Global Categorical Symmetries

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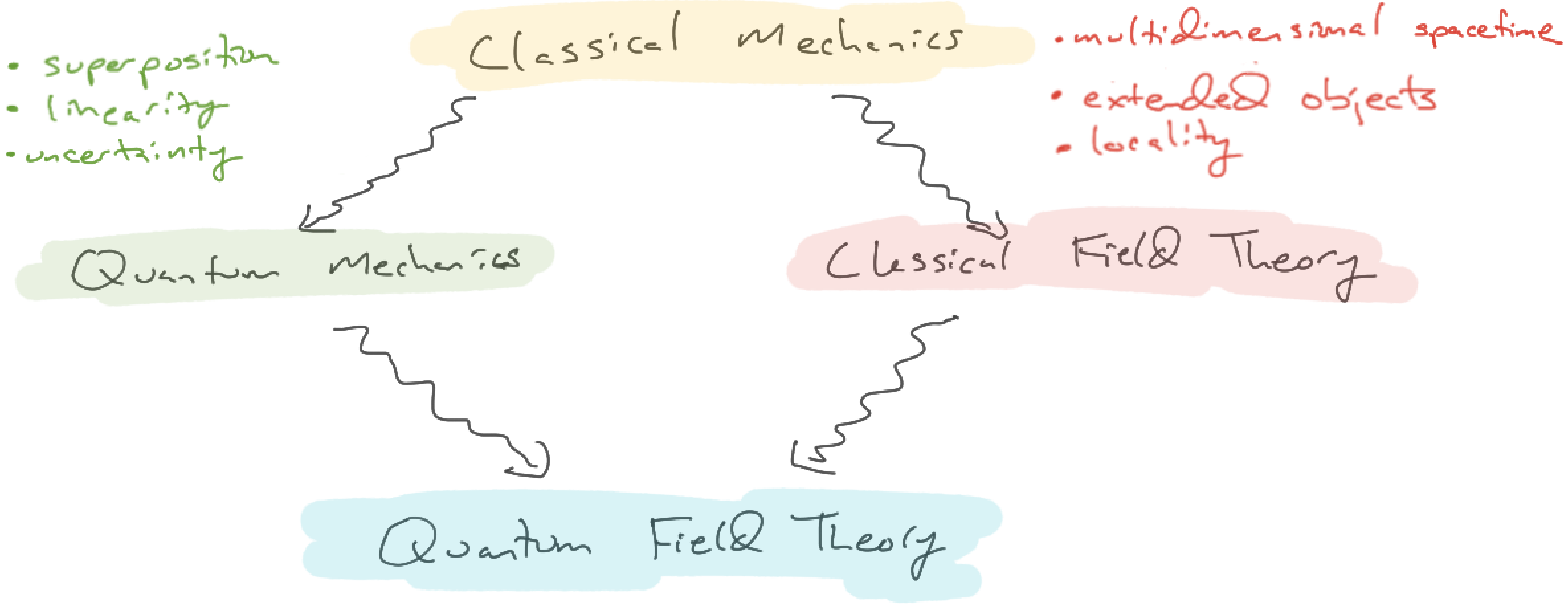
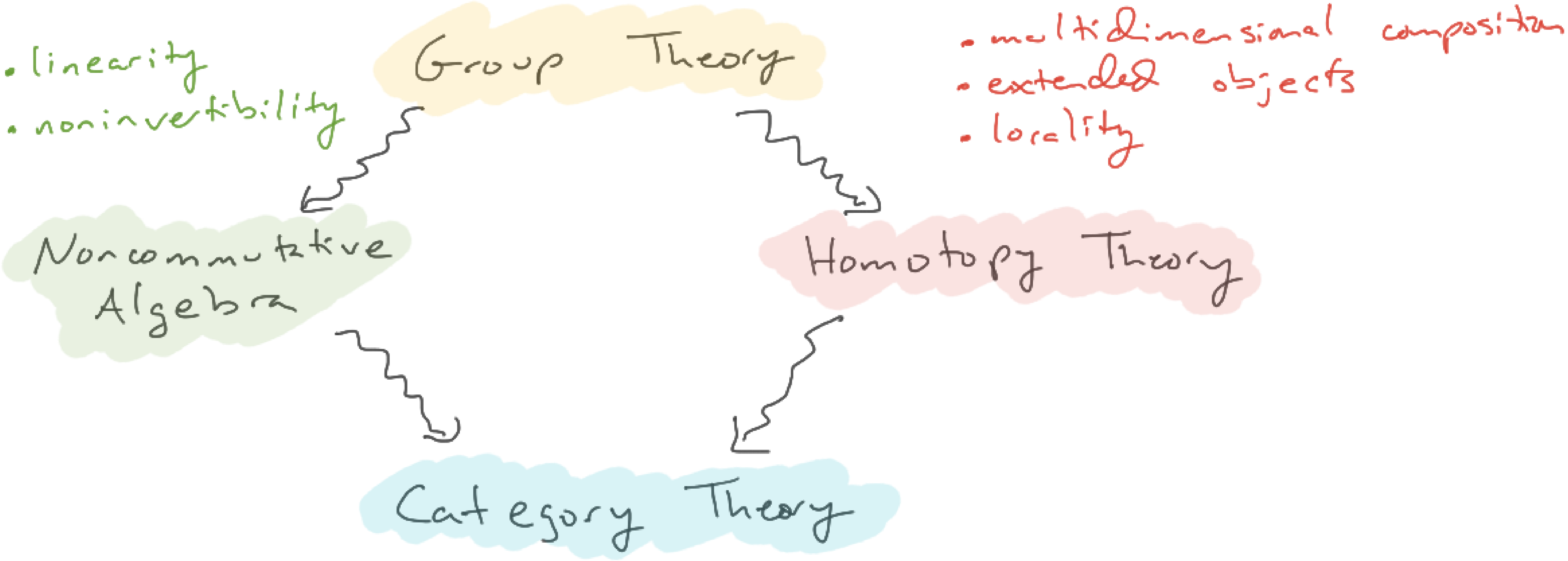
If you want to chat further, I'll be in town Oct 4–7. Or send an email.

These slides: categorified.net/PSI-ad.pdf

Symmetries are present in all subdisciplines of math + physics, and essentially every person at PI uses them in their research. I want to tell you about a kind of symmetry that is fundamentally quantum field theoretical. Recall that QFT generalizes classical mechanics in two orthogonal directions:



Classical symmetry operations are invertible, and the "temporal" order of composition matters. The abstract compositional structure of classical symmetries is controlled by a group. QFT provides two orthogonal generalizations:



The word "categorical" in GCS is doing double duty, refering to both of these generalizations at once.

The word "global" also does double duty. On one hand, it is a synonym for "flavour", as opposed to gauged/colour symmetries. But it also emphasizes an interest in the <u>global structure</u> of the symmetry "group". Ever since Noether, physicists have focused on infinitesimal symmetries, controlled by Lie algebras and implemented by Noether currents. Lie algebras do not see global structure, e.g. $SO(n)$ v.s. $O(n)$ v.s. $Spin(n)$.

I focus on an extreme case of this: <u>finite</u> symmetries, with <u>no</u>* infinitesimal structure.

* In characteristic zero. An amazing feature of finite group thy is that a finite gp <u>has</u> inf'l str in char dividing its order.

Even though finite symmetries lack Noether currents, they still have a Noether Theorem, because the "charge" operators

$$Q(\Sigma^d) = \text{“}\int_\Sigma \star J\text{”}$$

Noether current

Hodge star

surface on which to place Q.

still make sense. Noether: $d\star J = 0$, so $[Q, T^{\mu\nu}] = 0$.

In other words, Q is topological.

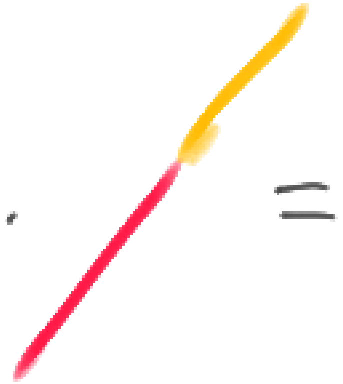
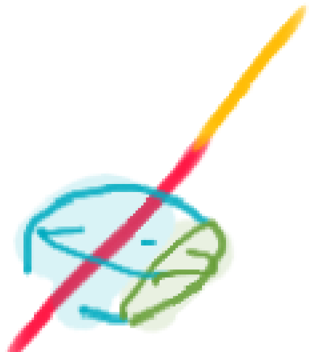
Defn: A p-form categorical symmetry is a topological, not nec. invertible, codim $= (p+1)$ operator.

It is called "p-form" because it assigns __charges__ to p-dim operators:

If  Q, a 2D top'l op. $= \mathbb{C} \ni \alpha \cdot$  , then Q acts on $\mathcal{O}$ with charge α.

$\mathcal{O}$, a 1D op. $\subseteq \mathbb{R}^4$ $\qquad \mathcal{O}$

__Lemma:__ Suppose $\mathcal{O}, \mathcal{O}'$ are connected by a ((p-1)-dim) interface, and Q, Q' are connected by a ((p+2)-codim) interface. Then charge$(Q, \mathcal{O})$ = charge$(Q', \mathcal{O}')$.

__Pf:__ $\alpha \cdot$  $=$  $=$  $=$  $=$  $=$  $= \alpha' \cdot$ 

Completeness of spectrum conjecture:

- If Q, Q' asign the same charges to all p-dim operators, then they are connected by a top'l interface.
- If $\mathcal{O}, \mathcal{O}'$ have the same charges under all p-form categorical symmetries, then they are connected by an interface.

Thm [JF–Reutter]:

This is true for $\underbrace{\text{topological}^*}$ quantum field theories.

$T^{\mu\nu}$ is central, so all operators are top'l.

To a great extent, GCS = "applied TQFT." E.g.:

"Anomaly Inflow": {QFTs w/ cat'l $\mathcal{C}$ symmetry}

nD

$\parallel$

{b.c.'s for "$\mathcal{C}$ gauge thy"}

an n+1 D TQFT.

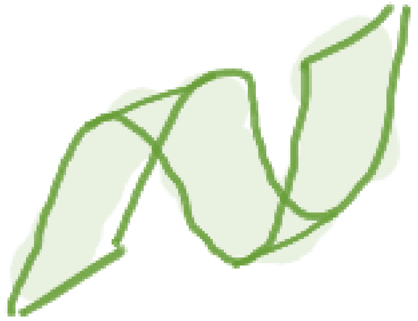
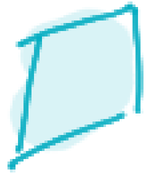
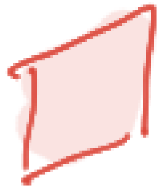
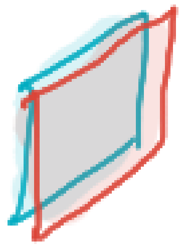
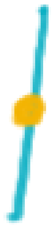
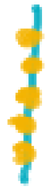
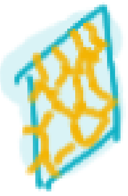
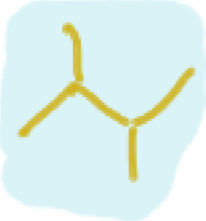
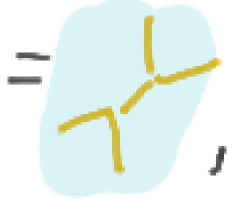
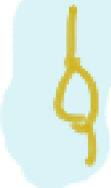
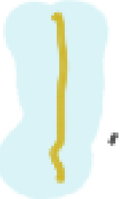
Sometimes TQFT methods can prove that a TQFT has no top'l b.c.s. This implies forced gaplessness for QFTs w/ $\mathcal{C}$ symmetry. E.g.:

Thm [JF-Reutter]: If a QFT has a framing anomaly (an invertible n+1 D framed thy) other than an Arf-Kervaire invariant, then it is gapless.

and these do have gapped b.c.s.

Finite global categorical symmetries are axiomatized by higher fusion categories. A fusion n-category is a collection of operators and interfaces of $\dim = 0, \ldots, n$ in $n+1$ ambient dimensions.

They should be:

- Fully topological. E.g.  is valid.
- Closed under $\otimes$: if  and  are op's, so is 
- Closed under $\oplus$ and higher idempotents: If  is s.t.  = , then  :=  exists. Ditto in higher dim, e.g.  if  = ,  = .
- Simple vacuum: Only 0-dim ops are multiples of 1.
- Homotopy-finite: All sets of form $\{\text{operators}\}/\{\text{interfaces}\}$ are finite.
- Semisimple: For every surface op., its cat of line ops is s.s. i.e. every line defect is a $\oplus$ of simples.

Active research questions

- Understand the "fusion rules" for higher fusion categories
- One source of higher fusion cats is homotopy-finite spaces (= finite higher gps) plus cohomology class plus equivariance data. How much more is there?
- Take constructions from homotopy theory, and implement the on higher fusion categories = "quantum homotopy types".
- Number Thy: Can higher fusion cats see beyond $\mathbb{Q}^{cyc}$?
- Analysis: Do higher fusion cats come from "higher subfactors"?
- Big Question: Move beyond the finite s.s. world.
- Many more!