# Higher Dagger Categories

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categorified.net/NYUADtalk.pdf        arXiv: 23XX.XXXXX

## Background

Definition (Selinger): A dagger category is a category $\mathcal{C}$ together with an assignment $(-)^\dagger : \hom(X,Y) \to \hom(Y,X)$ $\forall X,Y \in \mathrm{ob}\,\mathcal{C}$ s.t.: $f^{\dagger\dagger} = f$ and $(fg)^\dagger = g^\dagger f^\dagger$.

Ur-example: Hilb. $(-)^\dagger$ = adjoint bounded operator.

Other examples: * $\mathrm{Hilb}_{\mathbb{R}}$. Allow other signatures. Unitary rep thy.
** relations / spans / correspondences. *** etc.

Known "problem": Dagger structures are "evil" aka "incoherent": they do not transport across equivs of categories.

Selinger's answer: Dagger categories are just different. There is a perfectly good (2,1)-category of dagger categories, dagger functors, and unitary natural isos.

## Background

But to axiomatize "higher functional analysis" and unitary QFT, we need dagger $(\infty,n)$-categories. What should they be?

Last spring, I became aware of multiple groups nearing an answer to this question. I was worried that there could soon be competing definitions. (I was personally agnostic.)

To head this off, I organized a small Zoom workshop in June. Participants presented motivating examples and partial definitions. After four exciting days, we emerged with a consensus definition.

A few participants chose not to be authors on the report, but everyone contributed.

## Dagger $(\infty,1)$-categories

Part of the definition of dagger category was the functor $(-)^\dagger : \mathcal{C} \to \mathcal{C}^{op}$, and the requirement it be (anti)involutive. This is not evil:

a "Work In Progress edge"

Definition (ChatGPT): A wipedge category is a fixed point for the $\mathbb{Z}/2$-action $\mathcal{C} \mapsto \mathcal{C}^{op}$ on Cat.

The evil part was requiring $(-)^\dagger|_{ob(\mathcal{C})} = id$. Indeed, this is true only on the set of objects, not the groupoid of objects/isos.

A dagger category has a different natural groupoid: objects/unitaries. This groupoid is necessary data: it is what knows the difference between {Hilbert spaces} and {Hermitian spaces}.

Note: $\mathcal{C} \mapsto \mathcal{C}^{op}$ trivializes on $\mathrm{Gpoid} \subseteq \mathrm{Cat}$ by $g \mapsto g^{-1}$. So a wipedge category has $\left(\frac{\text{objects}}{\text{isos}}\right)^{\mathbb{Z}/2}$. $\frac{\text{objects}}{\text{unitaries}}$ is typically smaller.

# Dagger $(\infty,1)$-categories

**Definition** (Henry, Stehouwer–Steinebrunner): A coherent dagger $(\infty,1)$-category is a wipedge $(\infty,1)$-category $\mathcal{C}_1$ equipped with a $\infty$-groupoid $\mathcal{C}_0$ thought of as "objects/uniteries" and

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "\text{ess. surj.}"] \arrow[dr, "\text{f.f.}"'] & \mathcal{C}_1 \\
& \mathrm{ob}(\mathcal{C}_1)^{\mathbb{Z}/2} \arrow[u]
\end{tikzcd}$$

s.t. axioms in green.

**Theorem** (Stehouwer–Steinebrunner): There is an equiv of $(\infty,1)$-cats

$$\left\{\begin{array}{c}\text{coherent dagger}\\ 1\text{-categories}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{traditional dagger}\\ 1\text{-categories}\end{array}\right\}$$

# Dagger $(\infty,n)$-categories

Write $\mathrm{Cat}_{(\infty,n)}$ for the $(\infty,1)$-category of $(\infty,n)$-categories.

E.g. $\mathrm{Cat}_{(\infty,0)} = \mathrm{Spaces}$. A reason why coherent dagger categories are natural is that the $\mathbb{Z}/_2$-action $\mathcal{C} \mapsto \mathcal{C}^{op}$ supplies an iso $\mathrm{Aut}(\mathrm{Cat}_{(\infty,1)}) \simeq \mathbb{Z}/_2$. More generally:

<u>Theorem (Barwick – Schommer-Pries):</u> $\mathrm{Aut}(\mathrm{Cat}_{(\infty,n)}) \simeq (\mathbb{Z}/_2)^n$, with the $k^{th}$ $\mathbb{Z}/_2$ acting by oppositing the $k$-morphisms.

<u>Definition:</u> An $(\infty,n)$-category is (fully) wipedge if it is a (homotopy) fixed point for the $\mathrm{Aut}(\mathrm{Cat}_{(\infty,n)})$-action.

Remember: to be a fixed point is structure, not property!

A fixed point for $G \leq (\mathbb{Z}/_2)^n$ is a $G$-wipedge $(\infty,n)$-category.

# Dagger $(\infty,n)$-categories

As with the $n=1$ case, to enhance a wipedge structure to a dagger structure involves trivializing parts of it.

For $m \leq n$, let $\tau_m : Cat_{(\infty,n)} \to Cat_{(\infty,m)}$ denote the maximal sub-$m$-category (right adjoin to inclusion $Cat_{(\infty,m)} \subseteq Cat_{(\infty,n)}$).

Observe: If $\mathcal{C} \in Cat_{(\infty,n)}$ is wipedge, then $\tau_m \mathcal{C}$ is wipedge, and has an action by $(\mathbb{Z}/2)^{n-m}$.

Definition: A (fully) dagger $(\infty,n)$-category $\mathcal{C}$ is a sequence $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n$ with $\mathcal{C}_m$ a wipedge $(\infty,m)$-cat and wipedge maps $\mathcal{C}_m \to (\tau_m \mathcal{C}_{m+1})^{\mathbb{Z}/2}$ s.t.:

(1) $\mathcal{C}_m \to \tau_m \mathcal{C}_{m+1}$ is ess. surj on $(\leq m)$-morphisms.

(2) $\mathcal{C}_m \to (\tau_m \mathcal{C}_{m+1})^{\mathbb{Z}/2}$ is fully faithful on $(\geq m+1)$-morphisms.

# Dagger $(\infty,n)$-categories

$$
\begin{tikzcd}[column sep=small]
\mathcal{C}_0 \arrow[rr] \arrow[dr] & & \mathcal{C}_1 \arrow[rr] \arrow[dr] & & \mathcal{C}_2 \arrow[rr] & & \cdots \arrow[rr] & & \mathcal{C}_n \\
& (\mathcal{C}_0\mathcal{C}_1)^{\mathbb{Z}/2} \arrow[rr] \arrow[ur] \arrow[dr] & & (\mathcal{C}_1\mathcal{C}_2)^{\mathbb{Z}/2} \arrow[rr] \arrow[ur] \arrow[dr] & & \cdots \arrow[rr] & \arrow[dr] & (\mathcal{C}_{n-1}\mathcal{C}_n)^{\mathbb{Z}/2} \arrow[ur] & \\
& & (\mathcal{C}_0\mathcal{C}_2)^{(\mathbb{Z}/2)^2} \arrow[rr] \arrow[ur] \arrow[dr] & & \ddots & & (\mathcal{C}_1\mathcal{C}_n)^{(\mathbb{Z}/2)^{n-1}} \arrow[ur, dotted] & & \\
& & & \ddots \arrow[dr] & & \arrow[ur] & & & \\
& & & & (\mathcal{C}_0\mathcal{C}_n)^{(\mathbb{Z}/2)^n} \arrow[uurr] & & & &
\end{tikzcd}
$$

The step-by-step axioms imply more generally:

**Axioms:**

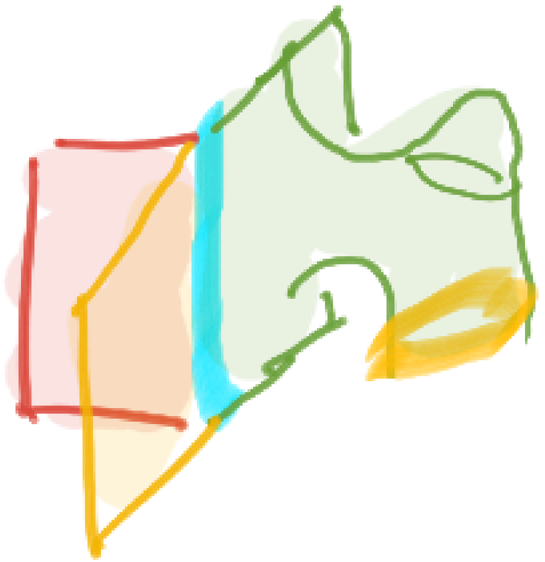
$$
\begin{tikzcd}
\mathcal{C}_k \arrow[rr, "\text{ess-surj on } (\leq k)\text{-mors}"] \arrow[dr] & & \mathcal{C}_\ell \\
& (\mathcal{C}_k\mathcal{C}_\ell)^{(\mathbb{Z}/2)^{\ell-k}} \arrow[ur] &
\end{tikzcd}
$$

f.f. on $(\geq \ell)$-mors

If $G \subseteq (\mathbb{Z}/2)^n$, a $G$-dagger category is similar but with $G \cap (\mathbb{Z}/2)^{\ell-k}$. Need the $k$-to-$\ell$ version if $G \neq \prod_{\text{subset of } \{1,\dots,n\}} \mathbb{Z}/2$.

# Unitary adjoints

An $(\infty,n)$-category **has adjoints** if every $k$-morphism, $0<k<n$, has both adjoints. Write $\mathrm{AdjCat}_{(\infty,n)} \subseteq \mathrm{Cat}_{(\infty,n)}$ the $(\infty,1)$-cat of $(\infty,n)$-categories with adjoints, all functors, and natural isos.



Expectation (cobordism hypothesis with singularities):
Any $\mathcal{C} \in \mathrm{AdjCat}_{(\infty,n)}$ determines a **graphical calculus** of framed embedded hypersurfaces in $\mathbb{R}^n$.

**Smoothing theory** says that framed smooth = framed PL.

$PL(n) := \{$piecewise-linear automorphisms of $\mathbb{R}^n\}$ acts on the space of diagrams, and hence we expect a map

$$PL(n) \longrightarrow \mathrm{Aut}(\mathrm{AdjCat}_{(\infty,n)}).$$

# Unitary adjoints

For comparison, smoothing theory does supply a map

$$PL(n) \longrightarrow \mathrm{Aut}(\mathrm{Bord}_n^{fr})$$

Theorem (Lurie, unpublished): Assuming the Cobordism Hypothesis, the map $PL(n) \to \mathrm{Aut}(\mathrm{Bord}_n^{fr})$ is an iso when $n \neq 4$. When $n=4$, it is equiv to the 4D PL Schoenflies conjecture.

Conjecture: There is an iso $PL(n) \xrightarrow{\sim} \mathrm{Aut}(\mathrm{AdjCat}_{(\infty,n)})$.

Definition assuming conjecture: A wipedge $(\infty,n)$-category with unitary adjoints is a fixed point for $PL(n) \circlearrowright \mathrm{AdjCat}_{(\infty,n)}$.

## Unitary adjoints

Restrict along $PL(n) \supset PL(k) \times PL(n-k)$. If $\mathcal{C} \in \mathrm{AdjCat}_{(\infty,n)}$ is wipedge, then $\tau_k \mathcal{C}$ is wipedge with a compatible $PL(n-k)$-action.

<u>Definition:</u> A dagger $(\infty,n)$-category with unitary adjoints is a diagram of equivariant wipedge objects with

$$\begin{tikzcd}
\mathcal{C}_k \arrow[rr, "\text{ess surj on}", "\leq k\text{-mors}"'] \arrow[dr, "\text{f.f. on}"', "\geq \ell\text{-mors}"'] & & \mathcal{C}_\ell \\
& (\tau_k \mathcal{C}_\ell)^{PL(\ell-k)} \arrow[ur] &
\end{tikzcd}$$

For $G \subset PL(n)$, can also talk about $G$-wipedge and $G$-dagger strs.

## Examples

$PL(2) = O(2) = SO(2) \rtimes \mathbb{Z}/2 = B\mathbb{Z} \rtimes \mathbb{Z}/2$. Action on bicats w/ adjs:

- $\mathbb{Z}/2$ acts by $\mathcal{C} \mapsto \mathcal{C}^{2op}$
- $B\mathbb{Z}$ acts by $(-)^{\vee\vee}$ (double right adjoint)

For $(\infty,2)$, there $\exists$ higher coherence data.

A dagger bicategory with unitary adjoints, in our sense, is equiv to a (bi)category enriched in dagger categories, plus a functorial choice of $f^\vee$, $ev_f$, $coev_f$ for each morphism $f$ such that $ev_f$, $coev_f$ are unitary.

Unitarity $\Rightarrow$ pivotality = trivialization of $(-)^{\vee\vee}$ that is id on objs.

Indeed, an $SO(2)$-dagger bicategory is a pivotal bicategory.

## Examples

A tangential structure on PL $n$-manifolds is a reduction of structure of $T_M : M \to BPL(n)$ through some $BH(n) \to BPL(n)$.

E.g.: Smoothing theory is the statement that smoothness is a PL tangential structure. ← this is extra data!

A tangential structure is stable if $BH(n) = BPL(n) \times_{BPL} BH$.

E.g.: Smoothness is not stable, but

$$\mathrm{Bord}_n^{\text{smooth}} \to \mathrm{Bord}_n^{\text{stably smooth}}$$

is an equiv after quotienting to $(n,n)$-categories.

**Theorem:** If $H$ is a stable tangential structure, then $\mathrm{Bord}_n^H$ is a dagger $(\infty,n)$-category with unitary adjoints.

## Examples

Chen, Ferrer, Hungar, Penneys, and Sanford have a soon-to-be-released theory of f.d. n-Hilbert spaces.

rigorous for $n \leq 3$. sketched for $n \geq 4$.

Theorem (CFHPS): $\mathrm{Hilb}_n^{f.d.}$ is dagger with unitary adjoints.

Their same construction also gives a super version.

Definition: For $H$ a stable tangential structure, a unitary fully-extended $H$-structured bosonic / fermionic $n$D TQFT is a functor

$$\mathrm{Bord}_n^H \longrightarrow \mathrm{Hilb}_n^{f.d.} \quad / \quad s\mathrm{Hilb}_n^{f.d.}$$

of symmetric monoidal dagger $(\infty,n)$-cats w/ unitary adjoints.