Bott periodicity via quantum Hamiltonian reduction

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The famous Morita equivalence $\mathrm{Cliff}(8) \simeq \mathbb{R}$ appears in many contexts, most notably as a manifestation of the eight-fold Bott periodicity of KO. It can be explained in many ways. The goal of this talk is to give yet another explanation, this time in terms of (super) symplectic geometry.

Clifford algebras have a natural (super) symplectic interpretation. Let $\mathbb{R}^{0|n}$ denote the "odd" manifold with coordinate ring $\mathcal{C}^\infty(\mathbb{R}^{0|n}) = \bigwedge^\bullet \mathbb{R}^n = \mathbb{R}[x_1, \dots, x_n]$, where the coordinate functions are Grassmann variables, so that $x_i x_j = -x_j x_i$ and $x_i^2 = 0$. Odd manifolds admit a calculus of differential forms fully analogous to the even case with one notable exception: since x_i is odd, the one-form $\mathrm{d}x_i$ is even, and so $\mathrm{d}x_i \wedge \mathrm{d}x_j = \mathrm{d}x_j \wedge \mathrm{d}x_i$ with no sign, and $\mathrm{d}x_i^{\wedge 2} \neq 0$ (as it has no reason to vanish). In particular, $\mathbb{R}^{0|n}$ admits a *positive definite symplectic form* $\omega = \frac{1}{2}\sum_i \mathrm{d}x_i^{\wedge 2}$. Since $\mathbb{R}^{0|n}$ is a vector space and ω translation-invariant, $(\mathbb{R}^{0|n}, \omega)$ admits a *canonical quantization* to its Weyl algebra, which in this case is nothing but the *Clifford algebra* $\mathrm{Cliff}(n) = \mathbb{R}\langle x_1, \dots, x_n\rangle / (x_i x_j + x_j x_i = 2\delta_{ij})$.

Linear symplectic geometry can explain Morita equivalences. Suppose that (M, ω) is a symplectic vector space with Weyl algebra $\mathrm{Weyl}(M) = \bigoplus (M^*)^{\otimes n} / ([x,y] = \{x, y\}, x, y \in M^*)$. Given a linear Lagrangian $L \subseteq M$ cut out by linear equations $L^\perp \subseteq M^*$, the corresponding *left Fock module* is $\mathrm{Fock}(L) = \mathrm{Weyl}(M)/L^\perp$. By construction, the commutant of $\mathrm{Weyl}(M)$ in $\mathrm{End}(L)$ is $\mathbb{R}$, and so up to issues of functional analysis that are absent in the purely-odd case, $\mathrm{Fock}(L)$ is a Morita trivialization of $\mathrm{Weyl}(M)$. This in particular "explains" the two-fold Bott periodicty of KU. Indeed, the complex Clifford algebra $\mathbb{C}\mathrm{liff}(2) = \mathrm{Cliff}(2) \otimes \mathbb{C}$ is the canonical quantization of the holomorphic symplectic manifold $\mathbb{C}^{0|2}$ with symplectic form $\frac{1}{2}(\mathrm{d}x^{\wedge 2} + \mathrm{d}y^{\wedge 2})$, which admits a holomorphic linear Lagrangian spanned by the lightlike vector $x + iy$. Linear symplectic geometry does not, however, explain any nontrivial Morita equivalences of real Clifford algebras, because the positive-definiteness of $\frac{1}{2}\sum_i \mathrm{d}x_i^{\wedge 2}$ prevents $\mathbb{R}^{0|n}$ from admitting Lagrangian sub-supermanifolds.

A *Hamiltonian action* of a connected and simply connected Lie group G on a symplectic manifold M is determined by a *comoment map* $\mu : \mathrm{Lie}(G) \to \mathcal{C}^\infty(M)$, considered as a Lie algebra under the Poisson bracket. The corresponding action is infinitesimally generated by the Hamiltonian vector fields $\{\mu(g), -\}, g \in \mathrm{Lie}(G)$. The *Hamiltonian reduction* $M//G$ is the space $\mu^{-1}(0)/G$ with coordinate ring $\big(\mathcal{C}^\infty(M)/(\mu(\mathrm{Lie}(G)))\big)^G$. Assuming the action of G on M is not too singular, $M//G$ is again a symplectic manifold and $\mu^{-1}(0)$ is a Lagrangian correspondence between M and $M//G$. The story of Hamiltonian reduction can be quantized. A *quantum Hamiltonian action* of G on an associative algebra A is determined by a map $\mu : \mathrm{Lie}(G) \to A$, considered as a Lie algebra under the commutator bracket; the corresponding action is infinitesimally generated by $[\mu(g), -], g \in \mathrm{Lie}(G)$. The *quantum Hamiltonian reduction* $A//G$ is the ring $\big(A/(\mu(\mathrm{Lie}(G)))\big)^G = \mathrm{End}_A\big(A/(\mu(\mathrm{Lie}(G)))\big)$, where $(\mu(\mathrm{Lie}(G)))$ now denotes the left ideal generated by the image of μ. By construction, the cyclic module $A/(\mu(\mathrm{Lie}(G)))$ is a bimodule between A and $A//G$. If the action is "not too singular," $A/(\mu(\mathrm{Lie}(G)))$ is a Morita equivalence.

When $M = \mathbb{R}^{0|n}$ with its positive-definite symplectic form, there is a subgroup of the full symplectomorphism group given by the linear symplectomorphisms $\mathrm{Sp}(0|n) \cong \mathrm{SO}(n)$. (The "metaplectic group" for $\mathbb{R}^{0|n}$ is $\mathrm{Spin}(n)$.) Thus representations of compact groups provide linear symplectic actions on odd symplectic manifolds, which are automatically Hamiltonian if the group is connected and simply connected. It is natural to focus on linear symplectomorphisms, as they canonically quantize. Linear actions never satisfy the Marsden–Weinstein condition — the classical moment map always has a quadratic singularity at the origin — but the quantum action is "not too singular" as soon as the reduction $\mathrm{Cliff}(n)//G$ is non-zero. The main results of the talk are the following calculations:

(1) $\mathrm{Cliff}(4)//\mathrm{Spin}(3) \cong \mathbb{H}$, the purely-even quaternion algebra, where $\mathrm{Spin}(3)$ acts on $\mathbb{R}^{0|4}$ via the real spin representation.
(2) $\mathrm{Cliff}(7)//G_2 \cong \mathrm{Cliff}(-1)$, the Clifford algebra with one generator and oppositive signature to that of $\mathrm{Cliff}(1)$, where the exceptional group G_2 acts on $\mathbb{R}^{0|7}$ via its defining representation.
(3) $\mathrm{Cliff}(8)//\mathrm{Spin}(7) \cong \mathbb{R}$, where $\mathrm{Spin}(7)$ acts on $\mathbb{R}^{0|8}$ via the real spin representation.

For comparison, the vector representation of $\mathrm{Spin}(n)$ on $\mathbb{R}^{0|n}$ is always "too singular." The calculations are explicit. For example, $G_2 \subseteq \mathrm{SO}(7)$ is by definition the stabilizer of the cubic function $\epsilon = x_1x_2x_7 - x_1x_3x_6 - x_1x_4x_5 - x_2x_3x_5 + x_2x_4x_6 + x_3x_4x_7 + x_5x_6x_7$ on $\mathbb{R}^{0|7}$, and $\mathrm{Spin}(7) \subseteq \mathrm{SO}(8)$ is the stabilizer of the quartic $\epsilon(x_8 + x_1x_2x_3x_4x_5x_6x_7) \in \mathrm{Cliff}(8)$.

How does this story relate to twisted functorial field theories and factorization algebras? My hope is that it can be used to explain the 576-fold periodicity of TMF. There is a conjectural analogy due in part to Stolz and Teichner and in part to Douglas and Henriques relating:

1-dim $\mathcal{N}=1$ SUSY QFT	KO	real Clifford algebras	$\mathrm{Cliff}(8) \simeq \mathbb{R}$
2-dim $\mathcal{N}=1$ SUSY QFT	TMF	free fermion chiral CFTs	$\mathrm{Fer}(576) \simeq \mathbb{R}$

The existence of such an analogy is conjectural, and also the lower right box is conjectural. With luck, quantum Hamiltonian reduction could establish the Morita equivalence conjectured in the lower right box. This would provide supporting evidence for the table as a whole.