On the classification of topological phases

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PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Plan for the talk:

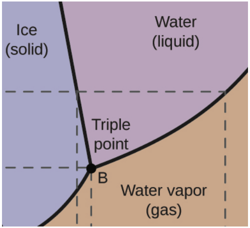
Phases = homotopy

Gapped condensation

Topological orders

Higher spin-statistics?

Phases = homotopy



(The actual phase diagram is much more complicated.)

What do we mean by phases of matter? As children, we learn a cartoon picture of {systems of water}, and define

$$\{\text{phases}\} = \pi_0\{\text{systems}\}.$$

π_0 means the set of connected components of the phase diagram. It is one measure of the homotopy of the space {systems}.

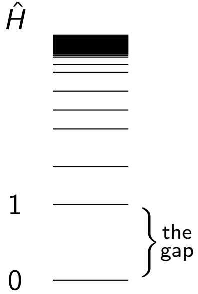
What distinguishes paths that cross phase transitions? At the phase transitions, the spectrum of the Hamiltonian undergoes a topological change: gaps between eigenvalues close and open.

Definition: A system of matter is gapped topological when there is a gap in the Hamiltonian, and the low-energy behaviour, i.e. the behaviour of the ground states, looks like a topological quantum field theory.

Warning: There is no complete definition of "gapped topological system." The problem is to make sense of "gapped" in a fully local way: what should be done about boundary conditions?

Goal: Classify gapped topological phases of matter.

Sub-goal: Classify invertible phases: gapped top'l phases that can be stacked with some other gapped top'l phase so that the result is in the trivial gapped top'l phase.



Cartoon of the spectrum of a gapped Hamiltonian. Only the behaviour near the ground state(s) matters.

Example: A topological insulator keeps its useful behaviour in spite of a bombardment from the environment because it is in a specific nontrivial gapped top'l phases.

Actually, it is in the trivial connected component of {gapped top'l systems}. It is nontrivial as a G-symmetric phases, i.e. it is G-Symmetry Protected Trivial, for $G = \mathrm{U}(1) \times \mathbb{Z}_2^T$. Environmental bombardment has trouble breaking that symmetry.

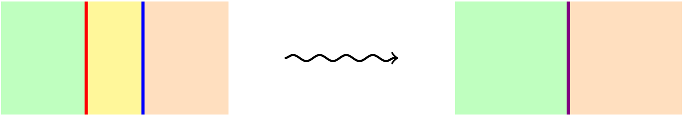
For any group G, there is a classifying space BG, and

$$\{\text{phases of } G\text{-symmetric systems}\} = \pi_0 \operatorname{maps}(BG, \{\text{systems}\}).$$

This probes the higher homotopy of the space {systems}.

Physical assumption: Given a gapped top'l system, gapped top'l modifications (operator insertions, defects, ...) in one region of spacetime do not affect the choices of gapped top'l modifications in macroscopically-separated regions.

Under this assumption, {(n+1)-dimensional gapped top'l systems} are the objects of an (n+1)-category. The 1-morphisms are interfaces. The 2-morphisms are junctions between interfaces. To compose morphisms, you place them next to each other, and then zoom out.



Theorem (Gaiotto-JF): This $(n+1)$-category is Segal-complete.
Corollary (G-JF, Kitaev): The spaces $\mathcal{I}^{n+1} := \{(n+1)\text{-dim'l invertible phases}\}$ compile into an Ω-spectrum.

Why this is useful: Ω-spectra are the abelian groups of stable homotopy theory. They are the same as generalized cohomology theories. Spaces are flimsy, but Ω-spectra are rigid, with lots of computational techniques available. The corollary implies:

$$\{G\text{-SPTs}\} = \text{reduced } \mathcal{I}^\bullet\text{-valued group cohomology of } G.$$

Moreover, the spectrum $\mathcal{I}^\bullet$ is coconnective. This implies that any finite piece $\mathcal{I}^{\leq n+1}$ is determined by finitely many "experiments": using other methods, work out $\{(n+1)\text{-dim'l } G\text{-SPTs}\}$ directly for finitely many groups G.

So to understand $\{G\text{-SPTs}\}$, we need to know the spectrum $\mathcal{I}^\bullet$. In low dimensions, we can prove:

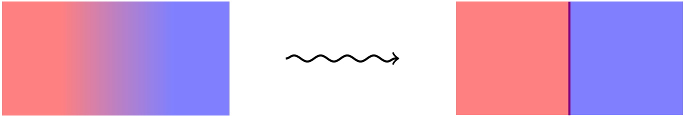
Conjecture (Kapustin):

- Bosonic $\mathcal{I}^\bullet = \Sigma I_{\mathbb{Z}}\mathrm{MSO}$.
- Fermionic $\mathcal{I}^\bullet = \Sigma I_{\mathbb{Z}}\mathrm{MSpin}$.

} The RHSs are well-studied objects in topology.

This conjecture is surprising. It implies that every $X \in \mathcal{I}^\bullet$ is invariant under rotations in Wick-rotated (Euclidean) spacetime. But condensed matter systems are typically defined on a lattice: no rotation symmetry at all! At the end of the talk I will propose an explanation/generalization of this conjecture.

Gapped condensation



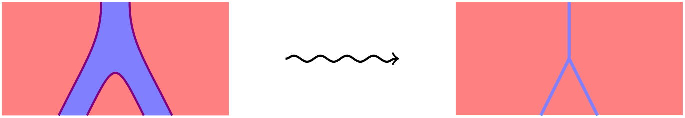
Suppose you have two gapped top'l phases X and Y and a way of "condensing" from X to Y. Run the procedure just on one half of the room. Now zoom out: you get an interface between X and Y.

As with any interface, it may support its own excitations.

Definition: The condensation procedure is gapped topological if the corresponding interface is gapped topological.

I am *not* saying that gaps do not open or close during a gapped top'l condensation. Rather, during gapped top'l condensation, only finitely many gaps open and close.

When this happens, the composition of interfaces $X|X := X|Y|X$ has an "algebra" structure: a junction between $X|X|X$ and $X|X$.



By filling X with a network of these interfaces, you can recover the phase Y. Axiomatizing this structure leads to the notion of condensation algebra in the category of X-X interfaces, and the notion of categorical condensation of a condensation algebra.

Theorem (G-JF): Condensation algs are the higher-cat version of idempotents: functions e s.t. $e^2 = e$. Categorical condensation is the higher-cat version of "image of e." The $(n+1)$-category of all $(n+1)$-dim'l gapped top'l phases is Karoubi-complete.

Theorem (G-JF): The following n-categories are equivalent:

(1) The Karoubi completion of the $(n+1)$-cat delooping of $\mathrm{V\scriptstyle{EC}}_{\mathbb{C}}$.

(2) The $(n+1)$-category of $(n+1)$-dim'l gapped top'l phases that arise from the trivial phase via gapped top'l condensation.

(3) The $(n+1)$-category of fully dualizable $\mathbb{C}$-linear n-categories, i.e. fully extended $(n+1)$-dim'l top'l field theories.

Moreover, this $(n+1)$-category can be realized by gapped top'l systems of commuting Hamiltonian projector type.

Warning: The equivalence (1) $\Leftrightarrow$ (2) is a physical statement. It is conditional on the Physical Assumption from earlier.

Warning: The equivalence (1) $\Leftrightarrow$ (3) is a mathematical statement. It is conditional on some basic facts about colimits in enriched higher cats that have yet to appear in the higher-cat literature.

Topological orders

Definition (Wen):

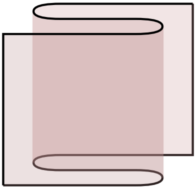
$$\{\text{physical topological orders}\} := \frac{\{\text{gapped top'l phases}\}}{\{\text{invertible phases}\}}.$$

Warning: The quotient both subtracts and adds equivalence classes. If $(n+1)$-dim'l invertible phases are considered trivial, then an interface between two different $(n+1)$-dim'l invertible phases should be considered an n-dim'l system.

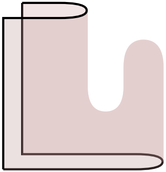
Why is this a reasonable definition? In an invertible phase, the algebra of extended top'l operators is trivial. So the algebra of extended top'l operators in a general top'l phase only depends on the corresponding top'l order.

Idea/Hope: $\{$topological orders$\}$ = $\{$algebras of extended top'l operators$\}$. These are more physically meaningful than actual Hilbert spaces and Hamiltonians.

Definition (JF): A fusion n-category $\mathcal{A}$ is a Karoubi-complete monoidal n-category/$\mathbb{C}$ which is at-least-twice dualizable: you can arrange, and then continuously unkink, the configurations on the right.



Theorem (JF) $\mathcal{A}$ is at-least-twice dualizable iff it is finite-dim'l and separable: multiplication $\mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$ is part of a cat condensation.



Theorem (JF): Fusion n-cats are fully dualizable: they determine $(n+2)$-dim'l TFTs. It is the "bulk TFT" with boundary operators $\mathcal{A}$.

Definition (JF): An $(n+1)$-dimensional algebraic topological order is a fusion n-category satisfying Remote Detectability: the (Drinfel'd) centre of $\mathcal{A}$ is trivial.

The objects in $\mathcal{A}$ are the n-dim'l operators in a physical top'l order. The 1-morphisms are $(n-1)$-dim'l operators. Etc.

Why Remote Detectability? The centre measures the operator insertions which cannot be detected by other operators.

Theorem (JF):

$$\{(n+1)\text{-dim'l alg top'l orders}\} = \frac{\{(n+1)\text{-dim'l TFTs}\}}{\{\text{invertible } (n+1)\text{-dim'l TFTs}\}}.$$

Can we classify (algebraic) topological orders?

(0+1)d: Over $\mathbb{C}$, all top'l orders are sums of invertible phases.

(1+1)d: Over $\mathbb{C}$, all top'l orders are sums of invertible phases.

(2+1)d: Modular tensor categories.

(3+1)d: Lan-(Kong-)Wen announced a classification in 2017-18: up to some issues about emergent fermions, all are top'l sigma models with target finite 1-groupoids $\mathcal{G}$. Why?

(a) Condense all the line operators. $\mathcal{G}$ is the groupoid of choices for how to do this.

(b) Result has no line operators, and so is invertible. These invertible phases compile to the sigma model Lagrangian.

Their argument was inspiring, but they did not claim it was completely rigorous.

(b) Result has no line operators, and so is invertible.

LKW argument: an $(n+1)$-dim'l top'l order with no lines also has no $(n-1)$-dim'l operators, because they would be undetectable. When $n = 3$, this means no surface operators. Similarly, no vertex operators should mean no n-dim'l operators.

Problem: you can always build m-dim'l operators as networks of $(m-1)$-dim'l operators. Let's call such networks trivial.

Theorem (JF): If an $(n+1)$-dim'l top order $\mathcal{A}$ has no nontrivial $(\leq d)$-dim'l ops, then it also has no nontrivial $(\geq n-d)$-dim'l ops.

Idea of proof: Compare dim'l reductions along various spheres.

(a) Condense all the line operators. Why is this possible?

Consider the (1+1)-dim'l situation, where every top'l phase is a sum of invertible top'l phases. **Why?** A top'l phase with a vacuum (=local ground state) degeneracy is unstable: you can add a small operator which condenses the top'l phase onto a single ground state. In (1+1)d, the condensed top'l phase has no nontrivial vertex operators, and hence is invertible by the Theorem. The original top'l phase is $\bigoplus_{v \in \text{vacua}}$($v$'th invertible phases).

This fails in the presence of a $\mathbb{Z}_2^T$-symmetry. **Why?** Without a $\mathbb{Z}_2^T$-symmetry, the algebra A of vertex operators is a f.d. commutative $\mathbb{C}$-algebra. A $\mathbb{Z}_2^T$-symmetry descends A to an $\mathbb{R}$-algebra $A_\mathbb{R}$ (i.e. $A = A_\mathbb{R} \otimes \mathbb{C}$). The operator needed to condense onto a single vacuum requires solving a polynomial equation. Over $\mathbb{C}$, but not $\mathbb{R}$, every polynomial equation has solutions.

There is a way to define 1-categorical polynomial equations, where the coefficients and variables are objects in a category.

Theorem (Deligne, interpretation by JF): Over $\mathrm{SV}\textsc{ec}_{\mathbb{C}}$, but not $\mathrm{V}\textsc{ec}_{\mathbb{C}}$, every 1-categorical polynomial equation has a solution.

This allows you to condense any fermionic ($\geq 3+1$)-dim'l top'l order to one with no vertex or line operators.

The collection of all solutions to the 1-categorical polynomial equation determined by $\mathcal{A}$ is the groupoid $\mathcal{G}$, the target space of the sigma model.

Descent: $\mathbb{R} \subset \mathbb{C}$ is selected by $\mathbb{Z}_2^T$-symmetry. $\mathrm{V}\textsc{ec}_{\mathbb{C}} \subset \mathrm{SV}\textsc{ec}_{\mathbb{C}}$ is selected by $\mathbb{Z}_2^f[1]$-symmetry in exactly the same way.

(3+1)d: Theorem (JF, following Lan–(Kong–)Wen): Every fermionic (3+1)-dim'l top'l order can be written canonically as an anomalous topological sigma model with target a finite groupoid $\mathcal{G}$. For bosonic (3+1)-dim'l top'l orders, work $\mathbb{Z}_2^f[1]$-equivariantly. (The anomaly for $\mathbb{Z}_2^f[1]$ measures the presence of emergent fermions.)

(4+1)d: Classification within sight. Calculations joint with M. Yu.

(5+1)d: Only (fermionic) sigma models, by using: **Theorem (Hopkins-JF, unpublished):** Every 2-categorical poly equation has a sol'n over $\mathrm{SC}\textsc{at}_{\mathbb{C}}$, the 2-cat of super cats.

(6+1)d: Looks feasible.

(7+1)d: Would have only sigma models, if you could solve 3-categorical polynomial equations. $2\mathrm{SC}\textsc{at}_{\mathbb{C}}$ does not contain solutions to all 3-cat poly eqns. Need beyond-fermionic stringons.

Higher spin-statistics?

Conjecture (Kapustin):

$$\text{bosonic } \mathcal{I}^\bullet = \Sigma I_{\mathbb{Z}}\mathrm{MSO}, \qquad \text{fermionic } \mathcal{I}^\bullet = \Sigma I_{\mathbb{Z}}\mathrm{MSpin}.$$

$\mathcal{I}^n$ = invertible phases, i.e. invertible objects in the *n*-category of all phases. Algebraically, those categories are

Bosonic with $\mathbb{Z}_2^T$: $\Sigma^\bullet\mathbb{R} := (\mathbb{R}, \mathrm{Vec}_{\mathbb{R}}, \mathrm{Cat}_{\mathbb{R}}, 2\mathrm{Cat}_{\mathbb{R}}, \dots)$

Bosonic: $\Sigma^\bullet\mathbb{C} := (\mathbb{C}, \mathrm{Vec}_{\mathbb{C}}, \mathrm{Cat}_{\mathbb{C}}, 2\mathrm{Cat}_{\mathbb{C}}, \dots)$

Fermionic: $\Sigma^\bullet\mathrm{SVec}_{\mathbb{C}} := (\mathbb{C}, \mathrm{SVec}_{\mathbb{C}}, \mathrm{SCat}_{\mathbb{C}}, 2\mathrm{SCat}_{\mathbb{C}}, \dots)$

Definition (JF): A tower is a sequence, like those above, in which each entry is Karoubi-complete and contains all earlier entries as morphism cats. Towers are an ∞-cat version of commutative alg.

Conjecture: Every tower $\mathcal{K}$ has an algebraic closure $\bar{\mathcal{K}}$. The *n*th entry $\bar{\mathcal{K}}_n$ is the *n*-category in which every *n*-categorical polynomial equation with coefficients in $\mathcal{K}_n$ has solutions.

$\Sigma^\bullet\mathbb{R}$, $\Sigma^\bullet\mathbb{C}$, and $\Sigma^\bullet\mathrm{SV\small{EC}}_\mathbb{C}$ all have the same algebraic closure $\mathcal{R}$, if it exists.

Theorem (JF, unpublished): Assuming $\mathcal{R}$ exists, its Ω-spectrum $\mathcal{R}^\times$ of invertible elements is equivalent to $I_{\mathbb{C}^\times}\mathrm{Mfr}$.

Definition: An extension $\mathcal{K} \subset \mathcal{L}$ of towers is Galois if it is selected by G-equivariance ($\mathcal{K} = \mathcal{L}^G$). G is the Galois group of $\mathcal{K} \subset \mathcal{L}$.

Examples:

- $\Sigma^\bullet \mathbb{R} \subset \Sigma^\bullet \mathbb{C}$ is Galois with Galois group $\mathbb{Z}_2^T$.
- $\Sigma^\bullet \mathbb{C} \subset \Sigma^\bullet \mathrm{SV{\scriptstyle EC}}_{\mathbb{C}}$ is Galois with Galois group $\mathbb{Z}_2^f[1]$.

Conjecture: $\Sigma^\bullet \mathbb{R} \subset \mathcal{R} = \overline{\Sigma^\bullet \mathbb{R}}$ is Galois.

Corollary: Suppose $\Sigma^\bullet \mathbb{R} \subset \mathcal{R}$ has Galois group G. Then $(\Sigma^\bullet \mathbb{R})^\times = (I_{\mathbb{C}^\times} \mathrm{Mfr})^G = I_{\mathbb{C}^\times} \mathrm{M} G$.

Conjecture (higher spin-statistics): In the real-analytic topology, the Galois group of $\Sigma^\bullet\mathbb{R} \subset \mathcal{R}$ is homotopy equivalent to $\mathrm{O}(\infty)$.

Warning: I do not have a definition of "the real-analytic topology" on a tower over $\mathbb{R}$. The Conjecture would require a good notion of positivity, similar to Freed–Hopkins "reflection positivity" for invertible phases.

Corollary: The Galois group of $\Sigma^\bullet\mathbb{C} \subset \mathcal{R}$ is homotopy equivalent to $\mathrm{SO}(\infty)$. The Galois group of $\Sigma^\bullet\mathrm{SV}\textsc{ec}_{\mathbb{C}} \subset \mathcal{R}$ is homotopy equivalent to $\mathrm{Spin}(\infty)$.

Corollary (Kapustin's conjecture): In the real-analytic topology,

$$\text{bosonic } \mathcal{I}^\bullet = (\Sigma^\bullet\mathbb{C})^\times \simeq \Sigma I_{\mathbb{Z}}\mathrm{MSO},$$
$$\text{fermionic } \mathcal{I}^\bullet = (\Sigma^\bullet\mathrm{SV}\textsc{ec}_{\mathbb{C}})^\times \simeq \Sigma I_{\mathbb{Z}}\mathrm{MSpin}.$$

Thank you!

Further details:

[arxiv:1507.06297] Spin, statistics, orientations, unitarity.

[arxiv:1712.07950] Symmetry protected topological phases and generalized cohomology.

[arxiv:1905.09566] Condensations in higher categories.

[arxiv:2003.06663] On the classification of topological orders.

[pirsa:20030111] Gapped condensation in higher categories.

[these slides] http://categorified.net/PI-Colloquium.pdf