EXCEPTIONAL MATHEMATICS:
from Egyptian fractions to heterotic strings

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Canada/USA Mathcamp Colloquium, 23 July 2019

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

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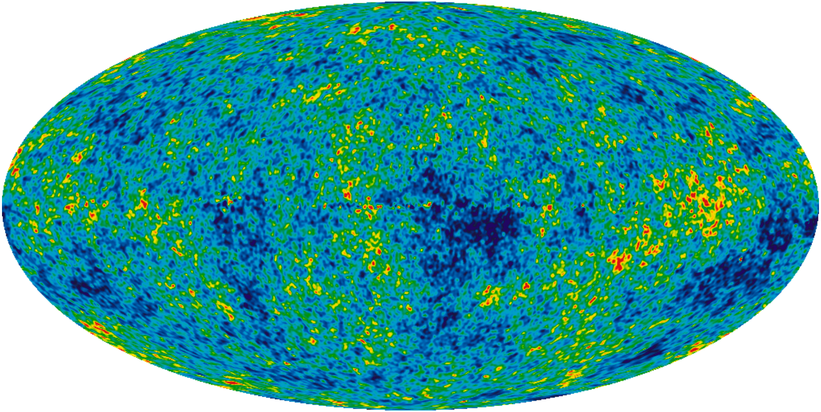
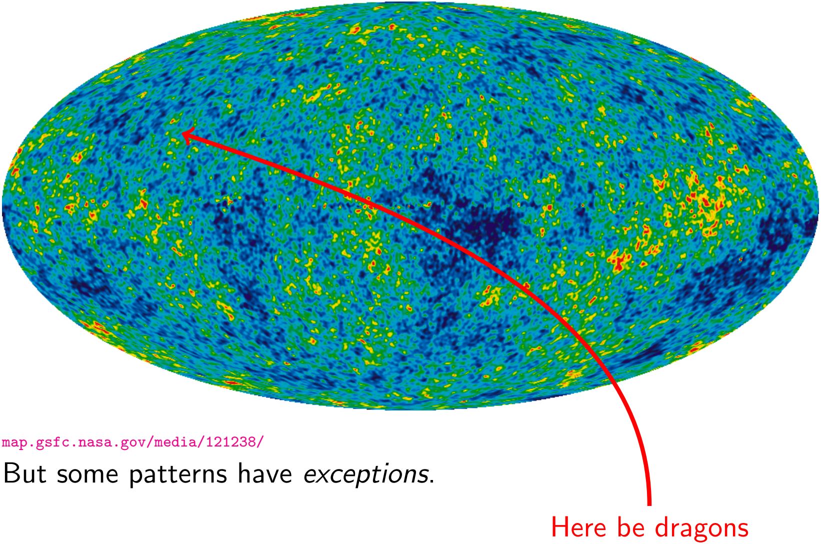


map.gsfc.nasa.gov/media/121238/

The mathematical universe is *infinitely* large. Most of what mathematicians do is look for regular, repeating, *infinite* patterns. Patterns provide ways that the universe is smooth and comprehensible: the laws of nature in one part of the mathematical universe are the same as in all other parts.



map.gsfc.nasa.gov/media/121238/

But some patterns have *exceptions*.

How do exceptional objects arise? Imagine plucking two violin strings that are almost, but not quite, in tune. You will here a throbbing noise, called a *beat*.

$$\underbrace{\cos(\alpha t) + \cos(\beta t)}_{\text{regular patterns}} = 2 \underbrace{\cos\left(\frac{\alpha - \beta}{2} t\right)}_{\text{throbbing ``beat''}} \underbrace{\cos\left(\frac{\alpha + \beta}{2} t\right)}_{\text{average pitch}}$$

At low volumes, the beat is just another repeating pattern. But at very high volumes, sound waves can scatter off other sound waves. The "beat" becomes a sonic boom, an exceptional object. Waves passing by the exceptional object get diffracted and irregularized.

Exceptional objects can be heard far away, with their strange, irregular calls. Exceptional objects are entertaining and enticing, like a Siren to Odysseus.

My goal for this talk is to explain the first exceptional object.

Five short chapters:

- From fractions to finite groups
- Species of finite groups: solvable and simple
- How to analyze a finite group I: spin representations
- How to analyze a finite group II: diagrams and lattices
- Applications of the E_8 lattice

From fractions to finite groups

wikimedia.org/wiki/File:Rhind_Mathematical_Papyrus.jpg

Its genesis begins in ancient Egypt. The Egyptians were interested in sums of simple fractions. What are all solutions to

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1, \qquad a \le b \le c \in \{2,3,4,\dots\}?$$

What are all solutions to

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1, \qquad a \leq b \leq c \in \{2, 3, 4, \ldots\}?$$

One infinite family:

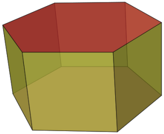
$$(a, b, c) = (2, 2, c), \quad c \in \{2, 3, 4, \ldots\}. \qquad \text{“Type D”}$$

Three exceptional solutions:

$$(2, 3, 3), \quad (2, 3, 4), \quad (2, 3, 5). \qquad \text{“Type E”}$$

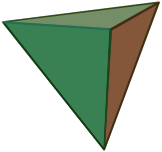
The lettering is from the 20th century. For later convenience, the Type D solutions are called “D_n” and the Type E solutions are called “E_n”, where $n = a + b + c - 2 = b + c$. E.g. the solution $(2, 3, 4)$ is called “E_7.”

The Greeks understood these solutions. Given any solution (a, b, c), take some regular c-gons, and attach them so that b of them meet at each vertex, and of course $a = 2$ faces meet at each edge. Together with their duals ($b \leftrightarrow c$), you will form all the (possibly degenerate) regular three-dimensional solids.

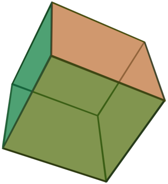


D_8

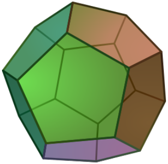
(shrink squares to edges)



E_6



E_7



E_8

wikimedia.org/wiki/File:Hexagonal_Prism_BC.svg
wikimedia.org/wiki/File:Tetrahedron.svg
wikimedia.org/wiki/File:Hexahedron.svg
wikimedia.org/wiki/File:Dodecahedron.svg

What are the rotational symmetries of a regular solid (a, b, c)?

- There is a symmetry α of order $a = 2$ which rotates the solid around an edge. Which edge? Pick one arbitrarily.
- There is a symmetry β of order b which rotates the solid around a vertex. Which vertex? One of the ends of the edge that you picked.
- There is a symmetry γ of order c which rotates the solid around a face. Which face? One of the sides of the edge that you picked.

Exercise: $\alpha \circ \beta \circ \gamma = 1$.

It turns out that *every* symmetry of the solid is a composition of these three symmetries. Furthermore, it turns out that the relation in the Exercise generates all relations between these symmetries. I.e. the full group of rotational symmetries is

$$G = \mathrm{Aut}(\text{solid}) = \langle \alpha, \beta, \gamma | \alpha^a = \beta^b = \gamma^c = \alpha\beta\gamma = 1 \rangle.$$

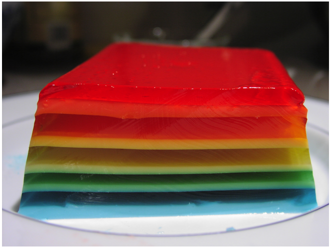
So the Egyptian puzzle led us to discover some finite groups.

A *group* is an abstract collection of "symmetries," which is closed under composition. By "abstract," I mean that the elements don't have to be symmetries of anything — what matters is the structural relations between the different elements of the group.

Species of finite groups

Solvable groups are built out of thin abelian layers. The layers are only loosely attached to each other.

It is impossible to classify all solvable groups. The problem is that there are a lot of ways to add a layer to the top of a solvable group, but the exact number depends on how all the other layers are attached to each other.



wikimedia.org/wiki/File:Rainbow-Jello-Cut-2004-Jul-30.jpg

Solvable groups are flexible, almost liquid. But if you push too much they shear apart. They feel like blocks of jello.

Simple groups, on the other hand, have no layers. They are rigid, crystalline, and highly structured.

wikimedia.org/wiki/File:Pyrite_60608.jpg

You could try to mix solvable and simple groups together, by including simple layers into your solvable jello.

It is near-impossible to layer a simple group on top of a solvable group: it sinks towards the bottom.

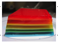
There are a few ways to layer a solvable group on top of a simple group, like jello resting on a crystalline foundation.

The "D" groups $G = \mathrm{Aut}(\text{D-type solid})$ are dihedral. They are solvable with only two layers.

layers are minimally attached

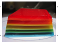
top layer

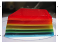
bottom layer

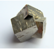
$$\mathrm{Aut}(D_n \text{ solid}) \cong C_{n-2} \rtimes C_2$$ ()

cyclic groups

The "E" groups are:

$$\mathrm{Aut}(E_6 \text{ solid}) \cong C_2^2 \rtimes C_3$$ ()

$$\mathrm{Aut}(E_7 \text{ solid}) \cong C_2^2 \rtimes C_3 \rtimes C_2$$ ()

$$\mathrm{Aut}(E_8 \text{ solid}) \cong \mathrm{Alt}(5)$$ ()

$\mathrm{Aut}(E_8 \text{ solid})$ is the smallest simple group. It has 60 elements.

Simplicity is *exceptional* among finite groups: if you choose a finite group at random, with 100% probability you will choose a solvable group. But the exceptional solutions to the Egyptian fraction problem led us to discover a simple group!

How to analyze a finite group I

To completely understand a group, you should understand its representations. A *representation* of a group is a way of assigning matrices to group elements, preserving the group law.

The group $\mathrm{SO}(3)$ of three-dimensional rotations can (almost!) be represented by 2×2 complex matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \mapsto \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \qquad x\text{-axis rotation}$$

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \mapsto \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \qquad y\text{-axis rotation}$$

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{e}^{i\theta/2} & 0 \\ 0 & \mathrm{e}^{-i\theta/2} \end{pmatrix} \qquad z\text{-axis rotation}$$

▸ formula for arbitrary rotation

It is only an "almost" representation because when $\theta = 2\pi$, you get $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. What you can actually represent is a *double cover* of $\mathrm{SO}(3)$, called "$\mathrm{Spin}(3)$". Every element in $\mathrm{SO}(3)$ corresponds to a pair of elements in $\mathrm{Spin}(3)$. In $\mathrm{Spin}(3)$, 360° rotation is not trivial, but 720° rotation is trivial.

You can realize $\mathrm{Spin}(3)$ by taking an object, and attaching some stretchy ribbons to it, with the other ends of the ribbons attached to the walls of the room. (You may use as many ribbons as you want.) If you rotate the object by 360°, the ribbons will twist up. But if you rotate by 720°, you can untwist the ribbons!

Watch this in action:
https://commons.wikimedia.org/w/index.php?title=File:Belt_Trick.ogv

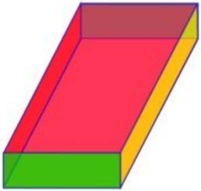
For each group $G = \mathrm{Aut(solid)}$, write $\tilde{G}$ for the set of elements in $\mathrm{Spin}(3)$ corresponding to rotations in G. Then $\tilde{G}$ is a *double cover* of G: each element of G corresponds to two elements in $\tilde{G}$.

Example: The D_4 solid is very degenerate. If you expand the degenerate faces, you get a rectangular prism, with

$$G(D_4) = \mathrm{Aut}(D_4 \text{ solid}) \cong C_2 \times C_2.$$

Its double cover is the *quaternion group*

$$\tilde{G}(D_4) \cong Q_8 = \{\pm 1, \pm i, \pm j, \pm k\},$$
$$i^2 = j^2 = k^2 = ijk = -1.$$

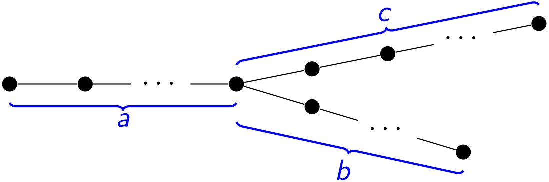


www.k6-geometric-shapes.com/rectangular-prisms.html

How to analyze a finite group II

Draw a graph whose vertices are the nontrivial indecomposable representations of $\tilde{G}$. Draw an edge between V and W if you get a standard 2×2 block when you decompose $V \otimes W$.

Remarkably, you will get a Y-shape graph. If you started with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$, then the arms of the Y have lengths a, b, and c.



To *decompose* a representation, find a basis making all matrices block-diagonal. The *tensor product* of representations is:

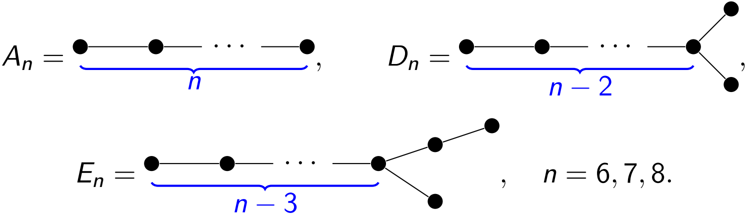
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} Aa & Ba & Ab & Bb \\ Ca & Da & Cb & Db \\ Ac & Bc & Ad & Bd \\ Cc & Dc & Cd & Dd \end{pmatrix}.$$

These are almost all the graphs Γ such that the *Cartan matrix*

$$2 \times (\text{identity matrix}) - (\text{adjacency matrix for } \Gamma)$$

is positive definite.

To get the complete set, you must allow the degenerate case $a = 1$. They are called "Type A." The finite group in this case is cyclic. The "A" solutions are an infinite, regular, unexceptional family.



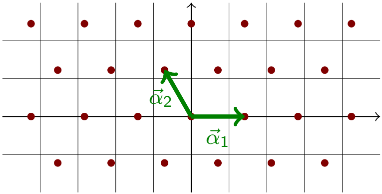
A matrix A is *positive definite* if $\sum_{ij} v_i A_{ij} v_j > 0$ for all nonzero vectors $\vec{v}$.

We can picture this as a basis $\{\vec{\alpha}_1, \ldots, \vec{\alpha}_n\}$ of $\mathbb{R}^n$, such that $\vec{\alpha}_i$ and $\vec{\alpha}_j$ are orthogonal if vertex i is not connected to vertex j, but they should be at 120° if they are connected. Since the dot product is

$$\vec{\alpha}_i \cdot \vec{\alpha}_j = |\vec{\alpha}_i|\,|\vec{\alpha}_j| \cos(\text{angle}),$$

and $\cos(120^\circ) = -\frac{1}{2}$, and since I don't like fractions, I will decide that each $\vec{\alpha}_i$ has length $|\vec{\alpha}_i| = \sqrt{2}$. Take all vectors that are *integer* sums of the $\vec{\alpha}$s. That set of vectors is the *lattice* of the given ADE type.

Example: The A_2 lattice is the triangular lattice in $\mathbb{R}^2$.



The distance between any two vectors in an ADE lattice is $\geq \sqrt{2}$. Indeed, the distance is always $\sqrt{\text{even number}}$. A lattice with this property is called *even*.

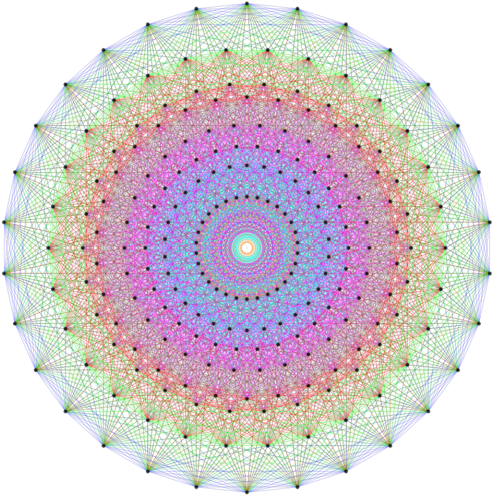
Fundamental fact: The density of an ADE lattice, i.e. the number of atoms per unit volume, is

$$\text{Density} = \frac{1}{\#\{\text{one-dimensional representations of } \tilde{G}\}}.$$

This number is < 1 if G solvable. The number of one-dimensional representations of a solvable group is equal to the size of its bottom layer.

But it is exactly 1 when G is simple. And that happens for exactly one example: the E_8 solid ($=$ dodecahedron).

Applications of the E_8 lattice



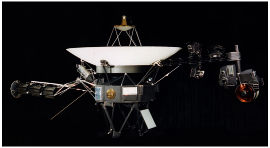
wikimedia.org/wiki/File:E8Petrie.svg

The E_8 lattice is a crystalline arrangement in eight dimensions such that there is one atom per unit volume, but any two atoms are at least $\sqrt{2}$-distance apart.

Each atom is adjacent to 240 neighbours. The picture on the left is the projection of these 240 neighbours to a two-dimensional plane.

Application 1 (Error correcting codes): The E_8 lattice determines (and can be built from) Hamming's error-correcting code Ham$(8,4)$. This is a way of transmitting four bits in an eight-bit channel. The 2^4 codewords are:

00000000	11111111
11110000	00001111
11001100	00110011
11000011	00111100
10101010	01010101
10100101	01011010
10011001	01100110
10010110	01101001



voyager.jpl.nasa.gov

Error-correcting codes are used extensively. An exceptional error-correcting code, called the *Golay code*, was how NASA communicated with the Voyager probes. The Golay code is the "beat" that leads to Monstrous Moonshine. For more details, take my class this week.

Application 2 ((1+1)-dimensional quantum matter):
Start with a (very cold) system of eight fermions $\psi_1, \ldots, \psi_8$ moving freely on a 1-dimensional wire, with no interactions. Slowly ("adiabatically") turn on a quartic potential energy, determined by the Hamming code:

$$\begin{aligned}\mathrm{PE} = {} & \psi_1\psi_2\psi_3\psi_4 + \psi_1\psi_2\psi_5\psi_6 + \psi_1\psi_2\psi_7\psi_8 + \psi_1\psi_3\psi_5\psi_7 \\ & - \psi_1\psi_3\psi_6\psi_8 - \psi_1\psi_4\psi_5\psi_8 - \psi_1\psi_4\psi_6\psi_7 - \psi_2\psi_3\psi_5\psi_8 \\ & - \psi_2\psi_3\psi_6\psi_7 - \psi_2\psi_4\psi_5\psi_7 + \psi_2\psi_4\psi_6\psi_8 + \psi_3\psi_4\psi_5\psi_6 \\ & + \psi_3\psi_4\psi_7\psi_8 + \psi_5\psi_6\psi_7\psi_8\end{aligned}$$

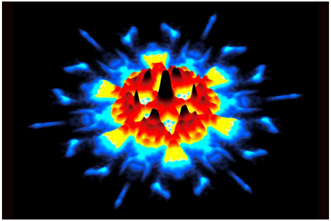
Each term is a (nontrivial) Hamming-code word. The signs are due to the fermionic nature of the particles.

Remarkably, this does not force the system to undergo a phase transition (e.g. from liquid to gas). Even more remarkably, the interacting system is *trivial*. This is because $G = \mathrm{Aut}(E_8 \text{ solid})$ is a simple group; if you started with a different group, the interacting system would look like the bottom layer of $\tilde{G}$.

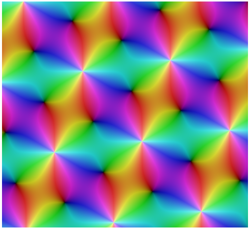
A system of eight (1+1)-dimensional fermions represents the same phase of matter as the trivial system. This can be seen in the laboratory.

It leads to myriad examples of 8-fold periodicity in mathematics. For example, the "homotopy groups" of many important topological objects exhibit an approximate 8-fold periodicity. It is only approximate because it is caused by an exceptional object.

Application 3 ((2+1)-dimensional quantum matter):
A similar construction in two spatial dimensions is responsible for the "thermal quantum Hall effect." It can be used to construct interesting "topological phases of matter." These are important for quantum computing.



www.nature.com/articles/d41586-018-05913-4

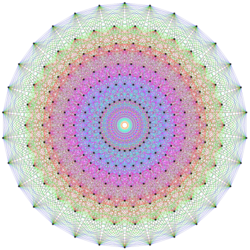


wikimedia.org/wiki/File:Weierstrass_elliptic_function_P.png

Application 4 (string theory):
Any n-dimensional lattice L determines an n-dimensional *torus* $T = \mathbb{R}^n/L$. Consider a (quantum) string propagating in T. Locally, any solution to the equations of motion factorizes into a "chiral" piece and an "antichiral" piece. Usually this factorization cannot be done globally. It only works globally if L is an even lattice of density $= 1$.

When the factorization works, the chiral piece is a "chiral string theory." The E_8 chiral string theory is important for the construction of "heterotic strings."

What are the symmetries of the E_8 string theory? There are *classical symmetries*: translating along $T = \mathbb{R}^8/(E_8 \text{ lattice})$. There are also *quantum symmetries* that convert a state into a superposition of other states (including states with nontrivial topology).



wikimedia.org/wiki/File:
E8Petrie.svg

It turns out that there is one dimension of quantum symmetries for each of the 240 neighbouring atoms to a given atom in the E_8 lattice.

Together with the classical symmetries, this produces a 248-dimensional group, called the "E_8 Lie group." It is an exceptional group, with exceptional properties. For instance, its smallest representation is itself!

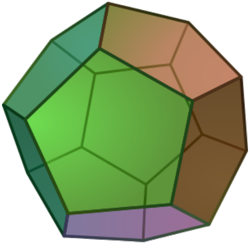
A *Lie group* is an infinite group which is also a smooth manifold. Just like for finite groups, most are solvable (layers of jello) and a few are simple (crystals). Remarkably, the classification of simple Lie groups is the same as the classification of solutions to $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1$, together with some decoration called "folding." There are four infinite families:

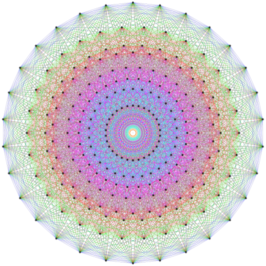
$$A_n, \quad D_n, \quad C_n \text{ (folded } A_n), \quad B_n \text{ (folded } D_n),$$

and five exceptions:

$$E_6, \quad E_7, \quad E_8, \quad G_2 \text{ (folded } E_6), \quad F_4 \text{ (folded } E_7).$$

E_8 is the most exceptional. That exceptional Egyptian "beat" $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}>1$ can be heard very far away!





$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}>1$$

Thank you!

$\mathrm{Spin}(3)$ is isomorphic to $\mathrm{SU}(2)$, which is by definition the (infinite) group consisting of the following complex matrices:

$$\begin{pmatrix} \lambda & \mu \\ -\bar{\mu} & \bar{\lambda} \end{pmatrix}, \qquad \lambda, \mu \in \mathbb{C}, \quad |\lambda|^2 + |\mu|^2 = 1.$$

Exercise: Check that $\mathrm{SU}(2)$ is closed under matrix multiplication.

Consider the function $f : \mathrm{SU}(2) \to \mathrm{SO}(3)$ which takes

$$f : \begin{pmatrix} \lambda & \mu \\ -\bar{\mu} & \bar{\lambda} \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{Re}(\lambda^2 + \mu^2) & \mathrm{Im}(\lambda^2 - \mu^2) & \mathrm{Im}(2\lambda\mu) \\ -\mathrm{Im}(\lambda^2 + \mu^2) & \mathrm{Re}(\lambda^2 - \mu^2) & \mathrm{Re}(2\lambda\mu) \\ \mathrm{Im}(2\lambda\bar{\mu}) & -\mathrm{Re}(2\lambda\bar{\mu}) & |\lambda|^2 - |\mu|^2 \end{pmatrix}.$$

Exercise: Show that $f(AB) = f(A)f(B)$ for any two matrices $A, B \in \mathrm{SU}(2)$, i.e. show that f is a *homomorphism*.

return to main slideshow