A 4D TQFT that is not $\underbrace{\quad}_{}$ a gauge theory

quite

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These slides:
http://categorified.net/Nongauge-SymSem.pdf

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Introduction: Goal of talk

The goal of this talk is to answer the following question:
Which 4D (=3+1D) TQFTs* are** gauge theories?

You might think this is a funny question. Didn't Lan–Kong–Wen and/or the speaker prove that the answer is "all of them"?

Not quite. Let's agree that a true gauge theory is anything which can be presented by a (finite) possibly-higher-form gauge group $\mathcal{G}$ and a Dijkgraaf–Witten Lagrangian $\omega \in \mathrm{H}^4_{\mathrm{gp}}(\mathcal{G}; \mathrm{U}(1))^{***}$. The path integral sums over $\mathcal{G}$-bundles.

Precifications

* Bosonic and semisimple, with a unique local vacuum and trivial gravitational anomaly. I know how to drop all assumptions except semisimplicity.
** Up to stacking with an invertible TQFT. I.e. up to changing the trivialization of the gravitational anomaly. I.e. up to considering two TQFTs to be "the same" when they have the same (extended) operators.
*** $:= \mathrm{H}^4(\mathrm{B}\mathcal{G}; \mathrm{U}(1))$. In higher dim, I would accept any Lag in $\Omega^{\bullet}_{\mathrm{SO}}(\mathrm{B}\mathcal{G})$.

Introduction: Lan–Kong–Wen classification

Theorem [LKW18, LW19, JF22]: Every 4D TQFT is either:

(AB) A true gauge theory for a finite 0-form group G.

(EF) A spin gauge theory for a 0-form group $G^f = \mathbb{Z}_2^f.G$, classified by $\kappa \in \mathrm{H}^2_{\mathrm{gp}}(G; \mathbb{Z}_2^f)$. When $G^f = \mathbb{Z}_2^f \times G$, the path integral sums over pairs (spin structure, G-bundle). In general, it sums over "twisted G^f-bundles," which are not flat but rather have curvature $(-1)^{w_2}$.

Moreover, the gauge group and the Lagrangian are determined canonically. The AB case occurs when all line operators are bosons. The EF case occurs when some line operators are fermions.

The slogan "All 4D TQFTs are gauge theories" is based on deciding that spin gauge theories do count as gauge theories.

Introduction: Dual gauge theory description?

That said, at least some spin gauge theories are true gauge theories for higher form groups. For example, spin-$\mathbb{Z}_2^f$ gauge theory is electromagnetic dual to true $\mathbb{Z}_2[1]$ gauge theory with Lagrangian $(-1)^{\mathrm{Sq}^2} \in \mathrm{H}^4_{\mathrm{gp}}(\mathbb{Z}_2[1]; \mathrm{U}(1)) \cong \mathbb{Z}_4$.

Suggestion: Any spin gauge theory for $G^f = \mathbb{Z}_2^f.G$ is dual to a true gauge theory for $\mathcal{G} = \mathbb{Z}_2[1].G$, via a version of Tachikawa's "gauging finite subgroups."

Does this suggestion work? If not, perhaps we can capture every TQFT by a true gauge theory for a higher group?

Introduction: Supercohomology

Main Claim: $\exists$ a spin gauge theory for $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_4^2$ which is not equivalent to a true gauge theory for any higher-form group.

Recall (c.f. Wang–Gu, Gaiotto–Kapustin) that 4D spin-G^f gauge theory Lagrangians are classified by twisted supercohomology $\mathrm{SH}^{4+\kappa}_{\mathrm{gp}}(G)$, where the twisting $\kappa \in \mathrm{H}^2_{\mathrm{gp}}(G; \mathbb{Z}_2^f)$ classifies the extension $G^f = \mathbb{Z}_2.G$. A cocycle for $\mathrm{SH}^{\bullet+\kappa}_{\mathrm{gp}}(G)$ consists of:

- A Majorana layer $\gamma \in \mathrm{C}^{\bullet-2}_{\mathrm{gp}}(G; \mathbb{Z}_2)$ solving $\mathrm{d}\gamma = 0$.
- A Gu–Wen layer $\beta \in \mathrm{C}^{\bullet-1}_{\mathrm{gp}}(G; \mathbb{Z}_2)$ solving $\mathrm{d}\beta = (\mathrm{Sq}^2 + \kappa\cup)\gamma$.
- A Dijkgraaf–Witten layer $\alpha \in \mathrm{C}^{\bullet}_{\mathrm{gp}}(G; \mathrm{U}(1))$ solving $\mathrm{d}\alpha = (-1)^{(\mathrm{Sq}^2+\kappa\cup)\beta} \times$ (something involving γ).

To construct the claimed TQFT, pick any class in $\mathrm{SH}^4_{\mathrm{gp}}(\mathbb{Z}_4^2)$ with Majorana layer $[\gamma] \neq 0 \in \mathrm{H}^2_{\mathrm{gp}}(\mathbb{Z}_4^2; \mathbb{Z}_2)$.

Introduction: Constructing a non-gauge theory

Main Claim: $\exists$ a spin gauge theory for $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_4^2$ which is not equivalent to a true gauge theory for any higher-form group.
To construct the claimed TQFT, pick any class in $\mathrm{SH}^4_{\mathrm{gp}}(\mathbb{Z}_4^2)$ with Majorana layer $[\gamma] \neq 0 \in \mathrm{H}^2_{\mathrm{gp}}(G; \mathbb{Z}_2)$.

To convince you that this works, I need to convince you:

- That a supercohomology class with nontrivial Majorana layer exists.
- That all true gauge theories give supercohomology classes with trivial Majorana layer.

For the rest of the talk, I'm going to explain some technical details of the proof. If you'd like to zone out, now's a good time.

Existence: What's the obstruction?

Question: Given G (e.g. $G = \mathbb{Z}_4^2$), what is the image of $\mathrm{Maj} : \mathrm{SH}^4_{\mathrm{gp}}(G^b) \to \mathrm{H}^2_{\mathrm{gp}}(G^b; \mathbb{Z}_2)$, $[(\gamma, \beta, \alpha)] \mapsto [\gamma]$?

Answer: There is an older version of supercohomology due to Gu–Wen, with only two layers:

- $\beta \in \mathrm{C}^{\bullet-1}_{\mathrm{gp}}(G; \mathbb{Z}_2)$ solving $\mathrm{d}\beta = 0$
- $\alpha \in \mathrm{C}^{\bullet}_{\mathrm{gp}}(G; \mathrm{U}(1))$ solving $\mathrm{d}\alpha = (-1)^{\mathrm{Sq}^2\beta}$.

I'll call it restricted supercohomology $\mathrm{rSH}^\bullet$.

One immediately gets a long exact sequence

$$\cdots \to \mathrm{SH}^\bullet_{\mathrm{gp}}(G) \xrightarrow{\mathrm{Maj}} \mathrm{H}^{\bullet-2}_{\mathrm{gp}}(G; \mathbb{Z}_2) \xrightarrow{\mathrm{obstr}} \mathrm{rSH}^{\bullet+1}_{\mathrm{gp}}(G) \to \mathrm{SH}^{\bullet+1}_{\mathrm{gp}}(G) \to \dots$$

The map obstr measures the obstruction to extending a class $[\gamma] \in \mathrm{H}^{\bullet-2}_{\mathrm{gp}}(G; \mathbb{Z}_2)$ to $[(\gamma, \beta, \alpha)] \in \mathrm{SH}^\bullet_{\mathrm{gp}}(G)$.

Existence: Gu–Wen layer of obstruction

Set $G = \mathbb{Z}_4^2$. Existence is equivalent to the claim is that $\ker\big(\mathrm{obstr} : \mathrm{H}^2_{\mathrm{gp}}(G;\mathbb{Z}_2) \to \mathrm{rSH}^5_{\mathrm{gp}}(G)\big) \neq 0$.

By Künneth formula, $\mathrm{H}^\bullet_{\mathrm{gp}}(G) = \mathbb{Z}_2[x, X, y, Y]/(x^2 = y^2 = 0)$, where $\deg x = \deg y = 1$, $\deg X = \deg Y = 2$. I specifically claim that $\mathrm{obstr}(xy) = 0 \in \mathrm{rSH}^5_{\mathrm{gp}}(\mathbb{Z}_4^2)$.

By definition of $\mathrm{rSH}^\bullet$, there is another long exact sequence

$$\dots \xrightarrow{\mathrm{Sq}^2} \mathrm{H}^\bullet_{\mathrm{gp}}(G;\mathrm{U}(1)) \to \mathrm{rSH}^\bullet_{\mathrm{gp}}(G) \to \mathrm{H}^{\bullet-1}_{\mathrm{gp}}(G;\mathbb{Z}_2) \xrightarrow{\mathrm{Sq}^2} \dots$$

recording the two layers (β, α). The composition

$$\mathrm{H}^{\bullet-2}_{\mathrm{gp}}(G;\mathbb{Z}_2) \xrightarrow{\mathrm{obstr}} \mathrm{rSH}^{\bullet+1}_{\mathrm{gp}}(G) \to \mathrm{H}^\bullet_{\mathrm{gp}}(G;\mathbb{Z}_2)$$

is Sq^2. Since $\mathrm{Sq}^2(xy) = 0$, $\mathrm{obstr}(xy) \in \mathrm{H}^5_{\mathrm{gp}}(G;\mathrm{U}(1))/\operatorname{im}(\mathrm{Sq}^2)$.

Existence: Dijkgraaf–Witten layer of obstruction

Our class $xy \in \mathrm{H}^2_{\mathrm{gp}}(G; \mathbb{Z}_2)$ is the pullback of a class living on $G' = \mathbb{Z}_4 \times \mathbb{Z}_2$. Already over G', $\mathrm{Sq}^2(xy) = 0$, so the obstruction over G' lives in $\mathrm{H}^5_{\mathrm{gp}}(G'; \mathrm{U}(1))/\operatorname{im}(\mathrm{Sq}^2)$. So the obstruction over G is the pullback of some such class. But, by direct computation,

$$\mathrm{H}^5_{\mathrm{gp}}(G'; \mathrm{U}(1))/\operatorname{im}(\mathrm{Sq}^2) \xrightarrow[=0]{\text{pullback}} \mathrm{H}^5_{\mathrm{gp}}(G; \mathrm{U}(1))/\operatorname{im}(\mathrm{Sq}^2) \cong \mathbb{Z}_2$$

Thus $\mathrm{obstr}(xy) = 0 \in \mathrm{rSH}^5(\mathbb{Z}_4^2)$, and so there exists a class in $\mathrm{SH}^4_{\mathrm{gp}}(\mathbb{Z}_4^2)$ with Majorana layer $xy \neq 0 \in \mathrm{H}^2_{\mathrm{gp}}(\mathbb{Z}_4^2; \mathbb{Z}_2)$.

True gauge theories: Initial simplification

Suppose you have a higher-group gauge theory with gauge group $\mathcal{G}$. Before gauging, any 2-form symmetries in $\mathcal{G}$ act by local operators; gauging kills the theory if they act nontrivially, and produces a (-1)-form dual description if they act trivially. So it suffices to consider $\mathcal{G} = A[1].G[0]$ merely a 2-group.

Pick $\omega \in \mathrm{H}^4_{\mathrm{gp}}(\mathcal{G}; \mathrm{U}(1))$. Look at

$q = \omega|_A \in \mathrm{H}^4_{\mathrm{gp}}(A[1]; \mathrm{U}(1)) = \{\mathrm{U}(1)\text{-valued quadratic forms on } A\}$.

The radical of q is

$$\mathrm{rad}(q) := \{a \in A \text{ s.t. } q(ab) = q(b) \,\forall b \in A\}$$

Then $\omega|_{\mathrm{rad}(q)[1]}$ is trivial, and (c.f. Tachikawa) the $\mathcal{G}$ gauge theory has a dual description as a gauge theory for a group of shape

$$(A/\mathrm{rad}(q))[1].(\mathrm{rad}(q)^*.G)[0]$$

True gauge theories: Further simplification

So WLOG $\mathrm{rad}(q) = 0$. Then either:

(AB) The quadratic form $q : A \to \mathrm{U}(1)$ is nondegenerate: the bilinear form $\langle a, b\rangle_q = q(ab)/q(a)q(b)$ is nondegenerate.

(EF) q is slightly degenerate: there is a $\mathbb{Z}_2 = \{1, e\} \subset A$ with $q(eb) = -q(b)\ \forall b \in A$, and $\langle , \rangle_q$ descends to a nondegenerate form on $A/\{1, e\}$.

In the AB case, $A[1]$ gauge theory is invertible, so the $A[1].G$ gauge theory is equivalent to some plain G gauge theory.

In the EF case, $A[1]$ gauge theory is equivalent to spin-$\mathbb{Z}_2^f$ gauge theory, so the $A[1].G$ gauge theory is equivalent to some spin-$\mathbb{Z}_2^f.G$ gauge theory. Our goal is to control its Majorana layer.

True gauge theories: Translation to fusion categories

$\mathcal{G} = A[1].G$ and $\omega \in \mathrm{H}^4_{\mathrm{gp}}(G; \mathrm{U}(1))$ together determine:

- a grouplike slightly-degenerate braided fusion category $\mathcal{B} = \mathrm{Vec}^q[A]$, i.e. an abelian spin Chern–Simons theory;
- an action of G thereon.

Our job is to understand the anomaly of this action.

This is something the fusion category theorists (e.g. Galindo–Venegas-Ramírez, Davydov–Nikshych, JF–Reutter) have studied in detail. For any slightly-degenerate braided fusion category $\mathcal{B}$, every automorphism of $\mathcal{B}$ lifts to a class in the Brauer–Picard group $\mathrm{Br}(\mathcal{B})$, with a $\mathbb{Z}_2 = \{\mathrm{Cliff}(0), \mathrm{Cliff}(1)\}$ ambiguity. The Majorana layer of the anomaly classifies this double cover

$$\{\mathrm{Cliff}(0), \mathrm{Cliff}(1)\} \to \mathrm{Br}(\mathcal{B}) \to \mathsf{Aut}(\mathcal{B}).$$

True gauge theories: Dimension trick

The Majorana layer of the anomaly classifies this double cover

$$\{\mathrm{Cliff}(0), \mathrm{Cliff}(1)\} \to \mathrm{Br}(\mathcal{B}) \to \mathsf{Aut}(\mathcal{B}).$$

But recall $\mathcal{B} = \mathrm{V{\scriptsize EC}}^q[A]$, so every algebra object $X \in \mathcal{B}$ has a dimension $\mathsf{dim}(X) \in \mathbb{N}$. And if X and Y represent the same Brauer–Picard class, then $\mathsf{dim}(X)\,\mathsf{dim}(Y)$ is a perfect square. In particular, writing $\mathsf{dim}(X) = 2^k \times \mathsf{odd}$, the parity is a multiplicative Brauer–Picard invariant $\mathrm{Br}(\mathcal{B}) \to \mathbb{Z}_2$.

But $\mathsf{dim}(\mathrm{Cliff}(1)) = 2$. So we can split $\mathrm{Br}(\mathcal{B}) \to \mathsf{Aut}(\mathcal{B})$ by choosing representatives of dimension $2^{\mathsf{even}} \times \mathsf{odd}$. Thus the Majorana layer of any action of G on $\mathcal{B}$ automatically vanishes.

Thus the Majorana layer of any true gauge theory vanishes.

True gauge theories: Correcting a slight lie

Actually, this Brauer–Picard description only works when $\kappa = 0$, where κ measures how the G-action on $\mathcal{B}$ fractionalizes on the fermion in $\mathcal{B}$. κ will become the extension $G^f = \mathbb{Z}_2^f.G$, and we can apply the argument only after pulling back along $G^f \twoheadrightarrow G$.

But actually, in the Lan–Kong–Wen classification, the Lagrangian $[(\gamma, \beta, \alpha)] \in \mathrm{SH}^{4+\kappa}_{\mathrm{gp}}(G)$ isn't quite canonical. The reason is that different spin-G^f theories can give the same TQFT: it turns out that there is some map $\{\text{Ising theories}\} = \mathbb{Z}_{16} \to \mathrm{SH}^{4+\kappa}_{\mathrm{gp}}(G)$, which depends on κ in a nonlinear way, and the TQFT is classified by the quotient $\mathrm{SH}^{4+\kappa}_{\mathrm{gp}}(G)/\operatorname{im}(\mathbb{Z}_{16})$.

The composition $\mathbb{Z}_{16} \to \mathrm{SH}^{4+\kappa}_{\mathrm{gp}}(G) \to \mathrm{H}^2_{\mathrm{gp}}(G; \mathbb{Z}_2)$ sends $1 \mapsto \kappa$. So what the TQFT actually determines is a Majorana layer in $\mathrm{H}^2_{\mathrm{gp}}(G; \mathbb{Z}_2)/\langle\kappa\rangle$. Happily, $\mathrm{H}^2_{\mathrm{gp}}(G; \mathbb{Z}_2)/\langle\kappa\rangle \subset \mathrm{H}^2_{\mathrm{gp}}(G^f; \mathbb{Z}_2)$!

True gauge theories: Summary and conclusion

Finally, if the Majorana layer does vanish, then indeed the earlier suggestion that you can dualize $\mathbb{Z}_2^f.G \rightsquigarrow \mathbb{Z}_2[1].G$ does work. All together, the refined classification of 4D TQFTs is:

Theorem: Suppose $\mathcal{Q}$ is a 4D TQFT.

(AB) If $\mathcal{Q}$ has only bosonic line operators, then $\mathcal{Q}$ is a true gauge theory for a 0-form group G.

(EF0) If $\mathcal{Q}$ has fermionic line operators and its Majorana layer vanishes in $\mathrm{H}^2_{\mathrm{gp}}(G;\mathbb{Z}_2)/\langle\kappa\rangle$, then $\mathcal{Q}$ is a true gauge theory for a 2-group $\mathbb{Z}_2[1].G$.

(EF1) If $\mathcal{Q}$ has fermionic line operators and its Majorana layer does not vanish in $\mathrm{H}^2_{\mathrm{gp}}(G;\mathbb{Z}_2)/\langle\kappa\rangle$, then $\mathcal{Q}$ is not a true gauge theory for any higher group.

Moreover, case EF1 can occur, for instance for $G = \mathbb{Z}_4^2$.

P.S.: I think EF1 cannot occur for $|G| < 16$. I think it occurs often for $|G| = 16$.