Orbifolds

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Akshay asked me to talk about orbifolds of vertex algebras, and it is a bit daunting because as you will find out, I barely know what is a vertex algebra and certainly don't know the finer points about C_2 cofiniteness and so on. Rather, I want to give a general description and I hope the experts will fill in the more technical steps during the discussion.

In quantum field theory, the word "orbifolding" is a synonym for "gauging a finite group of automorphisms." Here is a very high level algorithm that you could try to implement.

Whatever a "quantum field theory" is, it is supposed to be "local" in spacetime. Now suppose that you have a qft $\mathcal{Q}$ and a finite group G of automorphisms. If those automorphisms preserve locality, then you should be able to place $\mathcal{Q}$ on any spacetime M equipped with principle G-bundle $P : M \to \mathrm{B}G$. This principle G-bundle is called a "background gauge field," and a physicist would say that she has used the action of G on $\mathcal{Q}$ to *couple* $\mathcal{Q}$ to background gauge fields. For instance, perhaps $\mathcal{Q}$ is modelled by a factorization algebra on M, and you trivialize P in patches. Then you place $\mathcal{Q}$ in every patch, but glue the patches together by using the transition maps in P and the action of G on $\mathcal{Q}$. If $\mathcal{Q}$ were a space over $\mathrm{B}G$, then I would be describing the associated bundle $\mathcal{Q} \times_G P$.

So far, we have not "gauged G." We have just encoded the symmetry in some way. To *gauge* the G-action requires a second step, which is to somehow "dynamicalize" or "integrate out" the choice of principle G-bundle. What will happen is that this step can be obstructed, and if it is unobstructed, then there might be choices. Physicists call the obstruction an *anomaly*, and the choices are called *anomaly cancellation data* or, when G is a finite group, they are also called *discrete torsion*. These obstructions and choices arise because when a physicist says that she knows a qft, she usually means this *operationally*: she knows the algebra of operators/observables in the qft, the (normalized!) expectation values, etc. In particular, she has probably made some choices for how to parameterize things, like choosing an overall notion of "zero energy," but some of those choices are "unphysical" and so don't need to be preserved by the G-action. However, gauging the G-symmetry might require more of those choices to be rigidified.

1 Warm-up 1: quantum Hamiltonian reduction

Let me warm up by explaining this in a very down-to-earth example for mathematicians. By definition, 0+1D qft = quantum mechanics. Schrödinger told us that a good way to model a quantum mechanical system is to describe it by a Hilbert space $\mathcal{H}$ and a self-adjoint unbounded operator $\hat{H}$ called the *Hamiltonian* which measures the overall energy of a state. If you'd like to assume $\dim \mathcal{H} < \infty$, that is fine for what I am going to say.

What are the automorphisms of this system? A mathematician might assume that the automorphisms of the QM system modelled by $(\mathcal{H}, \hat{H})$ are the automorphisms of the model, i.e. the subgroup of $\mathrm{U}(\mathcal{H})$ that commutes with $\hat{H}$. But this is not correct. The reason is that $(\mathcal{H}, \hat{H})$ is a model — a coordinatization — of the QM system, and is not the QM system itself. A vector in $\mathcal{H}$ is not a state. Rather, the (pure) states in the QM system are the lines in $\mathcal{H}$ (and the mixed states are convex combinations of pure states), and so we shouldn't work with the unitary group $\mathrm{U}(\mathcal{H})$ but rather with the projective unitary group $\mathrm{PU}(\mathcal{H})$. There is another way to say this. The Hilbert space $\mathcal{H}$ is not itself physical — its projectivization $\mathrm{P}\mathcal{H}$ is, and another thing which is physical is the von Neumann algebra $\mathcal{A} := \mathcal{B}(\mathcal{H})$ of bounded operators (and various things related to it, like the Jordan algebra of self-adjoint operators, or various collections of unbounded operators). And indeed $\mathrm{PU}(\mathcal{H}) = \mathrm{Aut}(\mathcal{A})$. $\mathcal{A}$ determines $\mathcal{H}$ only up to *non-unique* isomorphism. So we really should work with the subgroup of $\mathrm{PU}(\mathcal{H})$ that commutes with the Hamiltonian.

Remark: Actually, there is no reason why an automorphism needs to commute with the Hamiltonian, because $\hat{H}$ itself isn't physical. This is because there is a fundamental ambiguity in the overall energy scale, which is related to the ambiguity in the phases of vectors representing pure states. So a symmetry only needs to satisfy $g\hat{H}g^{-1} = \hat{H} + E$ for some constant E (depending, of course, on g). For finite (or compact) groups of automorphisms this doesn't matter, but if you want to set up the right theory for infinite groups, it does. Also there could be antiunitary automorphisms, but those are importantly different: they involve reversing the direction of time, and so are not totally "local."

Now, suppose you have a group G of automorphisms of $(\mathcal{H}, \hat{H})$. If G really is represented linearly on $\mathcal{H}$, then the result of *gauging* aka *orbifolding* the G-symmetry would be modelled by the Hilbert space of (co)fixed points $\mathcal{H}/G$. Let me explain this by an example. Suppose that we are looking at a sigma model, i.e. a quantum mechanical particle moving in some *configuration space* X, with a second-order equations of motion. Then $\mathcal{H} = L^2(X)$. If G acts on X geometrically, then the "orbifold theory" should be just a particle moving in the orbifold X/G, and you expect $L^2(X/G) = \mathcal{H}/G$.

If, however, you have only a group acting projectively, then there is no "(co)fixed points." Rather, if you have $G \to \mathrm{Aut}(\mathcal{A}) = \mathrm{PU}(\mathcal{H})$, then in order to define the orbifold theory, you need to lift G along $\mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$. The fibre of this map is a $\mathrm{U}(1)$, and so the obstruction to doing this lifting is a class $\alpha \in \mathrm{H}^2_{\mathrm{gp}}(G; \mathrm{U}(1))$. This class α (or perhaps a cocycle for it) is called the *anomaly* for the G-action.

Let me say this again completely algebraically because I want to explain what happens if you don't have direct access to $\mathcal{H}$. If you have a von Neumann algebra $\mathcal{A}$, how do you know if it is isomorphic to $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$? The answer is that this happens if and only if $\mathcal{A}$ is *Morita invertible* in an appropriate world of von Neumann algebras and Hilbert bimodules. This is in turn the same as asking that the monoidal category $\mathrm{Bimod}(\mathcal{A})$ of $\mathcal{A}$-$\mathcal{A}$-bimodules should be equivalent to the trivial category Vec, i.e. we are asking that $\mathcal{A}$ has a unique simple bimodule (and we are asking for some complete reducibility). It is also equivalent to a dualizability condition on $\mathcal{A}$ together with $\mathcal{A}$ being central simple, i.e. that its centre is trivial. Central simplicity of $\mathcal{A}$ is basically the *Heisenberg uncertainty principle*: for every nontrivial operator, there is some other operator so that you must have some uncertainty in their joint spectrum.

You could of course make sense of these requests in pure algebra over any field (or even over any scheme), and then you will recover the notion of $\mathcal{A}$ being *Azumaya*. By the way, even in QM there is a good reason for working with Azumaya algebras which are not matrix algebras. As you know, an Azumaya algebra over some base is a thing which is locally a matrix algebra, but not globally. Geometrically, it is a bundle of projective spaces. In QM, this issue arises any time you have a family of QM systems parameterized by some base, again because an actual QM model is a projective space and not a Hilbert space, so projective bundles, not vector bundles, are what describe parameterized systems.

Ok, so let's say that $\mathcal{A}$ is an Azumaya algebra. Then the group $\mathcal{A}^\times$ of invertible elements of

$\mathcal{A}$ is a form of GL_n, and $\mathrm{Aut}(\mathcal{A}) = \mathcal{A}^\times/\mathbb{G}_m$ is a form of PGL_n. (**Exercise:** if $\mathrm{Bimod}(\mathcal{A}) = \mathrm{Vec}$, then $\mathcal{A}^\times \to \mathrm{Aut}(\mathcal{A})$ is surjective. I.e. every automorphism is inner[izable]. This statement "every automorphism is inner[izable]" is one of the many things that deserves the moniker *Noether's theorem*.) The anomaly $\alpha \in \mathrm{H}^2(G; \mathbb{G}_m)$ for a map $G \to \mathrm{Aut}(\mathcal{A})$ exactly measures whether the action of G can be innerized consistently.

Now I can tell you the result of gauging the G-action at the level of algebras. Suppose you have chosen a trivialization of α, i.e. a map $G \to \mathcal{A}^\times$, or equivalently a map $\mu : \mathbb{C}G \to \mathcal{A}$ from the group algebra. This choice is a *quantum comoment map*. Let's take the trivial G-module. Then you can look at the $\mathcal{A}$-module $\mathcal{A} \otimes_{\mathbb{C}G} \mathbb{C}$, and the *quantum Hamiltonian reduction* of $\mathcal{A} /\!\!/ G := \mathrm{End}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathbb{C}G} \mathbb{C})$. The point is: when $\mathcal{A} = \mathrm{End}(\mathcal{H})$ is a matrix algebra, then $\mathcal{A} /\!\!/ G = \mathrm{End}(\mathcal{H}/G)$ as desired.

This algebra can be described differently. Using just the map $G \to \mathrm{Aut}(\mathcal{A})$, you can define the fixed subalgebra $\mathcal{A}^G$. **Exercise:** there is an algebra homomorphism

$$\mathcal{A}^G \to \mathcal{A} /\!\!/ G.$$

It is a choice of way to promote the non-Azumaya algebra $\mathcal{A}^G$ to an Azumaya algebra $\mathcal{A} /\!\!/ G$.

In this case it is a quotient, but in the VOA case it will be an extension. Anyway, the kernel of this homomorphism depends on the choice of trivialization of α. One way to say it is the following. Let $\mathbb{C}^\alpha G$ denote the α-twisted group algebra, so that the map $G \to \mathrm{Aut}(\mathcal{A})$ selects canonically a map $\mathbb{C}^\alpha G \to \mathcal{A}$, and $\mathcal{A}^G$ is its commutator. (Specifically, you *define* $\mathbb{C}^\alpha G$ to be the sum of one-dimensional subspaces of $\mathcal{A}$, where the gth subspace is the subspace living over $g \in \mathrm{Aut}(\mathcal{A})$.) **Exercise:** since $\mathcal{A}$ is Azumaya, there is a canonical monoidal equivalence $\mathrm{Bimod}(\mathcal{A}^G) \cong \mathrm{Bimod}(\mathbb{C}^\alpha G)$. Now a choice of trivialization of α is the same as choice of homomorphism $\mathbb{C}^\alpha G \to \mathbb{C}$, which is the same as a choice of some specific algebra object in $\mathrm{Bimod}(\mathbb{C}^\alpha G)$, which is the same as the choice of some specific algebra object in $\mathrm{Bimod}(\mathcal{A}^G)$, which is the same as a choice of specific quotient/extension of $\mathcal{A}^G$.

2 Warm-up 2: characters

How would a high-energy physicist detect that some action of G on $\mathcal{A}$ has an anomaly, in the quantum mechanics case where $\mathcal{A} = \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$? High-energy physicists are very good at calculating *partition functions* aka *characters*. Given a Schrödinger model $(\mathcal{H}, \hat{H})$, the *partition function* is the function

$$Z(\tau) := \mathrm{Tr}_{\mathcal{H}}(\exp(-\tau \hat{H})).$$

What happened is that we have taken the *time evolution operator* $\exp(it\hat{H})$, and *Wick-rotated* to imaginary time $t = i\tau$. Depending on the details of your model, of course, $Z(\tau)$ might diverge. But for reasonable QM models, $\hat{H}$ is self-adjoint and bounded below and has discrete spectrum which grows not too slowly, and so $\exp(-\tau\hat{H})$ is trace-class if $\tau > 0$.

What is $Z(\tau)$ a function of? Well, $\tau \in \mathbb{R}_{>0}$. But really we should think of τ as the circumference of a Euclidean-signature circle $\mathbb{R}/\tau\mathbb{Z}$, and so $Z(\tau)$ is a function on the moduli space of circles: these circles are the possible "spacetimes" for the 0+1D qft, and $Z(\tau)$ is the value of the "path integral" over fields on the circumference-τ circle. Actually, it isn't really a function at all because as I mentioned above we already had to make some unphysical choice to write down $\hat{H}$. If you change $\hat{H} \mapsto \hat{H} + E$, then you change $Z(\tau) \mapsto e^{-\tau E} Z(\tau)$. So we should really think that there is some line bundle on the space of circles and $Z(\tau)$ is a section of it.

Suppose now that you have some group G of automorphisms. If G acts linearly on $\mathcal{H}$ — if you have trivialized the anomaly — then it is completely natural to ask about the character of the

G-action, i.e.

$$Z_g(\tau) := \mathrm{Tr}_{\mathcal{H}}(g \exp(-\tau \hat{H})).$$

A physicist might think of this as taking a circumference-τ circle and inserting g at some point along it, and might call it the *one-point function* of the operator g. The fact that g is a symmetry — that it commutes with $\hat{H}$ — means that it is a *topological operator*: even if you were to insert other operators from $\mathcal{A} = \mathcal{B}(\mathcal{H})$, the exact location of g wouldn't matter, but only its topological location relative to the locations of the other insertions. [In general, if you insert operator $a_i \in \mathcal{A}$ at time τ_i, for $i = 1, \dots, n$, with $\tau_1 < \tau_2 < \cdots < \tau_n$, then there is an *n-point function* $\mathrm{Tr}_{\mathcal{H}}(\exp(-\tau_1 \hat{H}) a_1 \exp(-(\tau_2 - \tau_1)\hat{H}) a_2 \dots \exp(-(\tau_n - \tau_{n-1})\hat{H}) a_n \exp(-(\tau - \tau_n)\hat{H}))$. It depends on the exact values of the τ_i except when the a_i are *topological*, in which case it only depends on their cyclic order. Also, you are forbidden from having $\tau_i = \tau_j$ by another of the things called "Heisenberg uncertainty." Well, if $[a_i, a_j] = 0$ then it's ok, but only then.]

Let's suppose that G is finite and that you want to gauge the G-action. Again let's start with the case when G is acting linearly i.e. without anomaly. Then the gauged=orbifold theory would have partition function

$$Z^{\mathrm{orb}}(\tau) := \frac{1}{|G|} \sum_{g \in G} Z_g(\tau) = \sum_{[g] \in G/G} \frac{1}{|C_G(g)|} Z_{[g]}(\tau).$$

This is a result of Frobenius. On the right-hand side, $G/G = \hom(S^1, \mathrm{B}G)$ means the set of conjugacy classes in G, and $C_G(g)$ is the centralizer of g in G. The physical interpretation is that we have summed over all G-bundles P on the circle, and the division records that G-bundles have automorphisms.

The *discrete torsion* appears for the following reason. Suppose you pick some 1-dimensional representation $\beta \in \mathrm{H}^1(G; \mathrm{U}(1))$. Then you can also consider

$$Z^{\mathrm{orb},\beta}(\tau) := \frac{1}{|G|} \sum_{g \in G} \beta(g) Z_g(\tau).$$

Note that this corresponds to changing the trivialization of the anomaly, but it does not change the action of G on $\mathcal{A}$. A physicist would think of $\beta(g)$ as being a *Dijkgraaf–Witten term in the path integral*.

Let me dwell a moment on this point that $G/G = \hom(S^1, \mathrm{B}G)$. It means that $Z_g(\tau)$, as a function of both g and τ, should be understood as a function on the moduli space of circles-equipped-with-a-G-bundle, where by "circle" I mean "S^1 with some circumference."

Now let's suppose that $G \to \mathrm{Aut}(\mathcal{A})$ has some anomaly α. Where lives an expression like "$\mathrm{Tr}_{\mathcal{H}}(g \exp(-\tau \hat{H}))$"? We do not know what g is as a matrix, but we do know what it is up to scale. So there is some line in which $\mathrm{Tr}_{\mathcal{H}}(g \exp(-\tau \hat{H}))$ lives. More precisely, there is a line bundle $\int_{S^1} \alpha \in \mathrm{H}^1(\hom(S^1, \mathrm{B}G); \mathrm{U}(1))$, and $\mathrm{Tr}_{\mathcal{H}}(g \exp(-\tau \hat{H}))$ lives in this line bundle. This line bundle goes by many names. One of them is the *transgression* and another is the *slant product*. See, for any $g \in G$ and for any cocycle representative of α, you can write down the cocycle $\imath_g \alpha := \alpha(g, -)/\alpha(-, g)$, which is a cocycle not on all of G but on the centralizer $C_G(g)$. This cocycle is precisely the monodromies of the line bundle $\int_{S^1} \alpha$ for loops in G/G based at the conjugacy class $[g]$.

A trivialization of α would trivialize this line bundle. And you need to trivialize this line bundle if you are going to make sense of the sum in Z^{orb}. So this line bundle being untrivializable would be a symptom of there being a nonzero anomaly, and hence of the problem being ungaugeable.

I should warn that there is a loss of information in the passage $\alpha \mapsto \int_{S^1} \alpha$. For example, I think that if $G = (\mathbb{Z}/2)^2$, then this map has kernel. But if G is an elementary abelian group of odd order, then $\int_{S^1}$ is faithful.

3 Holomorphic CFTs and twisted modules

Let's now look at 2D = 1+1D qft. There is not a complete definition of "2D qft," but there almost is. Instead of a single Hamiltonian, a 2D qft has a *energy-momentum* or *stress-energy* tensor. It is a "tensor" in the sense that it is a symmetric 2-tensor: it measures the response of the QFT to small changes to the metric: $T_{ij} = \partial \mathcal{Q}/\partial g^{ij}$. In 2D, a symmetric 2-tensor has three components. After polarizing (i.e. choosing lightlike coordinates $z, \bar{z}$), these components are $T := T_{zz}$, $\bar{T} := T_{\bar{z}\bar{z}}$, and $\mathrm{Tr}(T) := T_{z\bar{z}} \propto T_{ij}g^{ij}$. The third of these measures the response to rescaling the metric. The qft is *conformal* when $\mathrm{Tr}(T) = 0$. In this case T and $\bar{T}$ commute, and you can define the *chiral* and *antichiral* operators to be those operators which commute with $\bar{T}$ and T, respectively, and these are each VOAs, and you could try to define "cft" to mean "pair of VOAs compatible in some way," but no one has worked out the precise definition. The qft is *holomorphic* when $\bar{T}$ also vanishes. In this case, the only *operational* data in the qft is the VOA $\mathcal{A}$ of (necessarily chiral) operators. But, just as before, this operator cannot be arbitrary: physics requires that it be "Azumaya," which here is the definition of *holomorphic VOA*: a VOA for which the category $\mathrm{Rep}(\mathcal{A})$ of vertex modules is trivial. [I should also insist on niceness properties, like that $\mathcal{A}$ be C_2 cofinite and $\mathbb{N}$-graded and so on. Physics insists that $\mathcal{A}$ be unitary. These insistences are for the experts in the room, not for me.]

Note that Zhu's modularity does guarantee that there is a *partition function* $Z(\tau)$. What is this a function of? It is a holomorphic function of τ, and it transforms as a modular form. In other words, it is a holomorphic "function" on the moduli space of flat 2-tori moduli global rescaling, i.e. on the space of genus-1 curves $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. "function" is in quotes because it really is now a section of a not-necessarily-trivial line bundle.

Now let's suppose that we have some finite group G of automorphisms of $\mathcal{A}$. Analogy with the 1D case leads us to expect:

- Locality should allow us to place $\mathcal{A}$ on any worldsheet E equipped with a principle G-bundle $P : E \to \mathrm{B}G$. In particular, when $E = E_\tau$ is the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, then P is a pair of commuting elements $g, h \in G$, one for the A-cycle and one for the B-cycle, considered up to simultaneous conjugation. In particular, one should be able to define a partition function $Z_{g,h}(\tau)$ which is the expectation value of this configuration, i.e. of $\mathcal{A}$ *twisted* by g and *twined* by h. It might not be valued in numbers but rather in some line bundle.

- There might be an anomaly (aka obstruction) α to defining/constructing a gauge=orbifold theory $\mathcal{A} /\!\!/ G$. If the obstruction is trivializable, there might be choices in the trivialization. A symptom of such an anomaly might be the nontriviality of the line bundle where $Z_{g,h}$ takes values.

- If we can define $\mathcal{A} /\!\!/ G$, then its partition function will be formally an expression like

$$Z^{\mathrm{orb}}(\tau) = \frac{1}{|G|} \sum_{P \in \hom(E_\tau, \mathrm{B}G)} Z_{g,h}(\tau)$$

  Here (g, h) are monodromies of P around the A- and B-cycles in E_τ. The factor of $\frac{1}{|G|}$ accommodates the automorphisms of a G-bundle.

- If we can construct $\mathcal{A} /\!/ G$, its operator algebra will not be merely the G-fixed operators $\mathcal{A}^G$. Rather, it will be some new holomorphic VOA which receives a map

$$\mathcal{A}^G \to \mathcal{A} /\!/ G.$$

This map should be somehow "selected" by the trivialization of the anomaly α together with a characterization of $\mathrm{Rep}(\mathcal{A}^G)$ just in terms of G and α. (Unlike in the 1D case, now the map $\mathcal{A}^G \to \mathcal{A} /\!/ G$ will be an extension, not a quotient, because under any reasonable niceness hypotheses, $\mathcal{A}^G$ will be simple as a VOA.)

Before I continue, I want to make a remark about the language. In the VOA literature, the fixed-point subalgebra $\mathcal{A}^G$ is sometimes called the *orbifold* of $\mathcal{A}$ by G. Sometimes it is called the *chiral orbifold*. When you use those language, the gauged theory $\mathcal{A} /\!/ G$ is then called the *full orbifold* or *twisted orbifold*. There is a reason for this language. Suppose you have any nice enough VOA V. Then you can define a "full CFT" which is not holomorphic, but rather has chiral operators in V and antichiral operators in $\bar{V}$, and its full algebra of operators is something like

$$\mathcal{A} = \int_{I \in \mathrm{Rep}(V)} I \otimes \bar{I}$$

where the integral means a coend. This is called a *diagonal CFT*. Then $\mathrm{Aut}(\mathcal{A})$ will include $\mathrm{Aut}(V)$ as a (proper) subgroup, and the $\mathrm{Aut}(V)$ subgroup will be nonanomalous, and if you gauge $G \subset \mathrm{Aut}(V)$, the result will be the diagonal CFT for V^G. This is an important class of examples but it's not the class that arises in moonshine, where you really should think of $V^\natural$ as a holomorphic CFT and not as part of a diagonal CFT. Of course, the fourth bullet point means that studying the "chiral orbifold" $\mathcal{A}^G$ is very close to studying the "full orbifold" $\mathcal{A} /\!/ G$.

Let's start by guessing where the anomaly α should live. Well, to do this let's ask: where do the choices of anomaly trivialization data live? Changing the anomaly trivialization would correspond in particular to modifying the orbifold partition function to

$$\frac{1}{|G|} \sum_{P \in \hom(E_\tau, \mathrm{B}G)} \epsilon(g,h) Z_{g,h}(\tau)$$

for some numbers $\epsilon(g,h) \in \mathrm{U}(1)$. Actually, it would be better for me to call those numbers $\int_{E_\tau} \beta$ or something. If I only cared about writing down some function, then I could probably adjust these numbers pretty willy-nilly. But I want to think of this sum as a sum over G-gauge fields, and so the most natural thing would be to modify the sum by including a *Dijkgraaf–Witten term*, which in d dimensions is a term of the form $(M, P : M \to \mathrm{B}G) \mapsto \int_M P^*\beta \in \mathrm{U}(1)$ for some $\beta \in \mathrm{H}^d(\mathrm{B}G; \mathrm{U}(1))$. So in the 2D case we expect that the choices of anomaly trivialization — the *discrete torsion* — will live in $\mathrm{H}^2(\mathrm{B}G; \mathrm{U}(1))$, and so we expect that the anomaly itself will live in $\mathrm{H}^3(\mathrm{B}G; \mathrm{U}(1))$.

Remark: This expectation is correct for bosonic qfts in this dimension, but fails in higher dimensions. The reason is that it was overkill to demand that $P^*\beta$ be an ordinary $\mathrm{U}(1)$-valued cohomology class. All we need is for it to be something which can meaningfully be integrated over d-dimensional oriented manifolds. So really β should live in degree-d oriented *cobordism* of G, i.e. $\Omega^d_{\mathrm{SO}}(\mathrm{B}G) = \hom(\pi_d \mathrm{MT}(\mathrm{SO} \times G), \mathrm{U}(1)) = \mathrm{H}^d(\mathrm{B}G; I_{\mathrm{U}(1)}\mathrm{MTSO})$, which is to say it should be a cobordism invariant of oriented-manifolds-with-G-bundle. [There's some unitarity snuck into this statement which is why you care about MTSO and not MTSO(d).]

Now let's assume that we have trivialized the anomaly α and let's inspect the formula for Z^{orb} to see what we can learn about $\mathcal{A} /\!/ G$. Why is this reasonable? Because (the underlying vector space of) $\mathcal{A}$ is itself the Hilbert space that the qft assigns to S^1. This is a version of the *state-operator* *correspondence.* (In 1D, I'm saying that $\mathcal{B}(\mathcal{H})$ is the Hilbert space assigned to $S^0 = \{+,-\}$, and indeed $\mathcal{B}(\mathcal{H}) = \mathcal{H} \otimes \bar{\mathcal{H}}$ up to issues of functional analysis.) We can break up the sum as

$$Z^{\mathrm{orb}}(\tau) = \frac{1}{|G|} \sum_{g \in G} \sum_{h \in C_G(g)} Z_{g,h}(\tau) = \sum_{[g] \in G/G} \frac{1}{|C_G(g)|} \sum_{h \in C_G(g)} Z_{g,h}(\tau).$$

Then the inner sum $\frac{1}{|C_G(g)|} \sum_{h \in C_G(g)} Z_{g,h}(\tau)$ looks very much like it is projecting down to the fixed points of a $C_G(g)$-action on something.

Indeed, this is exactly what is happening. Given an A-cycle S^1 with a G-bundle $[g] \in G/G$, we can define a *g-twisted Hilbert space* $\mathcal{A}(g)$, aka *g-twisted module* (or at least we can do this when the anomaly is trivialized). It will automatically carry an action by $C_G(g)$, and we find that:

$$\mathcal{A} /\!\!/ G = \bigoplus_{[g] \in G/G} \mathcal{A}(g)^{C_G(g)}.$$

Note that the $g = e$ summand is the subalgebra $\mathcal{A}^G$, and the other summands are all $\mathcal{A}^G$-modules. This is the sense in which the extension $\mathcal{A}^G \to \mathcal{A} /\!\!/ G$ is controlled by $\mathrm{Rep}(\mathcal{A}^G)$.

Ok, now let me tell you what is the *twisted module.* Let's suppose that g is of order $n < \infty$. Consider the n-to-one cover $\mathbb{A}^1 \smallsetminus \{0\} \to \mathbb{A}^1 \smallsetminus \{0\}$, $z \mapsto z^n$. I'm going to place $\mathcal{A}$ upstairs, and descend it to the downstairs by using the automorphism g as my descent data. This gives a factorization algebra on $\mathbb{A}^1 \smallsetminus \{0\}$. If I draw in a branch cut, then I can think of this as being $\mathcal{A}$ away from the branch cut, and that I put a *g-symmetry defect* along the branch cut. The definition is: the g-symmetry defect is the topological defect so that if I move it over a vertex operator, the operator gets acted on by g.

Now there is some category of factorization algebra extensions of this factorization algebra over the origin.

If you did this in the language of VOAs, you'd find a category which is almost the category of $\mathcal{A}$-modules, except where the locality axiom is modified by g. Namely, in a vertex module M for $\mathcal{A}$, for any $m \in M$ and $a \in \mathcal{A}$, there should be a vertex operator $a(z)m(0) \in M((z^{\pm 1})) = M \otimes \mathbb{C}((z^{\pm 1}))$, and the integrality of the exponent means that you can move z around 0. Now I want instead to look at a vertex operator valued in $M \otimes z^{k/n}\mathbb{C}((z^{\pm 1}))$, where a is an eigenvector of g with eigenvalue $\exp(2\pi i k/n)$. Note that such an M is in particular an ordinary vertex module for $\mathcal{A}^G$.

So, up to analytic details like "admissibility" and the like, there is a category of these, the category of *g-twisted modules.* I think it was Lepowsky, but maybe it was Huang, who showed **Theorem:** If $\mathcal{A}$ is holomorphic, then the category of g-twisted modules is "one-dimensional" in the sense that it has a unique simple object (and good complete reducibility). **Warning:** this module is unique up to *nonunique* scalar. The module $\mathcal{A}(g)$ above is this simple module. The twisted-twined partition function is

$$Z_{g,h}(\tau) = \mathrm{Tr}(\mathcal{A}(g); hq^{L_0 - c/24}).$$

If you change $g \mapsto hgh^{-1}$, then you change this category is a controlled way. It follows that $\mathcal{A}(g)$ carries a *projective* action of $C_G(g)$. What controls the projectivity? Why, it is precisely the slant product of the anomaly $\alpha \in \mathrm{H}^3(\mathrm{B}G; \mathrm{U}(1))$ to $\imath_g \alpha \in \mathrm{H}^2(\mathrm{B}C_G(g); \mathrm{U}(1))$, given by the cocycle formula $(\imath_g \alpha)(h,k) = \alpha(g,h,k)\alpha(h,g,k)^{-1}\alpha(h,k,g)$. If you compile these together over all the $g \in G/G$ you get the *transgression* $\int_{S^1} \alpha \in \mathrm{H}^2(G/G; \mathrm{U}(1))$. [Or rather, it will be this after I explain how to really define α.]

4 Characters

How might a high-energy physicist measure an anomaly? She is very good at calculating partition functions, as we said already. So what she might do is to say: let me work out in what line bundle lives the twisted twined partition functions $Z_{g,h}(\tau)$, and whether it is or is not trivializable.

Where is this line bundle? It is a line bundle on the moduli space of elliptic-curves-with-G-bundle, or at least this is true $\mathbb{C}$-analytically. (I think if you want to work algebraically, you need to get some Galois twistings into place, and maybe make some other corrections.) Let me call this moduli stack $\mathcal{M}^G$. $\mathbb{C}$-analytically, it is the following. If we choose A- and B-cycles on the elliptic curve, then the curve is parameterized by a point $\tau \in \mathfrak{h}$ the upper half plane. If we also choose a basepoint on E_τ, then the monodromies are parameterized by pairs $(g, h) \in G \times G$ which commute, i.e. by $(g, h) \in \hom(\mathbb{Z}^2, G)$. Undoing the basepoint and the choice of cycles, we are looking at

$$\mathcal{M}^G(\mathbb{C}) = (\mathfrak{h} \times \hom(\mathbb{Z}^2, G))/(\mathrm{SL}_2(\mathbb{Z}) \times G).$$

The G-action simultaneously conjugates both g and h. The SL_2 action acts on $\tau \in \mathfrak{h}$ and also on the pair (g, h). Anyway, given $\alpha \in \mathrm{H}^3(\mathrm{B}G; \mathrm{U}(1))$, you can transgress it to a holomorphic line bundle $\int_{T^2} \alpha \in \mathrm{H}^1(\mathcal{M}^G; \mathrm{U}(1))$. A version of this construction is due to Ganter.

It is an interesting exercise to work out this line bundle. It is even an interesting exercise to work out the homotopy groups of various components in $\mathcal{M}^G$. I'm pretty sure that the map $\mathrm{H}^3(\mathrm{B}G; \mathrm{U}(1)) \to \mathrm{H}^1(\mathcal{M}^G; \mathrm{U}(1))$ has a small but nonzero kernel. Specifically, I think it is injective for many finite groups, including all abelian groups, but I think that it has a kernel of order 2 for the dicyclic (=binary dihedral) groups.

Let's focus on a special case where $G = \mathbb{Z}/n$ is cyclic. Then we expect the anomaly will like in $\mathrm{H}^3_{\mathrm{gp}}(\mathbb{Z}/n; \mathrm{U}(1)) \cong \mathbb{Z}/n$. If it is trivializable, then we expect no choices in the trivialization because $\mathrm{H}^2_{\mathrm{gp}}(\mathbb{Z}/n; \mathrm{U}(1)) = 0$. Pick a generator g of G and look at the (g, e) component of $\mathcal{M}^G$. Its homotopy group contains a boring copy of G. (It is boring because its monodromies in this line bundle are trivial.) What else does it contain? Well, it contains the stabilizer of (g, e) under the $\mathrm{SL}_2(\mathbb{Z})$-action, which is nothing other than $\Gamma_0(n)$.

I can never remember whether $\Gamma_0(n)$ is upper triangular or lower triangular. One version contains T and the other contains only T^n.

In any case, what am I telling you? I am telling you two things. First, I am telling you that $Z_{e,g}(\tau) = \mathrm{Tr}(\mathcal{A}; g q^{L_0 - c/24})$ should be modular for the copy of $\Gamma_0(n)$ that contains T. (Certainly it is valued in integral powers of q, and so is invariant under T-action.)

Applying an S-transformation gives $Z_{g,e}(\tau)$. This should be modular for the copy of $\Gamma_0(n)$ which contains T^n but not T. What does that mean? Well, it isn't modular on the nose: it is modular with some *multiplier*. This multiplier records the line bundle. In particular, $Z_{g,e}(\tau)$ will be an eigenvector for T^n with some eigenvalue. If the eigenvalue is trivial, that means that $Z_{g,e}(\tau) \in \mathbb{C}[[q^{1/n}]]$. If the eigenvalue is $\exp(2\pi i k/n)$, that means that $Z_{g,e}(\tau) \in q^{k/n^2}\mathbb{C}[[q^{1/n}]]$.

The neat calculation, which I learned from a paper of Gaberdiel, Persson, Ronellenfitsch, and Volpato, is the following:

Proposition: Suppose that the anomaly of the G-action is $k \in \mathrm{H}^3(\mathbb{Z}/n; \mathrm{U}(1)) \cong \mathbb{Z}/n$. Then $Z_{g,e}(\tau) \in q^{k/n^2}\mathbb{C}[[q^{1/n}]]$. I.e. T^n acts with multiplier $\exp(2\pi i k/n)$.

What do I mean by this as a proposition? One thing I mean is that it is a calculation of the line bundle that I told you. You still have to believe some statements about VOAs that I haven't explained.

5 Constructing the orbifold

Let's stick with the case when I'm just trying to gauge a cyclic group $G := \mathbb{Z}/n$. Then I can use a nontrivial statement of Carnahan and Miyamoto: **Theorem:** if $\mathcal{A}$ is nice, then so is the fixed subalgebra $\mathcal{A}^G$. Holomorphic is a special case of nice. (A lot but not all of what I'm about to say can be done without this theorem, and I don't know where the cut-off is exactly.)

Zhu's modularity tells you that for each $g \in G$, $Z_{g,e}(\tau) = \mathrm{Tr}(\mathcal{A}(g); q^{L_0 - c/24})$ and $Z_{e,g}(\tau) = \mathrm{Tr}(\mathcal{A}; gq^{L_0-c/24})$ are related by S-transformation and that the former is modular, with a multiplier, under $\Gamma_0(n)$. In particular you can unambiguously define the anomaly k because $L_0 - c/24$ acting on $\mathcal{A}$(generator) takes values in $\frac{k}{n^2} + \frac{1}{n}\mathbb{Z}$. The action is *nonanomalous* when $k = 0$.

Furthermore, from its definition, $\mathcal{A}(g)$ carries a canonical projective action by $G = \mathbb{Z}/n$. Every projective action of $\mathbb{Z}/n$ can be promoted to a linear action, but there are $n = \mathrm{H}^1(G; \mathrm{U}(1))$ choices. Moeller gave a direct proof of the following. Suppose that the action of G on $\mathcal{A}$ is nonanomalous. Then there is a unique way to resolve the G-action on the $\mathcal{A}(g)$s such that

$$\mathcal{A} /\!\!/ G := \bigoplus_{g \in G} \mathcal{A}(g)^G$$

carries a VOA structure extending the structure on $\mathcal{A}^G$. Moreover, this VOA structure is unique. (If the G-action were anomalous, then there would be nothing you could do: $\mathcal{A}(g)^G$ would not be integrally graded for any resolution of the projectivity. The unique promotion in the nonanomalous case is selected by wanting $\mathcal{A} /\!\!/ G$ to be integrally graded.)

Let me note, by the way, that this $\mathcal{A} /\!\!/ G$ carries a new action by the Pontryagin dual group $\mathrm{H}^1(G; \mathrm{U}(1))$ corresponding to the grading by G. Now let me change notation and write $G = \mathrm{Aut}(\mathcal{A})$ and $\mathbb{Z}/n = \langle g \rangle$. Then note that G acts on $\mathcal{A}$, but only the normalizer $N_G(g)$ acts on $\mathcal{A}^{\langle g \rangle}$, and actually it acts through the quotient $N_G(g)/\langle g \rangle$. Well, the thing that acts on $\mathcal{A} /\!\!/ \langle g \rangle$ is some new extension $\mathrm{H}^1(G; \mathrm{U}(1)).[N_G(g)/\langle g \rangle]$.

Example: Let $\mathcal{A} = V_{\mathrm{Leech}}$ denote the Leech lattice VOA. Then $\mathrm{Aut}(\mathcal{A})$ is a nonsplit extension $T.\mathrm{Co}_0$ where $T = \hom(\mathrm{Leech}, \mathrm{U}(1))$. Select a lift g of the central element in Co_0 — all lifts are conjugate. The centralizer of g is a nonsplit extension of shape $2^{24}.\mathrm{Co}_0 = (2^{24} \times 2).\mathrm{Co}_1$, where $2^{24} = \hom(\mathrm{Leech}, \mathbb{Z}/2)$. [Ivanov showed that there is a unique nonsplit extension like this.] The thing that acts on $\mathcal{A}^g$ is $2^{24}.\mathrm{Co}_1$. It is not too hard to show that when you extend to $\mathcal{A} /\!\!/ g$, the extension that you get of shape $2.(2^{24}.\mathrm{Co}_1)$ will contain a subextension of shape 2_+^{1+24}. But there are two possible total extensions $2.(2^{24}.\mathrm{Co}_1)$ like this, differing by how Co_1 is extended. It turns out that only one of them is a maximal subgroup of a simple group.

6 Nonabelian orbifolds

Time will not permit, but if it does, I will describe Kirillov Jr's result.