Quantum Homotopy Groups

Thematic Program in Field Theory and Topology

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Theo Johnson-Freyd, Dalhousie & Perimeter

Based on work in progress with David Reutter

these slides: categorified.net/NotreDame.pdf

Reconstructing Target Space

Recall:

π-finite space $X \rightsquigarrow$ $n+1$D sigma model $\sigma(X)$

$$M^{n+1} \xmapsto{\sigma(X)} |\mathrm{maps}(M, X)| = \sum_{\pi_0 \mathrm{maps}} \widehat{\prod_{i=1}} |\pi_i \mathrm{maps}|^{(-1)^i}.$$

Question:

How much about X can you recover from $\sigma(X)$?

Answer:

Not much. For example, electromagnetic duality: if X is an infinite loop space, then

$$\sigma(X) \cong \sigma(\Sigma^{n+1} I_{\mathbb{Z}} X).$$

So cannot even recover $\pi_* X$, not even $\pi_0 X$!

Canonical boundary conditions

The σ-model construction selects a boundary condition, called Neumann: The fields are unconstrained at ∂.

$$(M^{n+1}, \partial M) \xmapsto{(\sigma, N)} \left| \mathrm{maps}(M, X) \right| .$$

Any basepoint $x \in X$ selects a Dirichlet b.c., in which the fields have specified value $= x$ along ∂.

$$(M^{n+1}, \partial M) \xmapsto{(\sigma, D)} \left| \mathrm{maps}_*((M, \partial M), (X, x)) \right|$$

Electromagnetic duality exchanges Neum and Dir:

$$(\sigma(X), N) \cong (\sigma(\Sigma^{n+1} I_{\mathbb{Z}} X), D))$$
$$\not\cong (\sigma(\Sigma^{n+1} I_{\mathbb{Z}} X), N)$$

Reconstructing target space, revisited

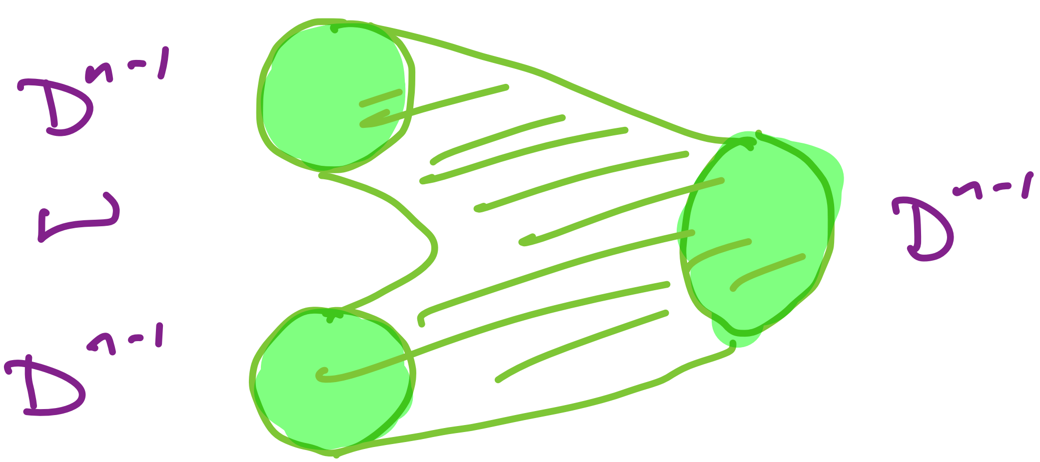
Revised Question:

How much about X can you recover from $\overbrace{(\sigma(X), N)}^{\mathcal{Q}}$?

Answer: A lot!

E.g.: Look at the 1-category $\mathcal{Q}(D^{n-1}, S^{n-2})$.

This is symmetric monoidal via the chaps.
↪ if $n \geq 4$ ↪ solid pants



Calculation:

As sym $\otimes$ cats,

$$\boxed{\mathcal{Q}(D^{n-1}) \cong \mathrm{Rep}_{\mathbb{K}}(\pi_{\leq 1} X).}$$

Tannakian Duality: Can recover $\mathcal{G}$ from $\mathrm{Rep}_{\mathbb{K}}(\mathcal{G})$.

Quantum fundamental groupoid

$n \geq 4$

Corollary: Let Q be any $n+1$D TQFT w/ nD b.c., at least once extended but not rec. fully extended, compact, etc. and framing is ok!

Then Q has a algebraic fundamental gpoid $\pi_{\leq 1} Q$, Tannakian-Dual to the sym $\otimes$ cat $Q(D^{n-1})$.

Often, depending on the coefs of Q, can prove that after tensoring w/ sVec $Q(D^{n-1})$ is Rep(some gpoid).

For other coefs, $Q(D^{n-1})$ defines $\pi_{\leq 1} Q$ the same way any commutative algebra defines some space (scheme).

Higher Homotopy groups

$\pi_i X$ is not a well-defined ab. gp. : you need to choose a basepoint. **Better**: remember this dependence by remembering $\pi_1(X,x) \circlearrowleft \pi_i(X,x)$.

Best: $\pi_i X \in \mathrm{Rep}_{\mathbb{Z}}(\pi_{\leq 1} X)$.

And we already know how who is $\mathrm{Rep}_{\mathbb{K}}(\pi_{\leq 1} X)$!

How to encode AbGp inside $\mathrm{Vec}_{\mathbb{K}}$?

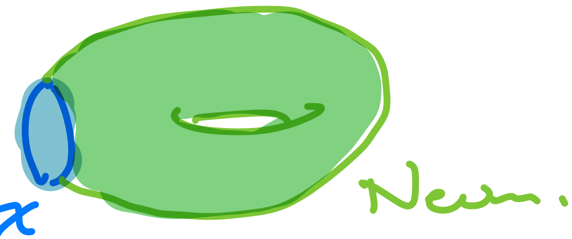
As commutative + cocommutative Hopf algebras!

Goal: Given a TQFT w/ b.c. Q, build Hopf algebras in $Q(D^{n-1})$. Recover $\pi_i X$ if $Q = (\sigma(X), N)$.

Higher homotopy groups

Strategy: In σ-model case, when you have both Neumann and Dirichlet-at-x boundary conditions,

$$\mathcal{O}(\pi_i(X,x)) = $$



$$= S^i \times D^{n-i}$$

with multiplication $= S^i \times$ chaps, comult $=$ copants $\times D^{n-i}$.

Theorem: Take a bite out of the donut to get a bordism-with-boundary

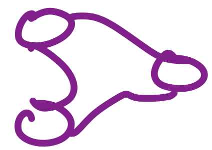
$$\emptyset \xrightarrow{S^i \times D^{n-i} \smallsetminus \text{bite}} D^{n-1}.$$

For any $n+1$D TQFT w/ b.c., this bordism selects a Hopf algebra object in $\mathcal{Q}(D^{n-1})$.

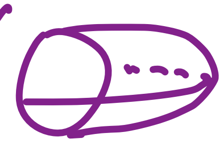
Proof outline

Suffice to prove in $\mathrm{Bord}^{\partial}_{n+1,n,n-1} \overset{\text{f.f.}}{\subset} \mathrm{Bord}^{\partial}_{n+1,\dots,0}$.

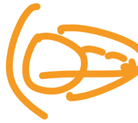
__Need to explain:__ multiplication, comult, and Hopf axiom.

__Question:__ Where does $\text{pants}^{k+1} =$  $: S^k \sqcup S^k \to S^k$ come from? Why is it coherently associative?

__Answer:__ $S^k = \underline{\mathrm{End}}(\text{hemisphere})$

$= \Big($  $\Big)^R \circ \Big($  $\Big)$

is a monad on $\emptyset$!

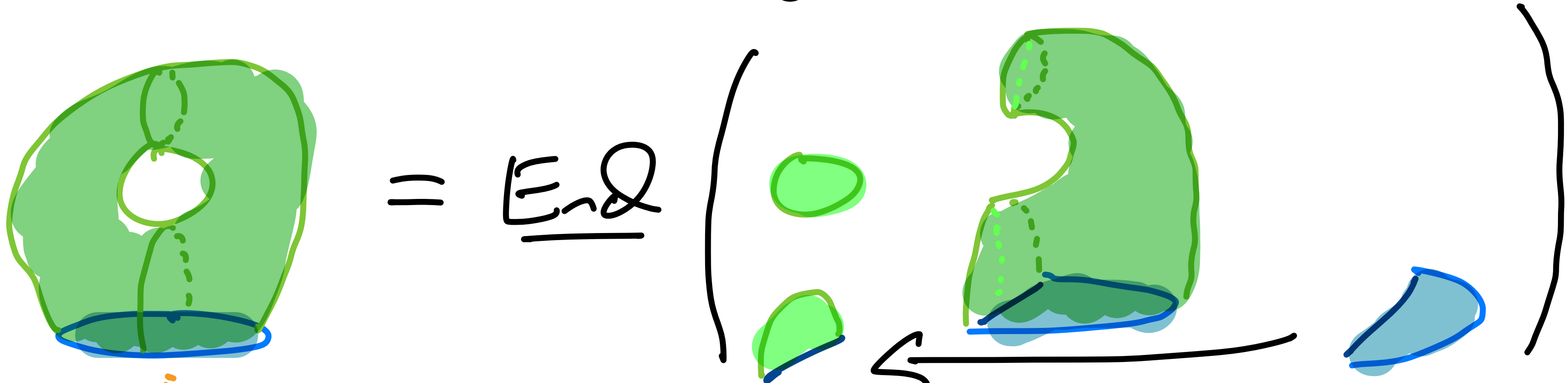
For copants, use $\Big($  $\Big)^L$.

Same thing for the chaps:

$$D^{k+1} = \underline{\mathrm{End}}(\text{hemidisk}).$$

Proof outline

Still works after removing a bite:



$S^i \times D^{n-i} \setminus \text{bite}$

$S^{i-1} \times D^{n-i} \setminus \text{bite}$

explains multiplication

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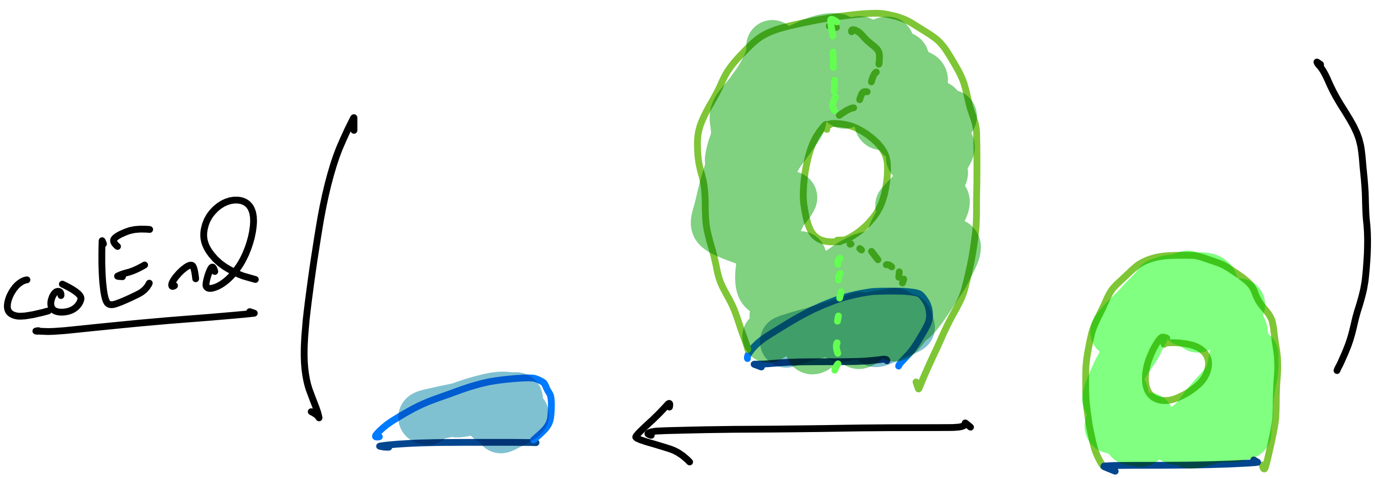


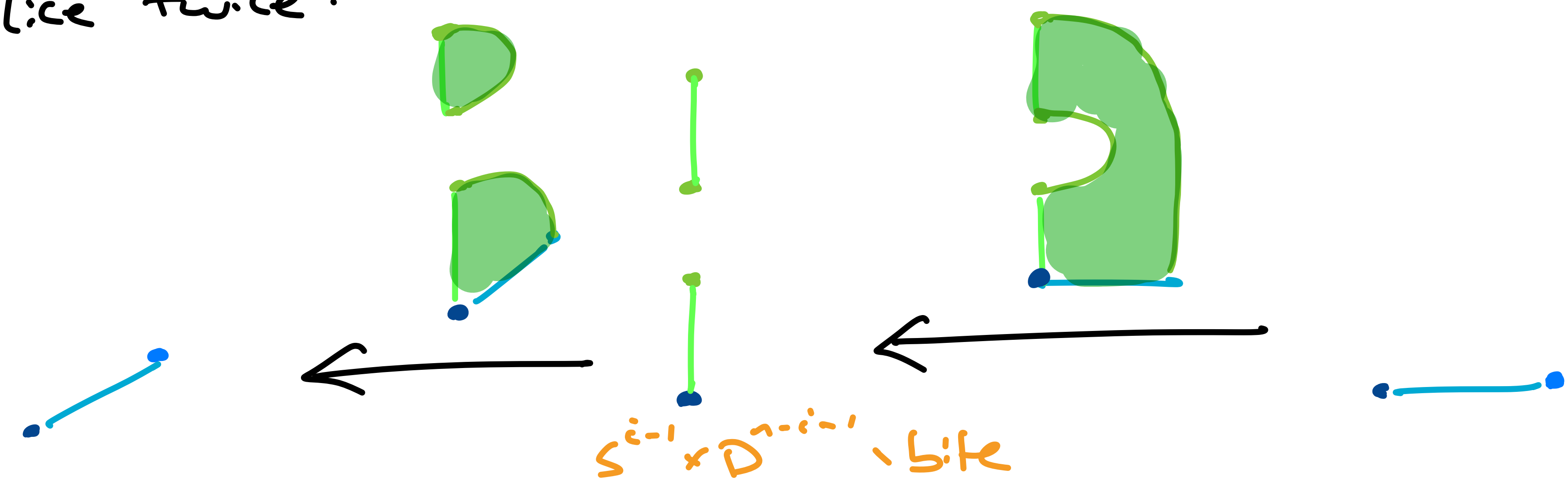
$S^i \times D^{n-i-1} \setminus \text{bite}$

explains comultiplication

what about bialg law

Proof outline

Slice twice:



This is a handle attachment followed by the cancelling handle attachment: it is a retract. That's the compatibility that leads to the bialg. axiom.

Theorem: Given any retract in any $(\infty, 3)$-cat with adjoints, you get not just a bialg but in fact a Hopf alg: it has an antipode!

Puppe sequence

Recall: the fibre F of a map $X \xrightarrow{f} Y$ is the local system of spaces $y \mapsto f^{-1}(y)$ on Y. If you want F to be a local system of pointed spaces (so to talk about $\pi_0 F$), you get instead a local system on X. **Upshot:** $\pi_k F$ makes sense as a functor $\pi_{\leq 1} X \xrightarrow{\pi_k F} \mathrm{AbGp}$.

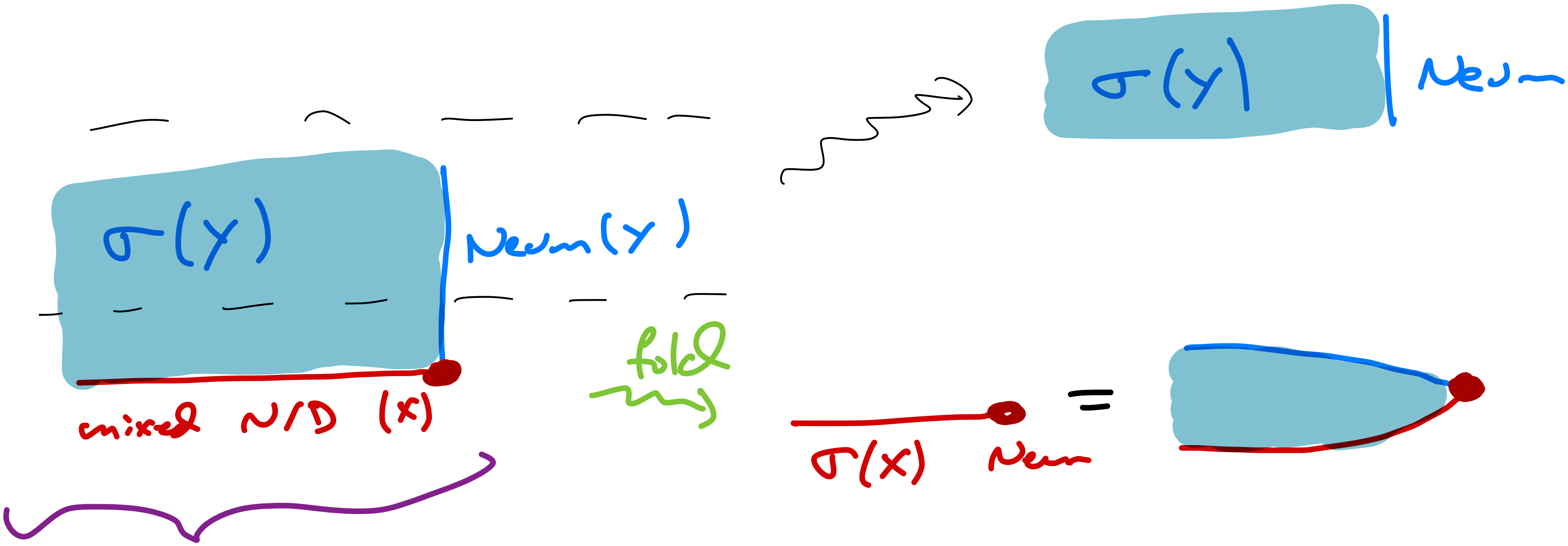
Also there is $\pi_{\leq 1} X \longrightarrow \pi_{\leq 1} Y \xrightarrow{\pi_k Y} \mathrm{AbGp}$.

Classical Puppe Sequence: There is a LES in $\mathrm{Rep}_{\mathbb{Z}}(\pi_{\leq 1} X)$ of the form

$$\cdots \longrightarrow \pi_k F \longrightarrow \pi_k X \longrightarrow \pi_k Y \longrightarrow \pi_{k-1} F \longrightarrow \cdots$$

Quantum fibre bundles

Can encode $X \xrightarrow{f} Y$ as a TQFT w/ two boundaries and a corner:

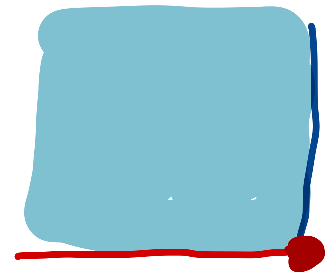


f equiv $F : Y \to \mathrm{Spaces}$.

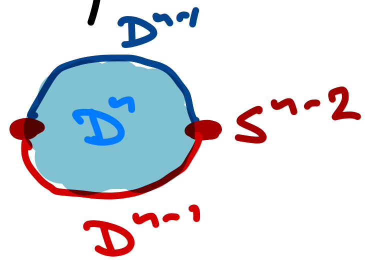
More generally, any bulk-boundary-boundary-corner TQFT is a "quantum fibre bundle".

Note: assymetric interpretation.

Quantum Puppe sequence



Theorem: For any $n+2$D TQFT with two boundaries and a corner, there is a long sequence of Hopf algebras in



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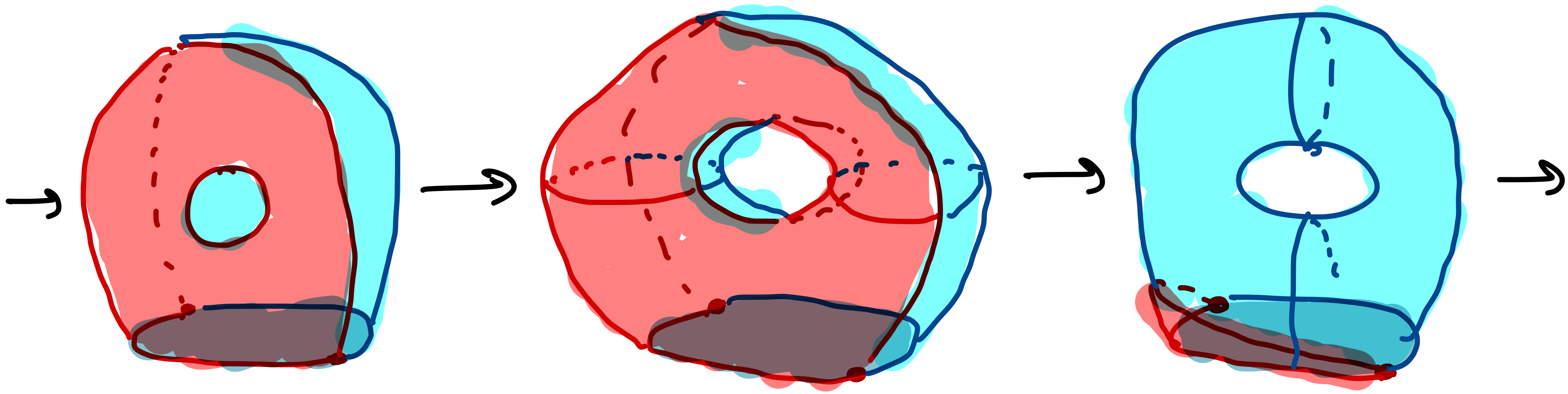

Moreover, this long sequence is often exact.

Exactness

Specifically, the quantum Puppe sequence is exact as soon as the Hopf algebras in it are all separable and coseparable. These conditions are automatic for extended TQFTs valued in 2Vec, but fail in Bord$^{@}$ and in target categories like Matrix Factorizations.

Behind the scenes: sep + cosep $\Longleftrightarrow$ antipode^2 is invertible.

If it is invertible, then you can write down an idemp which supplies a sort of quantum split exactness where the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & C \end{array}$$

is "Beck–Chevalley"...

THANKS!