Why the spaces of $\mathcal{N}=(0,1)$ SQFTs form a spectrum

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These slides available at:
http://categorified.net/Oxford-7mar22.pdf

Main Statement

wick-rotated partition function converges absolutely on closed spacetimes.

Let $SQFT_n := \left\{ \begin{array}{l} \text{compact } 1{+}1\text{D } \mathcal{N}=(0,1) \text{ sqfts with anomaly} \\ \text{equal to the anomaly of } n \text{ scalar multiplets} \end{array} \right\}$.

It is a pointed space with basepoint the "zero" theory, i.e. the sigma model with target the empty n-manifold.

My goal is to explain the following: there is a homotopy equivalence

$$SQFT_{n+1} \simeq \Omega\, SQFT_n .$$

Since $SQFT_n$ is not an object of mathematics, I will wear my physicist hat for the talk.

Remark: I think the statement holds in any spacetime dimension and any amount of susy. I will focus on the $1{+}1$D $\mathcal{N}=(0,1)$ case.

Superfield formalism

Super spacetime $= \mathbb{R}^{2|1}$. coords: $\underbrace{z, \bar{z}}_{\text{even}}, \underbrace{\theta}_{\text{odd}}$.

A **superfield** is a section of some bundle $\begin{matrix} V \\ \downarrow \\ \mathbb{R}^{2|1} \end{matrix}$

and a **super Lagrangian** is a diff op

$$\mathcal{L} : \Gamma(V) \to \{\text{volume forms}\}$$

$= C^\infty(\mathbb{R}^{2|1})\, dz\, d\bar{z}\, d\theta$, Berezinian integral

⚡ Not every sqft can be presented by fields + lagrangian. Those that can typically have multiple presentations

$\mathcal{N} = (0,1)$ susy is enforced by requiring equivariance for the **super-Poincaré group**, spanned by

$$\underbrace{\partial_z, \ \partial_{\bar{z}}, \ \partial_\theta - \theta\partial_{\bar{z}}}_{\text{supertranslations}}, \quad \underbrace{z\partial_z - \bar{z}\partial_{\bar{z}} - \frac{1}{2}\theta\partial_\theta}_{\text{Spin}(2)}.$$

The "$\frac{1}{2}$" makes θ into a right-handed spinor.

Note: Supertranslation group is noncommutative. It centralizes

$$D\!\!\!/ := \partial_\theta + \theta\partial_{\bar{z}} \quad \text{and} \quad \partial_z.$$

So these operators are available for use in $\mathcal{L}$.

Scalar supermultiplet

A scalar superfield is a function $\phi \in C^\infty(\mathbb{R}^{2|1})$.

Its components φ, ψ defined by

$$\phi(z,\bar z,\theta) = \varphi(z,\bar z) + \theta\psi(z,\bar z)$$

form the scalar supermultiplet. The kinetic energy is

$$\int_{\mathbb{R}^{2|1}} \partial_z\phi \, D\!\!\!/\,\phi \; dz\,d\bar z\,d\theta$$

$$= \int_{\mathbb{R}^{2|1}} (\partial_z\varphi + \theta\partial_z\psi)(\psi + \theta\partial_{\bar z}\varphi)\, dz\,d\bar z\,d\theta$$

modulo sign-rule errors

$$= \int_{\mathbb{R}^2} (\partial_z\varphi\,\partial_{\bar z}\varphi + \psi\,\partial_z\psi)\, dz\,d\bar z$$

Potential energy terms $V(\phi)$, e.g. mass terms $\sim \phi^2$, are forbidden by susy.

A "supermultiplet" is a representation of the susy alg.

$f(\phi)\partial_z\phi\, D\!\!\!/\,\phi$ is allowed by susy, but can be undone by $\phi \mapsto g(\phi)$.

Without a potential energy term, this sqft is NONCOMPACT.

Scalar supermultiplet

$$S(\Phi) = \int_{\mathbb{R}^{2|1}} \mathcal{L}(\Phi)\, dz\, d\bar{z}\, d\theta = \int_{\mathbb{R}^2} (\partial_z \varphi\, \partial_{\bar{z}} \varphi + \psi\, \partial_z \psi)\, dz\, d\bar{z}$$

The quantization is easy because the action is quadratic.

The equations of motion read:

$\partial_z \partial_{\bar{z}} \varphi = 0$, i.e. φ is harmonic.

$\partial_z \psi = 0$, i.e. ψ is antiholomorphic.

In components, the susy is

$$\varphi \mapsto \psi, \qquad \psi \mapsto \partial_{\bar{z}} \varphi .$$

This is generated by the supercurrent $:\varphi\psi:$.

This sqft is superconformal. $\mathcal{L}(\Phi)\, dz\, d\bar{z}\, d\theta$ is invariant under dilations $z\partial_z + \bar{z}\partial_{\bar{z}} + \frac{1}{2}\theta\partial_\theta$ and other superconformal transformations.

Chiral supermultiplet

$\hookleftarrow$ spin-statistics! $-\frac{1}{2} \notin \mathbb{Z}$!

A chiral superfield is a section λ of the odd one-dimensional v-bundle on which $\mathrm{Spin}(2) = z\partial_z - \bar{z}\partial_{\bar{z}} - \frac{1}{2}\theta\partial_\theta$ acts with eigenvalue $-1/2$. In components,

$$\lambda(z,\bar{z},\theta) = \alpha(z,\bar{z}) + \theta\beta(z,\bar{z})$$

where α is a left-handed spinor and β is a scalar.

The most general superlagrangian allowed by susy is

$$L(\lambda) = \lambda \not{D} \lambda + x\lambda$$

for some $x \in \mathbb{R}$.

This is superconformal only when $x = 0$.

Chiral supermultiplet

$$S(\lambda) = \int \mathcal{L}(\lambda)\, \mathcal{D}z\, \mathcal{D}\bar{z}\, \mathcal{D}\theta = \int (\lambda \not{D} \lambda + x\lambda)\, \mathcal{D}z\, \mathcal{D}\bar{z}\, \mathcal{D}\theta$$

$$= \int (\alpha + \theta\beta)(\beta + \theta \partial_{\bar{z}} \alpha + x)\, \mathcal{D}z\, \mathcal{D}\bar{z}\, \mathcal{D}\theta$$

$$= \int (\alpha \partial_{\bar{z}} \alpha + \beta^2 + x\beta)\, \mathcal{D}z\, \mathcal{D}\bar{z}.$$

So the equations of motion say:

α is holomorphic. $\beta = -\frac{x}{2}$ is constant

and the susy is $\alpha \mapsto \beta = -\frac{x}{2}$, $\beta \mapsto \partial_{\bar{z}} \alpha = 0$.

If $x \neq 0$, then supersymmetry is spontaneously broken, i.e. 1 is susy-exact, i.e. there are no susy ground states, i.e. the $\underbrace{\text{IR limit}}$ is the zero thy.

$\hookrightarrow |x| \to \infty$.

This is a CFT if you ignore the susy. The supercurrent is $-:\frac{x}{2}\beta:$, which has the wrong spin to be an SCFT.

Combining ingredients

Consider a theory w/ both a scalar multiplet Φ and a chiral multiplet λ. Then in addition to the kinetic terms

$$\partial_z \phi \, \not{D} \phi + \lambda \not{D} \lambda,$$

Shifting $\phi \mapsto \phi + x$ undoes an $x\lambda$ term.

the Lagrangian can contain the term $(\phi - x)\lambda$.

$$\int_{\mathbb{R}^{2|1}} \phi \lambda \, dz \, d\bar{z} \, d\theta = \int (\varphi + \theta\psi)(\alpha + \theta\beta) \, dz \, d\bar{z} \, d\theta = \int_{\mathbb{R}^2} (\varphi\beta + \alpha\psi) \, dz \, d\bar{z}$$

and so

$$S(\phi, \lambda) = \int_{\mathbb{R}^2} \underbrace{\left(\partial_z \varphi \, \partial_{\bar{z}} \varphi + \beta^2 + \beta\varphi\right)} + \left(\psi \partial_z \psi + \alpha \partial_{\bar{z}} \alpha + \alpha\psi\right)$$

modulo sign rules, constants, etc.

Now path-integrate the β field: $\int \left(\partial_z \varphi \, \partial_{\bar{z}} \varphi + \varphi^2\right)$

mass terms!

Both the scalar boson φ and the full fermion (α, ψ) are massive. In the far IR, the system is trivially gapped i.e. iso to the vacuum.

Lagrange multipliers

Suppose you have some sqft $\mathcal{Q}$ with a chosen operator Φ.
Build a new theory $\mathcal{Q} \otimes (\text{chiral multiplet } \lambda)$, where the
superlagrangian for λ is $\lambda \not{D} \lambda + \lambda(\Phi - x)$ ← "superpotential"

This energetically preferences $\boxed{\Phi = x}$. ← in other words, x is the "vev" of Φ

Examples: i.e. the "vacuum" sqft, aka "one".

* $\mathcal{Q} = \mathbb{1}$, $\Phi = 0$ $\longmapsto$ chiral multiplet w/ $\mathrm{susy}[\lambda] = x$,
which flows to 0 in the IR unless $x = 0$.
i.e. the "zero" sqft

** $\mathcal{Q}$ = scalar multiplet, $\Phi = \phi$

$\longleftrightarrow$ massive scalar + massive fermion,

which flows to $\mathbb{1}$ in the IR

Dynamicalization

Suppose you have a family of (s)qfts $Q(x)$, $x \in \mathbb{R}$. Does it make sense to allow x to vary over (super)spacetime?

The answer is unimpeachably YES if the dependence on x is through some coupling constant in a (super)lagrangian.

I claim it is also YES if "(s)qft" has a strong enough locality axiom built in. Then you should be able to give x different values on each little patch of (super)spacetime.

<u>Definition:</u> A one-parameter family $Q : \mathbb{R} \to \{sqfts\}$ is an sqft which couples to a nondynamical scalar field.

Nondynamical = not path-integrated over.

Dynamicalization

Definition: A one-parameter family $Q : \mathbb{R} \to \{\text{S}_{\text{T}}\text{fts}\}$ is an S_{T}ft which couples to a nondynamical scalar field.

Nondynamical = not path-integrated over.

The dynamicalization of such a family promotes the parameter x to a dynamical scalar multiplet ϕ:

$$\int D\phi \quad Q(\phi) \; \exp\left(\int_{\mathbb{R}^{2|1}} \partial_z \phi \, \cancel{D} \phi \right)$$

$\phi : \mathbb{R}^{2|1} \to \mathbb{R}$

sufficient conditions for this to be compact should be:

- each $Q(x)$ compact.
- $Q(x) \longrightarrow \text{zero}$ as $x \longrightarrow \pm\infty$

convergence should be not too slow

The natural loop spectrum structure on SQFT.

Lag. mult., setting $\Phi = x$.

$$\left\{ (Q, \Phi) : \begin{array}{l} Q \in SQFT_n \\ \text{and } \Phi \text{ an operator in } Q \end{array} \right\} \rightleftarrows \Omega\, SQFT_{n-1}$$

Dynamicalize x. Use Φ = field strength of ϕ.

$$\Omega\, SQFT_{n-1} = \{\text{one-parameter families } x \mapsto Q(x) \text{ s.t. } Q(\pm\infty) = \text{zero}\}$$

Forget Φ

homotopy equiv because fibres are vector spaces!

$$SQFT_n := \{\text{compact sqfts w/ same anomaly as } n \text{ scalar multiplets}\}$$

Claim: $\rightleftarrows$ are homotopy-inverses!

The natural loop spectrum structure on SQFT.

<u>Claim:</u> $\{(Q,\Phi)\} \underset{\text{dynamicalize}}{\overset{\text{Lag mult}}{\rightleftarrows}} \{Q(x)\}$ are homotopy inverses.

<u>Sketch of justification:</u> If you go around $\circlearrowleft$ or $\circlearrowright$, the result is to introduce to Q both a scalar boson multiplet ϕ and a chiral fermion multiplet λ, with superLagrangian

$$\mathcal{L} = (\text{coupling to } Q) + \partial\phi\not\partial\phi + \lambda\not\partial\lambda + \lambda(\phi - x \text{ or } \Phi)$$

But the ϕ, λ system is trivially gapped, i.e. flows to the vacuum $\mathbb{1}$. In that limit, ϕ $\underbrace{\text{is set to } x \text{ or } \Phi}_{\text{"vev"}}$.
And "flow" is a cont's deformation. $\square$