The CS-WZW correspondence

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Phil asked me to describe the correspondence between Wess-Zumino-(Novikov-)Witten theory and Chern-Simons(-Witten) theory, which is supposedly the best (only?) understood example of "ADS-CFT" correspondence, and is also supposed to be a warm-up for the mysterious maximally supersymmetric gauge theory in 6 dimensions. I won't have anything valuable to say about these generalizations, but I will try to get to WZW-CS in the second half of the talk, after the break.

To set the stage, though, I'd like to spend the first 45 minutes or so waxing philosophical about algebraic structures you really should expect to find in quantum field theory. Most of what I say will be based on work by Dan Freed and the people around him. Perhaps ε will be due to me.

I. Heisenberg-picture qft.

If you think back to your first course in quantum mechanics, you set up quantum mechanical systems in what's called the "Schrodinger picture": a system consisted of a "Hilbert space" $\mathcal{H}$ of "states" and a family of unitary "time evolution" operators satisfying a ~~group~~ group law. Let me emphasize that this is the data <u>of the system</u>, and not the data of the <u>questions</u> you wanted to ask about it — just like if I tell you the axioms and data of a group, I haven't told you if I want to know, say, numerics about its representation theory, or how lengths of words depend on a set of generators, or anything like that.

But if you paid careful attention, you learned that the "Schrödinger picture, although interesting mathematically, is not really "physical". Namely, the space of states is really not $\mathcal{H}$ but its projectivization $\mathbb{P}\mathcal{H}$. The Schrödinger picture suggests the Atiyah axioms:

Here's an example

$$\text{Spacetimes} \longrightarrow \text{Hilb}$$

Perhaps we should try

$$\text{Spaces} \longmapsto \mathbb{P}(\text{Hilb})$$

$$\text{Spacetimes} \longrightarrow \mathbb{P}U(\cdots)\ ?$$

This still isn't quite right. For a single system modeled by $\mathcal{H}$, its physical space of states is $\mathbb{P}\mathcal{H}$. But for combining systems, we want to end up with

$$\mathbb{P}(\mathcal{H}_1 \otimes \mathcal{H}_2), \text{ not } \mathbb{P}\mathcal{H}_1 \times \mathbb{P}\mathcal{H}_2.$$

and

$$\mathbb{P}(\mathcal{H}_1 \otimes \mathcal{H}_2), \text{ not } \dots \text{ ~~state~~ ???}$$

So, how to get the right answer? Think to your algebraic geometry class: there was some connection between projective bundles and Azumaya algebras and Brauer groups. How it goes is:

$$\boxed{\text{End}(\mathcal{H}) \text{ encodes } \mathbb{P}\mathcal{H}, \text{ not } \mathcal{H}.}$$

By $\text{End}(\mathcal{H})$ I mean the algebra of matrices, so "$\mathcal{B}(\mathcal{H})$" for Hilbert spaces.

So we might try:

<u>Schrödinger picture</u>

$N^{d-1} \longmapsto \mathcal{H}$

$M^d \longmapsto Z(M^d) : \mathcal{H}^{\otimes \partial M. \text{in}} \to \mathcal{H}^{\otimes \text{out}}$

$\sqcup \longmapsto \otimes$

<u>Heisenberg picture</u>

$N^{d-1} \longmapsto A$ e.g. $\text{End}(\mathcal{H})$.

$M^d \longmapsto ??$

$\sqcup \longmapsto \otimes$

What sort of maps? In the textbook QM example, $M^d = [0,t]$ and $Z(M^d) = \exp(\frac{i}{\hbar}\hat{H}t) = U_t$, and on the Heisenberg side you put the algebra automorphism "conjugate by U_t". But in higher dimensions, $Z(M^d)$ typically is not an BO.

A good guess is to ~~put a ... module.~~ use bimodules. Any homomorphism $f: A \to B$ gives a "modulation" $A \overset{f}{\curvearrowright} B \curvearrowleft B$, ~~and~~ But the modulation ~~is~~ of any conjugation morphism is isomorphic to trivial. To break this, you should use pointed bimodules. E.g. ${}_A B_B$ is pointed by $1 \in B$. Then in general given a linear map $V \xrightarrow{f} W$, you get a pointed bimodule ${}_{End(V)}Hom(V,W)_{End(W)} \ni f$. And sure enough for f an iso this is the modulation of "conjugate by f".

<u>Defn:</u> An (non-extended, affine) Heisenberg-picture qft is a symmetric monoidal functor

$$\text{SPACETIMES} \longrightarrow \text{ALG}_1 = \left\{\begin{array}{l}\text{algebras}\\ \text{pointed bimodules}\\ \text{pointed intertwiners}\end{array}\right\}$$

Two obvious generalizations:

- do everything derived, i.e. $\text{ALG}_1(\text{DG-Vect})$ in place of $\text{Alg}_1(\text{Vect})$.
- do it "k-extended", i.e.

$$\text{Alg}_k = \left\{\begin{array}{c}E_k\text{-algebras}\\ (k-1)\text{-algs}\\ \vdots\\ E_0\text{-algs}\end{array}\right\}.$$

<u>Thm (Scheimbauer + Calaque)</u> $\text{Alg}_k(S)$ is a sym $\otimes$ (∞,k)-cat for S an $(\infty,1)$-category satisfying mild conditions.

There's one further generalization I want to suggest. See, most things in nature are not affine — the natural map ~~X→Spec~~ $X \to \mathrm{Spec}(\mathcal{O}(X))$ is not an ISO. E.g. for $X = B\mathbb{G}_m$, $\mathcal{O}(X) = \mathbb{C}$ (even the derived space of global sections is just $\mathbb{C}$). On the other hand, most stacks are <u>2-affine</u> in the following sense: there is a good notion of "Spec $\mathcal{O}(\mathcal{C})$" for any sym $\otimes$ pres. cat. $\mathcal{C}$, and it is a stack in the fppf (or do I mean fpqc? I don't remember) topology, and you can ask if $X \to \mathrm{Spec}(\mathrm{QCoh}(X))$ is an ISO. (It isn't always. Underivedly, it fails if X has a closed pt w/ non-affine stabilizer.)

Symmetric monoidality is the structure enjoyed by modules for a <u>commutative</u> ring. For just an associative A, Mod_A is barely better than a category: it is pointed by A_A.

<u>Defn</u>: The 2-category of <u>2-affine</u> (but not nec. 1-affine) <u>quantum stacks</u> over $\mathbb{K}$ is $\mathrm{Alg}_0(\mathrm{Pres}_{\mathbb{K}})$, with:

- Obj. = pointed ~~lin~~ pres $\mathbb{K}$-lin cats.
- Mor = $(F, f): (\mathcal{C}, C) \to (\mathcal{D}, D)$ s.t. $F: \mathcal{C} \to \mathcal{D}$ is a $\mathbb{K}$-lin. pres functor and $f: D \to F(C)$.
- 2-mor = intertwiners of these.

A 2-affine quantum stack $(\mathcal{C}, C)$ is 1-affine iff C is a compact proj. generator of $\mathcal{C}$, as then $\mathcal{C} = \mathrm{Mod}_{\mathrm{End}(C)}$.

E.g.: $(\mathrm{Vect}, V) \simeq (\mathrm{Mod}_{\mathrm{End}(V)}, \mathrm{End}(V))$ if $\dim V < \infty$ and $V \neq 0$.

There's also a more topological version where "presentable over $\mathbb{C}$" gets replaced by a version of "presentability" for categories enriched over Hilb. Then $(\mathrm{Hilb}, \mathcal{H}) \simeq \mathrm{Mod}_{B(\mathcal{H})}$ where the RHS is in the sense of Von Neumann algebras.

<u>Defn</u>: A (non-extended, 2-affine) <u>Heisenberg picture QFT</u> is a sym $\otimes$ functor

$$\mathrm{SPACETIMES} \longrightarrow \mathrm{Alg}_0(\mathrm{Pres}_{\mathbb{K}}).$$

There's also extensions:

<u>Thm (—, Scheimbauer)</u>: if S is (∞, k), then $\mathrm{Alg}_{E_n}(S)$ is $(\infty, k+n)$.

<u>Defn</u>: A k-affine $(n+k-1)$-extended Heisenberg picture QFT is a functor into $\mathrm{Alg}_n(S)$ where S is an (∞, k)-cat that looks like Vect at the top, and Spacetimes should be an $(n+k-1)$-cat.

Warning: $\mathbb{1}$ is the only (n+k)-dualizable object of $\mathrm{Alg}_n S$ if S is (∞, k).

By taking "MOD" you can always see k-affine things as (k+1)-affine. What Freed calls "relative" is when $n=0$ and $k=$ # levels of SPACETIME.

Let me now give some topological examples. Actually, first a non-example:

~~Thm (~~ ~~, Chirvasitu-Brandenburg~~

Thm (Brandenburg–Chirvasitu–JF) If X is a scheme containing a closed proj. subscheme of positive dimension or if $X = BG$ for G an algebraic group without enough projective modules then $\mathrm{QCoh}(X)$ is not 1-dualizable in $\mathrm{Pres}_{\mathbb{K}}$.

So to capture these "classical 1-d TQFTs" you need derived categories (for $D^b\mathrm{QCoh}$, you do get dualizability by Ben-Zvi–Francis–Nadler).

OK, an example.

Thm (JF): Suppose $\mathcal{C} \in \mathrm{Alg}_2(\mathrm{Pres}_{\mathbb{K}})$ has:

(i) unit is compact proj.
(ii) every object is a colimit of dualizable objects.

Then $\mathcal{C}$ is 3-dualizable. The resulting theory is <u>3D Chern-Simons theory</u> for $\mathcal{C}$.
<u>E.g.</u>: $\mathcal{C}$ = Temperley-Lieb. More generally Rep (quantum group).

Oh, I almost forgot. Let's look at $Alg_0(S)$. There is a "forget" map to S. ~~If~~ It often happens that $X \in Alg_0(S)$ is interesting but forget$(X) \in S$ is invertible.
~~Example We're not Take $S = Mor(Pres_{\mathbb{K}})$, ie. ~~ ~~⊗ cats to do this honestly s.t~~

<u>Example (originally due to Walker)</u>: When you forget from $Alg_2(Pres_{\mathbb{K}})$, you should end up in a 4-cat with
- braided ⊗ cats.
- ⊗ cats.
- cats, un pointed
- functors
- natural transformations.

I don't know that this 4-cat has been written down, although the corresponding 3-cat w/ $Pres_{\mathbb{K}} \rightsquigarrow$ "finite abelian cats/$\mathbb{K}$" exists in Douglas-Schommer-Pries-Snyder.

In any case, <u>MTC</u>s are the <u>invertible</u> elements here.

Let's see in top dimensions what happens when that "forget" part is invertible.

<u>Notation</u>: "Forget" of a qft is its "anomaly".

So:

<u>top dim</u>: • an invertible vector space is a line.
• the "partition function" is section thereof. E.g.: Conformal field theory.

<u>(top - 1)</u>: • an invertible cat is the modules for an Azumaya algebra. Up to equiv, there's the Brauer group worth of these.
• The "Hilbert space" is a module thereof. (In "super land" this is the source of Clifford algebras, etc.)

<u>E.g.</u>: In <u>Conformal field theory</u>, $Z(\Sigma^{\text{closed}})$ is not a number, but rather a section of the line over $\mathcal{M}_{\text{curves}}$ ~~of sections of the trivial line over~~ given by:

$$\begin{array}{cl} \{\text{surfaces w/ metric}\} & \text{— take trivial line here} \\ \downarrow & \\ \mathcal{M} = \{\text{conformal curves}\} & \text{— over } \Sigma \in \mathcal{M}\text{, take sections} \\ & \text{of the line ~~on~~ upstairs that} \\ & \text{transform by } \exp(c \cdot \text{Liouville action}) \end{array}$$

it often happens that the anomaly is topological even when the qft isn't. (I wish I had a general explanation for this, but I don't.) If so, then the anomaly itself is a map

$$\mathrm{Bord} \longrightarrow \text{some sort of categorified group}$$

and so factors through the quotient of Bord in which everyone is forced to be invertible. But that quotient looks like

$\mathbb{Z}$ in dim $\equiv 0 \pmod 4$
0 in all other dimensions.

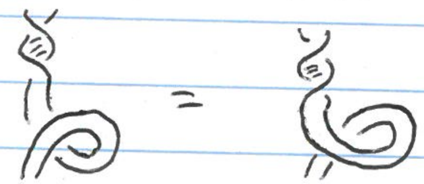
In dim 4, this is the signature. So this is why the signature shows up all over the place.

[Break here if I haven't already.]

II. Specifics on CS and WZW

Before the break I briefly mentioned Chern-Simons theory, which is a 2-affine, but ~~not~~ generally not 1-affine, Heisenberg picture of tft depending on a braided $\otimes$ cat $\mathcal{C}$. Actually, it is an <u>oriented</u> theory. This is more data than being a framed theory: you have to declare <u>how</u> the theory <u>doesn't</u> depend on the orientation. In this case that data is precisely a <u>ribbon</u> structure. Aron defined these last time: a natural iso

and



Anyway, in dimensions ≤ 2, this tft is "just" topological chiral homology with coeffs in $\mathcal{C}$. In dimension 3, it can be presented in terms of "relative skein modules."

I wish I could describe also WZW at this level of generality, but I can't. Here's the punchline that I'd like to explain:

$$\begin{tikzcd}
\mathrm{Bord}^{or}_{0123} \arrow[r, "CS"] & \mathrm{Alg}_2(\mathrm{Pres}_{\mathbb{k}}) \\
\mathrm{Bord}^{or}_{012} \arrow[u] \arrow[ur, phantom, "\Uparrow WZW^*"] & \\
\mathrm{Bord}^{conf}_{012} \arrow[u] \arrow[uur, bend right, "\mathbb{1}"'] &
\end{tikzcd}$$

The natural transformation is "oplax," meaning that naturality is only imposed by a morphism, not an $\mathcal{B}o$. (Ex: alg. objects in a $\otimes$ cat $\mathcal{C}$ are oplax transfs. $\mathrm{Assoc} \underset{1}{\overset{\mathcal{C}}{\rightrightarrows}} \mathrm{Cat}$.)

Explicitly, this means:

	CS	WZW
$\bullet$	$\beta \otimes$ cat	$\otimes$ cat it acts on
$\bigcirc$	$\otimes$ cat	pointed cat it acts on
	pointed cat	pointed object therein.

To get a feel for this, let's look at "classical" CS, where $\mathcal{C} = \mathrm{Rep}(G)$ (for G reductive$/\mathbb{C}$). Then in $\dim \le 2$, the category in all cases is $\mathrm{Qcoh}(\mathrm{Loc}_G(N))$ for manifold N, thought of as an $E_{2-\dim N}$ cat.

Instead of thinking of it this way, we can think of it as *the space* of local systems, aka the character stack (or variety...).

Classical fact: if $\dim \Sigma = 2$, $\partial\Sigma = \emptyset$, then $\mathrm{Loc}_G \Sigma$ is symplectic. Pf: At a flat connection A on bundle P, the derived tangent bundle is

$$T_A \mathrm{Loc}_G \Sigma = \underset{[-1]}{\Omega^0_\Sigma \otimes \mathrm{ad}_P} \xrightarrow{d_A} \underset{[0]}{\Omega^1_\Sigma \otimes \mathrm{ad}_P} \xrightarrow{d_A} \underset{[1]}{\Omega^2_\Sigma \otimes \mathrm{ad}}.$$

You can wedge in Ω and integrate, and use the metric on $\mathfrak{g}$ to pair in $\mathrm{ad}_P \cong \mathfrak{g}$.

I claim that any conformal structure on Σ

picks out a Lagrangian in $\mathrm{Loc}_G \Sigma$ consisting of pairs (a G local system, w/ a holomorphic section). This is the stackification of the groupoid $\dfrac{\{\text{holomorphic maps to } G\}}{\{\text{constant maps to } G\}}$.
I will call this $WZW^\chi_G(\Sigma)$.

Forgetting the section gives the map $WZW^\chi_G \to \mathrm{Loc}_G$.
What is the tangent space to $WZW^\chi_G(\Sigma)$? We should take the above groupoid and work it out. We get (near triv. connection):

$$\begin{tikzcd}[row sep=small]
{[-1]} & {[0]} & {[1]} & \\
\Omega^0\otimes\mathfrak{g} \arrow[r, "d"] \arrow[dr, "\mathrm{id}"'] & \Omega^1\otimes\mathfrak{g} \arrow[r, "d"] \arrow[dr, "\pi^{0,1}"] & \Omega^2\otimes\mathfrak{g} & \text{CS complex} \\
& \Omega^0\otimes\mathfrak{g} \arrow[r, "\bar\partial"] & \Omega^{0,1}\otimes\mathfrak{g} & \text{Dolbeault complex}
\end{tikzcd}$$

Sure enough, if you pull back the symplectic form, you get something exact.

Note: there's a cancelation along diags, so $T_0 WZW \simeq \Omega^{1,0}\otimes\mathfrak{g} \xrightarrow{\bar\partial} \Omega^{1,1}\otimes\mathfrak{g}$. (Everyone should really be twisted by connection A.) Then the map $WZW^\chi \hookrightarrow CS$ is inclusion $\Omega^{1,0} \xrightarrow{\bar\partial} \Omega^{1,1} \hookrightarrow \Omega^\cdot[1]$.

In any case, there's also an antichiral part with ∂ in place of $\bar\partial$. What is

$$WZW^\chi(\Sigma) \underset{CS(\Sigma)}{\times} WZW^{-\chi}(\Sigma)$$

the intersection of the two Lagrangians? We can compute its tangent bundle:

$$\begin{tikzcd}
{[-1]} & {[0]} & {[1]} \\
 & \Omega^0 \arrow[r, "\partial"] & \Omega^{1,0} \\
\Omega^0 \arrow[ur, "id"] \arrow[r, "d"] \arrow[dr, "id"'] & \Omega^1 \arrow[r, "d"] \arrow[ur, "\pi^{1,0}"'] \arrow[dr, "\pi^{0,1}"] & \Omega^2 \\
 & \Omega^0 \arrow[r, "\bar\partial"] & \Omega^{0,1}
\end{tikzcd}$$

Cancel the Ω^0 with ~~its~~ its (diagonal) image in $\Omega^0 \oplus \Omega^0$, and cancel Ω^1 w/ $\Omega^{1,0} \oplus \Omega^{0,1}$! You get

$$\begin{tikzcd}
\Omega^0 \arrow[r, "?"] & \Omega^2 \\
{[0]} & {[1]}
\end{tikzcd}$$

and chasing the differential gives $\Delta = \partial\bar\partial + \bar\partial\partial$. So this is "full WZW", i.e. harmonic maps.

If you have, in fact, the curving coming from G, you get actually not harmonic but the adjustment by the WZW term, I think.

That's (classically) what's going on at Σ w/ $\partial\Sigma = \emptyset$. What about in lower dim?

$$WZW^{\times}(pt) \simeq G[\![z]\!]/G$$ I.e. really it's germs of holomorphic maps mod constants. And $WZW^{\times}(S^1)$ is

$= \cancel{(LG)/G}$. something roughly the

$= G[z, z^{-1}]/G$ loop group.

Then I guess the claim is that

$$\frac{G[\![z]\!]}{G} \longrightarrow BG$$

is a Lagrangian for the (−2)-shifted symp structure on BG?

Anyway, that's as far as I understand things, so I'll stop here.