[After Douglas – Reutter]
JF – Gaiotto

Semisimple higher categories

I owe yo n-category version

A s.s. $\cancel{1}$ n-cat / $\mathbb{K}$ is

- linear
- additive and Karoubi complete

very mild version of abelian
just insists on split kernels + cokernels

- every object should decompose as a $\oplus$ of simple objects.

$0 \neq X$ is simple if any nonzero map $Y \to X$ is a split surjection.

$\exists$ s.t. $\circlearrowleft = \mathrm{id}_X$

$Y \twoheadrightarrow X$

I owe you

In n-cat $\mathcal{C}$, $X \twoheadrightarrow Y$ is the data of:

$$\begin{tikzcd} X \arrow[d, bend right, "f"'] \\ Y \arrow[u, bend right, "g"'] \end{tikzcd} \quad \text{and} \quad \left[fg \Rightarrow id_Y \right] \in \underbrace{End_{\mathcal{C}}(Y)}_{(n-1)\text{-cat}}$$

$$\begin{tikzcd} X \arrow[loop below, "e"] \end{tikzcd}$$

$\mathcal{C}$ is Karoubi complete if every idempotent e factors through a split surjection. $e \cong gf$.

Thm: if such fact exists it's unique

An idempotent is everything gf is if you have a split surjection.

The "split surjection" m is associative.

m, Δ is Frobenius.

You might add to your definition of "ss n-cat":
all morphisms are fully adjunctible.

$X \xrightarrow{f} Y$

In 1-cat, Y is simple
iff $\mathrm{End}(Y) = \mathbb{K}$ $\quad \mathbb{K} = \overline{\mathbb{K}}$.

$\mathcal{C}$, $Y \in \mathcal{C}$, based loop space $\Omega\mathcal{C} := \mathrm{End}_{\mathcal{C}}(Y)$, id_Y

$\Omega^n_Y \mathcal{C}$ is a E_n ring if $\mathcal{C}$ is an n-cat.

Suppose $\Omega^n_Y \mathcal{C} = \mathbb{K}$, and have lots of adjoints.
If lots of dualizability,
start building $X \xrightarrow{\partial} Y$,

...

Last step: win if $\Omega^{n-1}_Y \mathcal{C}$ a 1-cat.
is s.s.

Summary: lots of dualizability + local semisimplicity + $\oplus$ and Idem completn. $\Rightarrow$ global semisimplicity

$\underbrace{\phantom{+ \oplus \text{ and Idem completn.}}}_{\Leftrightarrow \text{ Cauchy completion.}}$

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s.s. cat $\Leftrightarrow$ Rep(compact gp).

finite s.s. cat $\Longleftrightarrow$ Rep(finite gp)

$\mathcal{C}$ is finite s.s. if

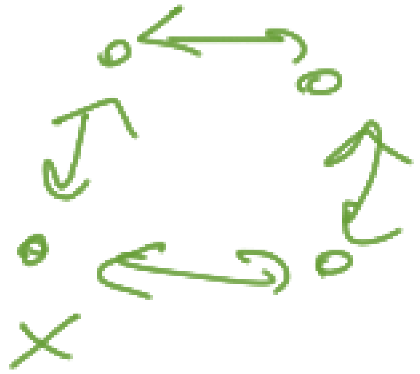
$\uparrow$
n-cat

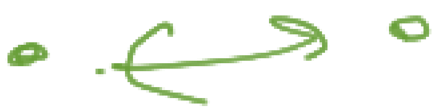
- $\oplus$, Ker completn
- locally finite s.s.
- ~~finitely many simples up to iso~~
- finitely many components.

It is very common for two non-iso simples to be related by a non-zero map.

↑ this is an equiv relation.

Thm (DR):





"components" $\pi_0 \mathcal{C}$

If $\mathcal{C} \ni X$ finite s.s. n-cat, $A = \Omega_X \mathcal{C}$ $\otimes$ $n-1$ cat.

"multifusion $n-1$ cat."

$A \otimes A \xrightarrow{m} A$ $\quad$ [m has adjoint among A-bilinear maps.] $\quad$ it must not $\Rightarrow$.

Conj: In characteristic zero, multifusion n-cats are separable.