha \in R \smallsetminus \{0\}} \frac{x_\alpha x_{-\alpha}}{(x_\alpha, x_{-\alpha})} = \sum_{i=1}^n u_i^2 + \sum_{\alpha \in R_+} \frac{x_\alpha x_{-\alpha} + x_{-\alpha} x_\alpha}{(x_\alpha, x_{-\alpha})} \tag{6.1.7}$$

Since M_λ is generated by its highest weight vector v_λ, and c is central, to understand the action of c on M_λ it suffices to compute $c v_\lambda$. We use the fact that for $\alpha \in R_+$, $x_\alpha v_\lambda = 0$; then

$$x_\alpha x_{-\alpha} v_\lambda = h_\alpha v_\lambda + x_{-\alpha} x_\alpha v_\lambda = h_\alpha v_\lambda = \lambda(h_\alpha)\, v_\lambda \tag{6.1.8}$$

where for each $\alpha \in R_+$ we have defines $h_\alpha \in \mathfrak{h}$ by $h_\alpha = [x_\alpha, x_{-\alpha}]$. Moreover, $(,)$ is $\mathfrak{g}$-invariant, and $[h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha$, where $\alpha(h_\alpha) \neq 0$. So:

$$(x_\alpha, x_{-\alpha}) = \frac{1}{\alpha(h_\alpha)}\big([h_\alpha, x_\alpha], x_{-\alpha}\big) = \frac{1}{\alpha(h_\alpha)}\big(h_\alpha, [x_\alpha, x_{-\alpha}]\big) = \frac{(h_\alpha, h_\alpha)}{\alpha(h_\alpha)} \tag{6.1.9}$$

We also have that $u_i v_\lambda = \lambda(u_i)v_\lambda$, and since $\{u_i\}$ is an orthonormal basis, $(\lambda, \lambda) = \sum_{i=1}^n \big(\lambda(u_i)\big)^2$. Thus:

$$cv_\lambda = \sum_{i=1}^n \big(\lambda(u_i)\big)^2 v_\lambda + \sum_{\alpha\in R_+} \frac{\lambda(h_\alpha)}{\frac{(h_\alpha,h_\alpha)}{\alpha(h_\alpha)}} v_\lambda = \left((\lambda, \lambda) + \sum_{\alpha \in R_+} \frac{\lambda(h_\alpha)\,\alpha(h_\alpha)}{(h_\alpha, h_\alpha)} \right) v_\lambda \tag{6.1.10}$$

We recall that h_α is proportional to $\alpha^\vee$, that $(\alpha, \alpha) = 4/(\alpha^\vee, \alpha^\vee)$, and that $\lambda(\alpha^\vee) = (\lambda, \alpha)/(\alpha, \alpha)$. Then $\frac{\lambda(h_\alpha)\,\alpha(h_\alpha)}{(h_\alpha,h_\alpha)} = (\lambda, \alpha)$, and so:

$$(\lambda, \lambda) + \sum_{\alpha\in R_+} \frac{\lambda(h_\alpha)\,\alpha(h_\alpha)}{(h_\alpha, h_\alpha)} = (\lambda, \lambda) + \sum_{\alpha\in R_+} (\lambda, \alpha) = (\lambda, \lambda + 2\rho) \tag{6.1.11}$$

Thus c acts on M_λ by multiplication by $(\lambda, \lambda + 2\rho)$. □

6.18 Lemma / Definition *Let X be a $\mathfrak{g}$-module. We weight vector $v \in X$ is* singular *if $\mathfrak{n}_+ v = 0$. In particular, any highest-weight vector is singular, and conversely any singular vector is the highest weight vector in the submodule it generates.*

6.19 Corollary *Let $\lambda \in P$, and M_λ the Verma module with weight λ. Then M_λ contains finitely many singular vectors, in the sense that their span is finite-dimensional.*

Proof Let C^λ be the set $C^\lambda \overset{\text{def}}{=} \{\mu \in P \text{ s.t. } (\mu + \rho, \mu + \rho) = (\lambda + \rho, \lambda + \rho)\}$. Then C^λ is a sphere in P centered at $-\rho$, and in particular it is a finite set. On the other hand, since $(\mu + \rho, \mu + \rho) = (\mu, \mu + 2\rho) + (\rho, \rho)$, we see that:

$$C^\lambda = \{\mu \in P \text{ s.t. } c \text{ acts on } M_\mu \text{ by } (\lambda, \lambda + 2\rho)\} \tag{6.1.12}$$

Recall that any module with highest weight μ is a quotient of M_μ. Let $v \in M_\lambda$ be a non-zero singular vector with weight μ. Then on the one hand $cv = (\lambda, \lambda + 2\rho)v$, since $v \in M_\lambda$, and on the other hand $cv = (\mu, \mu + 2\rho)$, since v is in a quotient of M_μ. In particular, $\mu \in C^\lambda$. But the weight spaces $(M_\lambda)_\mu$ of M_λ are finite-dimensional, and so the dimension of the space of singular vectors is at most $\sum_{\mu\in C^\lambda} \dim\big((M_\lambda)_\mu\big) < \infty$. □

6.20 Corollary *Let $\lambda \in P$. Then there are nonnegative integers $b_{\lambda,\mu}$ such that*

$$\operatorname{ch} M_\lambda = \sum b_{\lambda,\mu} \operatorname{ch} L_\mu \tag{6.1.13}$$

and $b_{\lambda,\mu} = 0$ unless $\mu \leq \lambda$ and $\mu \in C^\lambda$. Moreover, $b_{\lambda,\lambda} = 0$.

Proof We construct a filtration on M_λ. Since M_λ has only finitely many non-zero singular vectors, we choose w_1 a singular vector of minimal weight μ_1, and let $F_1 M_\lambda$ be the submodule of M_λ generated by w_1. Then $F_1 M_\lambda$ is irreducible with highest weight μ_1. We proceed by induction, letting w_i be a singular vector of minimal weight in $M_\lambda / F_{i-1} M_\lambda$, and $F_i M_\lambda$ the primage of the subrepresentation generated by w_i. This filters M_λ:

$$0 = F_0 M_\lambda \subseteq F_1 M_\lambda \subseteq \dots \tag{6.1.14}$$

Moreover, since M_λ has only finitely many weight vectors all together, the filtration must terminate:

$$0 = F_0 M_\lambda \subseteq F_1 M_\lambda \subseteq \dots \subseteq F_k M_\lambda = M_\lambda \tag{6.1.15}$$

By construction, the quotients are all irreducible: $F_i M_\lambda / F_{i-1} M_\lambda = L_{\mu_i}$ for some $\mu_i \in C^\lambda$, $\mu_i \leq \lambda$.

We recall that ch is additive for extensions. Therefore

$$\operatorname{ch} M_\lambda = \sum_{i=1}^{k} \operatorname{ch}(F_i M_\lambda / F_{i-1} M_\lambda) = \sum_{i=1}^{k} \operatorname{ch} L_{\mu_i} \tag{6.1.16}$$

Then $b_{\lambda,\mu}$ is the multiplicity of μ appearing as the weight of a singular vector of M_λ, and we have equation 6.1.13. The conditions stated about $b_{\lambda,\mu}$ are immediate: we saw that μ can only appear as a weight of M_λ if $\mu \in C^\lambda$ and $\mu \leq \lambda$; moreover, L_λ appears as a subquotient of M_λ exactly once, so $b_{\lambda,\lambda} = 1$. □

6.21 Definition *The coefficients $b_{\lambda,\mu}$ in equation 6.1.13 are the* Kazhdan-Luztig multiplicities.

6.22 Lemma *If $\lambda \in P_+$, $\mu \leq \lambda$, $\mu \in C^\lambda$, and $\mu + \rho \geq 0$, then $\mu = \lambda$.*

Proof We have that $(\mu + \rho, \mu + \rho) = (\lambda + \rho, \lambda + \rho)$ and that $\lambda - \mu = \sum_{i=1}^n k_i \alpha_i$, where all k_i are nonnegative. Then

$$\begin{aligned} 0 &= (\lambda + \rho, \lambda + \rho) - (\mu + \rho, \mu + \rho) \\ &= \big((\lambda + \rho) - (\mu + \rho), (\lambda + \rho) + (\mu + \rho)\big) \\ &= (\lambda - \mu, \lambda + \mu + 2\rho) \\ &= \sum_{i=1}^{n} k_i (\alpha, \lambda + \mu + 2\rho) \end{aligned}$$

But $\lambda, \mu + \rho \geq 0$, and $(\alpha_i, \rho) > 0$, so $(\alpha, \lambda + \mu + 2\rho) > 0$, and so all $k_i = 0$ since they are nonnegative. □

Proof (of Theorem 6.15) We have shown (Corollary 6.20) that $\operatorname{ch} M_\lambda = \sum b_{\lambda,\mu} \operatorname{ch} L_\mu$, were $b_{\lambda,\mu}$ is a lower-triangular matrix on $C^\lambda = C^\mu$ with ones on the diagonal. Thus it has a lower-triangular inverse with ones on the diagonal:

$$\operatorname{ch} L_\lambda = \sum_{\mu \leq \lambda, \mu \in C^\lambda} c_{\lambda,\mu} \operatorname{ch} M_\mu \tag{6.1.17}$$

But by equation 6.1.8 statement 1., $\operatorname{ch} L_\lambda$ is W-invariant, provided that $\lambda \in P_+$, thus so is $\sum c_{\lambda,\mu} \operatorname{ch} M_\mu$. We recall Example 6.11:

$$\operatorname{ch} M_\mu = \frac{x^\mu}{\prod_{\alpha\in R_+}(1-x^{-\alpha})} = \frac{x^{\mu+\rho}}{\prod_{\alpha\in R_+}(x^{\alpha/2}-x^{-\alpha/2})} \tag{6.1.18}$$

Therefore

$$\operatorname{ch} L_\lambda = \frac{\sum_{\mu\leq\lambda,\mu\in C^\lambda} c_{\lambda,\mu} x^{\mu+\rho}}{\prod_{\alpha\in R_+}(x^{\alpha/2}-x^{-\alpha/2})} \tag{6.1.19}$$

But the denominator if W-antisymmetric, and so the numerator must be as well:

$$\sum_{\mu\leq\lambda,\mu\in C^\lambda} c_{\lambda,\mu} x^{w(\mu+\rho)} = \sum_{\mu\leq\lambda,\mu\in C^\lambda} \epsilon(w) c_{\lambda,\mu} x^{\mu+\rho} \text{ for every } w\in W \tag{6.1.20}$$

This is equivalent to the condition that $c_{\lambda,\mu} = \epsilon(w) c_{\lambda,w(\mu+\rho)-\rho}$. By the proof of equation 6.1.8 statement 2., we know that P_+ is a fundamental domain of W; since $c_{\lambda,\lambda}=1$, if $\mu+\rho\in W(\lambda+\rho)$, then $c_{\lambda,\mu}=\epsilon(w)$, and so:

$$\sum_{\mu\leq\lambda,\mu\in C^\lambda} c_{\lambda,\mu} x^{\mu+\rho} = \sum_{w\in W}\left(x^{w(\lambda+\rho)} + \sum_{\substack{\mu<\lambda,\mu\in C^\lambda \\ \mu+\rho\in P^+}} c_{\lambda,\mu} x^{w(\mu+\rho)} \right) \tag{6.1.21}$$

But the rightmost sum is empty by Lemma 6.22. $\square$

6.23 Remark Specializing to the trivial representation L_0, Theorem 6.15 says that

$$1 = \frac{\sum_{w\in W}\epsilon(w) x^{w(\rho)}}{\prod_{\alpha\in R_+}(x^{\alpha/2}-x^{-\alpha/2})} \tag{6.1.22}$$

So we can rewrite equation 6.1.5 as

$$\chi^\lambda = \frac{\sum_{w\in W}\epsilon(w) x^{w(\lambda+\rho)}}{\sum_{w\in W}\epsilon(w) x^{w(\rho)}} \tag{6.1.23}$$

The following is an important corollary:

6.24 Theorem (Weyl Dimension Formula) *Let* $\lambda\in P_+$. *Then* $\dim L_\lambda = \prod_{\alpha\in R_+} \frac{(\alpha,\lambda+\rho)}{(\alpha,\rho)}$.

Proof The formula $\operatorname{ch}(L_\lambda) = \frac{\sum_{w\in W}\epsilon(w)\, x^{w(\lambda+\rho)}}{\prod_{\alpha\in R_+}(x^{\alpha/2}-x^{-\alpha/2})}$ is a polynomial in x. In particular, it defines a real-valued function on $\mathbb{R}_{>0}\times\mathfrak{h}$ given by $x^\alpha \mapsto a^{\alpha(h)}$ — when $a=1$ or $h=0$, the formula as written is the indeterminate form $\frac{0}{0}$, but the function clearly returns $\sum_\mu \dim\big((L_\lambda)_\mu\big) = \dim L_\lambda$. We will calculate this value of the function by taking a limit, using l'Hôpital's rule.

In particular, letting $x^\alpha \mapsto e^{t(\alpha,\lambda+\rho)}$ in equation 6.1.22 gives

$$\prod_{\alpha\in R_+}\left(e^{t(\alpha/2,\lambda+\rho)} - e^{-t(\alpha/2,\lambda+\rho)}\right) = \sum_{w\in W}\epsilon(w)e^{t(w(\rho),\lambda+\rho)} = \sum_{w\in W}\epsilon(w)e^{t(\rho,w(\lambda+\rho))} \tag{6.1.24}$$

where the second equality comes from $w \mapsto w^{-1}$ and $(w^{-1}x, y) = (x, wy)$. On the other hand, we let $x^\alpha \mapsto e^{t(\alpha,\rho)}$ in equation 6.1.5. Then

$$\operatorname{ch} L_\lambda|_{x=e^{t\rho}} = \frac{\sum_{w\in W}\epsilon(w)e^{t(w(\lambda+\rho),\rho)}}{\prod_{\alpha\in R_+}\left(e^{t(\alpha/2,\rho)} - e^{-t(\alpha/2,\rho)}\right)} \tag{6.1.25}$$

$$= \frac{\prod_{\alpha\in R_+}\left(e^{t(\alpha/2,\lambda+\rho)} - e^{-t(\alpha/2,\lambda+\rho)}\right)}{\prod_{\alpha\in R_+}\left(e^{t(\alpha/2,\rho)} - e^{-t(\alpha/2,\rho)}\right)} \tag{6.1.26}$$

$$= \prod_{\alpha\in R_+}\frac{\left(e^{t(\alpha/2,\lambda+\rho)} - e^{-t(\alpha/2,\lambda+\rho)}\right)}{\left(e^{t(\alpha/2,\rho)} - e^{-t(\alpha/2,\rho)}\right)} \tag{6.1.27}$$

Therefore

$$\dim L_\lambda = \lim_{t\to 0}\prod_{\alpha\in R_+}\frac{\left(e^{t(\alpha/2,\lambda+\rho)} - e^{-t(\alpha/2,\lambda+\rho)}\right)}{\left(e^{t(\alpha/2,\rho)} - e^{-t(\alpha/2,\rho)}\right)} \tag{6.1.28}$$

$$= \prod_{\alpha\in R_+}\lim_{t\to 0}\frac{\left(e^{t(\alpha/2,\lambda+\rho)} - e^{-t(\alpha/2,\lambda+\rho)}\right)}{\left(e^{t(\alpha/2,\rho)} - e^{-t(\alpha/2,\rho)}\right)} \tag{6.1.29}$$

$$\overset{\text{l'H}}{=} \prod_{\alpha\in R_+}\lim_{t\to 0}\frac{\left((\alpha/2,\lambda+\rho)e^{t(\alpha/2,\lambda+\rho)} + (\alpha/2,\lambda+\rho)e^{-t(\alpha/2,\lambda+\rho)}\right)}{\left((\alpha/2,\rho)e^{t(\alpha/2,\rho)} + (\alpha/2,\rho)e^{-t(\alpha/2,\rho)}\right)} \tag{6.1.30}$$

$$= \prod_{\alpha\in R_+}\frac{(\alpha,\lambda+\rho)}{(\alpha,\rho)} \tag{6.1.31}$$

□

6.25 Example Let us compute the dimensions of the irreducible representations of $\mathfrak{g} = \mathfrak{sl}(n+1)$. We work with the standard the simple roots be $\Delta = \{\alpha_1, \dots, \alpha_n\}$, whence $R_+ = \{\alpha_i + \alpha_{i+1} + \dots + \alpha_j\}_{1\le i<j\le n}$. Let us write λ and ρ in terms of the fundamental weights Λ_i, defined by $(\Lambda_i, \alpha_j) = \delta_{ij}$: $\rho = \sum_{i=1}^n \Lambda_i$ and $\lambda + \rho = \sum_{i=1}^n a_i\Lambda_i$. Then:

$$\dim L_\lambda = \prod_{\alpha\in R_+}\frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)} \tag{6.1.32}$$

$$= \prod_{1\le i\le j\le n}\frac{a_i + a_{i+1} + \dots + a_{j-1} + a_j}{j-i+1} \tag{6.1.33}$$

$$= \frac{1}{n!!}\prod_{1\le i\le j\le n}\sum_{k=i}^{j} a_k \tag{6.1.34}$$

where we have defined $n!! \overset{\text{def}}{=} n!\,(n-1)!\cdots 3!\,2!\,1!$. For example, the irrep of $\mathfrak{sl}(3)$ with weight $\lambda + \rho = 3\Lambda_1 + 2\Lambda_2$ has dimension $\frac{1}{2!!}2\cdot 3\cdot(2+3) = 15$.

6.2 Algebraic Lie Groups

We have classified the representations of any semisimple Lie algebra, and therefore the representations of its simply connected Lie group. But a Lie algebra corresponds to many Lie groups, quotients of the simply connected group by (necessarily central) discrete subgroups, and a representation of the Lie algebra is a representation of one of these groups only if the corresponding discrete normal subgroup acts trivially in the representation. We will see that the simply connected Lie group of any semisimple Lie algebra is algebraic, and that its algebraic quotients are determined by the finite-dimensional representation theory of the Lie algebra.

6.2.1 Guiding example: $\mathrm{SL}(n)$ and $\mathrm{PSL}(n)$

[8, Lecture 40]

Our primary example, as always, is the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, consisting of traceless 2×2 complex matrices. It is the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$, the group of 2×2 complex matrices with determinant 1.

6.26 Lemma / Definition *The group* $\mathrm{SL}(2,\mathbb{C})$ *has a non-trivial center:* $Z(\mathrm{SL}(2,\mathbb{C})) = \{\pm 1\}$. *We define the* projective special linear group *to be* $\mathrm{PSL}(2,\mathbb{C}) \overset{\text{def}}{=} \mathrm{SL}(2,\mathbb{C})/\{\pm 1\}$. *Equivalently,* $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{PGL}(2,\mathbb{C}) \overset{\text{def}}{=} \mathrm{GL}(2,\mathbb{C})/\{\textit{scalars}\}$*, the* projective general linear group*.*

6.27 Proposition *The group* $\mathrm{SL}(2,\mathbb{C})$ *is connected and simply connected. The kernel of the map* $\mathrm{ad} : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathfrak{sl}(2,\mathbb{C}))$ *is precisely the center, and so* $\mathrm{PSL}(2,\mathbb{C})$ *is the connected component of the group of automorophisms of* $\mathfrak{sl}(2,\mathbb{C})$*. The groups* $\mathrm{SL}(2,\mathbb{C})$ *and* $\mathrm{PSL}(2,\mathbb{C})$ *are the only connected Lie groups with Lie algebra* $\mathfrak{sl}(2,\mathbb{C})$*.*

Proof The only nontrivial statement is that $\mathrm{SL}(2,\mathbb{C})$ is simply connected. Consider the subgroup $U \overset{\text{def}}{=} \left\{\begin{bmatrix}1 & * \\ 0 & 1\end{bmatrix} \in \mathrm{SL}(2,\mathbb{C})\right\}$. Then U is the stabilizer of the vector $\begin{bmatrix}1\\0\end{bmatrix} \in \mathbb{C}^2 \smallsetminus \{0\}$, and $\mathrm{SL}(2,\mathbb{C})$ acts transitively on $\mathbb{C}^2 \smallsetminus \{0\}$. Thus the space of left cosets $\mathrm{SL}(2,\mathbb{C})/U$ is isomorphic to the space $\mathbb{C}^2\smallsetminus\{0\} \cong \mathbb{R}^4\smallsetminus\{0\}$ as a real manifold. But $U \cong \mathbb{C}$, so $\mathrm{SL}(2,\mathbb{C})$ is connected and simply connected. $\square$

6.28 Lemma *The groups* $\mathrm{SL}(2,\mathbb{C})$ *and* $\mathrm{PSL}(2,\mathbb{C})$ *are algebraic.*

Proof The determinant of a matrix is a polynomial in the coefficients, so $\{x \in \mathrm{M}(2,\mathbb{C}) \text{ s.t. } \det x = 1\}$ is an algebraic group. Any automorphism of $\mathfrak{sl}(2,\mathbb{C})$ preserves the Killing form, a nondegenerate symmetric pairing on the three-dimensional vector space $\mathfrak{sl}(2,\mathbb{C})$. Thus $\mathrm{PSL}(2,\mathbb{C})$ is a subgroup of $\mathrm{O}(3,\mathbb{C})$. It is connected, and so a subgroup of $\mathrm{SO}(3,\mathbb{C})$, and three-dimensional, and so is all of $\mathrm{SO}(3,\mathbb{C})$. Moreover, $\mathrm{SO}(3,\mathbb{C})$ is algebraic: it consists of matrices $x \in \mathrm{M}(3,\mathbb{C})$ that preserve the nondegenerate form (a system of quadratic equations in the coefficients) and have unit determinant (a cubic equation in the coefficients). $\square$

Recall that any irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ looks like a chain: e moves up the chain, f down, and h acts diagonally with eigenvalues changing by 2 from m at the top to $-m$ at the bottom:

$$
\begin{tikzcd}
v_0 \bullet \arrow[loop right, "h=m"] \arrow[d, bend left, "f=1"] \\
v_1 \bullet \arrow[u, bend left, "m=e"] \arrow[loop right, "h=m-2"] \arrow[d, bend left, "f=2"] \\
v_2 \bullet \arrow[u, bend left, "m-1=e"] \arrow[loop right, "h=m-4"] \arrow[d, bend left] \\
\vdots \arrow[u, bend left] \arrow[loop right] \arrow[d, bend left] \\
v_{m-1} \bullet \arrow[u, bend left] \arrow[loop right, "h=2-m"] \arrow[d, bend left, "f=m"] \\
v_m \bullet \arrow[u, bend left, "1=e"] \arrow[loop right, "h=-m"]
\end{tikzcd}
\tag{6.2.1}
$$

The exponential map $\exp : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$ acts on the Cartan by $th = \begin{bmatrix} t & \\ & -t \end{bmatrix} \mapsto \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix}$. Let $T = \exp(\mathfrak{h})$; then the kernel of $\exp : \mathfrak{h} \to T$ is $2\pi i\mathbb{Z}h$. On the other hand, when $t = \pi i$, $\exp(th) = -1$, which maps to 1 under $\mathrm{SL}(2,\mathbb{C}) \twoheadrightarrow \mathrm{PSL}(2,\mathbb{C})$; therefore the kernel of the exponential map $\mathfrak{h} \to \mathrm{PSL}(2,\mathbb{C})$ is just $\pi i\mathbb{Z}h$.

In particular, the $(m+1)$-dimensional representation V_m of $\mathfrak{sl}(2,\mathbb{C})$ is a representation of $\mathrm{PSL}(2,\mathbb{C})$ if and only if m is even, because $-1 \in \mathrm{SL}(2,\mathbb{C})$ acts on V_m as $(-1)^m$. We remark that $\ker\{\exp : \mathfrak{h} \to \mathrm{SL}(2,\mathbb{C})\}$ is precisely $2\pi i Q^\vee$, where $Q^\vee$ is the coroot lattice of $\mathfrak{sl}(2)$, and $\ker\{\exp : \mathfrak{h} \to \mathrm{PSL}(2,\mathbb{C})\}$ is precisely the coweight lattice $2\pi i P^\vee$

6.29 Remark This will be the model for any semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$. We will understand the exponential map from $\mathfrak{h}$ to the simply connected Lie group G corresponding to $\mathfrak{g}$, and we will also understand the map to $G/Z(G)$, the simplest quotient. Every group with Lie algebra $\mathfrak{g}$ is a quotient of G, and hence lies between G and $G/Z(G)$. The kernels of the maps $\mathfrak{h} \to G$ and $\mathfrak{h} \to G/Z(G)$ will be precisely $2\pi i Q^\vee$ and $2\pi i P^\vee$, respectively, and every other group will correspond to a lattice between these two.

Let us consider one further example: $\mathrm{SL}(n,\mathbb{C})$. It is simply-connected, and its center is $Z(\mathrm{SL}(n,\mathbb{C})) = \{n\text{th roots of unity}\}$. We define the *projective special linear group* to be $\mathrm{PSL}(n,\mathbb{C}) \overset{\text{def}}{=} \mathrm{SL}(n,\mathbb{C})/Z(\mathrm{SL}(n,\mathbb{C}))$; the groups with Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ live between these two, and so correspond to subgroups of $Z(\mathrm{SL}(n,\mathbb{C})) \cong \mathbb{Z}/n$, the cyclic group with n elements.

We now consider the Cartan $\mathfrak{h} \subseteq \mathfrak{sl}(n,\mathbb{C})$, thought of as the space of traceless diagonal matrices: $\mathfrak{h} = \{\langle z_1,\dots,z_n\rangle \in \mathbb{C}^n \text{ s.t. } \sum z_i = 0\}$. In particular, $\mathfrak{sl}(n,\mathbb{C})$ is of A-type, and so we can identify roots and coroots: $\alpha_i = \alpha_i^\vee = \langle 0,\dots,0,1,-1,0,\dots,0\rangle$, where the non-zero terms are in the $(i, i+1)$th spots. Then the coroot lattice $Q^\vee$ is the span of $\alpha_i^\vee$: if $\sum z_i = 0$, then we can write $\langle z_1,\dots,z_n\rangle \in \mathbb{Z}^n$ as $z_1\alpha_1 + (z_1+z_2)\alpha_2 + \cdots + (z_1+\cdots+z_{n-1})\alpha_{n-1}$, since $z_n = -(z_1+\cdots+z_{n-1})$. The coweight lattice $P^\vee$, on the other hand, is the lattice of vectors $\langle z_1,\dots,z_n\rangle$ with $\sum z_i = 0$ and with $z_i - z_{i+1}$ an integer for each $i \in \{1,\dots,n-1\}$. In particular, $\sum z_i = z_1 + \big(z_1 + (z_2 - z_1)\big) + \cdots + \big(z_1 + (z_2-z_1) + \cdots + (z_n - z_{n-1})\big) = nz_1 + \text{integer}$. Therefore $z_1 \in \mathbb{Z}\frac{1}{n}$, and $z_i \in z_1 + \mathbb{Z}$. So

$P^\vee = Q^\vee \sqcup (\langle \frac{1}{n}, \dots, \frac{1}{n} \rangle + Q^\vee) \sqcup \dots \sqcup (\langle \frac{n-1}{n}, \dots, \frac{n-1}{n} \rangle + Q^\vee)$. In this way, $P^\vee / Q^\vee$ is precisely $\mathbb{Z}/n$, in agreement with the center of $\mathrm{SL}(n, \mathbb{C})$.

6.2.2 Definition and General Properties of Algebraic Groups

[8, Lecture 41]

We have mentioned already (Definition 1.3) the notion of an "algebraic group", and we have occasionally used some algebraic geometry (notably in the proof of Theorem 5.27), but we have not developed that story. We do so now.

6.30 Definition *A subset $X \subseteq \mathbb{C}^n$ is an* affine variety *if it is the vanishing set of a set $P \subseteq \mathbb{C}[x_1, \dots, x_n]$ of polynomials:*

$$X = V(P) \overset{\text{def}}{=} \{x \in \mathbb{C}^n \text{ s.t. } p(x) = 0 \, \forall p \in P\} \tag{6.2.2}$$

Equivalently, X is Zariski closed *(see Definition 5.28). To any affine variety X with associate an ideal $I(X) \overset{\text{def}}{=} \{p \in \mathbb{C}[x_1, \dots, x_n] \text{ s.t. } p|_X = 0\}$. The* coordinate ring *of, or the* ring of polynomial functions *on, X is the ring $\mathscr{O}(X) \overset{\text{def}}{=} \mathbb{C}[x]/I(X)$.*

6.31 Lemma *If X is an affine variety, then $I(X)$ is a radical ideal. If $I \subseteq J$, then $V(I) \supseteq V(J)$, and conversely if $X \subseteq Y$ then $I(X) \supseteq I(Y)$. It is clear from the definition that if X is an affine varienty, then $V(I(X)) = X$; more generally, we can define $I(X)$ for any subset $X \supseteq \mathbb{C}^n$, whence $V(I(X))$ is the Zariski closure of X.*

6.32 Definition *A* morphism of affine varieties *is a function $f : X \to Y$ such that the coordinates on Y are polynomials in the coordinates of X. Equivalently, any function $f : X \to Y$ gives a homomorphism of algebras $f^\# : \mathrm{Fun}(Y) \to \mathrm{Fun}(X)$, where $\mathrm{Fun}(X)$ is the space of all $\mathbb{C}$-valued functions on X. A function $f : X \to Y$ is a morphism of affine varieties if $f^\#$ restricts to a map $f^\# : \mathscr{O}(Y) \to \mathscr{O}(X)$.*

6.33 Lemma / Definition *Any point $a \in \mathbb{C}^n$ gives an* evaluation map $\mathrm{ev}_a : p \mapsto p(a) : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}$*. If X is an affine variety, then $a \in X$ if and only if $I(X) \subseteq \ker \mathrm{ev}_a$ if and only if $\mathrm{ev}_a : \mathscr{O}(X) \to \mathbb{C}$ is a morphism of affine varieties.*

6.34 Corollary *The algebra $\mathscr{O}(X)$ determines the set of evaluation maps $\mathscr{O}(X) \to \mathbb{C}$, and if $\mathscr{O}(X)$ is presented as a quotient of $\mathbb{C}[x_1, \dots, x_n]$, then it determines $X \subseteq \mathbb{C}^n$. A morphism f of affine varieties is determined by the algebra homomorphism $f^\#$ of coordinate rings, and conversely any such algebra homomorphism determines a morphism of affine varieties. Thus the category of affine varieties is precisely the opposite category to the category of finitely generated commutative algebras over $\mathbb{C}$.*

6.35 Lemma / Definition *The category of affine varieties contains all finite products. The* product *of affine varieties $X \subseteq \mathbb{C}^m$ and $Y \subseteq \mathbb{C}^l$ is $X \times Y \subseteq \mathbb{C}^{m+l}$ with $\mathscr{O}(X \times Y) \cong \mathscr{O}(X) \otimes_{\mathbb{C}} \mathscr{O}(Y)$.*

Proof The maps $\mathscr{O}(X), \mathscr{O}(Y) \to \mathscr{O}(X \times Y)$ are given by the projections $X \times Y \to X, Y$. The map $\mathscr{O}(X) \otimes \mathscr{O}(Y) \to \mathscr{O}(X \times Y)$ is an isomorphism because all three algebras are finitely generated and the evaluation maps separate functions. $\square$

We recall Definition 1.3:

6.36 Definition *An* affine algebraic group *is a group object in the category of affine varieties. We will henceforth drop the adjective "affine" from the term "algebraic group", as we will never consider non-affine algebraic groups.*

Equivalently, an algebraic group is a finitely generated commutative algebra $\mathscr{O}(G)$ along with algebra maps

comultiplication $\Delta : \mathscr{O}(G) \to \mathscr{O}(G) \otimes_{\mathbb{C}} \mathscr{O}(G)$ *dual to the multiplication* $G \times G \to G$

antipode $\mathcal{S} : \mathscr{O}(G) \to \mathscr{O}(G)$ *dual to the inverse map* $G \to G$

counit $\epsilon = \mathrm{ev}_e : \mathscr{O}(G) \to \mathbb{C}$

The group axioms equations 1.1.1 to 1.1.3 are equivalent to the axioms of a commutative Hopf algebra (Definition 4.1).

6.37 Lemma / Definition *Let A be a Hopf algebra. An algebra ideal $B \subseteq A$ is a* Hopf ideal *if $\Delta(B) \subseteq B \otimes A + A \otimes B \subseteq A \otimes A$. An ideal $B \subseteq A$ is Hopf if and only if the Hopf algebra structure on A makes the quotient B/A into a Hopf algebra.*

6.38 Definition *A commutative but not necessarily reduced Hopf algebra is a* group scheme*.*

6.39 Definition *An affine variety X over $\mathbb{C}$ is* smooth *if X is a manifold.*

6.40 Proposition *An algebraic group is smooth.*

Proof Let $E = \ker \epsilon$. Since $e \cdot e = e$, we see that the following diagram commutes:

$$\begin{tikzcd} \mathscr{O}(G) \arrow[r, "\Delta"] \arrow[d, "\epsilon"] & \mathscr{O}(G) \otimes \mathscr{O}(G) \arrow[d, "\epsilon\otimes\epsilon"] \\ \mathbb{C} \arrow[r, leftrightarrow, "\sim"] & \mathbb{C} \otimes \mathbb{C} \end{tikzcd} \tag{6.2.3}$$

In particular,

$$\Delta E \subseteq E \otimes \mathscr{O}(G) + \mathscr{O}(G) \otimes E, \tag{6.2.4}$$

and so E is a Hopf ideal, and $\mathscr{O}(G)/E$ is a Hopf algebra. Moreover, equation 6.2.4 implies that $\Delta(E^n) \subseteq \sum_{k+l=n} E^k \otimes E^l$, and so Δ and $\mathcal{S}$ induce maps $\tilde{\Delta}$ and $\tilde{\mathcal{S}}$ on $R = \mathrm{gr}_E\, \mathscr{O}(G) \overset{\text{def}}{=} \bigoplus_{k \in \mathbb{N}} E^k/E^{k+1}$. In particular, R is a *graded Hopf algebra*, and is generated as an algebra by $R_1 = E/E^2$. Moreover, if $x \in R_1$, then x is primitive: $\Delta x = x \otimes 1 + 1 \otimes x$.

Since $R_1 = E/E^2$ is finitely dimensional, R is finitely generated; let $R = \mathbb{C}[y_1, \dots, y_n]/J$ where $n = \dim G$ and J is a Hopf ideal of the Hopf algebra $\mathbb{C}[y_1, \dots, y_n]$ with the generators

y_i all primitive. We can take the y_is to be a basis of R_1, and so $J_1 = 0$. We use the fact that $\mathbb{C}[y_1, \dots, y_n] \otimes \mathbb{C}[y_1, \dots, y_n] = \mathbb{C}[y_1, \dots, y_n, z_1, \dots, z_n]$, and that the antipode Δ is given by $\Delta : f(y) \mapsto f(y+z)$. Then a minimal-degree homogeneous element of J must be primitive, so $f(y+z) = f(y) + f(z)$, which in characteristic zero forces f to be homogeneous of degree 1. A similar calculation with the antipode forces the minimal-degree homogeneous elements $f \in J$ to satisfy $\mathcal{S} f = -f$.

In particular, $\mathrm{gr}_E \mathscr{O}(G)$ is a polynomial ring. We leave out the fact from algebraic geometry that this is equivalent to G being smooth at e. But we have shown that the Hopf algebra maps are smooth, whence G is smooth at every point. □

6.41 Corollary *An algebraic group over $\mathbb{C}$ is a Lie group.*

Recall that if G is a Lie group with $\mathscr{C}(G)$ the algebra of smooth functions on G, and if $\mathfrak{g} = \mathrm{Lie}(G)$, then $\mathcal{U}\mathfrak{g}$ acts on $\mathscr{C}(G)$ by left-invariant differential operators, and indeed is isomorphic to the algebra of left-invariant differential operators.

6.42 Definition *Let G be a group. A subalgebra $S \subseteq \mathrm{Fun}(G)$ is* left-invariant *if for any $s \in S$ and any $g \in G$, the function $h \mapsto s(g^{-1}h)$ is an element of S. Equivalently, we define the action $G \curvearrowright \mathrm{Fun}(G)$ by $gs = s \circ g^{-1}$; then a subalgebra is left-invariant if it is fixed by this action.*

6.43 Lemma *Let $S \subseteq \mathrm{Fun}(G)$ be a left-invariant subalgebra, and let $s \in S$ be a function such that $\Delta s = \{(x,y) \mapsto s(xy)\} \subseteq \mathrm{Fun}(G \times G)$ is in fact an element of $S \otimes S \subseteq \mathrm{Fun}(G) \otimes \mathrm{Fun}(G) \hookrightarrow \mathrm{Fun}(G \times G)$. Then let $\Delta s = \sum s_1 \otimes s_2$, where we suppress the indices of the sum. The action $G \curvearrowright S$ is given by*

$$g : s \mapsto \sum s_1(g^{-1}) s_2 \tag{6.2.5}$$

6.44 Corollary *Let u be a left-invariant differential operator and $s \in S$ as in Lemma 6.43, where $S \subseteq \mathscr{C}(G)$ is a left-invariant algebra of smooth functions. Then $us \in S$.*

Proof The left-invariance of u implies that $u(gs) = g\, u(s)$. Since $s(g^{-1}) \in \mathbb{C}$, we have:

$$u(gs)(h) = u\left(\sum s_1(g^{-1}) s_2\right)(h) = \sum s_1(g^{-1})\, u(s_2)(h) \tag{6.2.6}$$

Let $h = e$. Then $\sum s_1(g^{-1})\, u(s_2)(e) = u(gs)(e) = g(us)(e) = (us)(g^{-1})$. In particular:

$$(us)(g) = \sum s_1(g)\, u(s_2)(e) \tag{6.2.7}$$

But $(us_2)(e)$ are numbers. Thus $us \in S$. □

6.45 Corollary *Let G be an algebraic group, with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Then $\mathcal{U}\mathfrak{g}$ acts on $\mathscr{O}(G)$ by left-invariant differential operators.*

Since a differential operator is determined by its action on polynomials, we have a natural embedding $\mathcal{U}\mathfrak{g} \hookrightarrow \mathscr{O}(G)^*$ of vector spaces.

6.46 Lemma *Let $u, v \in \mathcal{U}\mathfrak{g}$, where G is an algebraic group. Then $uv\, s = \sum u(s_1)\, v(s_2)(e)$.*

Proof This follows from equation 6.2.7. □

6.47 Corollary *For each differential operator $u \in \mathcal{U}\mathfrak{g}$, let $\lambda_u \in \mathscr{O}(G)^*$ be the map $\lambda_u : s \mapsto u(s)(e)$. Then $\lambda_{uv}(s) = \sum \lambda_u(s_1)\, \lambda_v(s_2)$.*

6.48 Lemma *Let A be any (counital) coalgebra, for example a Hopf algebra. Then A^* is naturally an algebra: the map $A^* \otimes A^* \to A^*$ is given by $\langle \mu\nu, a\rangle \overset{\text{def}}{=} \langle \mu \otimes \nu, \Delta a\rangle$, and $\epsilon : A \to \mathbb{C}$ is the unit $\epsilon \in A^*$.*

6.49 Remark The dual to an algebra is not necessarily a coalgebra; if A is an algebra, then it defines a map $\Delta : A^* \to (A \otimes A)^*$, but if A is infinite-dimensional, then $(A \otimes A)^*$ properly contains $A^* \otimes A^*$.

6.50 Remark Following the historical precedent, we take the pairing $(A^* \otimes A^*) \otimes (A \otimes A)$ to be $\langle \mu \otimes \nu, a \otimes b\rangle = \langle \mu, a\rangle\langle \nu, b\rangle$. This is in some sense the wrong pairing — it corresponds to writing $(A \otimes B)^* = A^* \otimes B^*$ for finite-dimensional vector spaces A, B, whereas $B^* \otimes A^*$ would be more natural — and is "wrong" in exactly the same way that the "-1" in the definition of the left action of G on $\mathrm{Fun}(G)$ is wrong.

6.51 Proposition *The embedding $\mathcal{U}\mathfrak{g} \hookrightarrow \mathscr{O}(G)^*$ is given by the map $u \mapsto \lambda_u$ in Corollary 6.47, and is an algebra homomorphism.*

6.52 Definition *Let G be any group; then we define the* group algebra $\mathbb{C}[G]$ *of G to be the free vector space on the set G, with the multiplication given on the basis by the multiplication in G. The unit $e \in G$ becomes the unit $1 \cdot e \in \mathbb{C}[G]$.*

6.53 Lemma *If G is an algebraic group, then $\mathbb{C}[G] \hookrightarrow \mathscr{O}(G)^*$ is an algebra homomorphism given on the basis $g \mapsto \mathrm{ev}_g$.*

6.2.3 Constructing G from $\mathfrak{g}$

[8, Lectures 42 and 43]

A Lie algebra $\mathfrak{g}$ does not determine the group G with $\mathfrak{g} = \mathrm{Lie}(G)$. We will see that the correct extra data consists of prescribed representation theory. Throughout the discussion, we gloss the details, merely waving at the proofs of various statements.

6.54 Lemma / Definition *Let G be an algebraic group. A finite-dimensional module $G \curvearrowright V$ is* algebraic *if the map $G \to \mathrm{GL}(V)$ is a morphism of affine varieties.*

Any finite-dimensional algebraic (left) action $G \curvearrowright V$ of an algebraic group G gives rise to a (left) coaction *$V^* \to \mathscr{O}(G) \otimes V^*$:*

$$\begin{tikzcd}
V^* \arrow[r, "\text{coact}"] \arrow[d, "\text{coact}"] & \mathcal{O}(G) \otimes V^* \arrow[d, "\text{comult} \otimes \text{id}"] \\
\mathcal{O}(G) \otimes V^* \arrow[r, "\text{id} \otimes \text{coact}"] & \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes V^*
\end{tikzcd} \tag{6.2.8}$$

This in turn gives rise to a (left) action $\mathscr{O}(G)^* \curvearrowright V$, which specializes to the actions $G \curvearrowright V$ and $\mathcal{U}\mathfrak{g} \curvearrowright V$ under $G \hookrightarrow \mathbb{C}[G] \hookrightarrow \mathscr{O}(G)^*$ and $\mathcal{U}\mathfrak{g} \hookrightarrow \mathscr{O}(G)^*$.

We will take the following definition, referring the reader to [4] for the connections between rigid categories and Hopf algebras, and [1] and references therein for a thorough category-theoretic discussion.

6.55 Definition *A* rigid category *is an abelian category $\mathcal{M}$ with a (unital) monoidal product and duals. We will write the monoidal product as $\otimes$.*

A rigid subcategory *of $\mathcal{M}$ is a full subcategory that is a tensor category with the induced abelian and tensor structures. I.e. it is a full subcategory containing the zero object and the monoidal unit, and closed under extensions, tensor products, and duals.*

6.56 Definition *A rigid category $\mathcal{M}$ is* finitely generated *if for some finite set of objects $V_1, \dots, V_n \in \mathcal{M}$, any object is a subquotient of some tensor product of V_is (possibly with multiplicities). Of course, by letting $V_0 = V_1 \oplus \dots \oplus V_n$, we see that any finitely generated rigid category is in fact generated by a single object.*

6.57 Example For any Lie algebra $\mathfrak{g}$, the category $\mathfrak{g}$-MOD of finite-dimensional representations of $\mathfrak{g}$ is a tensor category; indeed, if U is any Hopf algebra, then U-MOD is a tensor category.

6.58 Definition *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$, and let $\mathcal{M}$ be a rigid subcategory of $\mathfrak{g}$-MOD. By definition, for each $V \in \mathcal{M}$, we have a linear map $\mathcal{U}\mathfrak{g} \to \operatorname{End} V$. Thus for each linear map $\phi : \operatorname{End} V \to \mathbb{C}$ we can construct a map $\{\mathcal{U}\mathfrak{g} \to \operatorname{End} V \xrightarrow{\phi} \mathbb{C}\} \in \mathcal{U}\mathfrak{g}^*$; we let $A_{\mathcal{M}} \subseteq \mathcal{U}\mathfrak{g}^*$ be the set of all such maps. Then $A_{\mathcal{M}}$ is the set of* matrix coefficients *of $\mathcal{M}$. Indeed, for each V, the maps $\mathcal{U}\mathfrak{g} \to \operatorname{End} V \to \mathbb{C}$ are the matrix coefficients of the action $\mathfrak{g} \curvearrowright V$. In particular, for each $V \in \mathcal{M}$, the space $(\operatorname{End} V)^*$ is naturally a subspace of $A_{\mathcal{M}}$, and $A_{\mathcal{M}}$ is the union of such subspaces.*

6.59 Lemma *If $\mathcal{M}$ is a rigid subcategory of $\mathfrak{g}$-MOD, then $A_{\mathcal{M}}$ is a subalgebra of the commutative algebra $\mathcal{U}\mathfrak{g}^*$. Moreover, $A_{\mathcal{M}}$ is a Hopf algebra, with comultiplication dual to the multiplication in $\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g}$.*

Proof The algebra structure on $A = A_{\mathcal{M}}$ is straightforward: the multiplication and addition stem from the rigidity of $\mathcal{M}$, the unit is $\epsilon : \mathcal{U}\mathfrak{g} \to \mathbb{C} \xrightarrow{\sim} \mathcal{C}$, and the subtraction is not obvious but is straightforward; it relies on the fact that $\mathcal{M}$ is abelian, and so contains all subquotients.

We will explain where the Hopf structure on A comes from — since $\mathcal{U}\mathfrak{g}$ is infinite-dimensional, $\mathcal{U}\mathfrak{g}^*$ does not have a comultiplication in general. But $\mathcal{M}$ consists of finite-dimensional representations; if $V \in \mathcal{M}$, then we send $\{\mathcal{U}\mathfrak{g} \to \operatorname{End} V \xrightarrow{\phi} \mathbb{C}\} \in A$ to $\{(\operatorname{End} V \otimes \operatorname{End} V) \xrightarrow{\text{multiply}} \operatorname{End} V \xrightarrow{\phi} \mathbb{C}\} \in (\operatorname{End} V)^* \otimes (\operatorname{End} V)^* \subseteq A \otimes A$.

That this is dual to the multiplication in $\mathcal{U}\mathfrak{g}$ comes from the fact that $\mathcal{U}\mathfrak{g} \to \operatorname{End} V$ is an algebra homomorphism. $\square$

6.60 Corollary *The map $\mathcal{U}\mathfrak{g} \to A^*$ dual to $A \hookrightarrow \mathcal{U}\mathfrak{g}^*$ is an algebra homomorphism.*

6.61 Proposition *Let $\mathcal{M}$ be a finitely-generated rigid subcategory of $\mathfrak{g}$-MOD. Then $A_{\mathcal{M}} = \mathscr{O}(G)$ for some algebraic group G.*

Proof If $\mathcal{M}$ is finitely generated, then there is some finite-dimensional representation $V_0 \in \mathcal{M}$ so that $(\operatorname{End} V_0)^*$ generates $A_{\mathcal{M}}$. Then $A_{\mathcal{M}}$ is a finitely generated commutative Hopf algebra, and so $\mathscr{O}(G)$ for some algebraic group G. $\square$

6.62 Lemma *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $\mathcal{M}$ a finitely-generated rigid subcategory of $\mathfrak{g}$-*MOD *, and G the algebraic group corresponding to the algebra $A_{\mathcal{M}}$ of matrix coefficients of $\mathcal{M}$. We will henceforth write $\mathscr{O}(G)$ for $A_{\mathcal{M}}$. Then G acts naturally on each $V \in \mathcal{M}$.*

Proof Let $\{v^1, \dots, v^n\}$ be a basis of V and $\{\xi_1, \dots, \xi_n\}$ the dual basis of V^*. For each i, we define $\lambda_i : V \to \mathscr{O}(G)$ by $v \mapsto \{u \mapsto \langle \xi_i, uv\rangle\}$ where $v \in V$ and $u \in \operatorname{End} V$. Then we define $\sigma : V \to V \otimes \mathscr{O}(G)$ a right coaction of $\mathscr{O}(G)$ on V by $v \mapsto \sum_{i=1}^n v^i \otimes \lambda_i(v)$. It is a coaction because $uv = \sum_{i=1}^n v^i \lambda_i(v)(u)$ by construction. In particular, it induces an action $G \curvearrowright V$. $\square$

6.63 Proposition *Let $\mathcal{M}$ be a finitely-generated rigid subcategory of $\mathfrak{g}$-*MOD *that contains a faithful representation of $\mathfrak{g}$. Then the map $\mathcal{U}\mathfrak{g} \to A_{\mathcal{M}}^*$ is an injection.*

Proof Let $\sigma : G \curvearrowright V$ as in the proof of Lemma 6.62. Then the induced representation $\operatorname{Lie}(G) \curvearrowright V$ is by contracting σ with point derivations. But $\mathfrak{g} \curvearrowright V$ and the map $\mathcal{U}\mathfrak{g} \to \mathscr{O}(G)^*$ maps $x \in \mathfrak{g}$ to a point derivation since $x \in \mathfrak{g}$ is primitive. Thus the following diagram commutes for each $V \in \mathcal{M}$:

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[d] & \mathscr{O}(G)^* \arrow[d] \\ \operatorname{Lie}(G) \arrow[r] & \mathfrak{gl}(V) \end{tikzcd} \tag{6.2.9}$$

The map $\mathfrak{g} \to \operatorname{Lie}(G)$ does not depend on V. Thus, if $\mathcal{M}$ contains a faithful $\mathfrak{g}$-module, then $\mathcal{U}\mathfrak{g} \hookrightarrow \mathcal{U}\operatorname{Lie}(G) \hookrightarrow \mathscr{O}(G)^*$. $\square$

6.64 Example Let $\mathfrak{g} = \mathbb{C}$ be one-dimensional, and let $\mathcal{M}$ be generated by one-dimensional representations V_α and V_β, where the generator $x \in \mathfrak{g}$ acts on V_α by multiplication by α, and on V_β by β. Then $\mathcal{M}$ is generated by $V_\alpha \oplus V_\beta$, and x acts as the diagonal matrix $\begin{bmatrix} \alpha & \\ & \beta \end{bmatrix}$. Let $\alpha, \beta \neq 0$, and let $\alpha \notin \mathbb{Q}\beta$. Then $\operatorname{Lie}(G)$ will contain all diagonal matrices, since $\alpha/\beta \notin \mathbb{Q}$, but $\mathfrak{g} \hookrightarrow \operatorname{Lie}(G)$ as a one-dimensional subalgebra. The group G is the complex torus, and the subgroup corresponding to $\mathfrak{g} \subseteq \operatorname{Lie}(G)$ is the irrational line.

6.65 Proposition *Let V_0 be the generator of $\mathcal{M}$ satisfying the conditions of Proposition 6.61, and let W be a neighborhood of $0 \in \mathfrak{g}$. Then the image of $\exp(W)$ is Zariski dense in G.*

Proof Assume that $\mathcal{M}$ contains a faithful representation of $\mathfrak{g}$; otherwise, mod out $\mathfrak{g}$ by the kernel of the map $\mathfrak{g} \to \operatorname{Lie}(G)$. Thus, we may consider $\mathfrak{g} \subseteq \operatorname{Lie}(G)$, and let $H \subseteq G$ be a Lie subgroup with $\mathfrak{g} = \operatorname{Lie}(H)$. Let $f \in \mathscr{O}(G)$ and $u \in \mathcal{U}\mathfrak{g}$; then the pairing $\mathcal{U}\mathfrak{g} \otimes \mathscr{O}(G) \to \mathbb{C}$ sends $u \otimes f \mapsto u(f|_H)(e)$. In particular, the pairing depends only on a neighborhood of $e \in H$, and hence only on a neighborhood $W \ni 0$ in $\mathfrak{g}$. But the pairing is nondegenerate; if the Zariski closure of $\exp W$ in G were not all of G, then we could find $f, g \in \mathscr{O}(G)$ that agree on $\exp W$ but that have different behaviors under the pairing. $\square$

6.66 Definition *Let $\mathcal{M}$ be a finitely-generated rigid subcategory of $\mathfrak{g}$-MOD, and let G be the corresponding algebraic group as in Proposition 6.61. Then $\mathfrak{g}$ is* algebraically integrable *with respect to $\mathcal{M}$ if the map $\mathfrak{g} \to \mathrm{Lie}(G)$ is an isomorphism. In particular, $\mathcal{M}$ must contain a faithful representation of $\mathfrak{g}$.*

6.67 Example Let $\mathfrak{g}$ be a finite-dimensional abelian Lie algebra over $\mathbb{C}$, and let $X \subseteq \mathfrak{g}^*$ be a lattice of full rank, so that $X \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{g}^*$. Let $\{\xi_1, \dots, \xi_n\}$ be a $\mathbb{Z}$-basis of X and hence a $\mathbb{C}$-basis of $\mathfrak{g}^*$, and let $\mathcal{M} = \{\bigoplus \mathbb{C}_\lambda \text{ s.t. } \lambda \in X\}$, where $\mathfrak{g} \curvearrowright \mathbb{C}_\lambda$ by $z \mapsto \lambda(z)\times$. Then $V_0 = \bigoplus \mathbb{C}_{\xi_i}$ is a faithful representation of $\mathfrak{g}$ in $\mathcal{M}$ and generates $\mathcal{M}$.

Then $G \subseteq \mathrm{GL}(V_0)$ is the Zariski closure if $\exp \mathfrak{g}$, and for $z \in \mathfrak{g}$, $\exp(z_1, \dots, z_n)$ is the diagonal matrix whose (i,i)th entry is $e^{\xi_i(z)}$. Thus G is a torus $T \cong (\mathbb{C}^\times)^n$, with $\mathcal{O}(T) = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. In particular, $\mathfrak{g}$ is algebraically integrable with respect to $\mathcal{M}$, since X is a lattice.

6.68 Proposition *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathcal{M}$ a finitely generated rigid subcategory of $\mathfrak{g}$-MOD containing a faithful representation. Suppose that $\mathfrak{g} = \bigoplus_{i=1}^r \mathfrak{g}_i$ as a vector space, where each $\mathfrak{g}_i$ is a Lie subalgebra of $\mathfrak{g}$; then $\mathcal{M}$ embeds in $\mathfrak{g}_i$-MOD for each i. If each $\mathfrak{g}_i$ is algebraically integrable with respect to (the image of) $\mathcal{M}$, then so is $\mathfrak{g}$.*

Proof Let G, G_i be the algebraic groups corresponding to $\mathfrak{g} \curvearrowright \mathcal{M}$ and to $\mathfrak{g}_i \curvearrowright \mathcal{M}$. Then for each i we have a map $G_i \to G$. Let $H \subseteq G$ be the subgroup of G corresponding to $\mathfrak{g} \subseteq \mathrm{Lie}(G)$. Consider the map $m : G_1 \times \cdots \times G_r \to G$ be the function that multiplies in the given order; it is not a group homomorphism, but it is a morphism of affine varieties. Since each $G_i \to G$ factors through H, and since H is a subgroup of G, the map m factors through H. Indeed, the differential of m at the identity is the sum map $\bigoplus \mathfrak{g}_i \to \mathfrak{g}$.

Thus we have an algebraic map m, with Zariski dense image. But it is a general fact that any such map (a *dominant morphism*) is dimension non-increasing. Therefore $\dim G \leq \dim(G_1 \times \cdots \times G_r) = \dim \mathfrak{g}$, and so $\mathfrak{g} = \mathrm{Lie}(G)$. $\square$

6.69 Theorem (Semisimple Lie algebras are algebraically integrable) *Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over $\mathbb{C}$, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be its Cartan subalgebra and Q and P the root and weight lattices. Let X be any lattice between these: $Q \subseteq X \subseteq P$. Let $\mathcal{M}$ be the category of finite-dimensional $\mathfrak{g}$-modules with highest weights in X. Then $\mathcal{M}$ is finitely generated rigid and contains a faithful representation of $\mathfrak{g}$, and $\mathfrak{g}$ is algebraically integrable with respect to $\mathcal{M}$.*

Proof Let $V \in \mathcal{M}$; then its highest weights are all in X, and so all its weights are in X since $X \supseteq Q$. Moreover, the decomposition of V into irreducible $\mathfrak{g}$-modules writes $V = \bigoplus L_\lambda$, where each $\lambda \in P_+ \cap X$. This shows that $\mathcal{M}$ is rigid. It contains a faithful representation because the representation of $\mathfrak{g}$ corresponding to the highest root is the adjoint representation, and the highest root is an element of Q and hence of X. Moreover, $\mathcal{M} = \{\bigoplus V_\lambda \text{ s.t. } \lambda \in P_+ \cap X\}$ is finitely generated: $P_+ \cap X$ is $\mathbb{Z}_{\geq 0}$-generated by finitely many weights.

We recall the triangular decomposition (c.f. Proposition 5.82) of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Then $\mathfrak{h}$ is abelian and acts on modules in $\mathcal{M}$ diagonally; in particular, $\mathfrak{h}$ is algebraically integrable by Example 6.67. On the other hand, on any $\mathfrak{g}$-module, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ act by strict upper- and strict lower-triangular matrices, and the matrix exponential restricted to strict upper- (lower-) triangular matrices is a polynomial. In particular, by finding a faithful generator of $\mathcal{M}$ (for example, the sum of the generators plus the adjoint representation), we see that $\mathfrak{n}_\pm$ are algebraically integrable. The conclusion follows by Proposition 6.68. $\square$

6.70 Theorem (Classification of Semisimple Lie Groups over $\mathbb{C}$) *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Any connected Lie group G with $\mathrm{Lie}(G)$ is semisimple; in particular, the algebraic groups constructed in Theorem 6.69 comprise all integrals of $\mathfrak{g}$.*

Proof Let $\tilde{G}$ be the connected and simply connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$; then any integral of $\mathfrak{g}$ is a quotient of $\tilde{G}$ be a discrete and hence central subgroup of $\tilde{G}$, and the integrals are classified by the kernels of these quotients and hence by the subgroups of the center $Z(\tilde{G})$. Let G_X be the algebraic group corresponding to X. Since $Z(G_P) = P/Q$, it suffices to show that G_P is connected and simply connected.

We show first that G_X is connected. It is an affine variety; G_X is connected if and only if $\mathscr{O}(G_X)$ is an integral domain. Since G_X is the Zariski closure of $\exp W$ for a neighborhood W of $0 \in \mathfrak{g}$, and $\exp W$ is connected, so is G_X.

Let $U_\pm$ be the image of $\exp(\mathfrak{n}_\pm)$ in G_X, and let $T = \exp(\mathfrak{h})$. But $\exp : \mathfrak{n}_\pm \to U_\pm$ is the matrix exponential on strict triangular matrices, and hence polynomial with polynomial inverse; thus $U_\pm$ are simply connected.

We quote a fact from algebraic geometry: the image of an algebraic map contains a set Zariski open in its Zariski closure. In particular, since the image of $U_- \times T \times U_+$ is Zariski dense, it contains a Zariski open set, and so the complement of the image must live inside some closed subvariety of G_X with complex codimension at least 1, and hence real codimension at least 2, since locally this subvariety is the vanishing set of some polynomials in $\mathbb{C}^n$. So in any one-complex-dimensional slice transverse to this subvariety, the subvariety consists of just some points. Therefore any path in G_X can be moved off this subvariety and hence into the image of $U_- \times T \times U_+$.

It suffices to consider paths in G_X from e to e, and by choosing for each such path a nearby path in U_-TU_+, we get a map $\pi_1(U_-TU_+) \twoheadrightarrow \pi_1(G_X)$. On the other hand, by the LU decomposition (see any standard Linear Algebra textbook, e.g. [15]), the map $U_- \times T \times U_+ \to U_-TU_+$ is an isomorphism. Since $U_\pm$ are isomorphic as affine varieties to $\mathfrak{n}_\pm$, we have:

$$\pi_1(U_-TU_+) = \pi_1(U_- \times T \times U_+) = \pi_1(T) \tag{6.2.10}$$

And $\pi_1(T) = X^*$, the co-lattice to X, i.e. the points in $\mathfrak{g}$ on which all of X takes integral values.

Thus, it suffices to show that the map $\pi_1(T) \twoheadrightarrow \pi_1(G_P)$ collapses loops in T when $X = P$. But then $\pi_1(T) = P^* = Q^\vee$ is generated by the simple coroots $\alpha_i^\vee$. For each generator $\alpha_i^\vee = h_i$, we take $\mathfrak{sl}(2)_i \subseteq \mathfrak{g}$ and exponentiate to a map $\mathrm{SL}(2,\mathbb{C}) \to G$. Then the loops in $\exp(\mathbb{R}h_i)$, which generate $\pi_1(T)$, go to loops in $\mathrm{SL}(2,\mathbb{C})$ before going to G. But $\mathrm{SL}(2,\mathbb{C})$ is simply connected. This shows that the map $\pi_1(T) \twoheadrightarrow \pi_1(G_P)$ collapses all such loops, and G_P is simply connected. $\square$

6.3 Conclusion

[8, Lecture 44]

In Chapter 5, we put semisimple Lie algebras over $\mathbb{C}$ into bijection with possibly-disconnected Dynkin diagrams, and classified all such diagrams. In the current chapter, we described how to compute the representation theory of any such diagram: the finite-dimensional irreducible representations correspond to elements of the weight lattice of the diagram. Then we showed that the integrals of any semisimple Lie algebra are algebraic, and correspond to lattices between the weight and root lattice: the simply connected group corresponds to the weight lattice, and the adjoint group corresponds to the root lattice. The category of representations of the Lie group corresponding to a given lattice is precisely the category of representations whose highest (and hence all) weights lie in the lattice. We remark that the index of the weight lattice on the root lattice — the size of the quotient — is precisely the determinant of the Cartan matrix.

The story we have told can be generalized. We will not justify it, but only sketch how it goes.

We recall Lemma/Definition 5.1. We say that a complex Lie group G is *reductive* if it is linear and if its finite-dimensional representations are completely reducible. We demand linearity to assure that the group have finite-dimensional representations: for example, an elliptic curve is a complex Lie group but it is compact, so any holomorphic function is constant, so its only finite-dimensional representations are trivial. Let G be a reductive Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. Then we have a Levi decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$, and if G is reductive, then $\mathfrak{r} = \mathfrak{z}$ is abelian and is the center of $\mathfrak{g}$, so $\mathfrak{g}$ is reductive as a Lie algebra.

Let $\mathfrak{g}$ be a reductive Lie algebra. Then the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is $\mathfrak{h}_{\mathfrak{s}} \oplus \mathfrak{z}$, where $\mathfrak{h}_{\mathfrak{s}}$ is the Cartan subalgebra $\mathfrak{s}$. We have a coroot lattice $Q^\vee \subseteq \mathfrak{h}_{\mathfrak{s}}$, and we pick a lattice $X^* \supseteq Q^\vee$ such that $X \supseteq Q$ and X^* spans $\mathfrak{h}$. Then let $\mathcal{M}$ be the category of finite-dimensional representations of $\mathfrak{g}$ with weights in X and on which $\mathfrak{h}$ acts diagonally — this is an extra condition, because $\mathfrak{z}$ need not act diagonally on a representation of $\mathfrak{g}$. Then the entire story of algebraic integrals applies. For any *root datum* — a choice of roots α and coroots $\alpha^\vee$ satisfying natural conditions — we get a Cartan matrix, the Cartan matrix of $\mathfrak{s}$.

In particular, as we will not explain here, the root data $(X, \alpha_i \in X, \alpha_i^\vee \in X^*)$ classify:

1. reductive Lie groups over $\mathbb{C}$
2. reductive Lie algebras over $\mathbb{C}$
3. compact real Lie groups
4. reductive algebraic groups over any algebraically closed field in any characteristic
5. *group schemes*, or "algebraic groups over $\mathbb{Z}$", and therefore *Chevalley groups*, the tensor products of group schemes with finite fields
6. finite groups of Lie type, essentially the source of the finite simple groups

Some of this story is outlined but mostly not proved in [18]. It would make a good second semester for this one-semester course, but in fact the second semester will tell a different story of quantum groups [17].

Exercises

1. Show that the simple complex Lie algebra $\mathfrak{g}$ with root system G_2 has a 7-dimensional matrix representation with the generators shown below.

$$e_1 = \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix} \qquad f_1 = \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&2&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{bmatrix} \qquad f_2 = \begin{bmatrix} 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&2&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0 \end{bmatrix} \tag{6.3.1}$$

2. (a) Show that there is a unique Lie group G over $\mathbb{C}$ with Lie algebra of type G_2.
   (b) Find explicit equations of G realized as the algebraic subgroup of $\mathrm{GL}(7,\mathbb{C})$ whose Lie algebra is the image of the matrix representation in Problem 1.

3. Show that the simply connected complex Lie group with Lie algebra $\mathfrak{so}(2n,\mathbb{C})$ is a double cover $\mathrm{Spin}(2n,\mathbb{C})$ of $\mathrm{SO}(2n,\mathbb{C})$, whose center Z has order four. Show that if n is odd, then Z is cyclic, and there are three connected Lie groups with this Lie algebra: $\mathrm{Spin}(2n,\mathbb{C})$, $\mathrm{SO}(2n,\mathbb{C})$ and $\mathrm{SO}(2n,\mathbb{C})/\{\pm I\}$. If n is even, then $Z \cong (\mathbb{Z}/2\mathbb{Z})^2$, and there are two more Lie groups with the same Lie algebra.

4. If G is an affine algebraic group, and $\mathfrak{g}$ its Lie algebra, show that the canonical algebra homormorphism $\mathcal{U}\mathfrak{g} \to \mathscr{O}(G)^*$ identies $\mathcal{U}\mathfrak{g}$ with the set of linear functionals on $\mathscr{O}(G)$ whose kernel contains a power of the maximal ideal $\mathfrak{m} = \ker(\mathrm{ev}_e)$.

5. Show that there is a unique Lie group over $\mathbb{C}$ with Lie algebra of type E_8. Find the dimension of its smallest matrix representation.

6. Construct a finite dimensional Lie algebra over $\mathbb{C}$ which is not the Lie algebra of any algebraic group over $\mathbb{C}$. [Hint: the adjoint representation of an algebraic group on its Lie algebra is algebraic.]

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