E_2, Gerstenhaber, BV, Formality

Student TQFT + String Topology

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①

Operads:

V = sym $\otimes$ cat.

An operad $\mathcal{O}$ in V consists of:

(1) an assignment $S \mapsto \mathcal{O}(S)$ to each finite set, functorial in bijections, i.e. $\mathcal{O}$: Gpoid of finite sets $\to V$

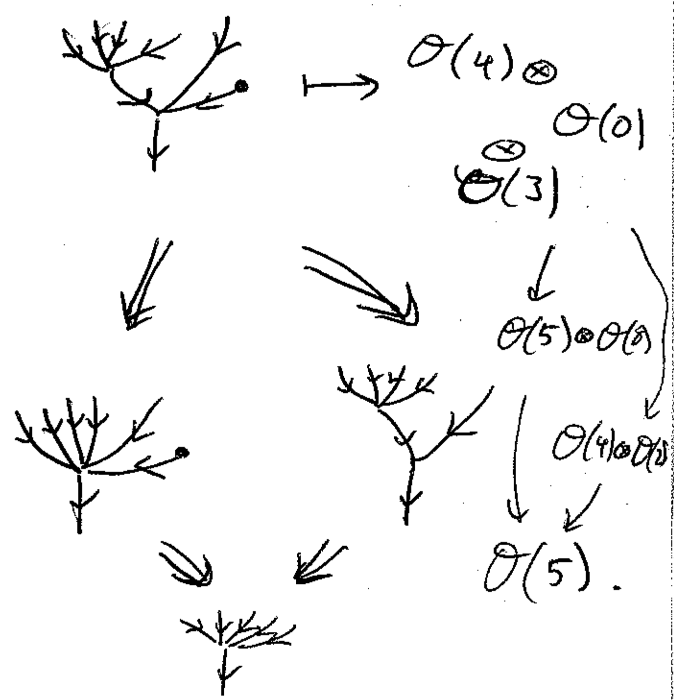
identify S = 

rooted vertex w/ leaves = S.

Think of $\mathcal{O}(S)$ = set/space of lables for an S-valent vertex.

$\Rightarrow$ ~~(2)~~ $\mathcal{O}$ extends to a functor

{labled rooted trees} $\to V$



(2) an assignment contraction of trees $\to$ maps in V

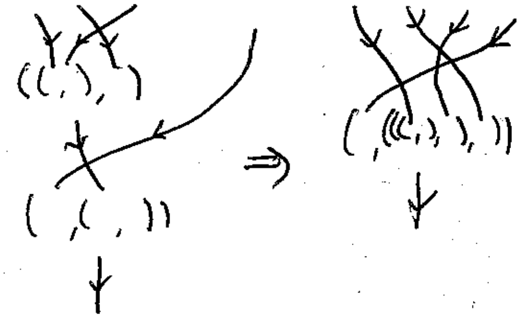
functorial in contractions

$\hookrightarrow$ associativity.

Examples:

(V = Set)

- $\mathcal{C}om(S) = \{*\}$.
- Paren(S) = {parenthesized permutations of S}



- If $\mathcal{C}$ is sym $\otimes$ (+V-enriched),

$\mathcal{H}om(X,Y)(S) = \hom(X^{\otimes S} \to Y)$

Main examples: $\mathcal{E}_2$ + $\tilde{\mathcal{E}}_2$

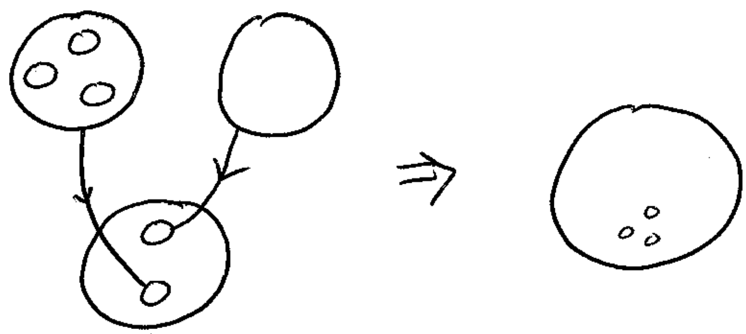
V = spaces.

$\{(x,y)\in\mathbb{R}^2 \text{ s.t. } x^2+y^2\leq 1\}$

$\mathcal{E}_2(S) = \{$  $\}$

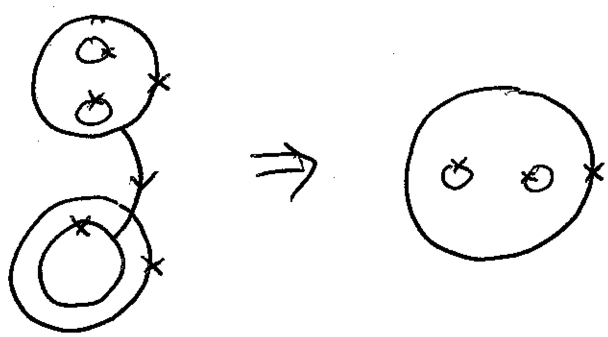
nonintersecting closed circular disks in bijection with S.

Gluing:



$\tilde{\mathcal{E}}_2(S) = \mathcal{E}_2(S)\times(S^1)^{\times S}$

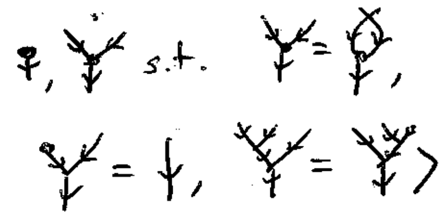
a marking on the perimeter of each little disk, Big disk marked at $(1,0)$.



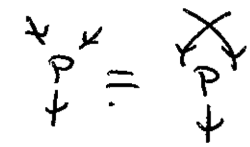
Goal: study $\mathcal{E}_2$, $\tilde{\mathcal{E}}_2$. Note: $\mathcal{E}_2 \to \tilde{\mathcal{E}}_2$ by $\circ \mapsto \circ$ is map of operads.

<u>Calculate homology (Q-coefs)</u>: $H_\bullet \mathcal{E}_2$ is an operad in GVECT.

$\mathcal{E}_2(S)$ is connected $\Rightarrow$ ~~$H_0(\mathcal{E}_2)$~~ $H_0 \mathcal{E}_2(S) = \mathbb{Q}$

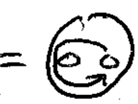
$\Rightarrow H_0 \mathcal{E}_2 = \mathbb{Q} \circ \mathrm{Com} = \langle$  $\rangle$

$\mathcal{E}_2(2) \simeq S^1$ ~~$H_\bullet \mathcal{E}_2(2$~~ $H_\bullet \mathcal{E}_2(2) = \mathbb{Q} \oplus [1]\mathbb{Q}$

basis: commutative binary multiplication; new binary operation "p" = 

$\mathcal{E}_2(3) \simeq S^1 \times \infty$.

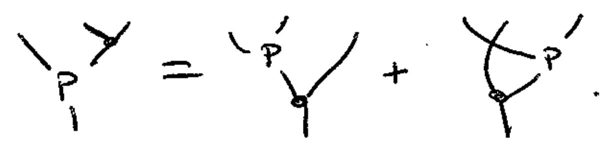
$$H_\bullet \mathcal{E}_2(3) = \mathbb{Q} \oplus [1]\mathbb{Q}^{\oplus 3} \oplus [2]\mathbb{Q}^{\oplus 2}$$

~~$\mathcal{E}$ plant~~ = .

$\Rightarrow$ there must be quadratic relations between m, p.

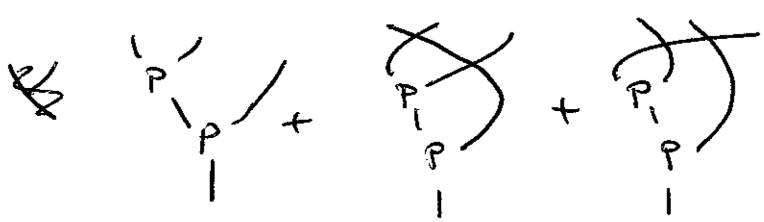
in grading 1:

i.e. $P(a, bc) = P(a,b)c + P(a,c)b$



in grading 2: some relation between $P(P(a,b),c)$ and cyclic permutations.

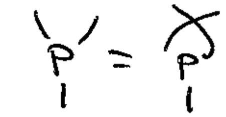
by sym, it must be

 $= 0$.

Defn:

The <u>Gerstenhaber operad</u> $\mathcal{G}_\bullet$ in GVECT is:

$$\mathcal{G}_\bullet = \langle \underbrace{\bullet,\ \curlyvee}_{\deg = 0},\ \underbrace{P}_{\deg = 1} \quad \text{s.t.} \quad \bullet, \curlyvee \text{ make a Com},$$

symmetry:  (!)

Leibniz: $P(a, bc) = \dots$

Jacobi: $\dots \quad \rangle$.

Thm [Getzler]

The map $\mathcal{G}_\bullet \longrightarrow H_\bullet \mathcal{E}_2$ of operads in GVECT is an iso.

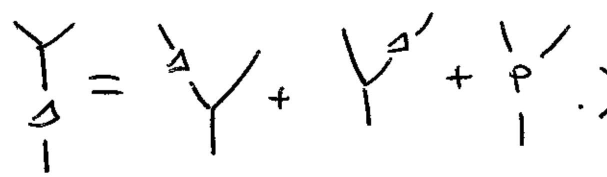
$$H_\bullet \tilde{\mathcal{E}}_2(s) = H_\bullet \mathcal{E}_2(s) \otimes (\mathbb{Q} \oplus [1]\mathbb{Q})^{\otimes s}.$$

$$H_1 \mathcal{E}_2(1) = \mathbb{Q} \qquad \text{basis} = \Delta =$$

It satisfies $\Delta(ab) = \Delta(a)\, b + a\, \Delta(b) + P(a,b)$.

Defn: The <u>Batalin-Vilkovisky operad</u> is

$$\tilde{\mathcal{G}}_\bullet = \langle \underbrace{\bullet\ \curlyvee}_{\deg\ 0},\ \underbrace{P \quad \Delta}_{\deg\ 1} \quad \text{s.t. (relations for } \mathcal{G}) \text{ and}$$

 $\rangle$

Cor [Getzler]: $H_\bullet \tilde{\mathcal{E}}_2 = \tilde{\mathcal{G}}_\bullet$ as operads in GVECT.

<u>Example:</u>

X = a manifold. ~~Define~~

$$\mathcal{C}^\infty([1]T^*X) = \text{gca w/ } (-k)^{th} \text{ part } = \Gamma(\Lambda^k T X).$$

pick local coords x^i on X. ~~Be~~ $\mathcal{C}^\infty([1]T^*X) \underset{\text{locally}}{\cong} \mathcal{C}^\infty(X) \otimes \Lambda^\bullet(p_i)$

$p_i = \text{"}\frac{\partial}{\partial x^i}\text{"} \in \Gamma(TX)$

define derivations $\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial p_i}$
"contract w/ dx^i".

<u>Lemma:</u> $P = \sum_i \left(\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial p_i} + \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial x^i} \right)$

is indep of choice of coords, and defines $\mathcal{I}$ str on $\mathcal{C}^\infty([1]T^*X)$.

Pick measure μ, and local coords x^i s.t. $\mu = dx^1 \dots dx^n$.

$\bigstar$ <u>Lemma:</u> $\Delta = \sum_i \frac{\partial^2}{\partial x^i \partial p_i}$ depends on μ

but not on choice of compatible coords, and defines $\tilde{\mathcal{I}}$ str on $(\mathcal{C}^\infty([1]T^*X), P)$.

<u>Better</u>: study operads in DGVECT.

$$C_\bullet \mathcal{E}_2 = \text{operad of singular chains in } \mathcal{E}_2 .$$

~~Defn: We find a~~

Spaces = ∞ gpoids. <u>Thm</u>: $\mathcal{E}_2$ is ~~a op~~ $K(G,1)$.

we find an equiv 1-gpoid:

Paren$\mathbb{B}$ = operad of gpoids:

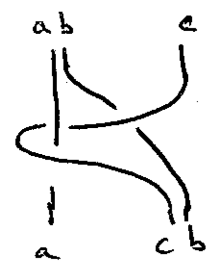
- Objects of Paren$\mathbb{B}(S)$ = Paren(S) = ~~$((,(,)),)$~~
- morphisms:

  some parenthesized permutation of S

  $\boxed{\text{a braid}}$

  some per- of S

  the braid should cover the permutation = each strand has same labels on top + bottm.

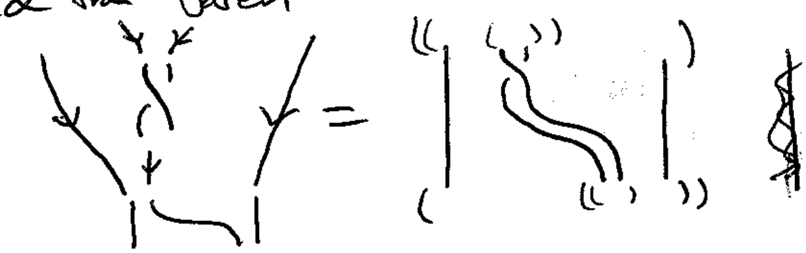
e.g. 

is a morphism $((ab)c) \downarrow (a(cb))$.

so Paren$\mathbb{B}(S) \simeq$ Pure braid gp on $|S|$ strands

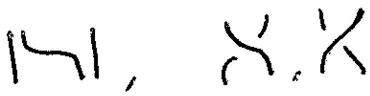
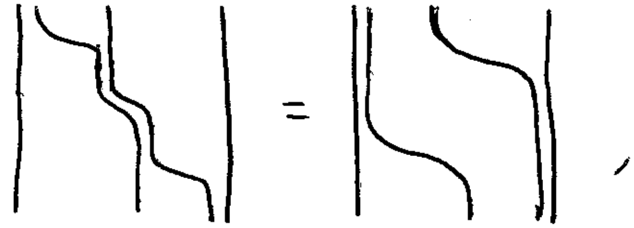
$\simeq$: equiv of gpoids; gp: one-object gpoid.

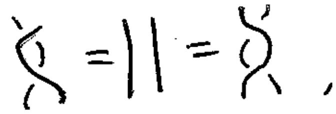
- operadic str: induced from Paren and cabling:

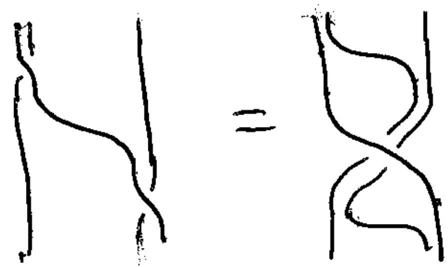


Thm [Mac Lane]

ParenB has presentation:

$\langle$  s.t.  ,

 ,

 , ditto w/  . $\rangle$.

Defn:
$C_\bullet$(a ~~gpoid~~ category) : an n-symplex = a string of n composable morphisms.

Lemma: $\exists$ $C_\circ X$ s.t. $C_\circ \mathcal{E}_2 \xleftarrow{\sim} C_\circ X \xrightarrow{\sim} C_\circ$ ParenB.

"Pf: $\mathcal{E}_2 \simeq$ ParenB is equiv of operads of ∞-gpoids."

Defn:
Given $\mathbb{Q}$-linear category $\mathcal{A}$ with linear functor $\varepsilon : \mathcal{A} \to \mathbb{Q}$ ← one-object cat ,

~~Def~~ $C_\bullet\mathcal{A}$ is chain complex w/

$$C_n\mathcal{A} = \{\text{composable } n\text{-tensors}\}$$

$$\begin{aligned}\partial(a_0 \otimes a_1 \otimes \dots \otimes a_n) = {} & \varepsilon(a_0)\, a_1 \otimes a_2 \otimes \dots \otimes a_n \\ & - a_0 a_1 \otimes a_2 \otimes \dots \otimes a_n \\ & + \dots + (-1)^n a_1 \otimes \dots \otimes a_{n-2} \otimes a_{n-1} a_n \\ & + (-1)^{n+1} a_1 \otimes \dots \otimes a_{n-1}\, \varepsilon(a_n).\end{aligned}$$

Then $C_\circ \mathcal{C} = C_\circ \mathbb{Q}\mathcal{C}$ for $\mathcal{C}$ a category ([illegible]).

So suffices to study $\mathcal{A} = \mathbb{Q}\cdot\mathrm{Paren}\mathcal{B} \doteqdot \cancel{\mathbb{Q}P\mathcal{B}}$

Would be equiv to $\mathbb{Q}\cdot\mathcal{C}om$ if $\overcrossing = \undercrossing$ — there's an ideal I spanned by $\overcrossing - \undercrossing$.

Get filtration

$$\mathcal{A} = \mathbb{Q}\cdot\mathrm{Paren}\mathcal{B} = I^0 \supseteq I^1 \supseteq I^2 \supseteq \dots$$

Study associated graded:

$$\operatorname{gr}\mathcal{A} = \frac{\mathcal{A}}{I} \oplus \frac{I}{I^2} \oplus \frac{I^2}{I^3} \oplus \dots$$

General question for any filtered algebraic gadget:

Does there exist homomorphism $\mathcal{A} \xrightarrow{f} \operatorname{gr}\mathcal{A}$ with $\cancel{\operatorname{gr} f}\ \{\operatorname{gr}\mathcal{A} \xrightarrow{\operatorname{gr} f} \operatorname{gr}\operatorname{gr}\mathcal{A} = \operatorname{gr}\mathcal{A}\} = id$?

~~General~~ Answers:

- for vector spaces: yes, but not canonical.
- in general: no, e.g. $U\mathfrak{g}$ vs. $\mathrm{Sym}\,\mathfrak{g}$ as algebras.
- sometimes yes: $U\mathfrak{g}$ vs. $\mathrm{Sym}\,\mathfrak{g}$ as $\mathfrak{g}$-~~modules~~ as coalgebras.
- In our case no !!.

But almost:

Define $\hat{\mathcal{A}}$ = I-adic completion of $\mathcal{A} = \varprojlim_n I^n$,

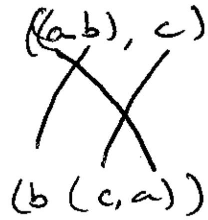
$$\operatorname{gr}\hat{\mathcal{A}} = \text{ditto} = \prod_{n=0}^{\infty} \frac{I^n}{I^{n+1}}.$$

let's compute gr $\mathcal{A}$:

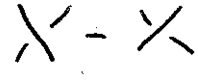
is a $\mathbb{Q}$-linear cat w/ objects = Paren.

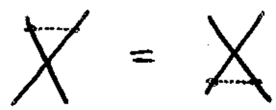
$gr\,\mathcal{A}_0 = \frac{\mathcal{A}}{I} \simeq \mathbb{Q}\cdot Com$: ~~two objs~~

~~think of basis~~ e.g.

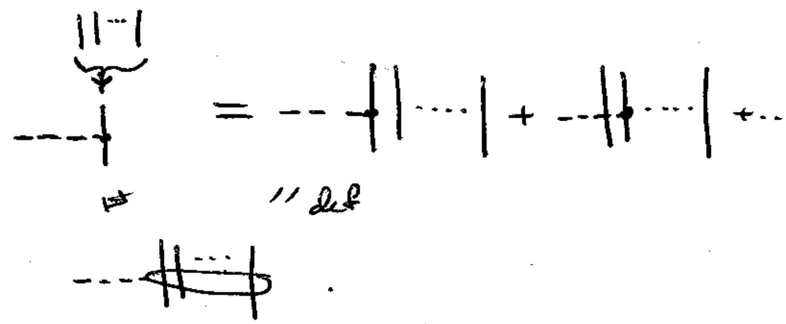


one morphism between any two objects.

~~gr A~~ in I/I^2 is image of  ;

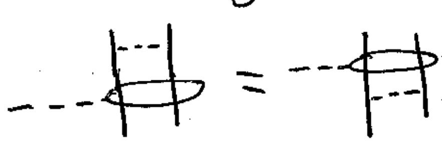
draw it as  .

~~Relations~~

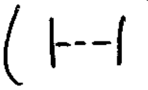
Operadic structure: 

Relations:

~~Distant things commute.~~

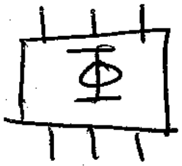
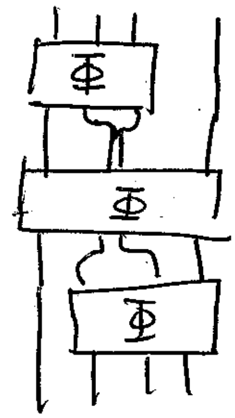
-  , what you expect for diagrams.

"braid relation"

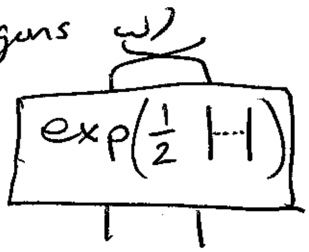
Counit: $\varepsilon($  $) = 1$ (no bridges) $\varepsilon($  $) = 0$.

~~Thm~~

<u>Defn</u>: A <u>Drinfel'd associator</u> is

 $\in \mathrm{gr}\,\hat{\mathcal{A}}(3)$ s.t.:  =

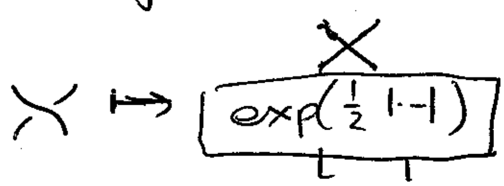
+ hexagons w/



<u>Thm: [Drinfel'd]</u>:

These exist. <u>Pf</u>: Solve Khizhnik-Zamolodchikov for $\mathbb{C}$ version.

<u>Cor</u>: $\exists$ iso $\hat{\mathcal{A}} \to \mathrm{gr}\,\hat{\mathcal{A}}$ of operads of $\mathbb{Q}$-linear categories:

 $\mapsto \Phi$, .

<u>Note</u>: $C_\bullet \mathcal{A} \simeq C_\bullet \mathcal{A} \hookrightarrow \hat{\mathcal{A}}$ induces

$C_\bullet \mathcal{A} \xrightarrow{\sim} C_\bullet \hat{\mathcal{A}}$ because H_n is ~~totally~~ finite-dim $\forall n$.

So we have

$$C_\bullet \mathcal{E}_2 \xleftarrow{\sim} C_\bullet X \xrightarrow{\sim} C_\bullet \mathrm{Paren}\mathcal{B} = C_\bullet \mathcal{A} \xrightarrow{\sim} C_\bullet \hat{\mathcal{A}} \underset{\Phi}{\cong} C_\bullet \mathrm{gr}\,\hat{\mathcal{A}} \xleftarrow{\sim} C_\bullet \mathrm{gr}\,\mathcal{A}.$$

Now, since $\mathrm{gr}\, \mathcal{A}_0 = \mathbb{Q}\cdot \mathrm{Com}$, there is <u>canonical</u> skeletalization

$$\mathrm{gr}\, \mathcal{A}_\bullet \xrightarrow{\sim} \text{an } \cancel{\text{alg}} \text{ operad of algebras}$$

$$\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\text{one-object cats}}$$

$$\overset{\shortparallel}{\cancel{\mathrm{End}}_{\mathcal{A}}}(\,|\,|\cdots|\,)\,.\ = A\,.$$

Then $C_n A = A^{\otimes n}$, and $C_\bullet$ computes $\mathrm{Tor}_A(\mathbb{Q},\mathbb{Q})$.

By inspecting defn, Hopf algebra A ~~≅~~ $= \mathcal{U}t$

for $t =$ ~~Lie~~ operad of Lie algebras $(\otimes = \oplus)$

~~with~~ ~~with:~~ ~~generato~~

$$t = \Big\langle\ \vdash\!\!-\!-\!\dashv \ \text{s.t.}\ \ [\underset{a\quad b}{\vdash\!\!-\!-\!\dashv}, \underset{c\quad d}{\vdash\!\!-\!-\!\dashv}] = 0 \ \text{ if } a,b,c,d \text{ all distinct}$$

$$[\underset{a\quad b}{\vdash\!\!-\!-\!\dashv}, \underset{b\quad c}{\vdash\!\!-\!-\!\dashv}] = [\underset{b\quad c}{\vdash\!\!-\!-\!\dashv}, \underset{c\quad a}{\vdash\!\!-\!-\!\dashv}] \Big\rangle.$$

Then ~~can com~~ $\mathcal{U}t$-mod $= t$-mod, so can compute $\mathrm{Tor}_A(\mathbb{Q},\mathbb{Q})$ by CE complex $C_\bullet t = \Lambda^\bullet t$.

And $t^{\wedge n} \hookrightarrow t^{\otimes n} \hookrightarrow (\mathcal{U}t)^{\otimes n}$ gives quasiiso $C_\bullet t \xrightarrow{\sim} C_\bullet \mathcal{U}t$.

<u>Finally</u>: $\mathcal{Z}_\bullet \xrightarrow{\sim} \Lambda^\bullet t$ by explicit map:

$$\underset{\mathcal{Z}_0}{\overset{\cap}{\text{multiplication}}} \longmapsto 1 \in \Lambda^0 t = \mathbb{Q}; \qquad \cancel{P(a,b)}\ P \in \mathcal{Z}_1(2) \mapsto\ \vdash\!\!-\!\dashv\ \in \Lambda^1 t(2)$$

$$\cancel{P(a,b)}\,t. \qquad\qquad = t(2).$$

Final Remarks:

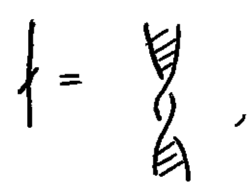
- Kontsevich used different argument, w/ periods of $\int$s; works over $\mathbb{R}$.

  Is equiv to this one if

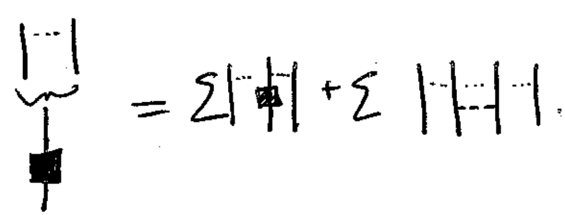
  Φ = Alekseev–Torossian associator.

- ~~I don't know a~~
- Kontsevich's argument handles all E_n; I don't think a similar algebraic argument exists for $n \geq 3$.

- Framed version is easy:

  use ribbon braids ,

  $\tilde{t} = t \oplus$  w/ relations:  commutes w/ everything,

I'll leave details of framed version as an exercise.