Quantum Groups

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Contents

Introduction

In the Spring of 2009, Nicolai Reshetikhin taught Math 261B: Quantum Groups. These are the notes from that class. The class met three times a week — Mondays, Wednesdays, and Fridays — from 10am to 11am. NR's website for the course is at `http://math.berkeley.edu/~reshetik/math261B.html`. In particular, on NR's website are links to his hand-written lecture notes.

This course was a continuation of the Fall 2008 course Math 261A: Lie Groups, taught by Prof. Mark Haiman. My notes from that course are available at `http://math.berkeley.edu/~theojf/LieGroups.pdf`, and Anton Geraschenko's notes from Reshetikhin et al.'s similar course in 2006 are at `http://math.berkeley.edu/~anton/written/LieGroups/LieGroups.pdf`. This course assumes 261A as a prerequisite; in particular, many definitions used in the class are supplied in those notes.

As with my other course notes, I typed these mostly for my own benefit, although I do hope that they will be of use to other readers. (It was Anton's excellent notes from a variety of classes — in addition to the Lie groups notes mentioned above, he has other notes on his website — that inspired me to type my own notes, and I have borrowed from his preamble.) I apologize in advance for any errors or omissions. Places where I did not understand what was written or think that I in fact have an error will be marked ****like this****. Please e-mail me (`mailto:theojf@math.berkeley.edu`) with corrections. For the foreseeable future, these notes are available at `http://math.berkeley.edu/~theojf/QuantumGroups.pdf`.

These notes are typeset using TEXShop Pro on a MacBook running OS 10.5; the backend is pdfLATEX. Pictures are drawn using PGF/Ti*k*Z, a graphics language far-superior to XY-pic. The raw LATEX sources are available at `http://math.berkeley.edu/~theojf/LieQuantumGroups.tar.gz`. These notes were last updated May 8, 2009.

0.1 Conventions and numbering

Each lecture begins a new "section", and if a lecture breaks naturally into multiple topics, I try to reflect that with subsections. Equations, theorems, lemmas, etc., are numbered by their lecture. Theorems, lemmas, and propositions are counted the same, but corollaries are assumed to follow from the most recent theorem/lemma/proposition. Definitions are not marked qua definitions, but *italics* always mean that the word is being defined by that sentence. All definitions are indexed in the index. Homework exercises are numbered consecutively throughout the course. Sometimes exercises are set off as their own "theorem", but just as often they are mentioned in the text. A list of all homework exercises is at the end of the document, with page numbers. A list of all theorems, propositions, etc., is also at the end of the document. To generate these lists and to format theorems, etc., I have used the package `ntheorem`. Better referencing is done by `cleveref`.

Lecture 1 January 21, 2009

1.1 Introduction

In this class, we will study Quantum Groups, which are neither quantum nor groups. They are non-commucative non-cocommutative Hopf algebras, deformations of $\mathcal{U}\mathfrak{g}$ and $C(G)$. Unless we say otherwise, G will always be an affine algebraic group over $\mathbb{C}$. But this is too general. Any G has real forms. Did you go over the classification of real forms? No. Well, we all know that there is $SL_n(\mathbb{C})$, but an important but only one of the real forms is SU_n, e.g. $SL_n(\mathbb{R})$ is also important.

So that's sort of the main subject. Fortunately, Mark laid down the foundations in 261A, focusing on algebraic groups and defining the *universal enveloping algebra*.

Well, formalities. It depends on how many people survive to the end of the class. NR won't try to make it difficult to survive, but it is a matter of taste. Depending on how many people make it to the end, NR will have a take-home final so that you feel like you get something out of the class. There will be homeworks, which NR will grade each month, but this is a graduate course, so grades don't matter that much. Only the homework will matter towards the grade.

NR will post hand-written lecture notes on the website, some links past `http://math.berkeley.edu/~reshetik`. Office hours will be Monday 4-5:30.

1.2 Let us start

Let $\mathfrak{g}$ be a Lie algebra. If there is a complex algebraic group G with $\mathfrak{g} = \mathrm{Lie}(G)$, then to $\mathfrak{g}$ we can associate two Hopf algebras $\mathcal{U}\mathfrak{g}$ and $C(G)$, and these are dual.

If $\mathfrak{g}$ is a Lie algebra, then let $\mathfrak{g}^*$ be the *dual vector space*, and let's fix a Lie algebra structure on $\mathfrak{g}^*$. So we have a pair of algebras, and subject to certain compatibility restrictions, this pair $(\mathfrak{g}, \mathfrak{g}^*)$ will be called a *Lie bialgebra*. Before we give the definition, we draw a map of quantum groups.

To the bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, then $\mathfrak{g}$ gives us a connected simply-connected Lie group G, and $\mathfrak{g}^*$ gives us another (connected simply-connected) group G^*. So we get a *dual pair of Lie groups* G and G^*, and out of this we can construct, assuming everything is algebraic, the pair of Hopf algras $\mathcal{U}\mathfrak{g}$ and $C(G)$, and also the pair $\mathcal{U}\mathfrak{g}^*$ and $C(G^*)$. Each is a pair of dual Hopf algebras, and the pairs are dual to each other in this other sense. Then we will have corresponding quantum groups $\mathcal{U}_q(\mathfrak{g})$ and $C_q(G^*)$, deformations in the category of Hopf algebras of $\mathcal{U}(\mathfrak{g})$ and $C(G)$, and also $\mathcal{U}_q(\mathfrak{g}^*)$ and $C_q(G)$. In fact, the algebras $\mathcal{U}_q(\mathfrak{g})$ and $C_q(G^*)$ are more or less the same — they are the same algebraically, but the topology is different. So the slang is that "after quantization, there is no difference between Universal Enveloping Algebra and Algebra of Functions."

The story will continue with Poisson geometry. Who knows what is a *Poisson manifold*? ****very few**** Well, the deformations above are essentially a reformulation of what you already know. But the general notion of quantization first appeared in physics, and then filtered to mathematics and eventually representation theory. The general idea is that to a *symplectic manifold* $(\mathcal{M}, \omega)$, and maybe using extra data, you can construct a family of associative algebras $A_\hbar$, but the center of $A_\hbar$ is usually trivial $(\mathbb{C} \cdot 1)$. But to a Poisson manifold $(\mathcal{P}, p)$, the family, which at least exists formally, can be very interesting. And there is a notion of *symplectic leaves*, and a general philosophy which is hard to formulate precisely, that to symplectic leaves we should associate irreducible representations.

So, we now start from the beginning: a definition of Lie bialgebra, and then a digression into Poisson geometry whence we will understand that G will have a natural Poisson structure, and whence the notion of *Poisson Lie group*. Why "Poisson Lie" and not "Lie Poisson" you'll have to ask Drinfeld. Probably because "Lie group" sounds like one word.

In Lie theory, we normally introduce the more natural notion of Lie group, then define Lie algebra by noticing that the tangent space at the identity has some natural structure. But we will go in the opposite direction, so to avoid having to know what a Poisson manifold is.

For now, we consider only finite-dimensional algebras over $\mathbb{C}$. A pair $(\mathfrak{g}, \delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g})$ is a *Lie bialgebra* if $\mathfrak{g}$ is a Lie algebra and δ satisfies:

1. δ is a *Lie cobracket*. We can understand this condition in two ways: either that $\delta^* : \bigwedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket, and also by the *co-Jacobi identity*:
$$\mathrm{Alt}(\delta \otimes \mathrm{id}) \circ \delta = 0 \tag{1.1}$$

2. a compatibility condition:
$$\delta([x, y]) = [x, \delta(y)] + [\delta(x), y] \tag{1.2}$$
This is a *cocycle property* of δ. We define $[x, y \wedge z] \stackrel{\text{def}}{=} [x, y] \wedge z + y \wedge [x, z]$.

Example 1.1 Let $\mathfrak{b}_+ = \mathbb{C}H \oplus \mathbb{C}X$ with $[H, X] = 2X$. Then you can check (this is **Exercise 1**) that $\delta(H) = 0$, $\delta(X) = H \wedge X$ makes $\mathfrak{b}_+$ into a Lie bialgebra. ◊

Question from the audience: I don't understand how to bracket an element of $\mathfrak{g}$ with a wedge product. **Answer:** It's the diagonal action on the exterior square: write $\delta(x) = \sum_i x^i \wedge x_i$, and then define $[y, \delta(x)] = \sum_i [y, x^i] \wedge x_i + \sum_i x^i \wedge [y, x_i]$.

We mentioned that the definition provides a "cocycle condition". Did Mark discuss Lie cohomology? ****audience says no****. Who knows BRST? Chevalley complex? ****one student each****.

Our problem is that we misspell the names. We define the *Chevalley complex* for a Lie algebra as the complex $C^\bullet(\mathfrak{g}, M) = \text{Linear Maps}(\bigwedge^\bullet \mathfrak{g} \to M)$, where M is a $\mathfrak{g}$-module. Let $C^n = \text{Hom}(\bigwedge^n \to M)$; then the differential $d : C^n \to C^{n+1}$ is given by

$$\begin{aligned} df(x_1, \ldots, x_{n+1}) = &\sum_{i<j} (-1)^{i+j-1} f([x_i, x_j], x_1, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_{n+1}) \\ &+ \sum_i (-1)^i x_i \cdot f(x_1, \ldots, \hat{x_i}, \ldots, x_{n+1}). \end{aligned} \tag{1.3}$$

Here $\cdot$ is the action of $\mathfrak{g}$ on M.

Exercise 2 *Show that $d^2 = 0$.*

Who knows the notion of Grassman algebra? Exterior algebra of a vector space? There is a very simple description of the Chevalley complex in terms of Grassman algebra $\bigwedge^\bullet \mathfrak{g}$. Let $\{c_i\}$ be a basis of $\mathfrak{g}$; then $\bigwedge^\bullet \mathfrak{g}$ is the associative algebra generated by the c^i subject to $c^i c^j + c^j c^i = 0$. For simplicity, let $M = \mathbb{C}$. Let f_{ij}^k be the structure constants. Then $d = \sum_{ijk} f_{ij}^k c^i c^j \frac{\partial}{\partial c^k}$.

Exercise 3 *Make sense of this. You need to be careful about upper and lower indices.*

See, when $M = \mathbb{C}$, then $C^* = \bigwedge^* \mathfrak{g}^*$. A mathematician (Chevalley) invented this, and the physicists reinvented it and called it BRST.

Now, let's consider $M = \mathfrak{g}$ with the adjoint action $x \cdot f = [x, f]$. Consider $(\bigwedge^\bullet(\mathfrak{g} \oplus \mathfrak{g}^*))^* = \bigwedge^\bullet(\mathfrak{g} \oplus \mathfrak{g}^*) \cong \bigwedge^\bullet \mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g}^*$. This is a bigraded vector space. The nth row is dual to the Chevalley complex: $M = \bigwedge^n \mathfrak{g}$. Probably we shouldn't have dualized earlier. So each row has different coefficients. But now let's say we had a bialgebra structure. Then we also have vertical maps from the cohomology of $\mathfrak{g}^*$.

$$\begin{array}{ccccccccccc}
\mathbb{C} & \xrightarrow{d_\mathfrak{g}} & \mathfrak{g}^* & \xrightarrow{d_\mathfrak{g}} & \bigwedge^2\mathfrak{g}^* & \xrightarrow{d_\mathfrak{g}} & \bigwedge^3\mathfrak{g}^* & \xrightarrow{d_\mathfrak{g}} & \cdots & & M=\mathbb{C}\\
\downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & & & \\
\mathfrak{g} & \xrightarrow{d_\mathfrak{g}} & \mathfrak{g}\otimes\mathfrak{g}^* & \xrightarrow{d_\mathfrak{g}} & \mathfrak{g}\otimes\bigwedge^2\mathfrak{g}^* & \xrightarrow{d_\mathfrak{g}} & \mathfrak{g}\otimes\bigwedge^3\mathfrak{g}^* & \xrightarrow{d_\mathfrak{g}} & \cdots & & M=\mathfrak{g}\\
\downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & & & \\
\bigwedge^2\mathfrak{g} & \xrightarrow{d_\mathfrak{g}} & \bigwedge^2\mathfrak{g}\otimes\mathfrak{g}^* & \xrightarrow{d_\mathfrak{g}} & \bigwedge^2\mathfrak{g}\otimes\bigwedge^2\mathfrak{g}^* & \xrightarrow{d_\mathfrak{g}} & \bigwedge^2\mathfrak{g}\otimes\bigwedge^3\mathfrak{g}^* & \xrightarrow{d_\mathfrak{g}} & \cdots & & M=\bigwedge^2\mathfrak{g}\\
\downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & & & \\
\bigwedge^3\mathfrak{g} & \xrightarrow{d_\mathfrak{g}} & \bigwedge^3\mathfrak{g}\otimes\mathfrak{g}^* & \xrightarrow{d_\mathfrak{g}} & \bigwedge^3\mathfrak{g}\otimes\bigwedge^2\mathfrak{g}^* & \xrightarrow{d_\mathfrak{g}} & \bigwedge^3\mathfrak{g}\otimes\bigwedge^3\mathfrak{g}^* & \xrightarrow{d_\mathfrak{g}} & \cdots & & M=\bigwedge^3\mathfrak{g}\\
\downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & \downarrow{\scriptstyle d_{\mathfrak{g}^*}} & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & & &
\end{array}$$

Question from the audience: So we have a bunch of Chevalley complexes, and if $\mathfrak{g}^*$ also has a Lie algebra structure, then this is a complex of complexes? **Answer:** Well, the map $[,]:\bigwedge^2\mathfrak{g}\to\mathfrak{g}$ gives the differential $d_\mathfrak{g}$, and Jacobi gives $d^2=0$. If we also have a cobracket with coJacobi, then we get the vertical maps $d_{\mathfrak{g}^*}$. It's a neat way of saying the bracket satisfies Jacobi: it's equivalent to saying that $d^2=0$. And what's the meaning of the compatibility? Well, we have a bicomplex, and the key question is when two differentials commute? Well, they never actually commute; when do they anticommute? I.e. when is $d_\mathfrak{g}d_{\mathfrak{g}^*}+d_{\mathfrak{g}^*}d_g=0$? The answer is that this happens exactly if δ dual to the cobracket gives a Lie bialgebra structure for $\mathfrak{g}$.

So this is the algebraical notion of a Lie bialgebra. Any questions? So this finishes the Lie bialgebra discussion, and next time we will have examples and the definition of Poisson structure.

Question from the audience: So δ gives the Lie algebra structure on $\mathfrak{g}^*$? **Answer:** Yes. It must both be a cobracket, and also be compatible.

Next time, we will see the following: If $\mathfrak{g}$ is a Lie algebra, it's a well-known fact that $\mathfrak{g}^*$ is a Poisson manifold. But in fact we will get a non-linear version of this on G, and many notions like the adjoint and coadjoint actions will all have nonlinear counterparts.

Lecture 2 January 23, 2009

2.1 Poisson algebras

We continue with the Lie bialgebras and Poisson Lie groups. Last time we gave the definition of a *Lie bialgebra*. We explained that to $\mathfrak{g}$ a Lie algebra we associate the *Chevalley complex* $C^\bullet(\mathfrak{g})\cong(\bigwedge^\bullet\mathfrak{g})^*$ with differential $d_\mathfrak{g}$. If we have a Lie bialgebra, we can construct $C^\bullet(\mathfrak{g}\oplus\mathfrak{g}^*)$ with two differentials $d_{\mathfrak{g}}$ and $d_{\mathfrak{g}^*}$, which anti-commute. But this doesn't explain the terminology: in what sense is $\delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ a 1-cocycle? Well, δ gives a map $\bigwedge^\bullet \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$, and δ gives a 1-cocycle in the complex $C^\bullet(\mathfrak{g}; \bigwedge^2 \mathfrak{g})$.

But these notions make more sense within the language of Poisson Lie groups. We begin with the definition of a *Poisson algebra*: this is a pair $(A, \{,\})$ such that

- A is a commutative algebra. ****unital?****
- The vector space A with $\{,\} : A^{\otimes 2} \to A$ is a Lie algebra.
- $\{,\}$ is a biderivation: $\{a, bc\} = \{a, b\}c + b\{a, c\}$.

So it is a mixture of Lie algebra and commutative algebra.

There are several reasons why this structure is very natural.

Example 2.1 Consider a family of associative multiplications $*_h : A^{\otimes 2} \to A$ such that

- $a *_0 b = ab$ is commutatiive.
- Let's assume that the multiplication is given by an analytic function in h: $a *_h b = ab + hm_1(a, b) + h^2 m_2(a, b) + O(h^3)$ as $h \to 0$. Of course, to say this precisely we must introduce some topology on A.

Proposition 2.1 $\{a, b\} \overset{\text{def}}{=} m_1(a, b) - m_1(b, a)$ *is a Poisson structure on a commutative algebra* A.

This is probably the reason Poisson algebra is so important: they describe the infinitesimal jets of deformations of commutative algebra.

Proof: It's a very nice and very easy **Exercise 4**. All you have to check is that the Jacobi identity on the commutator induces the Jacobi on $\{,\}$, and then you have to check the Leibniz rule. □

(Again, homeworks mostly will not be graded, but eventually NR will give homework that is graded. **Question from the audience:** Will there be pdfs with the homework? **Answer:** Yes, the handwritten notes are already there — don't skip lecture — and Theo is typing the notes.)

Since this is a deformation, it is in a sense "quantum", and it is in this sense that Quantum Groups are quantum. ♢

Example 2.2 A *Poisson manifold* is a pair (M, p) where $p \in \Gamma(\bigwedge^2 TM)$. We let M be a manifold (C^∞, affine algebraic variety, etc.; and whatever M is, we take the appropriate type of section p). So this p is a *bivector field*, and in local coordinates $p(x) = \sum_{ij} p^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$. Any time we choose local coordinates $\{x^i\}$ on M, then we let dx^i be the corresponding basis of $T_x^* M$ and $\frac{\partial}{\partial x^i}$ the basis in $T_x M$. Then we define $\{f, g\} \overset{\text{def}}{=} \langle p, df \wedge dg \rangle = \sum_{ij} p^{ij}(x) \frac{\partial f}{\partial x^i} \wedge \frac{\partial g}{\partial x^j}$, and we demand that $\{,\}$ is a Poisson bracket on $C(M)$ (the space of C^∞ or polynomial or whatever functions). That $\{,\}$ is a biderivation is trivial, since everything is a first-order operator. That it satisfies the Jacobi identity requires p to satisfy a non-trivial condition. ****I missed the equation it must satisfy.**** ♢

Example 2.3 Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^*$ its dual vector space. Since these are vector spaces, we have a canonical isomorphism $T_x^*\mathfrak{g}^* \cong \mathfrak{g}$ for each $x \in \mathfrak{g}^*$. ****finite-dimensional**** (If it is something more complicated, you may have to make a choice of isomorphism.) Thus, for any $f \in C(\mathfrak{g}^*)$ and $x \in \mathfrak{g}^*$, then $df(x) \in \mathfrak{g}$. So if $f, g \in C(\mathfrak{g}^*)$, we can define $[f(x), g(x)]$, and we define the Poisson bracket

$$\{f,g\}(x) \stackrel{\text{def}}{=} \langle x, [df(x), dg(x)]\rangle \tag{2.1}$$

This is the *Lie-Kirollov-Kostant bracket.* It was discovered by Lie, and used to study the representation of Lie algebras.

It's probably more instructive to see this is local coordinates. Let $\{e_i\}$ be a basis in $\mathfrak{g}$, and $\{x_i\}$ the coresponding coordinate functions on $\mathfrak{g}^*$. Then

$$\{x_i, x_j\} = \sum_k f_{ij}^k x_k \tag{2.2}$$

where f_{ij}^k are the *structure constants*: $[e_i, e_j] = \sum_k f_{ij}^k e_k$. $\Diamond$

If (M_1, p_1) and (M_2, p_2) are Poisson manifolds, we can define their *product* as $(M_1 \times M_2, p_{12})$ where p_{12} is the sum of p_1 and p_2. More explicitly, $T_{(x,y)}(M_1 \times M_2) = T_x M_1 \oplus T_y M_2$, so $\bigwedge^2 T_{(x,y)}(M_1 \times M_2) = \bigwedge^2 T_x M_1 \oplus T_x M_1 \otimes T_y M_2 \oplus \bigwedge^2 T_y M_2$, and we define $p_{12}(x,y) \stackrel{\text{def}}{=} p_1(x) \oplus 0 \oplus p_2(y)$.

Now, let's assume M_1, M_2 are affine algebraic, so that $C(M_1 \times M_2) = C(M_1) \otimes C(M_2)$. Then

$$\{f_1 \otimes f_2, g_1 \otimes g_2\} = \{f_1, g_1\} \otimes f_2 g_2 + f_1 g_1 \otimes \{f_2, g_2\} \tag{2.3}$$

If A_1, A_2 is a pair of Poisson manifolds, we define their *tensor product* $A_1 \otimes_{\mathbb{C}} A_2$ to be the Poisson algebra with the bracket defined by 2.3.

Question from the audience: Even if M_i are not algebraic, this still works for functions that are a product. **Answer:** Yes, but you won't get all functions.

If A_1 and A_2 are two Poisson algebra, then $\phi : A_1 \to A_2$ is a *morphism of Poisson algebras* if $\phi(ab) = \phi(a)\phi(b)$ and $\phi(\{a,b\}) = \{\phi(a), \phi(b)\}$. We define a map $\phi : (P_1, p_1) \to (P_2, p_2)$ to be a *morphism of Poisson manifolds* if ψ is a manifold map and the pullback ψ^* is a morphism of Poisson algebras.

So Poisson algebra is the Lie-algebraic enhancement of commutative algebra. We can ask what is the analogue of representations. One can argue that the correct analogue is the very important notion in Poisson geometry called *symplectic leaves*

2.2 Symplectic leaves

Who knows the definition of a "symplectic manifold"? ****Four or five hands.**** A *symplectic manifold* is a pair (M, ω) where M is a manifold (in your favorite category) and ω is a nondegenerate closed 2-form. More precisely:

Nondegeneracy in local coordinates $\omega(x) = \sum_{ij} \omega_{ij}(x) dx^i \wedge dx^j$, then we demand that $\det(\omega_{ij}(x)) \neq 0$ for every $x \in M$. Equivalently, the matrix ω_{ij} is invertible.

A corollary is that the dimension of M must be even. With physics, this is very easy to remember: symplectic manifolds are phase spaces.

We can introduce the *symplectic volume* $\omega^{\wedge(n/2)}$, where $n = \dim M$, and this just rewrites the determinant, so we demand that it never vanishes.

Closure $d\omega = 0$.

These were invented in the 19th century when trying to study classical mechanics.

Example 2.4 Let $M_{2n} = (\mathbb{R}^n) \oplus (\mathbb{R}^n)^*$, with coordinates p_i, q^i. Then $\omega = \sum_{i=1}^n dp_i \wedge dq^i$.

Example 2.5 Let N_n be a smooth n-dimensional manifold, and $M = T^*N$, which is $2n$-dimensional. Choose a chart $U \in N$ so that $T^*U \cong (\mathbb{R}^n)^* \times U$ and now $U \in \mathbb{R}^n$. Then in local coordinates $\omega = \sum_i dp_i \wedge dq^i$.

Exercise 5 *Find a natural 1-form θ on $M = T^*N$ such that $\omega = d\theta$.*

We are moving slowly but surely towards the interesting definitions.

A very important theorem, which again is a simple exercise but explains why we're talking about it, is the following:

Theorem 2.2 *Any symplectic manifold is a Poisson manifold: $\{f,g\} \overset{\text{def}}{=} \langle \omega^{-1}, df \wedge dg \rangle$.*

By the *inverse of a form* we mean the following: $\omega \in \Gamma^2(\bigwedge^2 T^*M)$ is nondegenerate, so it gives an isomorphism $\omega : TM \to T^*M$. In local coordinates, if $v = \sum v^i \frac{\partial}{\partial x^i} \in TM$ and $\omega = \sum \omega_{ij} dx^i \wedge dx^j$, then $\omega(v) \overset{\text{def}}{=} \sum \omega_{ij} dx^i v^j$. Then nondegeneracy implies that there is a bivector $\omega^{-1} \in \Gamma(\bigwedge^2 TM)$ giving the opposite map, which in coordinates is given by $\omega^{-1}(x) = \sum (\omega^{-1})^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$, where $(\omega^{-1})^{ij}$ is the inverse matrix to ω_{ij}.

Proof: Is this trivial? ****quiet class, without consensus**** We will leave it as **Exercise 6**, but the hint is that closure $d\omega = 0$ is equivalent to the Jacobi for $\{,\}$. The proof also works in the opposite direction: if a Poisson manifold is suitably nondegenerate, then the inverse of the bivector gives a symplectic structure. □

We mentioned classical mechanics several times. Now we introduce the notional of a *Hamiltonian vector field*. A vector field v on a Poisson manifold is *Hamiltonian* if there exists $H \in C(M)$ such that $v = p\langle dH \rangle$. Recall that $p \in \Gamma(\bigwedge^2 TM)$, so $p(x) : T_x^*M \to T_xM$. In local coordinates $p(x)\langle dH(x) \rangle = \sum_{ij} p^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$.

Then H is called the *Hamiltonian functino* for v. We get the flow lines of v_H, and these are called *Hamiltonian flow lines*. Then classical Hamiltonian mechanics, from this point of view, is exactly the study of dynamical systems defined by Hamiltonian vector fields. In other words, they are generated by H.

Just one last word. A *symplectic leaf* on (P, p) through x is the space of points on the manifold that you can reach by piecewise Hamiltonian flow starting at x. We will see that symplectic leaves are identical to co-adjoint orbits of ****missed****. Then we will study Poisson Lie groups, which are an enhancement of the geometry of Lie groups, and then we will deform everything.

Question from the audience: It's not obvious that you get a manifold doing this. **Answer:** Not at all. It is a theorem.

Lecture 3 January 26, 2009

3.1 More symplectic geometry

Last time, we discussed Poisson algebras, Poisson manifolds, symplectic manifolds, and symplectic leaves of Poisson manifolds. Today we begin by finishing this subject.

We recall the following definition. A *symplectic leaf* through $x \in (M, p)$, where (M, p) is a Poisson manifold, is a span of piecewise Hamiltonian flow lines through x.

Theorem 3.1 *A symplectic lead is a submanifold.*

The generic situation will be the following. There will be functions on M which Poisson-commute with everything. These are *Casimir functions*: $f \in C(M)$ such that $\{f, g\} = 0$ for all $g \in C(M)$. And level sets of these functions will be the symplectic leaves. This isn't quite true: a level surface need not be connected. But symplectic leaves will be contained within level sets of Casimir functions.

The analogy with representation theory is that, to construct an irreducible representation, we have to first fix all the central elements to be complex numbers.

Example 3.1 $G = SU_2$, the group of two-by-two unitary matrices with det $= 1$. This is the compact real form of $SL_2(\mathbb{C})$. Its Lie algebra $\mathfrak{g} = \text{Lie}(G)$ is a real three-dimensional algebra, so $\mathfrak{g}^*$ is a Poisson manifold. What are the symplectic leaves? They are the co-adjoint orbits of G acting on $\mathfrak{g}^*$.

More precisely, G acts on $\mathfrak{g}$ by the adjoint action. This is obvious for matrix algebras, whence this is conjugation. Who knows the invariant definition? ****most hands**** NR will skip definitions that most of the class knows, leaving it to the last one or two to look up. If G is a matrix group (so $G \subseteq GL(V)$), then $\mathfrak{g} = \text{Lie}(G)$ is a matrix Lie algebra (so $\mathfrak{g} \subseteq \mathfrak{gl}(V)$), and so the adjoint action is $\text{Ad}_g(x) \stackrel{\text{def}}{=} gxg^{-1}$, and we define the *co-adjoint action* by

$$\text{Ad}_g^*(l)(x) \stackrel{\text{def}}{=} l(\text{Ad}_{g^{-1}}(x)), \;\; l \in \mathfrak{g}^*, \; x \in \mathfrak{g}, \; g \in G \tag{3.1}$$

For SU_2, Ad^*-orbits through

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{pmatrix}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$. So this is a generic traceless Hermitian 2×2 matrix. The co-adjoint orbit of this is

$$\mathrm{Ad}^*_G \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{pmatrix} = \left\{ u \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{pmatrix} u^{-1} : u \in SU_2 \right\} \tag{3.2}$$

Question from the audience: That's the adoint orbit. **Answer:** Well, $\mathfrak{g}$ has a killing form $\langle x, y\rangle = \mathrm{tr}(xy)$, which is nondegenerate for $\mathfrak{su}_2$, so $\mathfrak{g} \cong \mathfrak{g}^*$ as an isomorphism of G-modules.

Now, what is the moduli space of conjugacy classes? Eigenvalues, module permutation. So

$$\mathrm{Ad}^*_G \cong \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right\} / \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \sim \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\} \tag{3.3}$$

Question from the audience: Hermitian or anti-Hermitian? **Answer:** Doesn't matter, because you can multiply by i.

Is this clear? The conjugation by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU_2$$

In any case, equation 3.3 gives just the ray $\mathbb{R}_{\geq 0}$.

Let's let H have eigenvalues $\pm\lambda$. Then $\mathrm{tr}(H^2) = 2\lambda^2$ is an invariant of the conjugation. Let's write in terms of the basis of Pauli matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -0 \end{pmatrix} \tag{3.4}$$

with coordinates x_1, x_2, x_3, then $\mathrm{tr}(H^2) = 2(x_1^2 + x_2^2 + x_3^2)$. So the Ad^*-orbits are the spheres, which are two-dimensional, and the special leaf at the origin. $\Diamond$

Question from the audience: Does this generalize to SU_n? **Answer:** We will later prove, for every Lie group:

Theorem 3.2 *Sumplectic leaves of $\mathfrak{g}^*$ are Ad^*-orbits of G acting on $\mathfrak{g}^*$.*

Example 3.2 What about SU_n? We identify $\mathfrak{su}_n \cong \mathfrak{su}_n^*$ as vector spaces via $\langle x, y\rangle = \mathrm{tr}(xy)$. The co-adjoint orbits are classified by the set of eigenvalues, so this is $\mathbb{R}^{n-1}/S_n$. It's $\mathbb{R}^{n-1}$ embedded as the hyperplane $\{\sum_i \lambda_i = 0\}$. These are the level surfaces of $c_i = \mathrm{tr}(x^i)$, $i = 2, \dots, n$. (Because $c_1 = 0$ on this hyperplane.) $\Diamond$

These two examples are of real Poisson manifolds. What if everything is complex-homolmorphic?

Example 3.3 $G = SL_2(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, which is non-canonically isomorphic to $\mathbb{C}^3$. But the Killing form gives a canonical isomorphism $\mathfrak{g}^* \cong \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. The theorem that symplectic leaves are Ad^*-orbits still holds, and we identify these with adjoint orbits. Let's understand the orbit:

$$C \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \left\{ g \begin{pmatrix} a & b \\ c & -a \end{pmatrix} g^{-1} : g \in SL_2(\mathbb{C}) \right\}$$

These come in various forms.

1. If the matrix is diagonalizable and non-zero, then the matrix is classified by its eigenvalues $\pm\lambda$. So the set of orbits through diagonalizable matrices is $(\mathbb{C} \setminus \{0\})/\mathbb{Z}_2$. Each orbit is 2-dimensional over $\mathbb{C}$.
2. If the matrix is not diagonalizable, it must have 0s on diagonal to be traceless. So, there's only one orbit, the one through $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. What is it's dimension? How do you compute the dimension of an orbit? You subtract the dimension of the stabilizer. And the stabilizer of x is the unipotent matrices
$$\exp(ax) = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\}$$
3. There is the 0-dimensional orbit through 0. ♢

You can do GL_n at home.

Exercise 7 *List all possible dimensions of conjugation orbits for $\mathfrak{sl}_3$, and describe the set of such orbits.*

In the complex-analytic case, symplectic leaves are complex homomorphic symplectic manifolds, and are algebraic. You can treat them like the real manifolds you're used to.

Exercise 8 *Consider the group of triangular matrices*
$$\begin{pmatrix} a & a_1 & b_1 \\ 0 & b & c_1 \\ 0 & 0 & c \end{pmatrix}$$

Any questions?

3.2 Poisson Lie groups

You should specify a category in which you want to work, and then be consistent within this category. For example, real smooth, affine algebraic, etc.

A pair (G, p) where G is a Lie group and $p \in \Gamma(\bigwedge^2 TG)$ is a Poisson structure on G is a *Poisson Lie group* if

1. Multiplication $\mu : G \times G \to G$ is a Poisson map.
2. $g \mapsto g^{-1}$ is also a Poisson map.

Exercise 9 *Check if $g \mapsto g^{-1}$ follows from the first condition. We will give the answer later.*

How does this relate to Lie bialgebras? We consider the tangent space at the identity.

We have two special geometric structures. The multiplication, and the Poisson structure, and these are compatible. Before we start further discussion, let's say a few words about tangent bundles.

If G is a group, then the tangent bundle TG is trivial: $TG \cong \mathfrak{g} \times G$, and we will always in this course choose the trivialization by left-translation. I.e. $\ell_g : G \to G$ is $x \mapsto gx$. Then $d\ell_g : TG \overset{\sim}{\to} TG$. It takes $T_h G \to T_{gh} G$, and in particular $d\ell_{h^{-1}} : T_h \overset{\sim}{\to} T_e G = \mathfrak{g}$. So $d\ell : TG \overset{\sim}{\to} \mathfrak{g} \times G$ by $(\xi, h) \mapsto (d\ell_{h^{-1}}(\xi), h)$.

So, the Poisson structure is a section $p \in \Gamma(\bigwedge^2 TG)$, which consists of section maps $G \to \bigwedge^2 TG$, but we can identify this with maps $G \to \bigwedge^2 \mathfrak{g}$. So if $x \in G$, then $p(x) \in \bigwedge^2 \mathfrak{g}$.

Exercise 10 $p(0) = 0$. ****Feb 9: certainly this should be $p(e) = 0$?****

Corollary 3.2.1 *$dp(e) : T_e G \to \bigwedge^2 T_0 \mathfrak{g}$; in other words this is a map $\mathfrak{g} \to \bigwedge^2 \mathfrak{g}$.*

We will choose this as our Lie bialgebra structure.

Lecture 4 January 28, 2009

We pick up where we left off last time. We stated the definition of the tangent Lie bialgebra to a Poisson Lie group. Recall, a *Poisson Lie group* is a Lie group G along with a compatible Poisson structure p. We take $\mathfrak{g} = \mathrm{Lie}(G) = T_e G$, and we want to use the Poisson structure p to construct a bialgebra structure on $\mathfrak{g}$. As we did last time, we trivialize $TG \cong \mathfrak{g} \times G$ by left translations, and then $p \in \Gamma(\bigwedge^2 TG)$ — which is a *section map*, i.e. a map $p : G \to \bigwedge^2 TG$ such that the natural projection $\pi : \bigwedge^2 TG \to G$ composes to that $\pi \circ p = \mathrm{id}_G$ — after trivialization, p becomes a map $p : G \to \bigwedge^2 \mathfrak{g}$. So a *Poisson structure on a Lie group* is a map $p : G \to \bigwedge^2 \mathfrak{g}$ with the following compatibility condition (equivalent to the earlier condition):

$$p(xy) = (\mathrm{Ad}_x \otimes \mathrm{Ad}_y) p(y) + p(x) \tag{4.1}$$

Exercise 11 *Verify this.*

****I think the equation should read $p(xy) = (\mathrm{Ad}_x \otimes \mathrm{Ad}_x) p(y) + p(x)$. The RHS of equation 4.1 is not obviously antisymmetric, and I don't believe it it internally consistent.****

****Actually, the formula is totally wrong. If you use *left*-translations to trivialize, the formula should read:**

$$p_l(xy) = p_l(y) + (\mathrm{Ad}_{y^{-1}} \otimes \mathrm{Ad}_{y^{-1}}) p_l(x) \tag{4.2}$$

and if you use *right*-translations, then you get

$$p_r(xy) = p_r(x) + (\mathrm{Ad}_x \otimes \mathrm{Ad}_x) p_r(y) \tag{4.3}$$

where each of p_l and p_r are functions $G \to \bigwedge^2 \mathfrak{g}$. Very precisely, $p_l(g) = (dl_{g^{-1}} \otimes dl_{g^{-1}})p(g)$ and $p_r(g) = (dr_{g^{-1}} \otimes dr_{g^{-1}})p(g)$, where l_g and r_g are the left- and right-translations of G by $g \in G$.**

Another name for this is to say that p is a 1-cocycle in the standard cohomology complex for G with coefficients in $\bigwedge^2 \mathfrak{g}$.

In any case, $dp : TG \to T(\bigwedge^2 \mathfrak{g}) \cong \bigwedge^2 T\mathfrak{g}$. At the identity $e \in G$, equation 4.1 becomes

$$p(ee) = p(e) + p(e) \tag{4.4}$$

and hence $p(e) = 0$. So $dp(e)$ is a map $T_eG \to T_{p(e)}(\bigwedge^2 \mathfrak{g}) = T_0(\bigwedge^2 \mathfrak{g}) = \bigwedge^2 \mathfrak{g}$ since $\mathfrak{g}$ is a vector space. So we define $\delta = dp(e) : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$.

Theorem 4.1 $(\mathfrak{g}, \delta)$ *is a Lie bialgebra.*

Proof: 1. **The Jacobi identity:** Something must satisfy Jacobi, since p did; we need to check that the dual map $[,]_{\mathfrak{g}^*} \overset{\text{def}}{=} \delta^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ is this something. Well, $\mathfrak{g}^*$ is the space of linear functions on $\mathfrak{g}$. How do we get these? Let's fix $f_1, f_2 \in C(G)$, and their differentials $df_1(e), df_2(e) \in T_e^*G = \mathfrak{g}^*$. Let's denote $df_i(e)$ by ξ_i. Then, letting $X \in \mathfrak{g}$, we have

$$[\xi_1, \xi_2]_{\mathfrak{g}^*}(X) = \langle dp(e)(X), df_1(e) \wedge df_2(e) \rangle \tag{4.5}$$

$$= \left.\frac{d}{dt}\right|_{t=0} \{f_1, f_2\}(e^{tX}) \tag{4.6}$$

Here $\{,\}$ is our Poisson bracket on G, and e^{tX} is the exponential map $\mathfrak{g} \to G$.

Exercise 12 *Prove equation 4.6. Try to do it invariantly, but if you cannot, do it in local coordinates. On the one hand, local coordinates are very messy, and on the other hand, by making your hands dirty, you can really see what you're doing.*

On the other hand, $\{f_1, f_2\}(e) = 0$. The Jacobi for $\{,\}$ says that $\{f_1, \{f_2, f_3\}\}(e^{tX}) + \text{cyclic} = 0$. So take $\frac{d^2}{dt^2} \left[\{f_1, \{f_2, f_3\}\}(e^{tX}) + \text{cyclic}\right]\Big|_{t=0}$ and conclue the Jacobi for $[,]_{\mathfrak{g}^*}$.

2. **The cocycle property:** The two-line proof says "p is a 1-cocycle for G, and so automatically induces a 1-cocycle for TG." We proceed to prove this:

We apply equation 4.1 twice, on a commutator:

$$p(y^{-1}zy) = \left(\mathrm{Ad}_{y^{-1}z} \otimes \mathrm{Ad}_{y^{-1}z}\right) p(y) + \left(\mathrm{Ad}_{y^{-1}} \otimes \mathrm{Ad}_{y^{-1}}\right) p(z) + p(y^{-1}) \tag{4.7}$$

All proofs in Lie algebras/groups feel like the first part. We apply equation 4.7 to $z = e^{tX}$, as $t \to 0$. The order-1 elements in t give

$$\delta\left(\mathrm{Ad}_{y^{-1}}(X)\right) = \left(\mathrm{Ad}_{y^{-1}} \otimes \mathrm{Ad}_{y^{-1}}\right)[X, p(y)] + \left(\mathrm{Ad}_{y^{-1}} \otimes \mathrm{Ad}_{y^{-1}}\right)\delta(X) \tag{4.8}$$

We've used that $\delta = dp(e)$. Now we take $y = e^{tY}$ and differentiate in t at $t = 0$:

$$\delta([X, Y]) = [X, \delta(Y)] - [Y, \delta(X)] \tag{4.9}$$

Here we used that $[X, Y \wedge Z] = [X, Y] \wedge Z + Y \wedge [X, Z]$. □

Did we ever supply examples of Poisson Lie groups?

Example 4.1 Suppose there is an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfying the *classical Yang-Baxter equation*:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{4.10}$$

This equation lives in $\mathcal{U}(\mathfrak{g})^{\otimes 3}$. $r_{12} \stackrel{\text{def}}{=} r \otimes 1$, where we have embedded $\mathfrak{g} \otimes \mathfrak{g} \hookrightarrow \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$, and $r_{23} \stackrel{\text{def}}{=} 1 \otimes r$. We leave it as an exercise to guess $r_1 3$.

Well, $r = \sum_{ij} r^{ij} e_i \otimes e_j$, where e_i is a basis of $\mathfrak{g}$. So r has $(\dim \mathfrak{g})^2$ variables, and equation 4.10 is $(\dim \mathfrak{g})^3$ equations, so it's entirely nonobvious why there would be any solutions to this equation. But, indeed, the "Drinfeld douple construction" says there are some.

For example, if $\mathfrak{g} = \mathfrak{sl}_2$, then $r = \frac{1}{4} H \otimes H + X \otimes Y$, where we have the basis X, Y, H with $[H, X] = 2X$, $[H, Y] = -2Y$, and $[X, Y] = H$. We will later see scientifically why this works.

We assume we have such an r. We define $\delta_r(x) \stackrel{\text{def}}{=} [r, x \otimes 1 + 1 \otimes x]$. This would be a natural candidate if it were obvious that the image is in the exterior square. **Question from the audience:** What is this commutator, and above? **Answer:** For example, $[A \otimes B, C \otimes 1] \stackrel{\text{def}}{=} [A, C] \otimes B$. ****So we extend $[,]$ to tensors by the Leibniz rule?****

Proposition 4.2 $\delta_r(x) \in \bigwedge^2 \mathfrak{g} \subseteq \mathfrak{g} \otimes \mathfrak{g}$

Proof: **Exercise 13**. Compare with notes when they appear online. □

Theorem 4.3 *(g, δ_r) is a Lie bialgebra.*

It has a special name: Drinfeld calls it *quasitriangular*, because there is a triangle in the Braid relation.

Now, let G be a Lie group such that $\mathfrak{g} = \mathrm{Lie}(G)$. Define $p_r(x) \stackrel{\text{def}}{=} (\mathrm{Ad}_x \otimes \mathrm{Ad}_x)(r) - r$, which gives a map $p : G \to \bigwedge^2 \mathfrak{g}$.

Theorem 4.4 *This is a Poisson Lie structure on G such that $(\mathfrak{g}, \delta_r)$ is the tangent Lie bialgebra.*

See, it's obvious that p_r is a 1-cocycle for G with coefficients in $\bigwedge^2 \mathfrak{g}$, but is't also a 1-coboundary. We don't want to go into group cohomology, but for example Fuchs' book *Cohomologies of Infinite-Dimensional Lie Algebras*, or any textbook with cohomology of groups and Lie groups, will explain this.

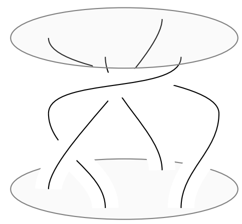
Question from the audience: Does it go the other way? If I have a 1-coboundary... **Answer:** No, a coboundary will not necessarily satisfy the Yang-Baxter equation. ♢

4.1 Braid groups

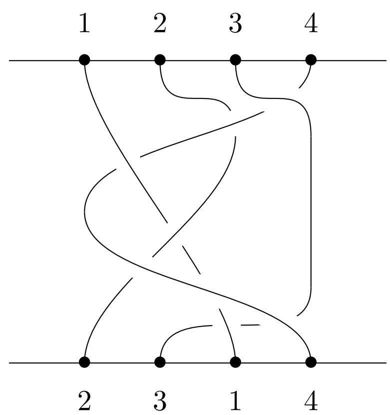
Let us see why we used the Yang-Baxter equation rather than something else. Let

$$X_n \stackrel{\text{def}}{=} \left\{ (x_1, \ldots, x_n) : x_i \neq x_j, x_i \in \mathbb{R}^2 \right\}.$$

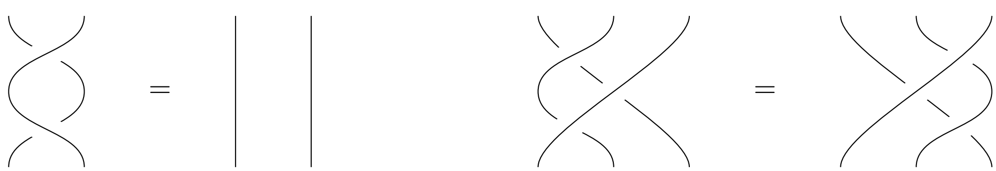
Then S_n acts on X_n, and we define $\tilde{X}_n \stackrel{\text{def}}{=} X_n/S_n$. The *braid group* is the fundamental group of this space $B_n \stackrel{\text{def}}{=} \pi_1(\tilde{X}_n)$. So what should happen is that you start with points, and they move around and end up where they started, up to a permutation. ****we let time be the downward direction, and draw the worldlines of the particles****



The more standard drawing: you pick the points on line, and project to the plane, with overcrossings and undercrossings. ****I don't guarantee that this is the same picture as above.****



In any case, in π_1, we should take paths, but only up to isotopy. We have two *Reidemeister moves* ****and their mirror versions****:



Question from the audience: What about Reidemeister 1? **Answer:** They are braids: paths only go down.

We define s_i, for $i = 1, \ldots, n-1$, to be the braid that is trivial on all strands except for i and $i+1$, and there is a single crossing between i and $i+1$.

Theorem 4.5 *The braid group has presentation:*

$$B_n \cong \langle s_i, i = 1, \ldots, n-1 \text{ s.t. } s_i s_j = s_j s_i \text{ if } |i-j| > 1, \text{ and } s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle \tag{4.11}$$

Well, any group has many representations. The relations are very local, so it's natural to look for representations of B_n on $V^{\otimes n}$, where

$$s_i = 1 \otimes \cdots \otimes S \otimes \cdots \otimes 1$$

where $S \in \mathrm{Aut}(V \otimes V)$, and it's acting in the i and $i+1$ spots.

Proposition 4.6 *$\pi : B_n \to \mathrm{Aut}(V^{\otimes n})$ is a representation if $(S \otimes 1)(1 \otimes S)(S \otimes 1) = (1 \otimes S)(S \otimes 1)(1 \otimes S)$.*

The whole reason for developing quantum groups, Poisson Lie groups, etc., was to study these equations. Except that this didn't evolve in Topology, but rather in Statistical Mechanics.

Lecture 5 January 30, 2009

The handwritten lecture notes are now online. These notes are also available, usually a few hours after class — they go up as soon as Theo has a chance to say "upload".

5.1 More on the Braid Group

Last time, we stopped at the Yang-Baxter equation. We have the *braid group*

$$B_n \stackrel{\text{def}}{=} \langle s_i \text{ s.t. } s_i s_j = s_j s_i \text{ if } |i-j| \geq 2, \text{ and } s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle \tag{5.1}$$

There are tensor-product representations $\pi : B_n \to \mathrm{Aut}(V^{\otimes n})$ where $s_i \mapsto 1 \otimes \cdots \otimes S \otimes \cdots \otimes 1$, where $S \in \mathrm{Aut}(V \otimes V)$ is acting in the $i, i+1$th spots. This satisfies the first condition, and satisfies the second iff S satisfies the *Yang-Baxter equation*:

$$(S \otimes 1)(1 \otimes S)(S \otimes 1) = (1 \otimes S)(S \otimes 1)(1 \otimes S) \tag{5.2}$$

This is a hugely over-determined system: there are $(\dim V)^6$ equations for $(\dim V)^4$ unknowns.

Example 5.1 $S = P : x \otimes y \mapsto y \otimes x$. This is a boring solution: the map factors through $B_n \to S_n$, hence ignores under- versus over-crossings. ♢

We should try to construct a family $S(h) = P(1 + hr + O(h^2))$ of solutions.

Proposition 5.1 *S satisfies the Yang-Baxter equation only if r satisfies:*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{5.3}$$

Proof: Expand equation 5.2 to order h^2; the order-h stuff cancels. □

This is an equation that involves only commutators. We should consider it as an equation in $\mathfrak{gl}(V)^{\otimes 3}$ for $r \in \mathfrak{gl}(V)^{\otimes 2}$.

Recall from 261A:

Theorem 5.2 (Ado) *Any finite-dimensional Lie algebra is a subalgebra of $\mathfrak{gl}(V)$ for some finite-dimensional V.*

So finding solutions to equation 5.3 is the same as classifying all solutions in $\mathfrak{g}^{\otimes 3}$ in arbitrary finite-dimensional $\mathfrak{g}$ a Lie algebra. This is why we are interested in Lie bialgebras if we are interested in knot theory.

So the philosophy is: we want to construct S satisfying equation 5.2, and we have one solution; we should perturb that solution in the direction r. So the general questions are

1. How to construct solutions to equation 5.3? I.e. how to construct Lie bialgebras.
2. "Quantization": How to construct S for a given r? The answer is in the construction of a special class of quantum groups.

This second question is the historical motivation for our subject. Similarly, the main motivation for Lie was to study the solutions of differential equations. This history is almost completely forgotten.

We gave an example last time of a solution to equation 5.3 for $\mathfrak{g} = \mathfrak{sl}_2$.

Let $\mathfrak{g}$ be a Lie algebra. Suppose that $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies equation 5.3. We consider $r_\pm : \mathfrak{g}^* \to \mathfrak{g}$ given by

$$r_+(l) \stackrel{\text{def}}{=} (l \otimes \text{id})r \tag{5.4}$$

$$r_-(l) \stackrel{\text{def}}{=} -(\text{id} \otimes l)r \tag{5.5}$$

The minus sign, we will see, is for later convenience.

Lemma 5.3 $\text{Im}(r_\pm) \stackrel{\text{def}}{=} \mathfrak{g}_\pm \subseteq \mathfrak{g}$ *are Lie subalgebras of* $\mathfrak{g}$*.*

Proof: We just look in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. By the definition, $r \in \mathfrak{g}_- \otimes \mathfrak{g}_+ \subseteq \mathfrak{g}^{\otimes 2}$: if $r = \sum_i r^i \otimes r_i$, then $r_+(l) = \sum_i l(r^i) r_i$ and $r_-(l) = -\sum_i r^i l(r_i)$, and by definition $\{r_i\} \in \mathfrak{g}_+$ span $\mathfrak{g}_+$ and $\{r^i\} \in \mathfrak{g}_-$ span $\mathfrak{g}_-$. ****there is some unhappiness**** This is general linear algebra. If $x \in V^* \otimes W$, then we get $x_+ : V \to W$ and $x_- : W^* \to V^*$.

Example 5.2 Let's do a small example. $\mathfrak{g} = \mathbb{C}^3$ with the basis $\{H, X, Y\}$, and $\mathfrak{g}^* = \mathbb{C}^3$ with the dual bases $\{H^\vee, X^\vee, Y^\vee\}$. We choose $r = \frac{1}{4} H \otimes H + X \otimes Y$. Then $r_+(l) = l(H)\frac{H}{4} + l(X)Y$. Since l can vary over all of $\mathfrak{g}^*$, then $\text{Im}(r_+) = \mathbb{C}H \oplus \mathbb{C}Y$. ◊

Question from the audience: What about the following:

Example 5.3 $r = H \otimes X + X \otimes X = (H + X) \otimes X$. Then $\text{Im}(r_+) = \mathbb{C}X$, and $\text{Im}(r_-) = \mathbb{C}(H + X)$, not the span $\mathbb{C}H + \mathbb{C}X$. ◊

Then we get different answers depending on how we write it. Do we need it to span? **Answer:** Ah, we were sloppy. By definition, $\mathfrak{g}_+ = \{\sum_i l(r^i) r_i \text{ s.t. } l \in \mathfrak{g}^*\}$. This is contained in the span of the r_i, but it's not equal.

Ok, so the earlier claim was wrong, but it's certainly the case that $r \in \mathfrak{g}_- \otimes \mathfrak{g}_+ \subseteq \mathfrak{g}^{\otimes 2}$. **Question from the audience:** Ok, but now I don't know why $r \in \mathfrak{g}_- \otimes \mathfrak{g}_+$? **Answer:** We will answer this next time, to save ourselves from thinking at the board. Maybe we have to impose more: in the relevant examples, it is true, and we thought it was obvious in all cases, but we may have to have more conditions. We will assume that $r \in \mathfrak{g}_- \otimes \mathfrak{g}_+$.

We continue with the proof. We look at equation 5.3:

$$\underset{[\mathfrak{g}_-,\mathfrak{g}_-]\otimes\mathfrak{g}_+\otimes\mathfrak{g}_+}{\underset{\cap}{[r_{12}, r_{13}]}} + \underset{\mathfrak{g}_-\otimes[\mathfrak{g}_+,\mathfrak{g}_-]\otimes\mathfrak{g}_+}{\underset{\cap}{[r_{12}, r_{23}]}} + \underset{\mathfrak{g}_-\otimes\mathfrak{g}_-\otimes[\mathfrak{g}_+,\mathfrak{g}_+]}{\underset{\cap}{[r_{13}, r_{23}]}} = 0 \tag{5.6}$$

So this is only possible if $[\mathfrak{g}_-, \mathfrak{g}_-] \subseteq \mathfrak{g}_-$, $[\mathfrak{g}_+, \mathfrak{g}_-] \subseteq \mathfrak{g}_+ + \mathfrak{g}_-$, and $[\mathfrak{g}_+, \mathfrak{g}_+] \in \mathfrak{g}_+$.

We will clarify this next time. □

We can also see, form the above proof, that there is a subspace $\tilde{\mathfrak{g}} \stackrel{\text{def}}{=} \mathfrak{g}_+ + \mathfrak{g}_- \subseteq \mathfrak{g}$. So we have another statement:

Lemma 5.4 $\tilde{\mathfrak{g}}$ *is a Lie subalgebra of* $\mathfrak{g}$*.*

And then we consider $t = r + \sigma(r)$, where σ is the permutation $x \otimes y \mapsto y \otimes x$. So t is symmetrized: $t \in S^2(\mathfrak{g})$. But since the only elements involved in the definition, in fact $t \in S^2\tilde{\mathfrak{g}}$.

Proposition 5.5 $t \in S^2(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ *— i.e. t is in the $\tilde{\mathfrak{g}}$-invariant part.*

Proof: We act by $(\sigma \otimes \mathrm{id})$ on equation 5.3, which just switches the indices 1 and 2, and add. So the last term cancels:

$$\sigma \otimes \mathrm{id} : [r_{21}, r_{13}] + [r_{21}, r_{23}] + [r_{23}, r_{13}] = 0 \tag{5.7}$$

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{5.8}$$

$$+ : [r_{12} + r_{21}, r_{13} + r_{23}] = 0 \tag{5.9}$$

But the first is t_{12}. So $[t \otimes 1, r^i \otimes 1 \otimes r_i + 1 \otimes r^i \otimes r_i] = 0$. This is equivalent to saying that for all l, $[t, \sum_i (r^i \otimes 1 + 1 \otimes r^i) l(r_i)] = 0$. And similarly for $\mathfrak{g}_+$. □

Theorem 5.6 $(\tilde{g}, \delta_r(x) \stackrel{\text{def}}{=} [r, x \otimes 1 + 1 \otimes x])$ *is a Lie bialgebra.*

Proof: We did this last time. We have to prove two prove two facts.

0. $\sigma \circ \delta_r(x) = [\sigma(r), x \otimes 1 + 1 \otimes x] = [t - r, x \otimes 1 + 1 \otimes x] = 0 - \delta(x)$, so δ_r lands in the exterior square.
1. cocycle: $\delta_r[x, y] = [r, [x, y] \otimes 1 + 1 \otimes [x, y]] = [x, \delta_r y] + [delta_r x, y]$ by Jacobi for $\tilde{\mathfrak{g}}$. Recall, $[x, y \wedge z] \stackrel{\text{def}}{=} [x, y] \wedge z + y \wedge [x, z]$.
2. co-Jacobi: $\mathrm{Alt}\big((\delta_r \otimes \mathrm{id}) \circ \delta_r\big) = 0$. This is equivalent to equation 5.3. □

We say that a Lie bialgebra $\mathfrak{g}$ is *factorizable* if there is a nondegenerate $t \in S^2(\mathfrak{g}) \subseteq \mathfrak{g} \otimes \mathfrak{g}$ (i.e. $t : \mathfrak{g}^* \to \mathfrak{g}$ is a linear isomorphism), and such that for any $x \in \mathfrak{g}$, there are unique $x_\pm \in \mathfrak{g}_\pm$ such that $x = x_+ + x_-$.

Example 5.4 Linear Gaussian factoraization $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_+ & b \\ 0 & d_+ \end{pmatrix} + \begin{pmatrix} a_- & 0 \\ c & d_- \end{pmatrix}$. ◊

****I wouldn't put a lot of faith, gentle reader, in this definition; I may have misheard it.****

Lecture 6 January 26, 2009

Recall from last time, we have $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfying

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{6.1}$$

Then we define $r_+(l) = (\mathrm{id} \otimes l)r$, and $r_-(l) = -(l \otimes \mathrm{id})r$.

Proposition 6.1 *1.* $r \in \mathfrak{g}_+ \otimes \mathfrak{g}_-$

2. $\mathfrak{g}_+$, $\mathfrak{g}_-$, *and* $\tilde{\mathfrak{g}} \overset{\text{def}}{=} \mathfrak{g}_+ + \mathfrak{g}_-$ *are Lie subalgebras in* $\mathfrak{g}$.

Proof: See notes. □

Lemma 6.2 *Let* $t = r + \sigma(r) \in S^2(\tilde{\mathfrak{g}}) \subseteq \mathfrak{g} \otimes \mathfrak{g}$. *Then* $t \in \S^2(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$, *i.e.* $[t, x \otimes 1 + 1 \otimes x] = 0$ *for* $x \in \tilde{\mathfrak{g}}$.

We assume that everything is in $\mathcal{U}(\mathfrak{g})$, and the bracket is a commutator ****extended to tensor products by the Leibniz rule****.

So, define $\delta_r : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} \wedge \tilde{\mathfrak{g}} \subseteq \tilde{\mathfrak{g}}^{\otimes 2}$ by $\delta_r(x) = [r, x \otimes 1 + 1 \otimes x]$. It's easy to check that this has the correct codomain, using the pervious lemma.

Proposition 6.3 $(\tilde{\mathfrak{g}}, \delta_r)$ *is a Lie bialgebra.*

Proof: 1. $\delta_r([x, y] = [r, [x, y] \otimes 1 + 1 \otimes [x, y]] = [x, \delta_r(r)] + [\delta_r(x), y]$ by the Jacobi identity.

2. Jacobi for δ_r^* follows from the classical Yang-Baxter equation. □

From now on, we will forget about tildes. The pair $(\mathfrak{g}, \delta_r)$ is a *quasitriangular* Lie bialgebra assuming

- $r + \sigma(r) \in S^2(\mathfrak{g})^{\mathfrak{g}}$
- Classical Yang-Baxter equation for r.

We say that $(\mathfrak{g}, \delta_r)$ is *factorizable* if $t \in \mathfrak{g} \otimes \mathfrak{g}$ defines a nondegenerate bilinear form on $\mathfrak{g}^*$ (where $\langle l, m\rangle_t \overset{\text{def}}{=} (l \otimes m)(t))$. If $(\mathfrak{g}, \delta_r)$ is factorizable, then we have $x = x_+ + x_-$ uniquely, where $x_\pm \in \mathrm{Im}(r_\pm)$ and $x_\pm = r_\pm(l)$ for some $l \in \mathfrak{g}^*$.

Any questions? It is better to move to semi-meaningless discussion — semi-meaningless on NR's part, because we asserted something wrong — than to leave something out.

Example 6.1 Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with standard basis H, X, Y (H is the Cartan, X, Y are the root elements): $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$. Then $r = \frac{1}{4}H \otimes H + X \otimes Y$ is a solution to equation 6.1, and $t = r + \sigma(r) = \frac{1}{2}H \otimes H + X \otimes Y + Y \otimes X \in S^2(\mathfrak{sl}_2)^{\mathfrak{sl}_2}$. This gives the *Casimir* element $c \stackrel{\text{def}}{=} \frac{H^2}{2} + XY + YX \in \mathcal{U}\mathfrak{sl}_2$. I.e. $c \in Z(\mathcal{U}\mathfrak{sl}_2)$, which is in fact freely generated by c: $Z(\mathcal{U}\mathfrak{sl}_2) = \mathbb{C}[c]$.

Then $t = \frac{1}{2}\Delta(c) - c \otimes 1 - 1 \otimes c$, where $\Delta : \mathcal{U}\mathfrak{sl}_2 \to \mathcal{U}\mathfrak{sl}_2^{\otimes 2}$ is a coassociative algebra homomorphism such that $\Delta x = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{sl}_2 \subseteq \mathcal{U}\mathfrak{sl}_2$. Anyway, so Δ is a homomorphism, and since $c \in Z(\mathcal{U}\mathfrak{sl}_2)$, we have $[t, \Delta x] = 0$ for each $x \in \mathfrak{sl}_2$. t is called a *mixed Casimir*, and r then is not so strange, being like half of the mixed Casimir.

Let's look at $\mathfrak{sl}_2^*$. We have our special basis of $\mathfrak{sl}_2$, so let's choose the dual basis: $\mathfrak{sl}_2^* = \mathbb{C}H^\vee \oplus \mathbb{C}X^\vee \oplus \mathbb{C}Y^\vee$, where we define $K^\vee$ (for $K = H, X, Y$) to be the linear functional that is 1 on K and 0 on the other two basis elements. Then $r_+(l) = \frac{H}{4}l(H) + X\,l(Y)$, where $l \in \mathfrak{sl}_2^*$, so $\text{Im}(r_+) = \mathbb{C}H \oplus \mathbb{C}X = \mathfrak{b}_+ \subseteq \mathfrak{sl}_2$ and $\text{Im}(r_-) = \mathfrak{b}_-$. We will see counterparts of this for all simple Lie algebras.

What about the kernels? $\ker(\mathfrak{r}_+) = \mathbb{C}X^\vee$ and $\ker(r_-) = \mathbb{C}Y^\vee$, so $\ker(r_+)^\perp$, which is the collection of all elements of $\mathfrak{sl}_2$ on which $X^\vee$ vanishes, is just $\mathfrak{b}_-$. See, the definition we were trying to sell last time and the correct definition from today coincide.

A little claim: $(\mathfrak{sl}_2, \delta_r)$ is a factorizable Lie bialgebra: t is given by the Killing form, which is nondegenerate for $\mathfrak{sl}_2(\mathbb{C})$.

Question from the audience: Can you say a bit more? It seems we don't have unique factorization. **Answer:** We're coming to it.

The cobracket $\delta_r : a \mapsto [r, a \otimes 1 + 1 \otimes a]$. In particular:

$$\begin{aligned}
\delta_r(H) &= [r, H \otimes 1 + 1 \otimes H] = 0 \\
\delta_r(X) &= \frac{1}{4}[H, X] \otimes H + \frac{1}{4}H \otimes [H, X] + X \otimes [Y, X] \\
&= \frac{1}{2}X \otimes H + \frac{1}{2}H \otimes X - X \otimes H \\
&= \frac{1}{2}H \wedge X \\
\delta_r(Y) &= \frac{1}{2}H \wedge Y
\end{aligned}$$

****check the signs**** ◊

The factorization: We have $t = r + \sigma(r) \in S^2(\mathfrak{g})^{\mathfrak{g}}$, defining a nondegenerate bilinear form on $\mathfrak{g}^*$, and so the linear map $t : \mathfrak{g}^* \to \mathfrak{g}$ defined by $l \mapsto (\mathrm{id} \otimes l)(t)$ is a linear isomorphism. Now,

$$\begin{aligned} t(l) &= (\mathrm{id} \otimes l)(r) + (\mathrm{id} \otimes l)(\sigma(r)) \\ &= (\mathrm{id} \otimes l)(r) + (l \otimes \mathrm{id})(r) \\ &= r_+(l) - r_-(l) \end{aligned}$$

Thus we had a sign error earlier with the definition of the factorization.

Proposition 6.4 *Because t is a linear isomorphism, any $x \in \mathfrak{g}$ has a unique presentation as $x = x_+ - x_-$ where $x_\pm = r_\pm(l)$ for some l.*

Now we check how this works for $\mathfrak{sl}_2$: $x = \alpha H + \beta X + \gamma Y$, then $x_+ = \frac{\alpha}{2} H + \beta X$ and $x_- = -\frac{\alpha}{2} H - \gamma Y$.

We finish with the Lie algebra structure on $\mathfrak{sl}_2^*$. By definition:

$$[H^\vee, X^\vee](a) = H^\vee \wedge X^\vee(\delta_r(a)) \tag{6.2}$$

Well, $\delta_r(a)$ had better be in $\mathbb{C}H \wedge X$, otherwise equation 6.2 is 0. So equation 6.2 is non-zero only if $a = cX$.

$$[H^\vee, X^\vee](X) = (H^\vee \wedge X^\vee)\left(\frac{1}{2} H \wedge X\right) \tag{6.3}$$

$$= \frac{1}{2}\langle H^\vee \wedge X^\vee, H \wedge X\rangle \tag{6.4}$$

$$= \frac{1}{2}\langle H^\vee \otimes X^\vee - X^\vee \otimes X^\vee, H \otimes X - X \otimes H\rangle \tag{6.5}$$

$$= \frac{1}{2}(2) \tag{6.6}$$

$$= 1 \tag{6.7}$$

So $[H^\vee, X^\vee] = X^\vee$ and $[H^\vee, Y^\vee] = Y^\vee$, and $[X^\vee, Y^\vee] = 0$. So this is a very different Lie algebra: it's not semisimple. This is the standard Lie bialgebra structure on $\mathfrak{sl}_2$. There is a classification of Lie bialgebra structures, and for $\mathfrak{sl}_2$ there is only one factorizable one.

Question from the audience: So there can be bialgebras that don't come from an r-matrix? **Answer:** Yes. **Question from the audience:** Then the notion of factorizability doesn't make sense? **Answer:** That's correct.

We say that $\mathfrak{g}_- \subseteq \mathfrak{g}_2$ is a *Lie sub-bialgebra* if $\delta(\mathfrak{g}_1) \subseteq \mathfrak{g}_1 \wedge \mathfrak{g}_1$.

Example 6.2 $b_\pm \subseteq (\mathfrak{sl}_2, \delta_r)$ is a Lie sub-bialgebra, because $\delta H = 0$ and $\delta X = \frac{1}{2} H \wedge X$, etc. These are not quasitriangular. ♢

Exercise 14 *Formulate the notion of Lie bialgebra ideal. You must decide on the correct condition on the cobracket.*

Next time we will begin the discussion of groups. For example, there are three real forms of $SL_2(\mathbb{C})$: $SL_2(\mathbb{R})$, $U(2)$, and the less-well-known one $U(1,1)$.

Lecture 7 February 4, 2009

Recall, if G is a Lie group, we can trivialize TG by right-translation: $dR : TG \cong \mathfrak{g} \times G$, where $dR_{h^{-1}} : T_hG \xrightarrow{\sim} T_eG \cong \mathfrak{g}$. Then $p_r(x) = -\operatorname{Ad}_x \otimes \operatorname{Ad}_x(r) + r \in \Gamma(\bigwedge^2 TG) \cong C(G \to \bigwedge^2 \mathfrak{g})$ is a *Poisson Lie structure* on G if the tangent Lie bialgebra $(\mathfrak{g}, \delta_r)$ has $\delta_r = dp(e) : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$, $\delta_r(x) = [e, x \otimes 1 + 1 \otimes x]$.

Then let's compute the Poisson bracket on $SL_2(\mathbb{C})$. We have coordinates:

$$SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ s.t. } ad - bc = 1 \right\} \tag{7.1}$$

And so $C(SL_2) = \mathbb{C}[a, b, c, d]/\sim$ is a commutative hops algebra. We compute the Poisson prackets between a, b, c, d: $\Delta a = a \otimes a + b \otimes x$, $\Delta b = a \otimes b + b \otimes d$, etc., from the multiplication of matrices.

Then the Poisson bracket is $\{f_1, f_2\}(g) = \langle p(q), df_1(g) \wedge df_2(g) \rangle$, and

$$p(g) = \sum_{\alpha,\beta} p^{\alpha\beta}(g) e_\alpha \otimes e_\beta \tag{7.2}$$

$$\langle e_\alpha, df(g) \rangle = \frac{d}{dt} f(e^{te_\alpha} g)\Big|_{t=0} \tag{7.3}$$

$$= \sum_{ij} \frac{d}{dt} (e^{te_\alpha} g)_{ij}\Big|_{t=0} \frac{\partial f}{\partial g_{ij}}(g) \tag{7.4}$$

$$= \sum_{ij} (e_\alpha g)_{ij} \frac{\partial f}{\partial g_{ij}} \tag{7.5}$$

To define the last line "$e_\alpha g$", we use the fact that SL_2 is a matrix group. We will always assume that all our groups are matrix groups, whence the exponential map really is matrix exponential. We did this with right-trivialization. If we had used left trivialization, then the formula would have included $f(ge^{te_\alpha})$.

Hence:

$$\{f_1, f_2\}(g) = \langle p(q), df_1(g) \wedge df_2(g) \rangle = 2 \sum_{\alpha,\beta,i,j,k,l} p^{\alpha\beta}(g) (e_\alpha g)_{ij} (e_\beta g)_{kl} \frac{\partial f_1}{g_{ij}} \frac{\partial f_2}{g_{kl}} \tag{7.6}$$

The 2 comes because from the wedge bracket, whence we should have subtracted $ij \leftrightarrow kl$, but everything is skew symmetric.

Well,

$$p^{\alpha\beta}(g) e_\alpha g \otimes e_\beta g = p(g)(g \otimes g) \tag{7.7}$$

$$= \left(-(g \otimes g)(r)(g^{-1} \otimes g^{-1}) + r \right) (g \otimes g) \tag{7.8}$$

$$= -(g \otimes g) r + r (g \otimes g) \tag{7.9}$$

and so, up to an unfortunate factor of 2, we have

$$\{f_1, f_2\} = 2\sum_{ijkl} [r, g \otimes g]_{ij,kl} \frac{\partial f_1}{\partial g_{ij}} \frac{\partial f_2}{\partial g_{kl}} \tag{7.10}$$

Let us find $\{g_{ij}, g_{kl}\}$ in SL_2; and we will organize this as a matrix in $\mathrm{End}(V^{\otimes 2})$, where $V = \mathbb{C}^2$. Using the formula, the derivatives are 1 (really δ functions), and so

$$\{g_{ij}, g_{kl}\} = 2[r, g \otimes g]_{ij,kl} \tag{7.11}$$

and so, assuming that our group is a matrix group, so that the formula makes sense:

$$\boxed{\{g \underset{,}{\otimes} g\} = 2[r^V, g \otimes g]} \tag{7.12}$$

Everywhere r should be r^V, which is the image of $r \in \mathfrak{g} \otimes \mathfrak{g}$ in $\mathrm{End}(V)^{\otimes 2}$.

****NR used the symbol $\{\cdot \underset{,}{\otimes} \cdot\}$ for the matrix of brackets. I couldn't tell quite what symbol he was using, and replaced $\underset{,}{\otimes}$ with either $\otimes$ or $,$. So some of the formulas from today are not what was on the board.****

Ok, so we introduce a particularly useful notation: $g_1 \overset{\text{def}}{=} g \otimes 1$, $g_2 = 1 \otimes g$, and $r_{12} = r \in \mathrm{End}(V)^{\otimes 2}$. ****NR writes "$I$" for the identity matrix, but says "one".**** Then $r_{12} \in \mathrm{End}(V)^{\otimes n}$ is $r \otimes 1 \otimes \cdots \otimes 1$ where the 1s are in positions $3, \dots, n$.

Ok, so $\{g_1, g_2\} = 2[r, g_1 g_2] = 2[r_{12}, g_1 g_2]$. Eventually, we may kill this 2. We can do this: we can rescale the Poisson bracket. The problem is the pairing of elements of the wedge product. ****are those the same 1 and 2 on either side of the equation?****

Ok, so let's prove the Jacobi identity:

$$\begin{aligned}
\{g_1, \{g_2, g_3\}\} &= \{g_1, [r_{23}, g_2 g_3]\} && (7.13)\\
&= [r_{23}, \{g_1, g_2 g_3\}] && (7.14)\\
\{g_1, g_2 g_3\} &= \{g_1, g_2\} g_3 + g_2 \{g_1, g_3\} && (7.15)\\
&= [r_{12}, g_1 g_2] g_3 + g_2 [r_{13}, g_1 g_3] && (7.16)
\end{aligned}$$

****I don't understand these indices.****

One more computation. Let $r^V = \frac{1}{4} H \otimes H + X \otimes Y$, considered as a matrix in $\mathrm{End}(\mathbb{C}^2)^{\otimes 2}$, where H, X, Y is the standard basis of $\mathfrak{sl}_2$. Let's choose a basis e_1, e_2 of $\mathbb{C}^2$, and so $e_{ij} = e_i \otimes e_j$ is a basis of $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$. Then if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$g \otimes g = \left(\begin{array}{cc|cc} aa & ab & ba & bb \\ ac & ad & bc & bd \\ \hline ca & \dots & & \end{array}\right), \quad r^V = \left(\begin{array}{cc|cc} 1/4 & 0 & 0 & 0 \\ 0 & -1/4 & 1 & 0 \\ \hline 0 & 0 & -1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{array}\right) \tag{7.17}$$

Hence,

$$[r^V, g\otimes g] = \left(\begin{array}{cc:cc} 0 & \frac{1}{2}ab & . & . \\ \frac{1}{2}ac & bd & 0 & . \\ \hdashline -\frac{1}{2}ac & 0 & . & . \\ . & . & \frac{1}{2}cd & . \end{array}\right) \tag{7.18}$$

and so we have the formulas $\{a,b\} = \frac{1}{2}ab$, $\{a,c\} = -\frac{1}{2}ac$, $\{a,d\} = bc$, $\{b,c\} = 0$, $\{b,d\} = -\frac{1}{2}bd$, and $\{c,d\} = \frac{1}{2}cd$. You would never guess these formulas.

Theorem 7.1 *$C(SL_2)$ with these brackets is a* Hopf Poisson algebra*, i.e.*

- *$C(SL_2)$ is a Hopf algebra.*
- *it is a Poisson algebra*
- *$\Delta(\{f_1, f_2\}) = \{\Delta f_1, \Delta f_2\}$*

To define this, we must defined the *tensor product of Poisson algebras*:

$$\{s\otimes t, u\otimes v\} \overset{\text{def}}{=} \{s,u\}\otimes tv + su\otimes\{t,v\} \tag{7.19}$$

Lemma 7.2 *If G algebraic is Poisson Lie, then $C(G)$ is Hopf Poisson.*

We now give a general remark. Suppose we have a group G and a subgroup $H\subseteq G$. Then we get two Hopf algebras $C(G)$ and $C(H)$. How do they relate? Let I_H be the vanishing ideal of H; since H is a subgroup, it is a Hopf ideal. Then $C(H) = C(G)/I_H$.

Now a definition: $H\subseteq G$ is a *Poisson Lie subgroup* if is a Lie subgroup and a Poisson submanifold. This is equivalent to saying that $C(H) = C(G)/I_H$ and $\{I_H, C(G)\}\subseteq I_H$, i.e. I_H is a *Poisson ideal*, and indeed a *Hopf Poisson ideal*.

Ok, so let's look back at SL_2. What are some natural ideals? What can you vanish without going into contradictions with the Poisson bracket. Can you vanish a? No, because $\{a,d\} = bc$, and if we vanished a, we'd have $0 = bc\neq 0$. Can we vanish c? Yes: there's no problem with $\{a,d\} = 0$. Indeed, if $c = 0$, since $ad - bc = 1$, we have $d = a^{-1}$, and so the bracket should vanish. Thus $c\equiv 0$ defined a Poisson Hopf ideal: $B_+\subseteq SL_2$ is given by $B_+ = \{\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix}\}$, with $\{a,b\} = \frac{1}{2}ab$, and here $I_H = \langle c\rangle$. We of course also have B_- with $I_H = \langle b\rangle$ and $\{a,c\} = -\frac{1}{2}ac$. Yet another subgroup: $I_H = \langle b,c\rangle$, then $H\subseteq B_\pm$ is the Cartan, and the Poisson structure is trivial.

Ok, so we started with $(\mathfrak{sl}, \delta_r)$, and last time we computed the dual $(\mathfrak{sl}_2^*, \delta_*)$, and discovered that this is spanned by $H^\vee, X^\vee, Y^\vee$, with $[X^\vee, Y^\vee] = 0$. Well, $\mathfrak{sl}_2^*$ is really pairs of matrices:

$$\mathfrak{sl}_2^* = \left\{\left(\begin{pmatrix} a & b\\ 0 & -a\end{pmatrix}, \begin{pmatrix} a & 0\\ c & -a\end{pmatrix}\right)\right\} \tag{7.20}$$

with $H^\vee = (H, -H)$, $X^\vee = (0, X)$, and $Y^\vee = (Y, 0)$. Thus SL_2^*, which should be the exponential of $\mathfrak{sl}_2^*$, is a subgroup of $B_+\times B_-$.

Question from the audience: How do those pairs pair with the matrices in $\mathfrak{sl}_2$? **Answer: **missed****

Next time we will repeat this, and then finish with SL_2^*, and then explain the Double construction and see how to obtain Poisson structures on all complex simple Lie groups, and bialgebra structures on complex simple Lie algebras, and also the story of symplectic leaves, which will give us a wonderful excuse to study the geometry of Lie groups. And after this we will quantize everything.

Lecture 8 February 6, 2009

Today we continue with the basic example of SL_2. We now try to understand the dual Poisson Lie group SL_2^*, the Poisson Lie group with $T_e SL_2^* = (\mathfrak{sl}_2^*, [,]^*_{\mathfrak{sl}_2})$. **Question from the audience:** The simply-connected one? **Answer:** Let's demand it being connected, but chose any Lie group; the choice is parameterized by π_1.

We know the Lie bialgebra: $\mathfrak{sl}_2^*$ is spanned by $H^\vee, X^\vee, Y^\vee$, with $[H^\vee, X^\vee] = X^\vee$, $[H^\vee, Y^\vee] = Y^\vee$, and $[X^\vee, Y^\vee] = 0$. We understand best how to exponentiate matrix algebras, so we first try to find a faithful representations. $\mathfrak{sl}_2^*$ cannot be written as 2×2 matrices, as is easy to see, but there is a 4-dimensional representation. Let I be the 2×2 identity matrix. Then

$$H^\vee = \frac{1}{2}(H \otimes I - I \otimes H), \; X^\vee = X \otimes I, \; Y^\vee = I \otimes Y \tag{8.1}$$

where H, X, Y are the 2×2 matrices giving the usual action of $\mathfrak{sl}_2$. Another way of writing this is as pairs, so

$$SL_2^* = \left\{ \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a^{-1} & 0 \\ c & a \end{pmatrix} \right) \right\} \tag{8.2}$$

Let us find the Poisson Lie structure on SL_2^* with $[,]^*_{\mathfrak{sl}_2} = dp(e)$.

Theorem 8.1 *Let* $b^+ = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ *and* $b^- = \begin{pmatrix} a^{-1} & 0 \\ c & a \end{pmatrix}$. *Then*

$$\begin{aligned} \{b^+ \overset{\otimes}{,} b^+\} &\overset{\text{def}}{=} [r, b^+ \otimes b^+] \\ \{b^+ \overset{\otimes}{,} b^-\} &\overset{\text{def}}{=} [r, b^+ \otimes b^-] \\ \{b^- \overset{\otimes}{,} b^-\} &\overset{\text{def}}{=} [r, b^- \otimes b^-] \end{aligned}$$

where $r = \frac{1}{4} H \otimes H + X \otimes Y \in \mathrm{End}(\mathbb{C}^2)^{\otimes 2}$ *in* $\mathbb{C}^2 \otimes \mathbb{C}^2$*, and if* M *is any Poisson manifold and* $A, B : M \to \mathrm{End}(V)$*, we define* $\{A \overset{\otimes}{,} B\}$ *to be the matrix* $\{A_{ij}, B_{kl}\}$ *where* i, j, k, l *range from* 1 *to* $\dim V$.

Well, this is so far an ad hoc definition with elusive meaning. But in fact it's a calculation that the above brackets give the correct Poisson Lie structure, or equivalently a Poisson Hopf structure on $C(SL_2^*)$. By "functions" $C(\cdot)$ we mean (possibly Laurant) polynomials.

So far we have simply taken the linear-algebraic duality $(\mathfrak{sl}_2^*, \delta_*) \leftrightarrow (\mathfrak{sl}_2, \delta_r)$, and integrated to get a duality $(SL_2^*, p_*) \leftrightarrow (SL_2, p_r)$.

Proof: We have to prove that $\Delta(\{A, B\}) = \{\Delta A, \Delta B\}$, where by definition $\{A \otimes B, C \otimes D\} \stackrel{\text{def}}{=} \{A, C\} \otimes BD + AC \otimes \{B, D\}$. Since the Poisson bracket acts by derivations, we need only check this on generators, which for us are the coordinate functions. This is a straightforward computation:

$$\Delta\{b^{\epsilon_1} \otimes b^{\epsilon_2}\} = \Delta\left([r, b^{\epsilon_1} \otimes b^{\epsilon_e}]\right)$$

where ϵ_i is $+$ or $-$. Ok, so we have two tensor products: the tensor product of Hopf algebras, and the tensor product of matrices, and we're using both of them. We adopt the following notation: $b_1 = b \otimes I$ and $b_2 = I \otimes b$. So we have:

$$\begin{aligned}
\Delta\{b_1^{\epsilon_1} \otimes b_2^{\epsilon_2}\} &= \Delta\left([r, b_1^{\epsilon_1} b_2^{\epsilon_e}]\right) \\
&= [r, \Delta(b_1^{\epsilon_1})\Delta(b_2^{\epsilon_2})] \\
&= [r, (b_1^{\epsilon_1} \otimes b_1^{\epsilon_1})(b_2^{\epsilon_2} \otimes b_2^{\epsilon_2})] \\
&= [r, b_1^{\otimes 1} b_2^{\epsilon_2} \otimes b_1^{\epsilon_1} b_2^{\epsilon_2}]
\end{aligned}$$

where this is not the $\otimes$ of Hopf algebras. **Question from the audience:** What is the co-product? **Answer:** On generators, $\Delta(b^\pm) = b^\pm \otimes b^\pm$. More precisely, $\Delta(b_{ij}) = \sum_k b_{ik} \otimes b_{kj}$. It's the matrix multiplication along with the tensor product of Hopf algebras.

$$\begin{aligned}
\{\Delta b_1^{\epsilon_1}, \Delta b_2^{\epsilon_2}\} &= \{b_1^{\epsilon_1} \otimes b_1^{\epsilon_1}, b_2^{\epsilon_2} \otimes b_2^{\epsilon_2}\} \\
&= \{b_1^{\epsilon_1}, b_2^{\epsilon_2}\} \otimes b_1^{\epsilon_1} b_2^{\epsilon_2} +_1^{\epsilon_1} b_2^{\epsilon_2} \otimes \{b_1^{\epsilon_1}, b_2^{\epsilon_2}\}
\end{aligned}$$

It is **Exercise 15** to complete this calculation. Please write it on a piece of paper and turn it in, and NR will check it. □

Question from the audience: So the $_i$ notation is just a convention to speed up calculations. **Answer:** Exactly. Any more questions?

Expanding out the above exercise:

Proposition 8.2 *The Poisson brackets between a, b, c are:*

$$\begin{aligned}
\{a, c\} &= -\frac{1}{2}ac \\
\{a, b\} &= \frac{1}{2}ab \\
\{c, b\} &= -a^2 + a^{-2}
\end{aligned}$$

The comultiplication is:

$$\begin{aligned}\Delta a &= a \otimes a \\ \Delta c &= a \otimes c + c \otimes a^{-1} \\ \Delta b &= a \otimes b + b \otimes a^{-1}\end{aligned}$$

If you don't believe that matrix proof, just check that Proposition 8.2 gives the correct Poisson Hopf algebra.

At e, $a = 1$, $b = 0$, and $c = 0$. In a neighborhood of e, $a = e^{\epsilon H/4}$, $b = \epsilon X$, and $c = \epsilon Y$. In the correct proof, you would need to consider the tangent space at the identity and everything, but we don't have the general theory, so we treat this intuitively. You can see that as $\epsilon \to 0$, we get the correct linear functions: $\{X, Y\} = H, \ldots$. **Question from the audience:** You're checking that $[,]^*_{\mathfrak{sl}_2} = dp(e)$? **Answer:** Yes. We do this by changing the algebra, and working infinitesimally.

Let's do this correctly. Consider a formal neighborhood of e, realized as the algbera $\mathbb{C}[H, X, Y] \otimes \mathbb{C}[[\epsilon]]$, completed in the ϵ-adic topology. In other words, it is the space of formal power series in ϵ with coefficients that are polynomials in H, X, Y. **Question from the audience:** So these are the functions on the formal neighborhood? **Answer:** Yes. Strictly speaking, the functions on a formal neighborhood should be power series in H, X, Y, but we want to consider a weaker version.

Ok, so the Proposition implies that $\{e^{\epsilon H/4}, \epsilon X\} = \frac{1}{4}\epsilon^2 e^{\epsilon H/4}\{H, X\}$. But the LHS is $\{a, b\} = \frac{1}{2}e^{\epsilon H/4}\epsilon X$, so $\{H, X\} = 2X$. ****ϵs don't line up, as we comment later.** Question from the audience:** How did you get the first equation? **Answer:** It is the Leibniz rule:

$$\{e^A, B\} = \sum_{n \geq 0} \frac{1}{n!}\{A^n, b\} = \sum_{n \geq 1} \frac{n}{n!} A^{n-1} B = e^A B$$

We are getting very close to the following strange statement, which we will make precise in a month, that this is the Poisson algebra whose quantization is the quantized universal enveloping algebra.

Finishing, the same argument gives $\{H, Y\} = -2Y$, and lastly

$$\{X, Y\} = \frac{e^{\epsilon H/2} - e^{-\epsilon H/2}}{\epsilon^2}$$

Actually, this is a problem, because the top is $O(\epsilon)$, but the bottom is $O(\epsilon^2)$. But we had intended to leave it as **Exercise 16**.

Question from the audience: So are we taking a family of associative algebras? How are we getting this structure? **Answer:** Not a family. We are taking the limit as $a \to 1$ and $b, c \to 0$. We are deforming the Poisson algebra itself: it's a family of Poisson brackets, parameterized by the formal variable ϵ.

Ah, we have the same problem above: $\{H, X\} = 2X/\epsilon$.

So the summary is we have a family of Poisson algebras parameterized by ϵ:

$$\begin{aligned}\{H,X\}_\epsilon &= \frac{2}{\epsilon}X \\ \{H,Y\}_\epsilon &= -\frac{2}{\epsilon}Y \\ \{X,Y\}_\epsilon &= \frac{e^{\epsilon H/2}-e^{-\epsilon H/2}}{\epsilon^2}\end{aligned}$$

And we define

$$\{,\}_0 \stackrel{\text{def}}{=} \lim_{\epsilon\to 0}\epsilon\{,\} \tag{8.3}$$

See, nobody said we had to take this Poisson structure. Well, we did.

Correction: Let $p : G \to \bigwedge^2\mathfrak{g}$ be a Poisson Lie structure on G. Then for any α, αp is also a Poisson-Lie strucuter, and this defined a family of Lie bialgebras $\delta = \alpha dp(e)$. So, anyway, we do the rescaling, and we see that

$$\begin{aligned}\{H,X\}_0 &= 2X \\ \{H,Y\}_0 &= -2Y \\ \{X,Y\}_0 &= H\end{aligned}$$

Exercise 17 *Show that $\delta_* = dp(e)$ in the usual way.*

Oh, before you all leave: if you have particular questions about Poisson Lie groups and quantum groups, send NR an e-mail: the syllabus is fluid.

Lecture 9 February 9, 2009

****We begin class with: I arrive a minute or two late, NR passes out a name-and-email sheet, and then NR's cell-phone rings.****

We have a Lie bialgebra $(\mathfrak{sl}_2, \delta_r)$, with $r = \frac{1}{4}H\otimes H + X\otimes Y$, and its dual $(\mathfrak{sl}_2^*, \delta = [,]^*_{\mathfrak{sl}_2})$. We exponentiate $\mathfrak{sl}_2$ to (SL_2, p_r) with $p_r = -\operatorname{Ad}_x\otimes\operatorname{Ad}_x(r)+r$, wheer we assume $TG\cong\mathfrak{g}\times G$ by right translations. We exponentiate $\mathfrak{sl}_2^*$ to (SL_2^*, p_*), which we described last time in coordinates. This example provides a definition of the ideal of a *dual pair of Poisson Lie groups.* Given a (dual pair of) Lie bialgebras, there is a unique dual pair of connected simply-connected Poisson Lie groups, but in fact we should think of the pair as parameterized by the π_1s.

Where did the formula $r = \frac{1}{4}H\otimes H + X\otimes Y$ come from?

9.1 The double construction of Drinfeld

We begin by constructing the bicross-product of Lie bialgebra. We let $\mathfrak{g}_1, \mathfrak{g}_2$ be finite-dimensional Lie algebras over $\mathbb{C}$.

****NR pauses to quiz people about what they are doing.**** We have a specific group with specific interests, and Quantum Groups is a huge subject. We could take a month to talk about $\mathcal{U}_q(\mathfrak{n}_+)$, but no one is interested in this. We suggest the following agenda:

- Poisson Hopf algebras, and then quantum-deform these into associative noncommutative algebras. We will do this by taking a family of ideals I_q in the free algebra F, and then form $A_q = F/I_q$, which we will identify as vector spaces. In this way we will form $\mathcal{U}_q(\mathfrak{g})$ and $C_q(G)$.
- We will spend some time on the real forms of these.
- Closer to NR's interest, we will end by studying affine ****missed**** algebras. For example, we have the diagonal action $SL_n \curvearrowright (\mathbb{C}^n)^{\otimes N}$ with centralizer S_N. The quantum version of this is that $\widehat{\mathcal{U}_q(\mathfrak{sl}_n)} \curvearrowright (\mathbb{C}^n)^{\otimes N}$, with centralizer the affine Hecke algebra $\widehat{H_N(q)}$. There are many reasons to be interested in this construction, including reasons from mathematical physics.

Anyway, the Drinfeld construction is well-known, and we describe it now.

Assume that $\mathfrak{g}_1$ acts by *derivations* on $\mathfrak{g}_2$, meaning that $\mathfrak{g}_2$ is a $\mathfrak{g}_1$-module, and also that $x \cdot [l, m] = [x \cdot l, m] + [l, x \cdot m]$, where $x \in \mathfrak{g}_1$, $l, m \in \mathfrak{g}_2$, $\cdot$ is the action, and $[,]$ is the bracket in $\mathfrak{g}_2$.

Then $\mathfrak{g}_1 \ltimes \mathfrak{g}_2 \overset{\text{def}}{=} \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as a vector space, with $[(x, l), (y, m)] = ([x, y], [l, m] + x \cdot m - y \cdot l)$ is a Lie algebra. It is the infinitesimal version of the semi-direct product of groups.

Oh, who knows the rule for which way to write the $\ltimes$? The open end points towards the thing being acted on: it's a pair of hands, twisting things around.

Example 9.1 Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^*$ a dual vector space with $[,]_{\mathfrak{g}^*} = 0$ trivial. Then $\mathfrak{g} \curvearrowright \mathfrak{g}^*$ wit hthe ad*-action, and so we form $\mathfrak{g} \ltimes \mathfrak{g}^*$. For example, $\mathfrak{g} = \mathfrak{so}(3)$, $\mathfrak{g}^* = \mathbb{R}^*$, then $\mathfrak{g} \ltimes \mathfrak{g}^* = \mathfrak{so}(3) \ltimes \mathbb{R}^3$ is the affine transformations. ♢

We now explain Drinfeld's construction, at least in the case when $\mathfrak{g}_2 = \mathfrak{g}_1^*$. So let $(\mathfrak{g}, \delta)$ be a Lie bialgerba, $(\mathfrak{g}^*, \delta_*)$ its dual. Then $\mathfrak{g} \curvearrowright \mathfrak{g}^*$ be $\mathrm{ad}^*_{\mathfrak{g}}$, and $\mathfrak{g}^* \curvearrowright \mathfrak{g}$ by $\mathrm{ad}^*_{\mathfrak{g}^*}$. We should look for a version of $\ltimes$ that is more symmetrical. We formulate the construction as a theorem:

Theorem 9.1 *There exists a unique Lie algebra structure on* $\mathcal{D}(\mathfrak{g}) \overset{\text{def}}{=} \mathfrak{g} \oplus \mathfrak{g}^*$ *such that*

- $\mathfrak{g}, \mathfrak{g}^* \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ *are Lie subalgebras.*
- *the natural bilinear form* $((x, l), (y, m)) = \langle x, m \rangle + \langle y, l \rangle$ *is* $\mathcal{D}(\mathfrak{g})$*-invariant. I.e.*

$$([\eta, \xi_1], \xi_2) + (\xi_1, [\eta, \xi_2]) = 0 \ \forall \eta, \xi_1, \xi_2 \in \mathcal{D}(\mathfrak{g}) \tag{9.1}$$

In other words, we have a pairing $\mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{g}) \to \mathbb{C}$, *with and* $\mathcal{D}(\mathfrak{g}) \curvearrowright \mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{g})$ *diagonally, and trivially on* $\mathbb{C}$, *and we require that the pairing is a* $\mathcal{D}(\mathfrak{g})$*-module homomorphism.*

We outline the proof, essentially computing the structure.

Proof (Outline): 1. Let $\{e_i\}$ be a basis in $\mathfrak{g}$, with structure constants $[e_i, e_j] = \sum_k C_{ij}^k e_k$ and $\delta e_i = \sum_{jk} f_i^{jk} e_i \wedge e_k$.

2. Let $\{e^i\}$ be the dual basis in $\mathfrak{g}^*$, so $\langle e_i, e^j \rangle = \delta_i^j$. Then the brackets are $[e^i, e^j] = \sum_k f_k^{ij} e^k$ and $\delta_* e^i = \sum_{jk} c_{jk}^i e^j \wedge e^k$.

3. Then the set $\{e_i, e^j\}$ form a basis in $\mathfrak{g} \oplus \mathfrak{g}^* = \mathcal{D}(\mathfrak{g})$, and the brackets in 1 and 2 define the brackets so that $\mathfrak{g}, \mathfrak{g}^*$ are subalgebras. All we have to do is define $[e^i, e_j] \in \mathfrak{g} \oplus \mathfrak{g}^*$. Let us use the invariant bilinear form to find this bracket. Of course, $\mathfrak{g}$ and $\mathfrak{g}^*$ are isotropic subspaces for this pairing, and $([e^i, e_j], e^k)$ will pick up the $\mathfrak{g}$-component. By invariance:

$$([e^i, e_j], e^k) + (e_j, [e^i, e^k]) = 0 \tag{9.2}$$

but $[e^i, e^k] = f_j ike^j$, and so we use the pairing. Hence

$$[e^i, e_j] = -\sum_k f_j^{ik} e_k + ? \tag{9.3}$$

where we need to compute $? \in \mathfrak{g}^*$. We repeat the method, and get that

$$([e^i, e_j], e_k) = C_{jk}^i \tag{9.4}$$

Hence

$$[e^i, e_j] = -\sum_k f_j^{ik} e_k + \sum_k C_{jk}^i e^k \tag{9.5}$$

The point is the mixed-bracket is kind of symmetric: it's the action of $\mathfrak{g}$ on $\mathfrak{g}^*$, plus the action of $\mathfrak{g}^*$ on $\mathfrak{g}$.

Exercise 18 *Check the Jacobi. We fear that if we say one more time that there is required homework, there will be no more people in the class. The bottom line of these advanced classes is that they are for your consumption, not the grade.*

We outline the homework, and also provide another way to think about the above construction. Remember that if $\mathfrak{g}$ is a Lie algebra, we have the Chevalley complex $\bigwedge^i \mathfrak{g}, d_{\mathfrak{g}}$, and the Jacobi identity for $[,]$ is equivalent to $d_{\mathfrak{g}}^2 = 0$. If we have a Lie bigalgebra $(\mathfrak{g}, \delta)$, then we get the bigraded complex $\bigoplus \bigwedge^i \mathfrak{g} \otimes \bigwedge^j \mathfrak{g}^* = \bigwedge^\bullet (\mathfrak{g} \oplus \mathfrak{g}^*)$, with maps $d_{\mathfrak{g}}$ and $d_{\mathfrak{g}^*}$. The bialgebra requirements include both Jacobis, so $d_{\mathfrak{g}}^2 = 0 = d_{\mathfrak{g}^*}^2$, and the bialgebra compatibility requiredment is that $d_{\mathfrak{g}} d_{\mathfrak{g}^*} + d_{\mathfrak{g}^*} d_{\mathfrak{g}} = 0$. But once we have this, then we have the diagonal differential of the total complex $\delta = d_{\mathfrak{g}} + d_{\mathfrak{g}^*}$, and $\delta^2 = 0$. So the Drinfeld double construction is very natural, as it exactly computes this double complex: $\delta = d_{\mathcal{D}(\mathfrak{g})}$. ****there are two δs in this paragraph, but it should be clear from context which is which. We should call the former "$[,]_{\mathfrak{g}^*}^*$".****

Question from the audience: There is something to check. Is it possible to define a Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ so that the original algebras are subalgebras, etc., but without the invariance of the scalar product? I.e. in the Chevalley complex, are we secretly using invariance? **Answer:** No. Indeed:

Theorem 9.2 *δ as defined in the previous paragraph defines a Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$, which is isomorphic to $\mathcal{D}(\mathfrak{g})$ from the previous theorem.*

Both constructions are useful: Jacobi comes for free from the Chevalley construction, but you don't see the scalar product, whereas it's central to the other construction (where Jacobi is obscured).

Theorem 9.3 *There is a natural Lie bialgebra structure on $\mathcal{D}(\mathfrak{g}) = \mathfrak{g} \bowtie \mathfrak{g}^*$ defined by requiring that the embeddings $\mathfrak{g}, \mathfrak{g}^* \hookrightarrow \mathcal{D}(\mathfrak{g})$ are Lie bialgebra embeddings.*

Exercise 19 *Prove this.*

Corollary 9.3.1 *$\mathcal{D}(\mathfrak{g})^* = \mathfrak{g}^* \oplus \mathfrak{g}$ is the direct sum of Lie algebras.*

Lecture 10 February 11, 2009

Lecture notes up through today are on the website. These are shorter than the actual lecture: Theo has a complete ****only mildly paraphrased, and occasionally annotated**** transcript.

Today we finish the Double construction, and then consider some examples.

Recall, $\mathfrak{g}$ is a Lie bialgebra. The *double* $\mathcal{D}(\mathfrak{g})$ of $\mathfrak{g}$ is the direct sum of two vector spaces $\mathfrak{g} \oplus \mathfrak{g}^*$ as a vector space. It is $\mathcal{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^{*\mathrm{op}}$ as a Lie coalgebra. ****I missed the word op, which controls the sign of the cobracket, last time.**** The Lie brackets are such that $\mathfrak{g}, \mathfrak{g}^* \hookrightarrow \mathcal{D}(\mathfrak{g})$ as Lie subalgebras, but $\mathcal{D}(\mathfrak{g})$ is not a direct sum. In terms of a basis:

$$[e^i, e_j] = \sum_k C^i_{jk} e^k - \sum_k f^{ik}_j e_k \tag{10.1}$$

Another word for the Double is the *bicross-product* $\mathcal{D}(\mathfrak{g}) = \mathfrak{g} \bowtie \mathfrak{g}^*$.

Theorem 10.1 *$\mathcal{D}(\mathfrak{g})$ is a quasitriangular Lie bialgebra with $r = \sum_i e^i \otimes e_i \in \mathfrak{g}^* \otimes \mathfrak{g} \hookrightarrow \mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{g})$. (Hence r does not depend on the basis: it is simply the identity map $id : \mathfrak{g} \to \mathfrak{g}$ thought of as an element of $\mathfrak{g} \otimes \mathfrak{g}^*$.)*

Proof: There is probably a basis-independent proof. We work in a basis.

$$\begin{aligned}
\delta_r(e_i) &= [r, e_i \otimes 1 + 1 \otimes e_i] && (10.2)\\
&= \sum_j [e^j, e_i] \otimes e_j + \sum_j e^j \otimes [e_j, e_i] && (10.3)\\
&= \sum_{jk} C^k_{ji} e_k \otimes e^j + \sum_j e_j \otimes \left(\sum_k C^i_{jk} e^k - \sum_k f^{ik}_j e_k \right) && (10.4)\\
&= \sum_k f^{jk}_i e_j \otimes e_k && (10.5)
\end{aligned}$$

In equation 10.5 we have reindexed and used the skew-symmetry to cancel two terms. ****Equation 10.4 is wrong: we realized a sign error, and have gone back and fixed it mostly.****

Question from the audience: You need also to check that r satisfies the Yang-Baxter equation. **Answer:** Yes. We need to prove that. Of course, the best way to prove the Yang-Baxter equation is to leave it as an exercise. How should you prove this? We are working in a basis, because this is useful when you compute examples.

In a basis, the first term $[r_{12}, r_{13}]$ of the Yang-Baxter equation is really

$$[r_{12}, r_{13}] = [e^i \otimes e_i \otimes 1, e^j \otimes 1 \otimes e_j] = [e^i, e^j] \otimes e_i \otimes e_j \tag{10.6}$$

The second term is $e^i \otimes e^j \otimes [e_i, e_j]$, and the last is $e^i \otimes [e_i, e^j] \otimes e_j$. So we should have

$$[e^i, e^j] \otimes e_i \otimes e_j + e^i \otimes e^j \otimes [e_i, e_j] + e^i \otimes [e_i, e^j] \otimes e_j \stackrel{?}{=} 0 \tag{10.7}$$

Sure enough, we have

$$f_k^{ij} e^k \otimes e_i \otimes e_j + C_{ij}^k e^i \otimes e^j \otimes e_j + e^i \otimes (-C_{ik}^j e^k + f_i^{jk} e_k) \otimes e_j = 0 \tag{10.8}$$

because everything cancels. □

So, given enough supply of Lie bialgebras, we can take their doubles to get a number of examples of quasitriangular.

Question from the audience: Is there a more invariant way to express the bracket? **Answer:** Yes. It should be something like $[(x,l),)y,m)] = ([x,y]_{\mathfrak{g}} + \mathrm{Ad}_x^* m, [l,m]_{\mathfrak{g}^*} + \mathrm{Ad}^* \dots)$. Well, this is a good question, and we didn't prepare this, so we will make it **Exercise 20**. Here's a way to get participation: each time we suggest a problem, someone will explain it the next time. Let's vote. ****7 to 2 in favor.****

In fact, it's better than quasitriangular:

Proposition 10.2 *$\mathcal{D}(\mathfrak{g})$ is facotrizable, and $r + \sigma(r) = \sum_i (e^i \otimes e_i + e_i \otimes e^i)$ defines a nondegenerate invariant scalar product $((x,l),(y,m)) = l(y) + m(x)$. Hence the map $t : \mathcal{D}(\mathfrak{g})^* \to \mathcal{D}(\mathfrak{g})$ by $l \mapsto r_+(l) - r_-(l)$ is a linear isomorphism, and so $\forall x$ there is a unique factorization $x = x_+ \oplus x_-$ where $x_\pm \in \mathfrak{g}_\pm$.*

Proof: In fact, in this form is is a tautology, since as a vector space $\mathcal{D}(\mathfrak{g})$ is defined as a direct sum. □

Example 10.1 We saw already there is a Lie algebra $\mathfrak{b}_+ \subseteq \mathfrak{sl}_2$ given by $[H,X] = 2X$, $\delta H = 0$, $\delta X = \frac{1}{2} H \wedge X$.

Let us describe $\mathcal{D}(\mathfrak{b}_+)$. We choose a dual basis $H^\vee, X^\vee$, so that $\mathfrak{b}_+^\vee = \mathbb{C}H^\vee \oplus \mathbb{C}X^\vee$ with $[H^\vee, X^\vee] = X^\vee$, $\delta H^\vee = 0$, and $\delta X^\vee = H^\vee \wedge X^\vee$. Then $\mathcal{D}(\mathfrak{b}_+) = \mathfrak{b}_+ \oplus \mathfrak{b}_+^\vee = \mathbb{C}H \oplus \mathbb{C}X \oplus \mathbb{C}H^\vee \oplus \mathbb{C}X^\vee$.

Question from the audience: Confused by notation, what is this δ on $\mathfrak{b}_+^\vee$? **Answer:** We would write $\mathfrak{b}^\vee = \mathfrak{b}^*$, and δ should be δ_*, dual to $[,]_{\mathfrak{b}}$.

If we think about Lie algebras alebraically, rather than geometrically, we can present it with a basis, hopefully.

Ok, so we want to compute the cross brackets, using the above formulas. We have, e.g. $C_{HX}^{X} = 2$, and f_X^{HX} is the only other non-zero structural constant.

$$[X^\vee, H] = \sum_a C_{Ha}^{X} a^\vee - \sum_b f_H^{Xb} b = 2X^\vee \tag{10.9}$$

$$[X^\vee, X] = \sum_a C_{Xa}^{X} a^\vee - \sum_b f_X^{Xb} b = -2H^\vee + H \tag{10.10}$$

$$[H^\vee, H] = 0 \tag{10.11}$$

$$[H^\vee, X] = -X \tag{10.12}$$

We should also write out the coalgebra. Remember that as a coalgebra $\mathcal{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^{*\mathrm{op}}$, so $\delta H^\vee = 0 = \delta H$, $\delta X = \frac{1}{2} H \wedge X$, and $\delta X^\vee = -H^\vee \wedge X^\vee$. Also, we have:

$$r = H^\vee \otimes H + X^\vee \otimes X \tag{10.13}$$

So, let's define $H' = \frac{1}{2}H - H^\vee$, $H'' = \frac{1}{2}H + H'$, $X' = X$, and $Y' = -\frac{1}{2}X^\vee$. Then H'' is in the center of the Lie algebra $\mathcal{D}(\mathfrak{b}_+)$, and in fact spans the center. Moreover, $\delta H'' = 0$, and so $\mathbb{C}H''$ is a Lie bialgebra ideal.

Also, H', X', Y' generate $\mathfrak{sl}_2 \hookrightarrow \mathcal{D}(\mathfrak{b}_+)$, and so $\mathcal{D}(b_+) = \mathbb{C}H'' \oplus \mathfrak{sl}_2$ as a Lie algebra. And $\mathbb{C}H''$ is a Lie bialgebra ideal, so the quotient $\mathcal{D}(\mathfrak{b}_+)/\mathbb{C}H''$ is a Lie bialgebra, whose algebra part is $\mathfrak{sl}_2$. But in fact the coalgebra part is our old friend: $\mathcal{D}(\mathfrak{b}_+)/\mathbb{C}H'' = (\mathfrak{sl}_2, \delta)$ is the standard Lie bialgebra structure on $\mathfrak{sl}_2$.

This also explains why our example is quasitriangular:

$$r = \frac{1}{2} H'' \otimes H'' + \frac{1}{2}\left(H' \otimes H'' - H'' \otimes H'\right) - \frac{1}{2}\left(\frac{1}{4} H' \otimes H' - Y' \otimes X'\right) \tag{10.14}$$

We will have to clean up signs and $\frac{1}{2}$s. The signs are always wrong, and the $\frac{1}{2}$s come from the problem of pairing exterior squares.

Anyway, that last term if the $\mathfrak{sl}_2$ r-matrix.

Question from the audience: Is that a general fact, that the quotient of a quasitriangular Lie bialgebra by a Lie bialgebra ideal is quasitriangular? **Answer:** Yes. $\Diamond$

Ok, so we didn't explain where $\mathfrak{b}_+$ with $[H, X] = 2X$, $\delta H = 0$, and $\delta X = \frac{1}{2} H \wedge X$ — we didn't explain where this Lie bialgebra comes from. But this algebra is vecy natural: on $\mathbb{C}[x]$, we let X act as x, and H by $2x\frac{\partial}{\partial x}$. So $\mathfrak{b}_+$ is the linear part of the derivations.

In fact, there is a generalization of this algebra to $\mathfrak{b}_+ \subseteq \mathfrak{g}_a$ for any symmetrizable Kac-Moody algebra. In particular, this works for

- simple Lie algebras
- central extensions of loop algebras $S^1 \to \mathfrak{g}$.
- other examples.

We said the words "Kac-Moody algebras". Who knows this word? ****two or so hands**** The usual suspects. We will explain this construction for simple algebras, and later explain what a KM algebra is.

Ok, so let $\mathfrak{g}$ be simple, and define $\mathfrak{b}_+$ in terms of generators and relations. For each $i \in \Gamma$ the Dynkin diagram, we have two basis elements H_i and X_i, and we let

$$[H_i, H_j] = 0,\ [H_i, X_i] = a_{ij} X_j,\ \text{and}\ \left(\mathrm{ad}_{X_i}\right)^{1-a_{ij}}(X_j) = 0 \text{ if } i \neq j \qquad (10.15)$$

The dimension is $r + |\Delta_+|$, and $X_i = X_{\alpha_i}$ are the corresponding simply roots. Then $\delta H_i = 0$, $\delta X_i = \frac{d_i}{2} H_i \wedge X_i$, where d_i is the length of the root $d_i = (\alpha_i, \alpha_i)/2$.

Exercise 21 $\mathcal{D}(\mathfrak{b}_+) \cong \mathfrak{g} \oplus \mathfrak{h}$, *where* $\mathfrak{h}$ *is a central copy of the Cartan, as a Lia algebra. The bialgebra you know.*

In particular, as before, $\mathcal{D}(\mathfrak{b}_+) \cong \mathfrak{sl}_2 \oplus \mathfrak{h}$.

Lecture 11 February 13, 2009

We begin with Matt presenting a basis-free description of the bracket on the Drinfeld Double $\mathcal{D}(\mathfrak{g})$ of a Lie bialgebra $\mathfrak{g}$. ****I was caught in this morning's hailstorm, rather than catching the bus. I will later include an invariant discussion of the Drinfeld Double and the Chevalley Complex in terms of Penrose et al.'s "birdtrack" stringy notation.****

We now turn to a few examples.

11.1 Kac-Moody algebras

Last time we gave an example of a canonical Lie bialgebra structure for any simple Lie algebra.

We now explain this in full generality. (Ten years ago, NR wrote a paper, never published, working a number of nice examples in Kac-Moody algebras.)

Let $\mathfrak{h}$ be a finite-dimensional vector space, $\mathfrak{h}^*$ its dual, and a collection $h_1, \ldots, h_n \in \mathfrak{h}$ of "co-roots" and $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}$ of "roots". We define the *generalized Cartan matrix* to be $a_{ij} = \langle \alpha_i, h_j \}$. We demand the following conditions:

- $a_{ii} = 2$,
- $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$

- If $a_{ij} \neq 0$, then $a_{ji} \neq 0$.
- There exists $d_1, \ldots, d_n \in \mathbb{Z}_{>0}$ diagonalizing the matrix, i.e. $d_i a_{ij} = a_{ji} d_i$ (no sum).
- $\dim \mathfrak{h} = n + \dim(\ker a)$.

Then we define the *Kac-Moody algebra* $\mathfrak{g}(a)$ to be the Lie algebra generated by $\mathfrak{h}$ (the Cartan), and e_i, f_i for $i = 1, \ldots, n$, with defining relations

$$[h, h'] = 0, \ [h, e_i] = \langle \alpha_i, h \rangle e_i, \ [h, f_i] = -\langle \alpha_i, h \rangle f_i, \text{ and } [e_i, f_j] = \delta_{ij} h_i \tag{11.1}$$

and also the *Serre relations*:

$$(\mathrm{ad}_{e_i})^{1-a_{ij}} (e_j) = 0 \text{ and } (\mathrm{ad}_{f_i})^{1-a_{ij}} (f_j) = 0 \tag{11.2}$$

It is clear that $\mathfrak{g}(a)$ is $\mathbb{Z}$-graded with $\deg(\mathfrak{h}) = 0$, $\deg(e_i) = 1$, and $\deg(f_i) = -1$.

Example 11.1 Assume a is nondegenerate, so that $n = \dim \mathfrak{h}$, and $\mathfrak{g}(a)$ is a simple finite-dimensional Lie algebra. $\Diamond$

Example 11.2 Assume that $\dim \ker a = 1$. Then $\dim \mathfrak{h} = 1 + n$, and we have

Theorem 11.1 (Galber, Kac) *Then the a has block form of an $(n-1) \times (n-1)$ nondegenerate thing, in columns $1, \ldots, n-1$, and a last column number 0, and the nondegenerate part generates a simple Lie algbera $\mathfrak{g}$. Then*

$$\mathfrak{g}(a) \cong \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}t \frac{d}{dt} \tag{11.3}$$

and the Cartan is $\mathfrak{h}_{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathbb{C}t\frac{d}{dt}$.

In particular, for $i = 1, \ldots, n-1$, then $e_i \mapsto e_i \in \mathfrak{g}$, and $f_i \mapsto f_i \in \mathfrak{g}$. Also, $e_0 \mapsto t f_\theta$, where θ is the longest root for $\mathfrak{g}$, and $f_0 \mapsto t^{-1} e_\theta$.

These are the most studied infinite-dimensional Lie algebras. On the one hand, they have a simple presentation, and the representation theory of simple Lie algebras transfers directly. On the other hand, they have a simply geometrical interpretation as a central extension to the loop algebra, along with derivatives. Connections to physics: two-dimensional gauge theories, CFTs.

****missed something about the grading**** $\Diamond$

In the 80s, people asked about whether there were interesting examples when $\dim \ker a \geq 2$. There don't seem to be.

NR learned from Richard Borcherds that you should think of the simple Lie algebras as spheres, and the affine ones ****when $\dim \ker a = 1$?**** as cylinders. The higher examples grow exponentially. There are difficult questions, because e.g. the Weyl groups of these generalized algebras are interesting. For example, $SL_2(\mathbb{Z})$ shows up. You think that it's a difficult problem, so there should be geniuses working on it, but no: geniuses look for difficult problems with easy solutions. So this is an open field of research.

Ok, so why are we doing this? Every Kac-Moody algebra has a Lie bialgebra question, and moreover a quantum counterpart. But in the generalized case, it's strange to ask these questions, because we know so little about the Lie algebra: why make it more complicated?

Proposition 11.2 *If $\mathfrak{g}(a)$ is a Kac-Moody algebra, then*

$$\delta h = 0,\ \delta e_i = \frac{d_i}{2} h_i \wedge e_i,\ \delta f_i = \frac{d_i}{2} h_i \wedge f_i \tag{11.4}$$

is a Lie bialgebra structure, and moreover it is quasitriangular.

We will not explain the quasitriangular structure, although there is a formula for it, because it is a good presentation. See, the dimension of the Cartan is larger than the rank of the algebra. So there are extra elements, and always they act as derivations, as in the example.

11.2 Real forms of Lie bialgebras

A *real Lie bialgebra* is a real vector space, real $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, and real $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, i.e. it is a Lie bialgebra over $\mathbb{R}$. When we have such a thing $\mathfrak{g}_{\mathbb{R}}$, we can define as always the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Then $\mathfrak{g}_{\mathbb{R}} \subseteq \mathfrak{g}_{\mathbb{C}}$ is invariant with respect to complex conjugation.

What we can do is complement the conjugation with an automorphism. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be a real automorphism of $\mathfrak{g}_{\mathbb{C}}$, so that $\sigma[a, b] = [\sigma a, \sigma b]$, and we demand that it be a $\mathbb{C}$-antilinear involution: $\sigma(\lambda a) = \bar{\lambda}\sigma a$, and $\sigma^2 = \mathrm{id}$.

A *real form* of $\mathfrak{g}_{\mathbb{C}}$ is a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ such that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Then the real form corresponding to σ is $\mathfrak{g}^{\sigma}$ = the set of fixed points of σ.

Exercise 22 *Real forms of $\mathfrak{g}_{\mathbb{C}}$ are classified by equivalence classes (modulo inner automorphism) of such σ. (C.f. Fulton and Harris)*

We can make all the same definitions for Lie bialgebras, additionally demanding that σ preserves the coalgebra: $(\sigma \otimes \sigma)\delta = \delta\sigma$. Then again $\mathfrak{g}^{\sigma}$ is a real Lie bialgebra with complexification $\mathfrak{g}_{\mathbb{C}}$, and this gives the classification of real forms.

Example 11.3 We start with $\mathfrak{sl}_2(\mathbb{C})$ and the standard Lie bialgebra structure. If we have $x = aH + bX + cY$, then we define the *Killing form* $(x, x) = \mathrm{tr}(\mathrm{Ad}_x \mathrm{Ad}_x) = 2a^2 + bc$. So this is a quadratic form on $\mathbb{C}^3$, and it's natural to ask what it looks like on the real three-dimensional subspace.

Compact real form Let $\sigma = (-1) \circ$ (Hermetian conjugation), i.e. $\sigma(H) = -H$, $\sigma(X) = -Y$, $\sigma(Y) = -X$. It's important that this is minus conjugation. If you think of $\mathcal{U}\mathfrak{g}_{\mathbb{C}}$, it has a $*$-operation from Hermetian conjugation, and this is not that. It's a combination of that with the antipode. We say this for those who know to much. Then iH, $X - Y$, and $i(X + Y)$ are the invariant elements, and the real form is $\mathfrak{su}_2 = \mathbb{R}(iH) \oplus \mathbb{R}(X - Y) \oplus \mathbb{R}i(X + Y)$, and we adopt coordinates s, t, u.

The Killing form is is $-2(s^2+t^2+u^2)$. There is a problem for the Lie coalgebra structure. We have $\delta X = \frac{1}{2}H \wedge X$. So $(\sigma \otimes \sigma)\delta X = \frac{1}{2}\sigma H \wedge \sigma X = -\frac{1}{2}H \wedge Y$, but $\delta \sigma X = \delta(-Y) = -\frac{1}{2}H \wedge Y$. So it's fine, but it's not supposed to be.

For Lie algebras, we have two real forms: $\mathfrak{su}_2$, $\mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{su}_{1,1}$, but for bialgebras these will split. $\Diamond$

Lecture 12 February 18, 2009

Last time we began discussing real forms of complex Lie algebras. We gave the most natural definition for Lie bialgebras. Recall, a *real Lie bialgebra* is simple a Lie bialgebra over $\mathbb{R}$, so that the bracket and cobracket are real linear maps. Such an algebra has a complexification, and then we have:

Theorem 12.1 (Fulton, Harris) *Equivalence classes of real forms of Lie bialgebras are in bijection with outer automorphisms σ such that $\sigma^2 = id$, $\sigma(\lambda a) = \bar{\lambda}\sigma(a)$, and σ is an automorphism of the Lie bialgebra* ****over $\mathbb{Z}$****: $\sigma([a,b]) = [\sigma a, \sigma b]$ *and* $(\sigma \otimes \sigma)\delta(a) = \delta(\sigma a)$.

If we just said "automorphism of the Lie algebra", we'd get that classification, and there's the decorated Dynkin diagrams with which to compute these. The classification of real forms of bialgebras — we don't know where it is.

Example 12.1 $\mathfrak{sl}_2(\mathbb{C})$, and there are two nonequivalent real forms. See, there's the bilinear nondegenerate form $(,)$, the Killing form, on $\mathfrak{sl}_2(\mathbb{C})$. If $\mathfrak{g}$ is a real form of $\mathfrak{sl}_2(\mathbb{C})$, then $(,)_\mathfrak{g}$ is a real Killing form, quadratic on $\mathbb{R}^3$, and so it is classified by the signature: $(+++)$ or $(++-)$. So we have $\mathfrak{su}_2$ ($+++$ or $---$) and $\mathfrak{su}_{1,1} \cong \mathfrak{sl}_2$ ($+--$ or $-++$). It is a matter of taste the sign of the Killing form.

$\mathfrak{su}_2$: Then $\sigma(H) = -H$, $\sigma(X) = -Y$, and $\sigma(Y) = -X$. This is a bialgebra automorphism of the standard Lie bialgebra structure ($\delta H = 0$, $\delta X = \frac{1}{2}H \wedge X$, and $\delta Y = \frac{1}{2}H \wedge Y$).

We make an aside: the $\frac{1}{2}$s are a mess, because of the confusion about how to define the bracket of exterior products. If you embed in the tensor product, you get $(l \wedge m, x \wedge y) = 2(l(x)m(y) - m(x)l(y))$. We will fix the notes.

Oh, and there's another error. If you just have a solution to CYB, the arguments we gave don't work to give a quasitriangular structure.

So back to $\mathfrak{su}_2$. We have $\mathfrak{su}_2 = \mathfrak{sl}_2^\sigma = \mathbb{R}iH \oplus \mathbb{R}i(X+Y) \oplus \mathbb{R}(X-Y)$. Let's look at the cobracket. We'd have $\delta(iH) = 0$, $\delta(i(X+Y)) = i\frac{H}{2} \wedge (X+Y) = -i\frac{iH}{2} \wedge i(X+Y)$. This i is a big problem, but not really: $\lambda\delta$ is a Lie cobracket for $\mathfrak{sl}_2(\mathbb{C})$ for any complex λ. So we take $\tilde{\delta} = i\delta$, whence $(\mathfrak{su}_2, \tilde{\delta})$ is a real Lie bialgebra, with complexification $(\mathfrak{sl}_2(\mathbb{C}), \tilde{\delta})$. ****Jacobi and compatibility are both homogeneous, so for any bialgebra, you can scale the bracket and cobracket independently****

$\mathfrak{sl}_2(\mathbb{R})$: $\sigma(H) = H$, $\sigma(X) = X$, and $\sigma(Y) = Y$. Then $\mathfrak{sl}_2^(\mathbb{C})^\sigma = \mathbb{R}H \oplus \mathbb{R}X \oplus \mathbb{R}Y = \mathfrak{sl}_2(\mathbb{R})$. So $(\mathfrak{sl}_2(\mathbb{R}), \delta)$ is a real Lie bialgebra.

$\mathfrak{su}_{1,1}$: Lastly, we consider $\mathfrak{su}_{1,1}$, which happens to be isomorphic to $\mathfrak{sl}_2$; for $\mathfrak{sl}_n$, there will be a family of real forms $\mathfrak{su}_{p,q}$, which are not isomorphic to $\mathfrak{sl}_n$. In any case, $\sigma(H) = -H$, $\sigma(X) = Y$, and $\sigma(Y) = X$. You can see that the Cartan is imaginary as for $\mathfrak{su}_2$, but there is no minus sign on X, Y, which will change the signature. We have $\mathfrak{su}_{1,1} = \mathbb{R}iH \oplus \mathbb{R}i(X - Y) \oplus \mathbb{R}(X + Y)$.

Let's check the signature, looking at the value of the Killing form with respect to this basis, which is special, since this basis is orthonormal. Rather than working over the adjoint representation, let's work over $\mathbb{C}^2$, and we'll get a multiply of the basis. Then

$$iH = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \, i(X - Y) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \, iH = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{12.1}$$

and the signature of the Killing form is

$$\operatorname{tr}_{\mathbb{C}^2}\left(\left(\alpha iH + \beta i(X - Y) + \gamma(X + Y)\right)^2\right) = -2\alpha^2 + 2\beta^2 + 2\gamma^2 \tag{12.2}$$

Looking at the cobracket, we see that $\delta(iH) = 0$, but we have the same problems as with $\mathfrak{su}_2$: $\delta(i(X - Y)) = i\frac{H}{2} \wedge (X - Y) = -i\frac{iH}{2} \wedge i(X - Y)$, and $\delta(X + Y) = -i\frac{iH}{2} \wedge (X + Y)$. So we take $\tilde{\delta}$, and then $(\mathfrak{su}_{1,1}, \tilde{\delta})$ is a real Lie bialgebra, and its complexification is not $(\mathfrak{sl}_2, \delta)$ but rather $(\mathfrak{sl}_2, \tilde{\delta})$.$\Diamond$

In general in this class $\mathfrak{sl}_2$ will be our main hero, and you can try $\mathfrak{sl}_n$ based on the literature out there. But we make some general remarks.

If $\mathfrak{g}$ is simple, we let it be generated by (H_i, X_i, Y_i) for roots $i = 1, \dots, r = \operatorname{rank}(\mathfrak{g})$, and there's the Chevalley-Serre relations. Then we have the "standard Lie bialgebra structure" — there are other choices —

$$\delta H_i = 0, \, \delta X_i = \frac{d_i}{2} H_i \wedge X_i, \, \delta Y_u = \frac{d_i}{2} H_i \wedge Y_i \tag{12.3}$$

Then we have *compact real forms*: $\sigma(H_i) = -H_i$, $\sigma(X_i) = -Y_i$, and $\sigma(Y_i) = -X_i$. Then we get

$$\mathfrak{g}_{\mathbb{R}}^{\text{compact}} = \bigoplus_{j=1}^{r} \mathbb{R}iH_j \oplus \bigoplus_{\alpha \in \Delta_+} \left(\mathbb{R}i(X_\alpha + X_{-\alpha}) \oplus \mathbb{R}(X_\alpha - X_{-\alpha})\right) \tag{12.4}$$

For $\mathfrak{g} = \mathfrak{sl}_n$, this is $\mathfrak{su}_n$. We take the notation that $j = \alpha_j$ for the (enumerated) simple roots. Anyway, $(\mathfrak{g}_{\mathbb{R}}^{\text{compact}}, \tilde{\delta} = i\delta)$ is a real compact Lie bialgebra.

For $\mathfrak{sl}_n$, $\Delta_+ = \{\epsilon_i - \epsilon_j\}_{i<j}$, where $\epsilon_1, \dots, \epsilon_n$ is a basis in $\mathbb{R}^n$, and $X_{\epsilon_i - \epsilon_j} = e_{ij}$, $X_{-\epsilon_i + \epsilon_j} = e_{ji}$, and $H_i = e_{ii} - e_{i+1,i+1}$ for $i = 1, \dots, n-1$.

Ok, so we have the Lie bialgebra $(\mathfrak{g}_{\mathbb{R}}^{\text{compact}}, \tilde{\delta})$. Let's try to recognize the dual as a known Lie algebra, so that perhaps we can recognize the double.

Let's start with $\mathfrak{su}_2$, with basis $iH, i(X + Y), (X - Y)$. Then $\mathfrak{su}_2^*$ has the dual basis h, e, f, with the bracket $[h, e] = e$, $[h, f] = f$, and $[e, f] = 0$, a real Lie algebra. The easiest faithful representation of this algebra is .... Can we do this real algebra as 2×2 matrices? Yes:

$$h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \tag{12.5}$$

So we can write this as triangular traceless 2×2 matrices with real diagonal but complex upper part:

$$\mathfrak{su}_2^* \cong \left\{ \begin{pmatrix} \lambda & c \\ 0 & -\lambda \end{pmatrix} \text{ s.t. } \lambda \in \mathbb{R}, c \in \mathbb{C} \right\} \tag{12.6}$$

We write it in this way so that we can exponentiate:

We have the pairing $(\mathfrak{su}_2, \tilde{\delta})$ with $(\mathfrak{su}_2^*, \delta_*)$, and we can recognize $\mathfrak{su}_2^* \cong \mathfrak{h}^{\mathbb{R}} \oplus \mathfrak{n}_+^{\mathbb{C}}$. We can exponentiate this pairing: (SU_2, p) versus (SU_2^*, p_*), where

$$SU_2^* = \left\{ \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \text{ s.t. } a > 0, c \in \mathbb{C} \right\} \tag{12.7}$$

Now we turn to SU_n. The most amazing thing is that it seems different, but is ultimately the same. Did we ask this exercises last time?

Exercise 23 *The Lie bialgebra structure is given by equation 12.3, and so we have $\mathfrak{sl}_n^*$ generated by $H_i^\vee$, $X_i^\vee$, and $Y_i^\vee$, and the bracket*

$$[H_i^\vee, H_j^\vee] = 0, [H_i^\vee, X_j^\vee] = \delta_{ij} \frac{d_i}{2} X_j, [H_i^\vee, Y_j^\vee = \delta_{ij} \frac{d_i}{2} Y_j^\vee \tag{12.8}$$

and Serre's relations for $X_i^\vee$ and $Y_i^\vee$.

Then $\mathfrak{sl}_n^* \subseteq \mathfrak{b}_+ \oplus \mathfrak{b}_-$ as $\{(h \oplus x, -h \oplus y)\}$ where $h \in \mathfrak{h}$, $x \in \mathfrak{n}_+$, and $y \in \mathfrak{n}_-$.

You know, there are two levels. The first level is when a name appears in a subject, and the second level is where "gaussian" is written with a small "g". We were tempted to write "serre".

Ok, so let's take $\sigma(H_i) = -H_i$, $\sigma(X_i) = -Y_i$, and $\sigma(Y_i) = -X_i$.

Theorem 12.2 $\mathfrak{su}_n^* = \mathfrak{sl}_n^{*\,\sigma}$ *is isomorphic as a Lie algebra to the space of traceless triangular matrices with complex upper part but pure-real diagonal.*

Proof: Exercise 24. The proof is almost the same as for $\mathfrak{su}_2$. □

And so we have

$$SU_n^* = \left\{ \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \text{ s.t. } a_i > 0, a_1 \dots a_n = 1, * \in \mathbb{C} \right\} \tag{12.9}$$

Next time we will start looking at symplectic leaves

12.1 Bruhat decomposition

We assume that everyone knows that if W is the Weyl group of $\mathfrak{g}$ (of the root system of $\mathfrak{g}$), then it acts naturally on $\mathfrak{h}$. In particular, for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, we have $\mathfrak{h} = \mathbb{C}^{n-1}$ embedded in $\mathbb{C}^n$ as the hyperplane perpendicular to $(1,\dots,1)$. Then W is the symmetric group S_n acting naturally on $\mathbb{C}^n$ and fixing $(1,\dots,1)$.

Well, G acts on $\mathfrak{g}$ be the Adjoint action, e.g. $SL_n \curvearrowright \mathfrak{sl}_n$ by conjugation of matrices. The question is: do we have a natural embedding of $W \hookrightarrow SL_n$? The answer is no: rather, $N(H)$, the normalizer of the Cartan as a subgroup $H \leq G$, and then $N(H) \cong H \rtimes W$. So $N(H)$ acts naturally on $\mathfrak{sl}_n$, and that's how W acts on the whole algebra.

The goal: find symplectic leaves of SU_n, which will be the Shubert cells of SU_n/T.

Lecture 13 February 20, 2009

13.1 Bruhat Decomposition

Last time we began to discuss Bruhat Decomposition. We will see that this provides a cell decomposition of a Poisson Lie group into symplectic leaves.

First we recollect the *Weyl group*. Suppose we have a root system $\Delta = \{\alpha \in \mathbb{R}^n\}$. We will be dealing only with semisimple finite-dimensional Lie algebras. We pick $\Gamma \subseteq \Delta$ the simple roots, so that $\Delta = \Delta_+ \cup \Delta_-$ and $\Gamma \subseteq \Delta_+$. It will be convenient to enumerate the simple roots: $\Gamma = \{\alpha_1,\dots,\alpha_r\}$ where r is the rank of Δ (i.e. the rank of $\mathfrak{g}$).

To each root $\alpha \in \Delta_+$, we associate a reflection $s_\alpha : x \mapsto x - 2(\alpha,x)/(\alpha,\alpha)$, which is reflection with respect to the hyperplane $\alpha^\perp$. We define the *Wely group* to be the group generated by these reflections, and it is a property of root systems that this is a finite group.

We let $s_i = s_{\alpha_i}$. Then we have a well-known result (c.f. Fulton and Harris):

Theorem 13.1 *$W \cong \langle s_i,\ i = 1,\dots,r$ s.t. $s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1\rangle$, where m_{ij} is related to the Cartan matrix.*

Example 13.1 For SL_n, $\Delta = A_{n-1}$, and so $(s_i s_j)^2 = 1$ if $i \neq j \pm 1$, and $(s_i s_{i+1})^3 = 1$. Thus $W_{A_{n-1}} = S_n$. If $\Delta_+ = \{\epsilon_i - \epsilon_j\}_{i<j}$, then $\Gamma = \{\epsilon_i - \epsilon_{i+1}\}_{i=1}^{n-1}$, and W acts by permutations on ϵ_i. $\Diamond$

If Δ is the root system of $\mathfrak{g}$, we let $\mathfrak{h} \subseteq \mathfrak{g}$ be the Cartan subalgebra, and so $\mathfrak{h} = \bigoplus_{i=1}^r \mathbb{R}\alpha_i^*$ (roots are in $\mathfrak{h}^*$). Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, where $\mathfrak{n}_\pm$ are nilpotent and correspond to $\Delta_\pm$.

W acts naturally on $\mathfrak{h}$. Does it act naturally on $\mathfrak{g}$? Not quite. Strictly speaking, the answer is No. But anytime the answer is No, you can ask "if not this, then what?"

I'm making all these pseudo jokes, and you are putting them online. You can skip most of them.

So, let G be the Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$. Then G acts on G by conjugation, and thus G acts on $\mathfrak{g} = T_e G$ by the Adjoint action. Can W naturally embed into G? If it can, then it has a natural action on $\mathfrak{g}$.

Let $H \subseteq G$ by the Cartan subgroup. **Question from the audience:** Is G connected? Or does it matter? **Answer:** It probably doesn't matter, but let's always assume that G is connected, and most of the time that G is simply connected.

So $H \subseteq G$, and we can construct $N(H) \subseteq G$ the normalizer of H in G. In other words, $N(H) = \{g \in G \text{ s.t. } gHg^{-1} \subseteq H\}$.

Theorem 13.2 *$N(H) \cong W \ltimes H$.*

Proof: For all G, see Fulton and Harris. For SL_n, it is obvious: Suppose d is diagonal, and so is $gdg^{-1} = d'$. So $gd = d'g$, and if $g_{ij} \neq 0$, then $d_i = d'_j$. So $d'_i = d_{\sigma(i)}$, where σ is a permutation on the indices $1, \dots, n$. When $g_{ij} = 0$, no conditions, but of course $\det g = 1$. Thus g must be a monomial matrix (one non-zero entry in each row and each column).

Question from the audience: How does this give us the semidirect product? **Answer:** $N(H)$ is the monomial matrices. For example,

$$\begin{pmatrix} & & a \\ b & & \\ & c & \end{pmatrix} = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \begin{pmatrix} b & & \\ & c & \\ & & a \end{pmatrix} \in S_n \ltimes \text{diagonal} = W \ltimes H \tag{13.1}$$

□

Corollary 13.2.1 *1. $N(H)$ acts naturally by conjugation on G.*

2. $N(H)$ acts naturally by Ad *on $\mathfrak{g}$.*

Corollary 13.2.2 *If we choose representations of W in $N(H)$, i.e. we chose a section $W \hookrightarrow N(H) : w \mapsto \dot{w}$, when $\dot{w}$ acts on $\mathfrak{g}$.*

This is not canonical, but it is up to the action of the Cartan. We will have the same story for quantum Weyl group.

Exercise 25 *Find the action of*

$$\dot{s}_i = \left(\begin{array}{ccc|cc|ccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ \hline & & & 0 & 1 & & & \\ & & & -1 & 0 & & & \\ \hline & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right) \tag{13.2}$$

on $\mathfrak{sl}_n$. On generators: $T_i \stackrel{\text{def}}{=} \mathrm{Ad}_{\dot{s}_i}$, and find $T_i(H_j = e_{jj} - e_{j+1,j+1})$, $T_j(e_{i,i+1} = e_i)$, and $T_j(e_{i+1,i} = f_i)$.

If you can do this, then quantizing is easy: you judiciously add qs.

Ok, so We have G a complex Lie group, and a fix a Borel subgroup $B \subseteq G$. A priori there are many Borel subgroups, but for simple groups we can list all of them, and they are related by conjugation.

Theorem 13.3 $G = \bigsqcup_{w \in W} BwB$.

$BwB = B\dot{w}B = \{b\dot{w}b' \text{ s.t. } b, b' \in B\}$, where $\dot{w}$ is any representative of w in $N(H)$. We remark that $B\dot{w}B$ does not depend on the choice of $\dot{w} \in N(H)$. If $\dot{w}$ and $\ddot{w}$ are two representatives, then $\dot{w} = \ddot{w}h$ for $h \in H$.

Ok, so for SL_n, we take B the upper triangular matrices. Then

Theorem 13.4 $G = \bigsqcup_{w \in W} B_- w B_- = \bigsqcup_{w \in W} B_- w B_+ = \bigsqcup_{w \in W} B_+ w B_+$

Hence, we can define *double Bruhat cells* $G_{u,v} = BuB \cap B_- v B_-$. These are very important in the analysis of totally positive subsets of SL_n.

Proof: For all G, look in textbooks, c.f. Cherry ****?**** and Preston. We will do SL_n.

1. Let us fix $g \in SL_n$. We want to put it in one cell. We chose $b \in B$ such that bg^{-1} has maximal number of zeros in the left side of each row. For example, any matrix can be multiplied by a triangular from the left to get into the form $\begin{pmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix}$. This is backwards row-eschelon reduction: if at any time two rows have the same number of 0s, then we can multiply by an upper triangular to create another 0.

   So each row will have different number of zeros, we can find $\sigma \in S_n$ such that $\dot{\sigma} b g^{-1}$ is upper triangular.

   So for any g there is σ, b so that $\dot{\sigma} b g^{-1} = b' \in B$, and so $g = (b')^{-1} \dot{\sigma} b \in B\sigma B$.

2. We now must show that these cells do not intersect. So assume that $b\dot{\sigma}b' = \tilde{b}\dot{\tau}\tilde{b}'$ for some bs$\in B$, $\sigma, \tau \in S_n$. Letting $\beta = \tilde{b}^{-1}b$ and $\beta' = \tilde{b}'(b')^{-1}$, we have $\beta\dot{\sigma} = \dot{\tau}\beta'$. But $\dot{\sigma}, \dot{\tau}$ are monomial matrices, and so the only possibility is that $\sigma = \tau$ and $\beta, \beta' \in H$. □

Exercise 26 *Do it by hand for SL_2.*

Exercise 27 *Describe the closure $\overline{BwB}$.*

Then you will see why it's called "Bruhat cells": there's something called Bruhat ****?**** for permutations.

A few remarks:

- SL_n/B is naturally isomorphic to the *flag variety*, in other words the collection of chains of subspaces $0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n$, where $\dim V_i = i$. See, SL_n includes all changes of bases, and the Borel changes the basis in each flag.

- **Theorem 13.5** *Consider G/B as a real manifold. Then $G/B \cong K/T$, where K is the compact real form of G and $T \subseteq K$ is the maximal torus, i.e. the Cartan in K.*
- G/B is called the *generalized flag variety.* It is $(\bigsqcup BwB)\,\{b\dot{w}b'B\} = \{b\dot{w}B \text{ s.t. } b \in B\}$. But often $\dot{w}$ conjugates b to another upper-triangular, so in fact $\{b\dot{w}B \text{ s.t. } b \in B\} = \{u\dot{w}B \text{ s.t. } u \in B \text{ and } \dot{w}^{-1}u\dot{w} \in U^-\}$, where U^- are the unipotent lower-triangular matrices.

  The remarkable fact is that $\dim_{\mathbb{C}}(U_w^-) = \ell(w)$ is the length of the permutation.

  $\dim_{\mathbb{C}}(BwB/B) = \ell(w)$. Hence the corresponding cell of K/T has real dimension twice the complex dimension, hence $2\ell(w)$. And any time the dimension is even, you should expect a natural symplectic structure. This will happen.

Lecture 14 February 23, 2009

****I was ten minutes late. I pick up when I arrived.****

14.1 Shubert Cells

We have $K/T = \bigsqcup_{w\in W} C_w$, where $C_w = U_w$ is regarded as a real manifold. We have "almost coordinates" on U_w, for which everything is algebraic over $\mathbb{C}$, meaning we have coordinates on a Zariski open subset of U_w.

See, $U = \exp(\mathfrak{n}_+)$ and $\mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta_+} \mathbb{C}e_\alpha$. Let $e_i = e_{\alpha_i}$ are generators corresponding to simple roots, whence $\mathfrak{n}_+$ is the Lie algebra generated by e_i with standard relations.

Example 14.1 $\mathfrak{n}_+ \subseteq \mathfrak{sl}_3$, generated by e_1, e_2 with $[e_1,[e_1,e_2]] = [e_2,[e_1,e_2]] = 0$. This is equivalent to $[e_1,e_2] = e_{12}$ ∈center. ♢

How do you get the coordinates? We formulate as a theorem, and skip the proof. It is not difficult, but involves involved computations.

Theorem 14.1 *The mapping $\phi_{\tilde{w}} : \mathbb{C}^{\ell(w)} \to U_w$ gives an almost coordinate system on U_w (meaning a coordinate system on a Zariski open subset). Here:*

1. *$\tilde{w}$ is a reduced decomposition $s_{i_1}\dots s_{i_l}$ of w. I.e. it is a factorization of w into simple reflections which is minimal in length. E.g. $s_1s_2s_1$ is reduced in S_3, but $s_1s_2s_1s_2s_1s_2$ is not (it equals 1). Reduced decompositions are not unique, but do all have the same length $l = \ell(w)$.*
2. *Fix such a decomposition $\tilde{w}$ of w. Define $\phi_{\tilde{w}} : \mathbb{C}^{\ell(w)} \to U_w$ by*

$$(t_1,\dots,t_l) \mapsto \exp(t_1e_{i_1})\dots\exp(t_le_{i_l}) \tag{14.1}$$

Example 14.2 $\mathfrak{sl}_3$, and $w = w_0$. So U_w is the subset of all upper triangular matrices so that when you conjugate, you get a lower-triangular matrix. But we can take $\dot{w}_0$ to be $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, whence the conjugation of any upper-triangular is lower-triangular. So $U_{w_0} = U$.

There are two reduced recompositions of w_0: $w_0' = s_1 s_2 s_1$, and $w_0'' = s_2 s_1 s_2$. Then:

$$\phi_{w_0'}(t_1, t_2, t_3) = \begin{pmatrix} 1 & t_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{14.2}$$

When you multiply these matrices, you will get some Zariski open subset of U. For the other one:

$$\phi_{w_0''}(t_1, t_2, t_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{14.3}$$

$\Diamond$

One can ask: do these different coordinate systems fit together to form a chart? This is a very deep subject, and quickly gets you into the topic of *Cluster algebras*.

One can now ask: Does there exist a similar coordinate system on G? Answer: yes, on each BwB there exists a similar system.

In fact, these become not just almost-coordinate systems, but coordinate systems on $G(\mathbb{R})_{\geq 0}$, the space on non-negative elements in the split real form.

We will make a detour advertising the split real form. Everyone knows the compact real form; the nonnegative part of the split real form is important too.

Example 14.3 $G = SL_n$. Then $SL_n(\mathbb{R})_{\geq 0}$ is the space of matrices $g \in SL_n(\mathbb{R})$ such that all minors are non-negative. $\Diamond$

This has been studied since the nineteenth century, and interesting results were discovered in the 50s and then forgotten, and the subject has come back again in representation theory with the results of Lusztig and ****?****.

Ok, so let's look at the above example from a more theoretical point of view. What is a minor?

Let $V = \mathbb{C}^n$. This is the first fundamental representation of $\mathfrak{sl}_n$, and it is the representation V_{ω_1} of the fundamental weight ω_1. There are other representations: $V_{\omega_i} = \bigwedge^i V$, where g acts diagonally: $g(x_1 \wedge \cdots \wedge x_i) = gx_1 \wedge \cdots \wedge gx_i$ ****for $g \in G$; for $g \in \mathfrak{g}$, it acts as a derivation?****. Ok, let $\{e_i\}_{i=1}^n$ be a basis in $\mathbb{C}^n$. Then $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{i_1 < \cdots < i_k}$ is a basis of $\bigwedge^k \mathbb{C}^n$. There is not a natural dot product, but there is an SL_n-invariant pairing $\bigwedge^{n-k} \mathbb{C}^n \otimes \bigwedge^k \mathbb{C}^n \to \mathbb{C}$.

$$\begin{aligned} g(e_{i_1} \wedge e_{i_k}) &= ge_{i_1} \wedge \cdots \wedge ge_{i_k} \\ &= \sum_{j_1 < \cdots < j_k} g_{i_1 \ldots i_k}^{j_1 \ldots j_k} e_{j_1} \wedge e_{j_k} \end{aligned}$$

Then $g^{j_1 \dots j_k}_{i_1 \dots i_k}$ is (the determinant of the) $k \times k$ minor of $\mathfrak{g}$.

So $g \in SL_n$ is nonnegative exactly if it has nonnegative matrix elements in the monomial basis for the fundamental representation ****meaning the full $\bigwedge^\bullet V$?****.

Ok, but in representation-theory, we should not talk about the basis e_i as we have, but rather as the weight basis. And there is a unique weight basis up to rescaling:

Theorem 14.2 (Lusztig) *In each finite-dimensional representation V_λ, there exists a unique basis in which ll matrix elements of non-negative $g \in SL_n$ are non-negative. This is called the* canonical basis *or the* Crystal basis*, and was originally discovered in $\mathcal{U}_q\mathfrak{g}$. The theorem is true for any G, although we didn't define what it means.*

The non-negative matrices are not a subgroup: if you take inverses, you do not remain nonnegative. But it is a subsemigroup. In terms of Hopf algebras, it is not a sub-Hopf algebra, but it is a subbialgebra over $\mathbb{R}_{\geq 0}$.

We now make the definition of nonnegative element for any group. **Question from the audience:** What is the split real form? **Answer:** Given $\mathfrak{g}_\mathbb{C}$, we define $\mathfrak{g}_\mathbb{R}$ the *split real form* to be the real form corresponding to $\sigma = \mathrm{id}$ on generators are $\sigma(\lambda a) = \bar{\lambda}\sigma(a)$. So it is the real span of the generators, and $G(R) = \exp(\mathfrak{g}_\mathbb{R})$. In any case, we define $G(\mathbb{R})_{\geq 0}$ to be the subset of $G(\mathbb{R})$ where "matrix elements" in fundamental representation are non-negative. C.f. Fomin and Zelevinsky.

We will see soon that as we pass to quantization, we will deform the Poisson algebra to an associative algebra. If it is over $\mathbb{R}$, we will deform the real algebra to a complex $*$-algebra. The natural question: what exactly gives the quantization of this positive part? There are some works on this, but this direction is still largely open. It's some kind of non-compact quantum groups with this positivity condition: Not Hopf algebras, but some well-structured bialgebras.

Next time we start the description of symplectic leaves.

14.2 Kac-Moody algebras — mini-presentation by Chul-hee Lee

We show the quasi-triangularity of Kac-Moody algebras. For simplicity, we restrict to affine KM algebras. The idea: we learned the double construction of Lie bialgebras, and Kac-Moody algebras have Borel subalgebras, and if we apply the double construction to the Borel subalgebra, we will get back the KM algebra. We halve and then double.

Review: If we have a Lie bialgebra $(\mathfrak{g}, \delta)$, then as we did in class, we have $[e_i, e_j] = \sum_k C^k_{ij} e_k$ and $\delta(e_i) = \sum_{jk} f^{jk}_i e_j \wedge e_k$. Then on $\mathfrak{g}^*$ we have $[e^i, e^j]_* = \sum f^{ij}_k e^k$ and $\delta_*(e^i) = \sum C^i_{jk} e^j \wedge e^k$. So on $\mathcal{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^{*\mathrm{op}}$ we have $[e^j, e_i] = \sum_k C^i_{jk} e^k - \sum_k f^{ik}_i e_k$.

Ok, so let A be a generalized Cartain matrix. It is $n \times n$, and on an index-set $i, j \in I = \{0, \dots, n-1\}$ we have $A = (a_{ij})$. We are in the *affine* case if $\mathrm{rank}(A) = n - 1$. We also assume that (a_{ij}) is already symmetric: we are in the A,D, or E cases.

Ok, so the Borel subalgebra is $\mathfrak{b}_+(A) = \mathfrak{h} \oplus \mathfrak{n}_+(A)$, where $\mathfrak{h}$ is generated by h_i for $i \in I$ and also a new symbol d, and $\mathfrak{n}_+$ by e_i for $i \in I$, with relations $[h_i, h_j] = [h_i, d] = 0$, and $[h_i, e_j] = a_{ij} e_j$, and $[d, e_j] = \delta 0, j e_j$, and also $(\mathrm{ad}_{e_i})^{1-a_{ij}}(e_j) = 0$.

So the Lie bialgebra structure on $\mathfrak{b}_+(A)$ is given by $\delta h_i = \delta d = 0$ and $\delta e_i = \frac{1}{2} h_i \wedge e_i$. So $\mathcal{D}(\mathfrak{b}_+(A))$ is generated by $h_i, d, e_i, h_i^*, d^*, e_i^*$.

Theorem 14.3 *$\mathcal{D}(\mathfrak{b}_+) \cong \mathfrak{g}(A) \oplus H$ is the direct sum of Lie algebras. Where $\mathfrak{g}(A)$ is generated by $H_i = \sum_{j=0}^{n-1} h_j^* a_{ij} + h_i + d^*$, and also by e_i, $f_i = e_i^*$, and $D = d + h_0^*$. The H part is generated by $\tilde{H}_i = -\sum_{j=0}^{n-1} h_j^* a_{ij} + h_i - d^*$ and by $\tilde{D} = d - h_0^*$. Then $H_i/2$ and $D/2$ satisfy the relations above. The $\tilde{H}$ and $\tilde{D}$ are central. Moreover, the quotient map $\mathcal{D}(\mathfrak{b}_+) \to \mathfrak{g}(A)$ lets us push the quasitriangular element from the double to the Kac-Moody algebra.*

Notes on this last part will be online.

Lecture 15 February 25, 2009

Recall, we have $G/B \cong K/T$ as a real space, and a decomposition as the disjoint union $\bigsqcup_{w \in W} U_w$, where $U_w = \{u \in U \text{ s.t. } \dot{w}^{-1} u \dot{w} \in U^-\}$, where $U = \exp(\mathfrak{n}_+)$ are the upper-triangular unipotent matrices. Then $\dim_{\mathbb{C}} U_w = \ell(w)$. A better name of U_w is C_w, and $K = \bigsqcup_{w \in W} K_w$ with the fibration $C_w \to K_w \to T$. We will see that G/B has a natural Poisson structure, and these are the symplectic leaves.

15.1 Symplectic leaves of Poisson Lie groups

Let P be a Poisson manifold, $G \times P \to R$ an action of the Lie group G on P. If G is a Poisson Lie group, the action is called a *Poisson Lie action* if $G \times P \to P$ is a Poisson map.

Example 15.1 $G : P \to P$, so that the Poisson-structure is G-invariant. Then if G is given the trivial Poisson structure, then $G \times P \to P$ is Poisson. ♢

Example 15.2 $G \times G \to G$ the group multiplication, if G is Poisson Lie. Both left- and right-actions are Poisson Lie. ♢

When we have a Poisson Lie action $G \curvearrowright P$, we can look at the quotient manifold P/G, which will be a Poisson manifold, but we will say this again later — we don't need it yet.

General fact: If P is a Poisson manifold with Poisson tensor π; i.e. $\{f, g\} = \pi(df \wedge dg)$. Let's say that $\omega, \omega', \omega''$ are one-forms. We can define the usual pairing $\pi(\omega \wedge \omega')$. If we have a triple, we define

$$\pi(\omega, (\omega' \wedge \omega'')) \stackrel{\text{def}}{=} \pi(\omega \wedge \omega') \wedge \omega'' - \pi(\omega \wedge \omega'') \wedge \omega' \tag{15.1}$$

Theorem 15.1 (Koszul) *If π is a Poisson tensor, then*

$$[\omega_1, \omega_2] = d\pi(\omega_1, \omega_2) + d\pi(\omega_1, d\omega_2) - \pi(d\omega_1, \omega_2) \tag{15.2}$$

if a Lie bracket on $\Omega^1(P)$.

Proof: Exercise 28. □

Ok, so we will use this in the following way. We have $G \curvearrowright \mathfrak{g}^*$ by the coadjoint action Ad^*. We will look for a nonlinear version of this. More precisely, if we have a pair of Poisson Lie groups, $G^* \curvearrowright \mathfrak{g} = \mathrm{Lie}(G)$ by $\mathrm{Ad}^*_{G^*}$, and we want at least a local action of $G^* \curvearrowright G$. It is unrealistic to look for a global action.

A *local action* of G^* on G is a Lie algebra homomorphism $\mathfrak{g}^* \to \mathrm{Vect}(G)$. If this distribution is integrable, then we will have a global map.

Theorem 15.2 (Weinstein) *If G is a Poisson Lie group, then:*

1. *The space of left-invariant one-forms on G is a Lie subalgebra in $\Omega^1(G)$ with the bracket as in equation 15.2.*
2. *We trivialize the cotangent bundle $T^*G \cong G \times \mathfrak{g}^*$ by right-translations. This gives a natural embedding $\mathfrak{g}^* \hookrightarrow \Omega^1(G)$, with image exactly the left-invariant forms, which is a homomorphism of Lie algebras. With the Poisson tensor p, we take $p : \Omega^1(G) \to \mathrm{Vect}(G)$. Then $\mathfrak{g}^* \hookrightarrow \Omega^1(G) \xrightarrow{p} \mathrm{Vect}(G)$ is a Lie algebra homomorphism.*

Question from the audience: Which partial maps are Lie algebra homomorphisms? **Answer:** $\mathfrak{g}^* \hookrightarrow \Omega^1(G)$ is, and picks out the space of left-invariant forms. The map $p : \Omega^1(G) \to \mathrm{Vect}(G)$ is not a Lie algebre homomorphism, but its restriction to $\mathfrak{g}^*$ is.

This gives the action of $\mathfrak{g}^* \curvearrowright G$ by vector fields. The map $\mathfrak{g}^* \to \mathrm{Vect}(G)$ is the (local) *dressing action* of G^* on G. You may ask "Why is it called Dressing? What does it dress?" The name came from the theory of solitons, which are waves that propagate without loosing shape. The theory from the 70s is that waves are an infinite-dimensional completely integrable system, and the solitons really matter. Examples: KdV, nonlinear Schrodinger, and other examples from nonlinear physics.

In order to talk more about integrable systems, let's review very quickly Hamiltonian mechanics. You have a symplectic manifold (M, ω), and $H \in C^\infty(M)$. You invert ω to get the Poisson structure, and then define $v_H = \omega^{-1}(dH)$. Then we have a Lagrangian fibration $L_n \hookrightarrow M_{2n} \to B_n$, and this reduces the dimension of the action. Anyway, the language is that there are subgroups of the group that create solitons — they dress soliton solutions into multiple solitons. So that's were the terminology comes from: from the action of Poisson Lie groups on phase space of integrable systems.

In any case, if the dressing action is integrable, then it gives the global dressing action. ****I think that "integrable" only depends on the Lie algebra homomorphism. To integrate to a Lie group action, we should assume that G^* is connected and simply-connected.****

So, we have $\mathfrak{g}^* \to \mathrm{Im}(p) \subseteq TG$, by $\alpha \mapsto \alpha_l \mapsto p(\alpha_l)$, where $\alpha \in \mathfrak{g}^*$, α_l is a left-invariant one-form, and $p(\alpha_l) \in \mathrm{Vect}(G)$.

Ok, remember that to each function $H \in C^\infty(G)$ we define a *Hamiltonian vector field* $v_H = p(dH)$. Then to each $x \in G$, we get a symplect manifold through x by taking every point we can reach by piece-wise Hamiltonian flow lines. But $\operatorname{Im} p \subseteq TG$, and $S \subseteq G$ is a symplectic leaf if $T_x S = \operatorname{Im}(p)_x$ for $x \in S$.

Corollary 15.2.1 (STS, Weinstein and Lu) *Symplectic leaves are orbits of the dressing action.*

The nicest case is when the action is global. Then symplectic leaves are just $\{G^* g\}_{g \in G}$, i.e. orbits of G^* in G.

Question from the audience: So the map $\mathfrak{g}^* \to \operatorname{Im}(p)$ surjects? **Answer:** Yes. **Question from the audience:** I'm not sure what you mean? **Answer:** We have trivialized TG. By the image of p, we mean the subbundle in $G \times \mathfrak{g}$ given by $\{g, \operatorname{Im}(p_q)\}$. It may drop in dimension for special g. ****I would say that $\mathfrak{g}^*$ is mapping to *sections of* $\operatorname{Im}(p)$, where $p : T^*G \to TG$.****

So, if G is a Poisson Lie group, remember that it has is a connected simply-connected dual Poisson Lie group G^*, because G has a tangent Lie bialgebra $(\mathfrak{g}, \delta)$, with dual bialgebra $(\mathfrak{g}^*, \delta_*)$. But we can also construct the double $\mathfrak{g} \bowtie \mathfrak{g}^*$, which we can exponentiate to a Lie group: $\exp(\mathfrak{g} \bowtie \mathfrak{g}^*) \overset{\text{def}}{=} \mathcal{D}(G)$ is the connected simply-connected exponentiation, which we consider as the double of G. Just as we had embeddings $\mathfrak{g}, \mathfrak{g}^{*\text{op}} \hookrightarrow \mathfrak{g} \bowtie \mathfrak{g}^*$, ****$\mathfrak{g}^{*\text{op}}$ is $\mathfrak{g}^*$ as a Lie algebra but with the opposite coalgebra****, then we have Lie group embeddings $i : G \hookrightarrow \mathcal{D}(G)$ and $j : G^{*\text{op}} \hookrightarrow \mathcal{D}(G)$. **Question from the audience:** We must assume that G is simply connected? **Answer:** Yes. All of G, G^*, and $\mathcal{D}(G)$ are connected and simply connected.

Ok, what is $i(G) \cap j(G^{*\text{op}}) = \Sigma \subseteq \mathcal{D}(G)$? Well, since $i(\mathfrak{g}) \cap j(\mathfrak{g}^{*\text{op}}) = 0$, we have that Σ is a discrete subgroup of $\mathcal{D}(G)$.

Let's look at examples. Well, we have only one minute, so only one example:

Example 15.3 If $G = K$ is the compact real form with the standard Poisson Lie structure on $G_{\mathbb{C}}$. Let's say for definiteness that $G_{\mathbb{C}} = SL_n(\mathbb{C})$. Then $G^* = AU$, where $U =$ complex upper-triagular unipotent matrices, and A are the real positive diagonal unimodular matrices. ♢

Theorem 15.3 $\mathcal{D}(K) = G_{\mathbb{C}} = KAU^-$, *where we consider* $G_{\mathbb{C}}$ *as a real manifold.*

Next time we will finish this and then see why the Shubert cells are symplectic leaves of K/T.

Lecture 16 February 27, 2009

Since the best way to learn things is by explaining — indeed, NR will do this later — we suggest that someone learn and explain about the following: Schonten bracket, Poisson cohomology, and lots of other things. This appears in the quantization of gauge theories, where it comes up in BV quantization. The minimalist part of the project is to report on the proof of the theorem from last time, and broader is to report on everything. We repeat the theorem from last time:

Theorem 16.1 (K) *1. The bracket on $\Omega^1(M)$ defined by:*

$$[\omega, \omega'] \stackrel{\text{def}}{=} d\pi(\omega \wedge \omega') + \pi\omega(d\omega') - \pi\omega'(d\omega) \tag{16.1}$$

where π is the Poisson bivector field on M, and $\pi\omega(\omega' \wedge \omega'') \stackrel{\text{def}}{=} \pi(\omega, \omega')\omega'' - \pi(\omega, \omega'')\omega'$, is a Lie bracket.

2. The mappting $\pi : \Omega^1(M) \to \mathrm{Vect}(M)$ is a Lie algebra homomorphism.

Proof: Matt will present this next week. □

It's not clear how the bracket in 1. came up, but if you try to pull back the bracket from vector fields, you will see that it is what it must be. But perhaps its source is also from the more general technology.

Theorem 16.2 (W) *The natural mapping $\mathfrak{g}^* \to \Omega^1(G)_{left}$ induced by the trivialization of T^*G by right translations is a Lie algebra isomorphism.*

Proof: Theo will present this next week. □

Corollary 16.2.1 *We have $\mathfrak{g}^* \to \mathrm{Vect}(G)$, a local action $G^* \curvearrowright G$.*

Question from the audience: So we have that TG is foliated by the image of $\mathfrak{g}^*$. The fact that these are the symplectic leaves follows from that the image of $\mathfrak{g}^*$ is in Hamiltonian vector fields? And these are all of them? **Answer:** Yes, well the Hamiltonian vector fields are exactly the image under π of the exact forms, so you have to prove that this is the image of the left-invariant things.

Corollary 16.2.2 *The symplectic leaf through $x \in G$ is the orbit of the above action through x.*

Ok, so let's now give an algebraic description of these orbits. First, some preliminaries. Let M be a Poisson manifold, and G and Poisson Lie group acting via a *Poisson Lie action*: the action map $G \times M \to M$ is a Poisson map. An example is whenever M is Poisson, G preserves the Poisson structure on M, and G has the trivial Poisson structure. Another example is that if $H \subseteq G$ is a subgroup, the left and right actions $H \times G \to G$ and $G \times H \to G$ are Poisson-Lie.

Proposition 16.3 *The functions preserved by G — $C(M)^G \subseteq C(M)$ — make a Poisson subalgebra.*

Proof: Let $\alpha : G \times M \to M$, then $\alpha^* f(g, m) \stackrel{\text{def}}{=} f(\alpha(g, m))$. Well, $\alpha^*(\{f, g\}_M) = \{\alpha^* f, \alpha^* g\}_{G \times M}$, and G-invariance means $\alpha^* f = 1 \otimes f$. Let's make the assumption that everything is algebraic, so that $C(G \times M) = C(G) \otimes C(M)$. Then if $f \in C(M)^G$, then $\alpha^* \{f, g\}_M = 1 \otimes \{f, g\}_M$, and so $\{f, g\}_M$ is G-invariant. □

Ok, so one more thing, which is an obvious corollary, but indeed isn't even: it's just what we said. If M/G is a manifold — this is non-trivial, it may be quite singular, but suppose that it is or that it can naturally be made so — then the projection $M \to M/G$ is a Poisson map. This is just what we said above, but not in terms of Poisson algebras but in terms of their spectra.

Corollary 16.3.1 *If $H \subseteq G$ is a Poisson Lie subgroup, then both $H\backslash G$ and G/H are Poisson manifolds with Poisson maps $G \to G/H$ and $G \to H\backslash G$.*

Question from the audience: Why is the action of G on G/H Poisson? **Answer:** $G \curvearrowright G$ is Poisson, and you push it down.

Ok, so consider the double $\mathcal{D}(G)$, the connected simply-connected Lie group with algebra $\mathfrak{g} \bowtie \mathfrak{g}^*$. Then we have natural embeddings $G \underset{i}{\hookrightarrow} \mathcal{D}(G) \underset{j}{\hookleftarrow} G^{*\mathrm{op}}$. Because $i(\mathfrak{g}) \cap j(\mathfrak{g}^{*\mathrm{op}}) = \{0\}$, we see that $\Sigma \stackrel{\mathrm{def}}{=} i(G) \cap j(G^{*\mathrm{op}})$ is a discrete subgroup of $\mathcal{D}(G)$.

We have already proven:

Proposition 16.4 *i and j are inclusions of Poisson Lie groups.*

Corollary 16.4.1 *$\mathcal{D}(G) \to \mathcal{D}(G)/j(G^{*op})$ is Poisson and commutes with the left $\mathcal{D}(G)$-action. Ditto for $\mathcal{D}(G) \to \mathcal{D}(G)/i(G)$.*

So, we have a sequence of Poisson maps:

$$G \stackrel{i}{\hookrightarrow} \mathcal{D}(G) \to \mathcal{D}(G)/G^{*\mathrm{op}}$$

and $G^{*\mathrm{op}}$ acts on the double by left multiplication (as a subgroup of $\mathcal{D}(G)$), and so also acts on the right term, and the diagram commutes:

$$\begin{array}{ccc} G \stackrel{i}{\hookrightarrow} \mathcal{D}(G) & \to & \mathcal{D}(G)/G^{*\mathrm{op}} \\ \circlearrowleft & & \circlearrowleft \\ G^{*\mathrm{op}} & = & G^{*\mathrm{op}} \end{array}$$

Then $G \to \mathcal{D}(G)/G^{*\mathrm{op}} \curvearrowleft G^{*\mathrm{op}}$. In particular, this gives $\mathfrak{g}^* \to \mathrm{Vect}(\mathcal{D}(G)/G^{*\mathrm{op}})$.

Theorem 16.5 *f commutes with $\mathfrak{g}^*$ action (dressing on G, and natural on $\mathcal{D}(G)/G^{*op}$).*

Corollary 16.5.1 *Symplectic leaves of G are connected components of G^{*op} orbits on $\mathcal{D}(G)/G^{*op}$. I.e. they are double cosets $G^{*op}xG^{*op}$ for $x \in \mathcal{D}(G)$.*

This is a general fact, and of course it is the same for G^*, because there is this symmetry. We have to use $G^* \to \mathcal{D}(G)/G$.

Example 16.1 Let K be the compact real form of $G_{\mathbb{C}}$ simple. Think of $K = SU(n)$. Let's assume the standard Poisson Lie structure on $G_{\mathbb{C}}$. We already had the discussion that $K^* = AU \subseteq B \subseteq G$, the subgroup of the Borel subgroup of G, where U is the subgroup of complex unipotent matrices in B, and A is the real positive diagonal matrices. If you want: A is the split real form of the Cartan $H \subseteq B$; it is the one for this particular Borel. $\diamondsuit$

The *Iwasawa decomposition* of $G_{\mathbb{C}}$ is $G = KAU$.

Theorem 16.6 *$\mathcal{D}(K) \cong G$ as a real manifold, and the Iwasawa decomposition agrees with natural embeddings $i : K \hookrightarrow G$ and $j : K^{*op} \hookrightarrow G$.*

We are not that careful about K^* and $K^{*\mathrm{op}}$, but it doesn't matter: it's upper- versus lower-triangular matrices.

In this particular case, $i(K) \cap j(K^{*\mathrm{op}}) = \{1\}$.

So the Double of a complex simple Lie group has a natural description: you just have to complexify ****missed****.

Corollary 16.6.1 *$\mathcal{D}(K)/K^{*op} \cong K$, and so we have a global action of K^{*op} on K.*

Thus, symplectic leaves of K are preimages of the double cosets $K^{*\mathrm{op}}xK^{*\mathrm{op}} \subseteq G$ with respect to the natural embedding $K \hookrightarrow G = KAU$.

So what we have to do now is to describe the double cosets. What do we know about G? We must describe $AUxAU$.

Well, we know that $G = \bigsqcup_{w\in W} BwB$, the Bruhat decomposition. So let's choose $x \in BwB$.

Ok, so let's consider $AUb\dot{w}b'AU$. How many parameters are left? Can we absorb b into AU? Obviously not. But we can write $b = auk$, where $k \in$compact part of H, so it is in $(S^1)^r$, where r is the rank. Ok, so we have $AUk\dot{w}k'AU$ — we have also used $b' = k'a'u'$ —, and we move k' past $\dot{w}$ to get $k'_w\dot{w}$. Ok, so what is $kk'_w\dot{w}$? It is an element where? We claim that it is naturally in $N(T) \subseteq K$. Because anything else can be absorbed.

So we have proved: The space of double cosets $K^{*\mathrm{op}}\backslash\mathcal{D}(K)/K^{*\mathrm{op}}$ is naturally isomorphic to $N(T)$.

So let's take the subset $N_w(T) \subseteq N(T) \cong T \rtimes W$ corresponding the $w \in W$. Then:

Theorem 16.7 *$K \cong \bigsqcup_{w\in W} K_w$, where each K_w is homogeneous Poisson submanifold, fibered over the torus $T \subseteq K$, such that fibers are symplectic leaves isomorphic to Shubert cells of K/T.*

The idea is you take a point in the symplectic leaf, and the statement is that you can assemble these into a homogeneous Poisson submanifold: i.e. it is the union of leaves of the same size. We know that $K/T \cong G/B \cong \bigsqcup_w C_w$, where G/B is the generalized flag variety, and $C_w \subseteq U_w$, the elements mapped to the negative ****something**** when conjugated by w.

The point is that because $\mathcal{D}(K) \cong G \cong KAU$, the space of double cosets really parameterizes symplectic leaves, because it is the space of dressing orbits $K^{*\mathrm{op}}\backslash K$. But some symplectic leaves have the same dimension, and the wonderful thing is that you can assemble leaves of the same dimension into these subvarieties.

Example 16.2 In particular, you take $SU(2)$. How many cells are there in $SU(2)/T$? It is $C_{w_0} \sqcup C_1$, of dimensions 2 and 0. (How many even numbers are there less than 3? $\dim_{\mathbb{R}}(C_w) = 2\ell(w)$.) And this is $Bw_0B/B \sqcup \{\mathrm{pt}\}$. So it is easy to show that $C_{w_0} \cong S^2$, and so the result is a symplectic manifold $(S^2, \omega_{SU(2)})$.

Do we have any Poisson forms on the sphere? Well, there's the area form, (S^2, ω), coming from the coadjoint orbit of $SU(2)$ on $SU(2)^*$. But this is a different one. $\lozenge$

Anyway, all the C_2 have symplectic structure.

Lecture 17 March 2, 2009

17.1 Matt presents a proof from last time

We will show that the bracket defined last time on $\Omega^1(P)$ is a Lie bracket, and that we get a homomorphism into vector fields. Precisely: we have a Poisson manifold (P, π), and from this we define: $\tilde{\pi} : \Omega^1(P) \to \mathrm{Vect}(P)$ defined by $\langle \eta, \tilde{\pi} \rangle = \pi(\eta, \omega)$. This may be the negative of last time: we keep the one-forms in the same order. From this we define a bracket on $\Omega^1(P)$, by:

$$[\omega_1, \omega_2]_{\Omega^1(P)} = d\Big(\pi(\omega_1, \omega_2)\Big) - i_{\tilde{\pi}\omega_1}(d\omega_2) + i_{\tilde{\pi}\omega_2}(d\omega_1) \tag{17.1}$$

We remark that if ω_1, ω_2 are closed, then $[\omega_1, \omega_2]$ is exact. We will show that $-\tilde{\pi}$ is a Lie algebra homomorphism.

Lemma 17.1 $\tilde{\pi}(df) = X_f$*: the image of an exact one-form is a Hamiltonian vector field.*

Proof: $\langle dg, \tilde{\pi}(df)\rangle \overset{\text{def}}{=} \pi(dg, df) \overset{\text{def}}{=} \{g, f\} \overset{\text{def}}{=} \langle dg, X_f\rangle$. $\square$

We remark that we have chosen to define Hamiltonian vector fields by putting in the second coordinate. This is a sign choice, and with this convention $[X_f, X_h] = -X_{\{f,h\}}$. Otherwise, we'd have a sign somewhere else.

Lemma 17.2 $[f\omega_1, \omega_2] = f[\omega_1, \omega_2] + \pi(df, \omega_2) \cdot \omega_1$*, where $\cdot$ is the action* ****?****

Proof: Exercise 29. $\square$

Proposition 17.3 $[,]$ *is a Lie bracket on* $\Omega^1(P)$.

Proof: Antisymmetry is clear. We check Jacobi, by checking it on exact forms and then using the derivation property lemma 17.2.

$$[df, dg] = d\Big(\pi(df, dg)\Big) = d\{f, g\} \tag{17.2}$$

$$[[df, dg], dh] + \text{c.p.} = [d\{f, g\}, dh] + \text{c.p.} = d\{\{f, g\}, h\} + \text{c.p.} = 0 \tag{17.3}$$

$\square$

Proposition 17.4 $-\tilde{\pi} : \Omega^1(P) \to \mathrm{Vect}(P)$ *is a Lie algebra homomorphism.*

Proof: Again we check on exact things, and then show that everything performs well under multiplication by functions.

$$-\tilde{\pi}([df, dg]) = -\tilde{\pi}(d\{f, g\}) = -X_{\{f,g\}} = [-X_f, -X_g] = [-\tilde{\pi}(df), -\tilde{\pi}(dg)] \tag{17.4}$$

For the general case, observe that $[fV, W] = f[V, W] - (Wf) \cdot V$. □

17.2 Symplectic leaves of compact groups

Last time, we discussed: we have $K \subseteq G_{\mathbb{C}}$ the compact real form of G, with standard Poisson Lie structure. Last time we came to the conclusion that $K = \bigsqcup_w K_w$, each of which is a Poisson homogeneous space. And each K_w is fibered over the torus, with fibers C_w the Shubert cells.

Last time we gave an example of SU_2, with a small mistake. We have $SU_2 \cong S^3$, and we were talking about SU_2/T. Of course, $K/T = \bigsqcup_w C_w$. In our case, $SU_2/T = S^3/T = C_{w_0} \sqcup C_1$. The first is two-dimensional and the latter zero-dimensional.

Question from the audience: How do you build the C_ws? **Answer:** The way we did before, we took $K/T \cong G/B \cong \bigsqcup_w BwB/B \cong \bigsqcup_w U_w$, and $C_w = U_w = \{u \in U \text{ s.t. } \dot{w}^{-1} u \dot{w} \in U^-\}$. So in the case of SU_2, $U_{w_0} = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \text{ s.t. } a \neq 0\} = U \smallsetminus \{e\}$, and $U_1 = \{e\}$. **Question from the audience:** So it's the sphere without two points, and then another point. How is this compact? **Answer:** We will return to it.

Moving on, we had two facts:

1. $\mathcal{D}(K) = G_{\mathbb{C}} = KAU$, and $AU = K^{*\mathrm{op}}$.
2. $K \hookrightarrow \mathcal{D}(K) \to \mathcal{D}(K)/K^{*\mathrm{op}}$, and the actions of right-multiplication compute. So: Symplectic leaves are connected components of pre-images of $K^{*\mathrm{op}}$-orbits, where $K^{*\mathrm{op}}$ acts by left-multiplicaiton.

Now we can also ask about symplectic leaves of $K^{*\mathrm{op}}$. Think about the analogy. When we have Lie algebras, it looks like we have two inputs of symplectic geometry: structures on K and K^*. When we quantize, and get a bialgebra, we will have two representation theories: one of "representations" and one of "corepresentations".

So the counterpart will be dual pairs of Lie bialgebras. So far we have worked finite-dimensionally, except for a short aside on KM algebra. Infinite-dimensional spaces have canonical duals only in the presence of topology. Algebraically, we will talk not about a space and its dual, but rather a *dual pair*, which is a pair of spaces and a nondegenerate pairing.

Anyway, so we have:

$$K^{*\mathrm{op}} \hookrightarrow \mathcal{D}(K) \to \mathcal{D}(K)/K \tag{17.5}$$

and symplectic leaves of $K^{*\mathrm{op}}$ are preimages of K-orbits in $\mathcal{D}(K)/K$, where K acts by left multiplication.

Exercise 30 *Provide an explicit description of the symplectic leaves. We will return to this after studying symplectic leaves of $G_{\mathbb{C}}$ and $G^*_{\mathbb{C}}$. Hint: $G = KAU = UAK$ and $G = KH$ are two decompositions. In the latter, this is "compact" times "hermitian". In SL_n, this is $SU_n \cdot H$, where H really is the space of Hermitian matrices. In this language the symplectic leaves will be simply*

Hermitian orbits. There is a natural identification H with AU. **Question from the audience:** *This is the* polar decomposition*?* **Answer:** *Yes.*

17.3 Symplectic Leaves of $G^*_{\mathbb{C}}$?

Let's investigate the following question. Suppose we have $(\mathfrak{g}, \delta_r)$ a factorizable Lie bialgebra. For example: 1. $\mathfrak{g}$ simple with the standard r-matrix; 2. $\mathcal{D}(\mathfrak{b})$ for any $\mathfrak{b}$ a Lie bialgebra is factorizable. We haven't really talked about what happens if you take the double of the double, but you don't get anything new.

So, $t = r + \sigma(r0 \in S^2(\mathfrak{g})^{\mathfrak{g}}$ is a nondegenerate invariant bilinear form on $\mathfrak{g}^*$. If $\mathfrak{g}$ is simple, t is the dual to the Killing form.

Then we defined $r_\pm : \mathfrak{g}^* \to \mathfrak{g}$ by $r_+ : l \mapsto (\mathrm{id} \otimes l)r$ and $r_- : l \mapsto -(l \otimes \mathrm{id})r$. So we define $\tilde{t} : \mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}$, and then $\tilde{t}(l) = (\mathrm{id} \otimes l)t = r_+(l) - r_-(l)$. So we have the *factorization property*: for all $x \in \mathfrak{g}$, there is a unique decomposition $x = x_+ - x_-$ such that for some $l \in \mathfrak{g}^*$, $x_\pm = r_\pm(l)$.

Now, let's consider the map $i : \mathcal{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{g} \oplus \mathfrak{g}$ by $(x, l) \mapsto (x + r_+(l), x - r_-(l))$.

Theorem 17.5 *This map is a Lie algebra isomorphism.*

Proof: First we show that this is a Linear isomorphism. Well, $(x_1, x_2) \mapsto ? \in \mathfrak{g} \oplus \mathfrak{g}^*$. We have $x = (x_1 + x_2)/2$, and we want to extract l. But $x_1 - x_2 = r_+(l) - r_-(l) = \tilde{t}(l)$, so $l = \tilde{t}^{-1}(x_1 - x_2)$. By the definition of factorizability, this defines l uniquely, and so we have the inverse to the above map.

Second, it's clear that i is a Lie algebra homomorphism. Indeed, the diagonal map $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ is a Lie homomorphism, and so are $r_\pm$. □

Corollary 17.5.1 *When $\mathfrak{g}$ is factorizable, we have:*

- $\mathcal{D}(G) \cong \mathfrak{g} \oplus \mathfrak{g}$ *as a Lie algebra.*
- $\mathcal{D}(G) \cong \mathfrak{g} \oplus \mathfrak{g}^*$ *as a coalgebra.*

and so $\mathcal{D}(\mathcal{D}(\mathfrak{b}))$ has no new data for any bialgebra $\mathfrak{b}$.

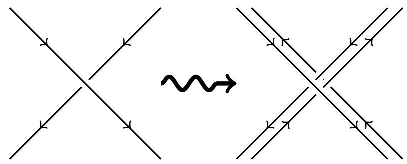
Now we take G corresponding to our factorizable $\mathfrak{g}$, and by the corollary we can take $\mathcal{D}(G) \cong G \times G$ and $\mathcal{D}(G)^* \cong G \times G^*$.

Exercise 31 *Let $\mathfrak{g}$ be factorizable. Take $r^{\mathcal{D}}$ to be the r-matrix for the double of $\mathfrak{g}$. This is canonical:*

$$r^{\mathcal{D}} = \sum_i e_i \otimes e^i \in \mathfrak{g} \otimes \mathfrak{g}^* \hookrightarrow \mathcal{D}(\mathfrak{g})^{\otimes 2} \tag{17.6}$$

Ok, so let's take $(i \otimes i)(r^{\mathcal{D}}) \in (\mathfrak{g} \oplus \mathfrak{g})^2$. Compute the image in terms of i.

This should be an entertaining exercise. See, if we have a tangle, we can double the strands:



Then this respects the Reidemeister moves.

Ok, so we have a very good description:

$$G \hookrightarrow \mathcal{D}(G) = G \times G \to \mathcal{D}(G)/G^{*\mathrm{op}} \tag{17.7}$$

and $G^{*\mathrm{op}} \subseteq B_+ \times B_- \subseteq G \times G$ for simple G.

Then symplectic leaves of $G_{\mathbb{C}}$ are preimages of connected components of $G^{*\mathrm{op}}$ orbits on $(G \times G)/G^{*\mathrm{op}}$. So we get $G^{*\mathrm{op}}\backslash(G \times G)/G^{*\mathrm{op}}$.

We will see that these have to do with *double Bruhat cells* $B_+wB_+ \cap B_-vB_- = G_{w,v}$.

We will moreover see that if we take $G^{*\mathrm{op}}$, we can naturally take it to G by $(b_+, b_-) \to b_+b_-^{-1}$. Then symplectic leaves will go to conjugation orbits in G.

So the difference: symplectic leaves of G will be related to double Bruhat cells, which are nice varieties; whereas those of $G^{*\mathrm{op}}$ are related to conjugation orbits in G, which are notoriously bad. This will all have quantum analog.

Lecture 18 March 4, 2009

Recall, we have $\mathcal{D}(G) = G \times G$. When $G = G_{\mathbb{C}}$ is a simple Lie group, then G is factorizable, meaning that $\mathfrak{g}$ is. If $\mathfrak{g}$ is spanned by H_i (simple roots), e_i, f_i, with $[H_i, e_j] = a_{ij}e_j$ (no sum), then we have the fundamental weights H^i given by $[H^i, e_j] = \delta^i_j e_j$. Then the standard r-matrix is $r = \frac{1}{2}\sum_{ij} H_i \otimes H_j b_{ij} + \sum_{\alpha>0} e_\alpha \otimes f_\alpha$. Here $b = a_{\mathrm{sym}}^{-1}$, where a_{sym} is the symmetrized Cartan matrix $d_i a_{ij}$. So $r + \sigma(r) = t$ is the bilinear form on $\mathfrak{g}^*$ dual to the Killing form.

Our primary example of a factorizable group is a simple one, but there are others. The full classification of factorizable structures on G is given by Belavin and Drinfeld, and would make a good 30-minutes presentation.

Ok, so we have two embeddings:

$$G \underset{i}{\hookrightarrow} G \times G = \mathcal{D}(G) \underset{j}{\hookleftarrow} G^{*\mathrm{op}} \tag{18.1}$$

where i is the diagonal embedding, and $G^{*\mathrm{op}} = \{(b_+, b_-) \in B_+ \times B_- \text{ s.t. } [b_+]_0 = [b_-]_0^{-1}\}$, where $[b_\pm]_0$ is the Cartan part of the element. Then $j : (b_+, b_-) \mapsto (b_+, b_-) \in G \times G$, since $B_\pm \subseteq G$.

18.1 Symplectic Leaves in G^*

$G^{*\mathrm{op}}$ is G^* with the opposite Poisson structure. But we will for a while talk only about the group structure, keeping the Poisson structure only up to non-zero constant multiple. So we will write G^*.

So, we have $G^* \underset{j}{\hookrightarrow} G \times G \to (G \times G)/G$, and symplectic leaves in G^* are exactly connected components of preimages of left-G-orbits in $(G \times G)/G$.

So, first of all, what is $(G \times G)/G$? As a set, it is the set of cosets $\{(g_1, g_2)G \text{ s.t. } g_i \in G\}$. So $(g_1, g_2)G = \{(g_1 g, g_2 g) \text{ s.t. } g \in G\}$. When we have a set with orbits, a good way to parameterize them is to take a cross section. One way to do this is make one of them the identity: $g = g_2^{-1}$, whence $(G \times G)/G$ is in bijection with $G \times \{e\}$.

Here's another way to understand this set. Let's consider G-invariant functions. I.e., we look at the function $\pi : (G \times G) \to G$ by $(g_1, g_2) \mapsto g_1 g_2^{-1}$. Then $G \curvearrowright (G \times G)$ diagonally, and this pushes forward to the trivial action $G \curvearrowright G$. So, G acts transitively on fibers $\pi : (G \times G) \to G$, and so we can identify the set of cosets with the base: $\tilde{\pi} : (G \times G)/G \cong G$.

Ok, so what about the left action? We have $h(g_1, g_2)G = (hg_1, hg_2)G \xrightarrow{\pi} h(g_1 g_2^{-1})h^{-1}$. Ok, so the left action of G on $(G \times G)/G$ corresponds under $\tilde{\pi}$ to the conjugation action of G on G. In particular, $\tilde{\pi} : \{\text{left } G \text{ orbits on } (G \times G)/G\} \cong \{\text{conjugation orbits of } G\}$.

Theorem 18.1 *Symplectic leaves of G^* are connected components of preimages of conjugation orbits in G with respect to the map $G^* \overset{j}{\hookrightarrow} G \times G \xrightarrow{\pi} G$. This map is $(b_+, b_-) \mapsto (b_+, b_-) \to b_+(b_-)^{-1}$.*

Now look at what happens to diagonal elements. If both $b_\pm$ are in the Cartan, well $[b_+]_0 = [b_-]_0^{-1}$, so elements in the Cartan are mapped to their square. We mentioned the group $\Sigma = i(B_+) \cap j(B_-)$, and we have shown that $|\Sigma| = 2^r$, because Σ is the kernel of the map, which is also the diagonal things with only ± 1s.

We did the symplectic leaves of K, the compact real form of G. We will now try to do K^*. Since G is Poisson-Lie, $\mathfrak{k} = \mathrm{Lie}(K)$ is the compact real form of $\mathfrak{g}$, and inherits the Poisson $\rightsquigarrow$ bialgebra structure. So K^* is the real form of the corresponding dual algebra, and is a real form of G^*. ****It is not necessarily compact.**** Indeed, we say that $\mathfrak{k}$ is spanned by $iH_j, e_\alpha - f_\alpha, i(e_\alpha + f_\alpha)$. And $K^* \cong AU$ where A is the real positive form of $H_{\mathbb{C}} \subseteq G_{\mathbb{C}}$.

Proposition 18.2 *$AU \cong \mathscr{H}$ as real varieties, where $\mathscr{H}$ is the Hermetian unimodular matrices with positive eigenvalues.*

Proof: We map $b \mapsto b^* b$. This is obvious on the diagonal matrices, and the rest is **Exercise 32**.□

Here b^* is the Hermetian conjugation of b.

Proposition 18.3 *$K^* \cong \{(b, (b^*)^{-1})\} \subseteq G^*_{\mathbb{C}}$. This is a real form of $G^*_{\mathbb{C}}$ dual to $K \subseteq G_{\mathbb{C}}$.*

Proof: **Exercise 33**. □

So, now we compute the symplectic leaves of K^*, simply taking the real forms of what we did for G^*.

Proposition 18.4 *1. The real form of $G^* \hookrightarrow G \times G \to G$ is $K^* \to \mathscr{H}$ given by $(b, (b^*)^{-1}) \mapsto bb^*$, and this is a diffeomorphism of real manifolds.*

2. The dressing action of K on K^* translates to the conjugation action of K on $\mathscr{H}$.

See, if we want to take the real form of $G^* \hookrightarrow G \times G \to G$, we can take K^* for the first part, but it's not clear which real form for the second parts. The proposition says which: it's not the compact form, but the Hermetian ones.

Corollary 18.4.1 *Symplectic leaves of K^* are in bijection with*

$$\left\{ \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \text{ s.t. } x_i > 0, \prod x_i = 1 \right\} / W$$

where as always W is the Weyl group.

See, the standard Poisson structure is really quite special. It intertwines the standard geometrical structures of Lie groups.

One more remark: For factorizable groups, we have a mapping $\pi \circ j : G^* \to G$. The left is a Poisson variety.

Proposition 18.5 *1. If G is factorizable, then there exists a Poisson structure p_* on G such that $\pi \circ j$ is a Poisson map. Warning: (G, p_*) is just a Poisson variety, not a Poisson Lie group.*

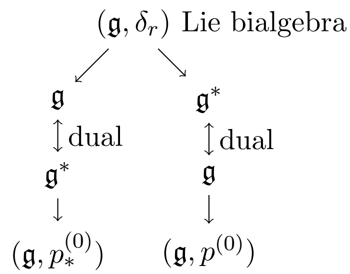
2. Symplectic leaves of (G, p_*) are exactly conjugation orbits of $G \curvearrowright G$.

There is a good use of this Poisson structure, which we will probably never come to because we will not have enough time, but there is a natural quantum analog.

Question from the audience: Is there any direct relationship between p_* and the original Poisson structure? **Answer:** Not really. The relationship is that by PBW, $\mathcal{U}\mathfrak{g} \cong \mathcal{U}\mathfrak{n}_+ \otimes \mathcal{U}\mathfrak{b}_-$ as vector spaces. On this space, there are two natural algebra structures. This is roughly what's going on with the two Poisson structures. Put another way, the linearization of p_* at the identity is the Kirilov-Kostant-Lie structure on $\mathfrak{g}$, where symplectic leaves are coadjoint orbits.

Important remark: $\mathfrak{g}$ is simple, so we use the Killing form to identify $\mathfrak{g} \cong \mathfrak{g}^*$.

We have (G, p) and (G, p^*). Near e, (G, p) linearizes to the KKL Poisson structure for $\mathfrak{g}$, the Lie algebra adjoint to $\mathfrak{g}$. (G, p^*) linearizes to the KKL structure on $\mathfrak{g}^*$ corresponding to the Lie algebra on $\mathfrak{g}$, and we move this back to $\mathfrak{g}$ via the Killing form.



Lecture 19 March 6, 2009

Last time we classified symplectic leaves of G^* and K^*, corresponding to G and K — K is the compact real form of G, and K^* is its dual group. Today we will classify symplectic leaves of G, and this will conclude our classical-geometry discussion of Poisson Lie groups.

We use the same strategy. We embed $G \hookrightarrow \mathcal{D}(G) \to \mathcal{D}(G)/G^*$ as a Poisson Lie group, where as always G is simple and complex. We want to understand connected components of preimages of left G^*-orbits on $\mathcal{D}(G)/G^*$, which are orbits of the dressing action on G^*, which will be symplectic leaves. As before, but the groups are different.

As we saw before, G is factorizable if it has the standard PL structure, and so as $\mathcal{D}(G) = G \times G$.

Ok, so we have the diagonal embedding $G \hookrightarrow G \times G$, and we mod out by $G \times G \to (G \times G)/G^*$. ****We write G^* rather than $G^{*\mathrm{op}}$ because we are only interested in the group structure.**** And $G^* = \{(b_+, b_-) \text{ s.t. } b_\pm \in B_\pm, [b_+]_0 = [b_-]_0^{-1}\} \hookrightarrow G \times G$. What are the G^* orbits on $(G \times G)/G^*$?

Recall, we have the Bruhat decomposions $G = \bigsqcup_w BwB$, but we actually have four different ones: each B can be a B_+ or a B_-. So let's take $G = \bigsqcup_w B_- w B_-$. So, we have:

$$(G \times G) \curvearrowleft G^* \hookrightarrow B_+ \times B_- \tag{19.1}$$

And it's natural to take different decompositions. Indeed, if $(g, h) \in G \times G$, we should take $g \in B_+ u B_+$ and $h \in B_- v B_-$. Then this is designed for studying the action of $B_+ \times B_-$.

Oh, let's take a correction. We don't want to study G^*-orbits on $G \times G$, but rather on the diagonal embedding mod G^*. I.e. we want to understand:

$$G^* \backslash \operatorname{diag}(G)/G^*$$

Ok, so we have $(g, g) \in G \times G$, with $g \in B_+ u B_+ \cap B_- v B_- \overset{\text{def}}{=} G^{uv}$, a *double Bruhat cell.* Then $G^*(g, g)G^* =? \; G^* = \{(u_+ h, u_- h^{-1}) \text{ s.t. } u_\pm \in U_\pm B_\pm, h \in H\}$. We remark that by the Bruhat decomposition, we can write $g = u_+h_1\bar{u}u'_+ = u_-h_2\bar{v}u'_-$, where $u_+, u'_+ \in U_+$, $u_-, u'_- \in U_-$, and $h_1, h_2 \in H$, where $\bar{u}, \bar{v} \in G$ are special representatives of $u, v \in W$ in $N(H)$.

Question from the audience: What does "special" mean? **Answer:** It means we choose a convention from the beginning. For example, for SL_2, $W = \{1, s\}$, and we may choose $\bar{s} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, or as $\bar{s} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, because in either case $\bar{s}^2 = 1$, but we make the choice at the beginning.

In any case, we have:

$$G^*(g,g)G^* = \left\{\left(U_+hu_+h_1\bar{u}u'_+h'U_+, U_-h^{-1}u_-h_2\bar{v}u'_-{h'}^{-1}U_-\right)\right\} \tag{19.2}$$

We now simplify. When you commute unipotent with cartan, the cartan remains, and the unipotent changes, but then we will absorb the unipotent into the set: $hU = Uh$. So:

$$G^*(g,g)G^* = \left\{\left(U_+hh_1\bar{u}h'U_+, U_-h^{-1}h_2\bar{v}{h'}^{-1}U_-\right) \text{ s.t. } h, h' \in H\right\} \tag{19.3}$$

Now we commute $\bar{u}$ and $\bar{v}$ past the h':

$$G^*(g,g)G^* = \left\{\left(U_+hh_1h'_u\bar{u}U_+, U_-h^{-1}h_2{h'_v}^{-1}\bar{v}U_-\right) \text{ s.t. } h, h' \in H\right\} \tag{19.4}$$

Let's now describe the orbit through (g,g), and we will then describe the whole set of such things.

$$G^*(g,g)G^* = \left\{\left(hh_1h'_u\, U_+\bar{u}U_+,\ h^{-1}h_2{h'_v}^{-1}\, U_-\bar{v}U_-\right) \text{ s.t. } h, h' \in H\right\} \tag{19.5}$$

But we have already described the set $U_+\bar{u}U_+$ when we talked about Bruhat cells. It is $U_+^uU_+$, where $U_+^u = \{v \in U_+ \text{ s.t. } \bar{u}^{-1}v\bar{u} \in U_-\}$. See, $v\bar{u} = \bar{u}(\bar{u}^{-1}v\bar{u})$, and the latter part is in U_-, so we are stuck. **Question from the audience:** So you are pulling as much as you can form one U_+ to the other? **Answer:** Yes, because we want to understand the orbit. So we have $U_+\bar{u}/U_+ \cong U_+^u$. Ok, so:

$$G^*(g,g)G^* = \{(hh_1h'_uU_+^u\bar{u}U_+, h^{-1}h_2{h'_v}^{-1}U_-^v\bar{v}U_-) \text{ s.t. } h, h' \in H\} \tag{19.6}$$

Choose h'_u such that $hh_1h'_u = 1$, so $h'_u = h^{-1}h_1^{-1}$, so $h' = u^{-1}(h)^{-1} \cdot u^{-1}(h_1)^{-1}$, where now $u^{-1}(-)$ is the action of the element u^{-1} in the Weyl group. So

$$h^{-1}h_2{h'}^{-1} = h^{-1}h_2{h'_v}^{-1} = h^{-1}h_2(h_{u^{-1}})_v((h_1)_{u^{-1}})_v \tag{19.7}$$

So this is the Cartan part of the second component, assuming the Cartan part of the first is 1. We can do this because the action on the first component is transitive.

But equation 19.6 is just $h_2(h_1)_{u^{-1}v}h^{-1}h_{u^{-1}v}$. See, as h goes through the Cartan, the $h^{-1}h_{u^{-1}v}$ part only goes through part of the Cartan. So the span of this when $h \in H$ is $\exp(\mathrm{Im}(uv^{-1} - \mathrm{id})) \subseteq H$. So the dimension is

$$\dim \mathcal{O} = \dim(U_+^u) + \dim(V_-^v) + \dim(\mathrm{Im}(uv^{-1} - \mathrm{id})) \tag{19.8}$$

Of course, $\dim(U_+^u)$ counts the length of u. The miracle is that the above number is even.

19.1 Theo presents a proof of Weinstein's theorem

****Naturally, I could not take notes and speak at the board. Here are my prepared remarks:****

This section consists of a proof of Theorem 16.2 stated in the February 27th lecture of Nicolai Reshetikhin's seminar on Quantum Groups. We first restate the theorem, and then proceed with the proof. The proof is largely the same as Weinstein's original, available in:

> A. Weinstein. Some remarks on dressing transformations. *J. Fac. Sci. Univ. Tokyo.* Sect. 1A, Math. 35 (1988), 163-167.

Theorem 19.1 (Alan Weinstein) *Let G be a Poisson Lie group, with Poisson bivector field π. Define a Lie bracket on $\Omega^1(G)$ by*

$$[\omega,\omega']_{\Omega^1(G)} \overset{\text{def}}{=} d\Big(\pi(\omega\wedge\omega')\Big) + \pi\Big(\omega(d\omega')\Big) - \pi\Big(\omega'(d\omega)\Big) \tag{19.9}$$

where $\pi\Big(\omega(\omega'\wedge\omega'')\Big) \overset{\text{def}}{=} \pi(\omega,\omega')\,\omega'' - \pi(\omega,\omega'')\,\omega'$. Then the natural mapping $\mathfrak{g}^* \to \Omega^1(G)_{left}$ induced by the trivialization of T^*G by right translations is a Lie algebra isomorphism.

Proof: We have seen already (Theorem 16.1) that $[,]_{\Omega^1(G)}$ is a valid Lie bracket, and that $\mathfrak{g}^* \to \Omega^1(G)_{\text{left}}$ is an isomorphism of vector spaces. Let $w_1, w_2 \in \mathfrak{g}^*$, and choose functions any one-forms $\omega_1,\omega_2 \in \Omega^1(G)$ so that $\omega_i(e) = w_i$. Then $[\omega_1,\omega_2]_\Omega = d\big(\pi(\omega_1,\omega_2)\big)(e)$, since $\pi(e) = 0$. But this is just $d\pi(e)\big(\omega_1(e),\omega_2(e)\big) = [w_1,w_2]_{\mathfrak{g}^*}$, since $\pi(e) = 0$. So it suffices to show that the bracket of two left-invariant forms is left-invariant.

We now make an aside about sign conventions and notation. The "insert" contraction $\rfloor$ used by the mathematicians is horrendous: do you insert into the first spot, the last, ??? The physicists' indices are better, as long as you're not taking derivatives: when you do, you have to remember that it's not all tensorial. For our purposes, we adopt the following conventions. We will always write sections of wedge products of the tangent bundle before sections of wedge products of the cotangent bundle, and we will work in terms of the pairing $\langle,\rangle : \text{Vect}(G)\otimes\Omega^1(G) \to C(G)$. Then π is our Poisson field, and we will define $\tilde\pi : \Omega^1(G) \to \text{Vect}(G)$ by:

$$\langle\tilde\pi\omega_1,\omega_2\rangle = \langle\pi,(\omega_1,\omega_2)\rangle = \pi(\omega_1,\omega_2) \tag{19.10}$$

The philosophy is always to keep all the symbols in the same order.

Moreover, given a vector field X, we have the *Lie derivative* $\mathscr{L}_X$ acting on tensor fields (and preserving their type), defined by $\mathscr{L}_X f = \langle X, df\rangle = X[f]$, $\mathscr{L}_X Y = [X,Y]$ the bracket of vector fields, and on one-forms:

$$\langle Y,\mathscr{L}_X\omega\rangle = \langle (Y,X), d\omega\rangle + \langle Y, d(\langle X,\omega\rangle)\rangle \tag{19.11}$$

In general, $\mathscr{L}_X$ is a derivation across any tensor contraction.

In this notation, the bracket $[,]_\Omega$ defined in equation 19.9 is determined by:

$$\langle Y, [\omega_1, \omega_2]_\Omega \rangle = \langle Y, d(\pi(\omega_1, \omega_2)) + \langle (Y, \tilde{\pi}\omega_1), d\omega_2 \rangle - \langle (Y, \tilde{\pi}\omega_2), d\omega_1 \rangle \tag{19.12}$$

In light of equation 19.11, we now perform a long sequence of manipulations, simplifying the above expression.

$$\begin{aligned} \langle (Y, \tilde{\pi}\omega_1), d\omega_2 \rangle &= \langle Y, \mathscr{L}_{\tilde{\pi}\omega_1} \omega_2 \rangle - \langle Y, d(\langle \tilde{\pi}\omega_1, \omega_2 \rangle \\ &= \langle Y, \mathscr{L}_{\tilde{\pi}\omega_1} \omega_2 \rangle - \langle Y, d(\pi(\omega_1, \omega_2)) \rangle \qquad (19.13) \\ -\langle (Y, \tilde{\pi}\omega_2), d\omega_1 \rangle &= -\langle Y, \mathscr{L}_{\tilde{\pi}\omega_2} \omega_1 \rangle + \langle Y, d(\langle \tilde{\pi}\omega_2, \omega_1 \rangle \\ &= -\langle Y, \mathscr{L}_{\tilde{\pi}\omega_2} \omega_1 \rangle - \langle Y, d(\pi(\omega_1, \omega_2)) \rangle \qquad (19.14) \end{aligned}$$

Therefore, making heavy use of cut-and-paste:

$$\begin{aligned} \langle Y, [\omega_1, \omega_2]_\Omega \rangle &= -\langle Y, d(\pi(\omega_1, \omega_2)) \rangle + \langle Y, \mathscr{L}_{\tilde{\pi}\omega_1} \omega_2 \rangle - \langle Y, \mathscr{L}_{\tilde{\pi}\omega_2} \omega_1 \rangle \qquad (19.15) \\ &= -\mathscr{L}_Y \big(\pi(\omega_1, \omega_2) \big) + \langle Y, \mathscr{L}_{\tilde{\pi}\omega_1} \omega_2 \rangle - \langle Y, \mathscr{L}_{\tilde{\pi}\omega_2} \omega_1 \rangle \qquad (19.16) \\ &= -(\mathscr{L}_Y \pi)(\omega_1, \omega_2) - \pi(\mathscr{L}_Y \omega_1, \omega_2) - \pi(\omega_1, \mathscr{L}_Y \omega_2) \\ &\quad + \langle Y, \mathscr{L}_{\tilde{\pi}\omega_1} \omega_2 \rangle - \langle Y, \mathscr{L}_{\tilde{\pi}\omega_2} \omega_1 \rangle \qquad (19.17) \\ &= -(\mathscr{L}_Y \pi)(\omega_1, \omega_2) - \pi(\mathscr{L}_Y \omega_1, \omega_2) - \pi(\omega_1, \mathscr{L}_Y \omega_2) \\ &\quad + \mathscr{L}_{\tilde{\pi}\omega_1} \langle Y, \omega_2 \rangle - \langle \mathscr{L}_{\tilde{\pi}\omega_1} Y, \omega_2 \rangle - \mathscr{L}_{\tilde{\pi}\omega_2} \langle Y, \omega_1 \rangle + \langle \mathscr{L}_{\tilde{\pi}\omega_2} Y, \omega_1 \rangle \qquad (19.18) \\ &= -(\mathscr{L}_Y \pi)(\omega_1, \omega_2) - \pi(\mathscr{L}_Y \omega_1, \omega_2) - \pi(\omega_1, \mathscr{L}_Y \omega_2) \\ &\quad - \langle [\tilde{\pi}\omega_1, Y], \omega_2 \rangle + \langle [\tilde{\pi}\omega_2, Y], \omega_1 \rangle + \mathscr{L}_{\tilde{\pi}\omega_1} \langle Y, \omega_2 \rangle - \mathscr{L}_{\tilde{\pi}\omega_2} \langle Y, \omega_1 \rangle \qquad (19.19) \\ &= -(\mathscr{L}_Y \pi)(\omega_1, \omega_2) - \pi(\mathscr{L}_Y \omega_1, \omega_2) - \pi(\omega_1, \mathscr{L}_Y \omega_2) \\ &\quad + \langle [Y, \tilde{\pi}\omega_1], \omega_2 \rangle - \langle [Y, \tilde{\pi}\omega_2], \omega_1 \rangle + \mathscr{L}_{\tilde{\pi}\omega_1} \langle Y, \omega_2 \rangle - \mathscr{L}_{\tilde{\pi}\omega_2} \langle Y, \omega_1 \rangle \qquad (19.20) \\ &= -(\mathscr{L}_Y \pi)(\omega_1, \omega_2) - \pi(\mathscr{L}_Y \omega_1, \omega_2) - \pi(\omega_1, \mathscr{L}_Y \omega_2) \\ &\quad + \langle \mathscr{L}_Y(\tilde{\pi}\omega_1), \omega_2 \rangle - \langle \mathscr{L}_Y(\tilde{\pi}\omega_2), \omega_1 \rangle + \mathscr{L}_{\tilde{\pi}\omega_1} \langle Y, \omega_2 \rangle - \mathscr{L}_{\tilde{\pi}\omega_2} \langle Y, \omega_1 \rangle \qquad (19.21) \\ &= -(\mathscr{L}_Y \pi)(\omega_1, \omega_2) - \pi(\mathscr{L}_Y \omega_1, \omega_2) - \pi(\omega_1, \mathscr{L}_Y \omega_2) \\ &\quad + \langle (\mathscr{L}_Y \tilde{\pi})\omega_1 + \tilde{\pi}\mathscr{L}_Y \omega_2, \omega_2 \rangle - \langle (\mathscr{L}_Y \tilde{\pi})\omega_2 + \tilde{\pi}\mathscr{L}_Y \omega_2, \omega_1 \rangle \\ &\quad + \mathscr{L}_{\tilde{\pi}\omega_1} \langle Y, \omega_2 \rangle - \mathscr{L}_{\tilde{\pi}\omega_2} \langle Y, \omega_1 \rangle \qquad (19.22) \\ &= (\mathscr{L}_Y \pi)(\omega_1, \omega_2) + \mathscr{L}_{\tilde{\pi}\omega_1} \langle Y, \omega_2 \rangle - \mathscr{L}_{\tilde{\pi}\omega_2} \langle Y, \omega_1 \rangle \qquad (19.23) \end{aligned}$$

Now let ω_1, ω_2, and Y be left-invariant. Then $\langle Y, \omega_i \rangle$ is a constant, and so the corresponding derivatives vanish. We win if we can show that $\langle Y, [\omega_1, \omega_2]_\Omega \rangle$, or equivalently that $(\mathscr{L}_Y \pi)(\omega_1, \omega_2)$ is a constant, or that $\mathscr{L}_Y \pi$ is left-invariant whenever Y is.

But this is almost immediate. The condition that G be Poisson-Lie is that π satisfy the following cocycle identity:

$$\pi(gh) = dl_g^{\otimes 2} \pi(h) + dr_h^{\otimes 2} \pi(g) \tag{19.24}$$

We have used the left- and right-actions $l_g, r_g : G \to G$ given by $l_g h = gh$ and $r_g h = hg$. Then for any $g, h \in G$, l_g and r_h commute, $l.$ is a homomorphism, and $r.$ an antihomomorphism. The differentials $dl_g, dr_g : TG \to TG$, and so the tensor square of a differential acts on a bivector.

Let Y be a left-invariant vector field, and $y = Y(e) \in \mathfrak{g}$. Then Y generates the flow $r_{\exp(ty)}$ (and conversely the generator of $r_{\exp(ty)}$ for any $y \in \mathfrak{g}$ must be left-invariant, as $r_{\exp(ty)}$ commutes with any l_g). So let $h = \exp(ty)$ in equation 19.24, and differentiate in t at $t = 0$. This corresponds to taking the Lie derivative $\mathscr{L}_Y$, and we get an equation in T_gG:

$$\mathscr{L}_Y \pi(g) = dl_g^{\otimes 2} \mathscr{L}_y \pi(e) + 0 \tag{19.25}$$

By construction, $\mathscr{L}_Y dr_{\exp(ty)}^{\otimes 2} \pi = 0$; indeed, if Z is any vector field, generating a flow $\zeta_t : G \to G$, then for any vector $v \in T_gG$, we have $\mathscr{L}_Z d\zeta_t v = 0$ tautologically.

But equation 19.25 says exactly that $\mathscr{L}_y \pi$ is left-invariant.

□

****After I presented a condensed version of the above, NR make the following remark:****

The proof is trivial if you trivialize by right translations at the outset. Indeed, if ω, ω' are right-invariant, then in the trivialization they are constant maps $G \to \mathfrak{g}^*$. The Poisson bivector field is a matrix of functions $\pi^{ij} : G \to \bigwedge^2 \mathfrak{g}$, and the cocycle identity equation 19.24 becomes:

$$\pi(gh) = \mathrm{Ad}_g^{\otimes 2} \pi(h) + \pi(g) \tag{19.26}$$

Now the invariance of $\mathscr{L}_Y \pi$ follows from differentiating in h, whence the $\pi(g)$ terms vanishes. But moreover, if ω, ω' are right-invariant, then they are constants, and the right-hand-side of equation 19.9 becomes (using indices to track the contractions):

$$\partial_i\big(\pi^{jk} w_j w'_k\big) + \pi^{jk} w_j \big(\partial_k w'_i - \partial_i w'_k\big) - \pi^{jk} w'_j \big(\partial_k w_i - \partial_i w_k\big) \tag{19.27}$$

But w_i, w'_i are constant, so their derivatives vanish, and we are left only with $[w, w']_i = \partial_i \pi^{jk} w_j w'_k$. Of course, $\partial_i \pi^{jk}(e)$ is precisely the bracket on $\mathfrak{g}^*$, and again if we fix g and differentiate equation 19.26 in $\partial/\partial h^i$ at $h = e$, we see that

$$\partial_i \pi^{jk}(g) = \mathrm{Ad}_g^{\otimes 2} \partial_i \pi^{jk}(e) \tag{19.28}$$

where everything is a matrix of functions. So $\partial_i \pi^{jk}$ is left-invariant, and hence $\partial_i \pi^{jk} w_j w'_k$ is.

****I had thought about trying to make this argument, although I tried it in the wrong order. But my worry is this: ∂_i is defined in terms of a coordinate system, and we have one — the exponential map $\exp : \mathfrak{g} \to G$ gives one near e, and we can move it by right-translations — but it itself is not obviously good under translations, in the sense that it's not obvious that the right-translate of the derivative of a tensor field should have much to do with the derivative of the right-translate of a tensor field. I think that it turns out not to matter, but I'm worried about $(d\omega)_{ij} = \partial_i \omega_j - \partial_j \omega_i$. This should be true, and perhaps it's obvious, but there's a reason mathematicians avoid coordinates.****

Lecture 20 March 9, 2009

We begin by repeating some material from last time.

We let G be a simple complex Poisson Lie group with the standard structure. Recall that the symplectic leaves of G are given by the map $G \hookrightarrow \mathcal{D}(G) = G \times G \to \mathcal{D}(G)/G^*$, and $G^* \hookrightarrow G \times G$ as $\{(b_+, b_-) \text{ s.t. } b_\pm \in B_\pm, [b_+]_0 = [b_-]_0^{-1}\}$. In this language, the symplectic leaves are connected components of preimages of left G^* orbits in $\mathcal{D}(G)/G^*$.

We have two related Bruhat decompositions: $G = \bigsqcup_{u \in W} B_+ u B_+ = \bigsqcup_{u \in W} B_- u B_-$ and we define the *double Bruhat cells* $G^{u,v} = B_+ u B_+ \cap B_- v B_-$. Let $g \in G^{u,v}$; we want to study the G^* orbit through $(g,g)G^*$.

Well, $B_\pm = HU^\pm$. We fix $\bar{u}$ and $\bar{v}$ representing u and v; then $g \in B_+ u B_+$ so $g = u_+ h_1 \bar{u} u'_+$, and by the same token $g = u_- h_2 \bar{v} u'_-$ ****shame we're using "u" in two different ways****. Then he orbit through $(g,g)G^*$ is

$$\mathcal{O}_{(g,g)G^*} = \{U_+ h \underbrace{u_+ h_1 \bar{u} u'_+}_{g} h' U_+, U_- h^{-1} \underbrace{u_- h_2 \bar{v} u'_-}_{g} h'^{-1} U_-\} \tag{20.1}$$

So, we now want to commute things past each other, so as to understand what part of G is the stabilizer, and what part is transitive on the orbit. We explained last time that $U_+ \bar{u} U_+ = U_+^u \bar{u} U_+$, where $U_+^u \overset{\text{def}}{=} \{x \in Y \text{ s.t. } \bar{u}^{-1} x \bar{u} \in U_-\}$; this is the limit of our ability to commute an upper-triangular past u. For SL_n, $\bar{u}$ and $\bar{v}$ are monomimal $n \times n$ matrices, i.e. exactly one nonzero element in each row and each column. I.e. these are permutation matrices up to a diagonal matrix. Similarly, $U_- \bar{v} U_- = U_-^v \bar{v} U_-$, where $U_-^v \overset{\text{def}}{=} \{x \in U_- \text{ s.t. } \bar{v}^{-1} x \bar{v} \in U_+\}$.

So,

$$\mathcal{O} = \Big\{\{U_+^u h h_1 h'_u \bar{u} U_+, U_-^v h^{-1} h_2 (h'_v)^{-1} \bar{v} U_-\} \text{ s.t. } h, h' \in H\Big\} \tag{20.2}$$

We make a change of variables: $h'_u = h''_u h_1^{-1} h^{-1}$, whence $(h'_v)^{-1} = \left(h''_v (h_1)^{-1}_{vu^{-1}} h^{-1}_{vu^{-1}}\right)^{-1}$. ($H$ is commutative.) Hence the orbit is:

$$\mathcal{O} = \Big\{\{U_+^u h''_u \bar{u} U_+, U_-^v h^{-1} h_2 (h_1)_{vu^{-1}} h_{vu^{-1}} h''^{-1}_v \bar{v} V_-\} \text{ s.t. } h'', h \in H\Big\} \tag{20.3}$$

We can always move elements of the Cartan past U_+, because it acts by conjugation. Then we get:

$$\begin{aligned}
\mathcal{O} &= \Big\{\Big\{U_+^u \bar{u} U_+ h'', U_-^v \left(h^{-1} h_{vu^{-1}}\right) [h_2 (h_1)_{vu^{-1}}] \bar{v} U_- h''^{-1}\Big\} \text{ s.t. } h, h'' \in H\Big\} && (20.4)\\
&= \Big\{\left(U_+^u \bar{u}, U_-^v \left(h^{-1} h_{vu^{-1}}\right) [h_2 (h_1)_{vu^{-1}}] \bar{v}\right) \text{ s.t. } h \in H\Big\} G^* && (20.5)
\end{aligned}$$

Now, as we vary the Cartan h, the part $\left(h^{-1} h_{vu^{-1}}\right)$ is not the full Cartan H. Indeed, we define $H_w = \{h^{-1} h_w\} = \exp(\mathrm{Im}(w - \mathrm{id}))$, where $(w - \mathrm{id}) : \mathfrak{h} \to \mathfrak{h}$ naturally. This is often smaller dimension than all of H. For example, when $w = \mathrm{id}$, $H_w = \{e\}$. But also, $w_0^2 = e$, where w_0 is the longest element, and so the matrix is filled with ± 1; thus even for $w = w_0$ this dimension may be less than r, the rank. So this dimension doesn't correspond to length.

Anyway, so in equation 20.5 we can easily see the dimension:

$$\dim_{\mathbb{C}}(\mathcal{O}_{(g,g)}) = \dim_{\mathbb{C}}(U_+^u) + \dim_{\mathbb{C}}(U_-^v) + \dim_{\mathbb{C}}(\mathrm{Im}_{\mathfrak{h}}(vu^{-1} - \mathrm{id})) = \ell(u) + \ell(v) + \text{something} \quad (20.6)$$

Corollary 20.0.1 *This number is even.*

Because the preimage is a symplectic leaf. One can show that there is no loss of dimension moving from the symplectic leaf to the orbit — the symplectic leaf may be complicated, some cover with different components, but it has finite fiber. That the number is even is absolutely clear when $u = v$.

We make a claim, which we will not prove, but someone might make into a presentation. It is related to "cluster algebras" and other things in modern research that people are doing now.

Proposition 20.1 *U_+^w has an* almost coordinate system*, meaning it's a coordinate system on a Zariski-open subset. (That's the idea of "cluster variety": you have almost-coordinates on two Zariski subsets, and you play a while and discover that the transition functions are simple nice Laurent polynomials, and with this you can cover the whole manifold.) The description is: let $w = s_{i_1} \dots s_{i_l}$ be a reduced decomposition. Then $U_+^w \supseteq \{\exp(t_1 e_{i_1}) \dots \exp(t_l e_{i_l})$ s.t. $t_i \in \mathbb{C}\}$.*

See,

$$\exp(t_1 e_{i_1}) \dots \exp(t_l e_{i_l}) \bar{s}_{i_1} \dots \bar{s}_{i_l} = \bar{s}_{i_1} \dots \bar{s}_{i_l} u^- \quad (20.7)$$

for some $u^- \in U^-$, by the definition of U_+^w. Think about it for SL_n, where $\exp(te_i)$ is a mostly-zero matrix with 1s on the diagonal and t in the ith row, $(i+1)$th column.

Ok, so what have we proved?

Theorem 20.2 *1. $G^{u,v}$ is fibered over $H^{vu^{-1}}$, where $H^w = \exp(\ker(w - id))$. The fibers are the orbits $\mathcal{O}^{u,v}$, which are isomorphic to $U_+^u \times U_-^v \times H^{vu^{-1}}$.*

2. $G^{u,v}$ is a homogeneous Poisson subvariety*, i.e. a fiber bundle whose fibers are symplectic leaves.*

Proof: Exercise 34

Using finite-dimensional representation theory, its very easy to describe exactly this fibering, along with coordinates, etc. We won't do it, but if anyone wants to learn it, there is good literature.

Rather, we make a short return to the discussion of the compact real form of G. Let G be a complex algebraic Poisson manifold, and $K \subseteq G$ is the compact real form of the Lie group and of the Poisson manifold. Then $K = G^\sigma$, the set of fixed points with respect to the involution σ on G. When $G = SL_n$, $\sigma(g) = g^{*-1}$, the complex conjugation. Then the symplectic leaves of K should be the fixed points of the symplectic leaves.

You can see that $(U_+^u)^* = U_-^u$. Hence, we leave as **Exercise 35**:

Proposition 20.3 *The only symplectic leaves of G which have σ-fixed points are $\mathcal{O}^{u,u}$, whence $(\mathcal{O}^{u,u})^\sigma \cong C_u =$ the Shubert cell corresponding to $u \in W$.*

Then $(G^{u,u})^\sigma = K_u$, and the description of the symplectic leaves is as we had: $K = \bigsqcup_u K_u$, and $K_u/T = C_u$. Then the projection $(G^{u,u})^\sigma \to H^\sigma$ is precisely the projection $K_u \to T$.

So, the moral of the story: we have the description of the symplectic leaves of algebraic groups, and the diagonal ones gives the symplectic leaves of the compact real forms.

We make one last comment. Any simple G has two distinguished real forms:

Compact $K = G^\sigma$, e.g. $SL_n \rightsquigarrow SU_n$.

Split-real $G_\mathbb{R}$, e.g. $SL_n(\mathbb{C}) \rightsquigarrow SL_n(\mathbb{R})$.

This latter one has a very interesting subset, namely $(G_\mathbb{R})_{\geq 0}$, where all minors are nonnegative, and on $(G_\mathbb{R}^{u,v})_{\geq 0}$, then the almost-coordinate system gives a coordinate system. When all minors are positive, $(G_\mathbb{R}^{u,v})_{>0} \cong \mathbb{R}^{\ell(u)+\ell(v)} \times \mathbb{R}_{>0}^{\dim(H^{uv^{-1}})}$.

Lecture 21 March 11, 2009

21.1 Geometry review

We begin with a short summary of the class:

- We have the notion of a *Lie bialgebra* $(\mathfrak{g}, \mathfrak{g}^*)$, which may or may not be *factorizable*.
- We thus can exponentiate to two *Poisson Lie groups* (G, p) and (G^*, p_*), and these make a *dual pair* in this sense.
- When G is simple, we explained its *standard Poisson Lie structure*, constructed its dual as a product of *Borel subgroups* of G, and described the *symplectic leaves* of G — double Bruhat cells — and of G^* — conjugacy classes in G.
- We described two important real forms of a simple Poisson Lie group:

  Compact real form (K, p), with dual (K^*, p_*), and we saw that $K^* \cong AU$ as a group, and $K^* \cong \mathscr{H}$, the Hermitian unimodular matrices, as a space. (Well, we say this for SU_n, but it's the same for other groups.) The symplectic leaves of K^* are K-conjugacy orbits in $\mathscr{H}$. The symplectic leaves of K are given by the decomposition $K = \bigsqcup_w K_w$, where each K_w is fibered over T, the maximal torus in the complex Cartan subgroup $H_\mathbb{C}$, with fibers C_w the *Shubert cells*. So symplectic leaves are Shubert cells.

  Split real form $G_\mathbb{R} \subseteq G_\mathbb{C}$, e.g. $SL_n(\mathbb{R}) \subseteq SL_n(\mathbb{C})$. The important part is the *totally non-negative* part $(G_\mathbb{R})_{\geq 0}$. Here the double Bruhat cells are $G_{>0}^{u,v} \cong \mathbb{R}^{\ell(u)+\ell(v)} \times \mathbb{R}_{>0}^r$, where $r = \operatorname{rank}(G)$.

These two compact forms are in an important sense opposite.

What's amazing is that the geometry given by the standard Poisson Lie structure matches these purely algebraic structure. We remark that there is a classification of factorizable Poisson Lie structures, on which there will be a presentation after the break.

There are other interesting geometries, coming from other groups. E.g. the Borel group in SL_n is the upper-triangular matrices; we could take a *parabolic subgroup* of SL_n, which sits between these, and study the geometry relative to it. But going in this direction will lead to a class on Poisson Lie geometry, and that's not our overall topic.

21.2 Algebra review

Recall that if P is a Poisson manifold, then $C(P)$ is a *Poisson algebra*: it is a commutative algebra along with a Lie bracket $\{,\}$ which satisfies the *Leibniz rule* $\{a, bc\} = \{a, b\}c + b\{a, c\}$.

On the other hand, if G is a group, then $C(G)$ is a *Hopf algebra*: it is an associative unital algebra (commutative in the case of $C(G)$, but not part of the definition) along with a *comultiplication* $\Delta : A \to A^{\otimes 2}$, which is a homomorphism of unital algebras. Δ should be *coassociative* — $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ — and there should be a *counit* — a linear functional $\epsilon : A \to \mathbb{C}$ that is a homomorphism of algebras satisfying $(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta$. For a $A = C(G)$, where G is finite or algebraic, we have $(\Delta f)(x, y) = f(xy)$, and $\epsilon(f) = f(1)$. So far we have defined a *bialgebra*. A *Hopf algebra* is a bialgebra along with a map $S : A \to A$ that is a bialgebra antiautomorphism — $S(ab) = S(b)S(a)$ and $(S \otimes S) \circ \Delta = \Delta^{\mathrm{op}} \circ S$, where $\Delta^{\mathrm{op}} = \sigma \circ \Delta$, where σ is the canonical "flip" map $X \otimes Y \to Y \otimes X$. Moreover, S must satisfy the following commutative diagram:

****pentagon****

In the case of a group, $S(f)(x) = f(x^{-1})$, and the diagram is the statement that $f(xx^{-1}) = f(x^{-1}x) = f(1)$. So a Hopf algebra is an algebraic description of a *group*.

Ok, so now let's assume that G is algebraic. We make this assumption because there are all sorts of completions one has to make to deal with functions that are not polynomials. This is possible, but one needs to go deep into functional analysis.

Ok, so if G is an algebraic Poisson Lie group, then $A = C(G)$ is a Poisson algebra and also a Hopf algebra, and moreover the comultiplication is a homomoprhism of Poisson algebras. **Question from the audience:** You have to prove that the bracket of polynomials is polynomails? **Answer:** Yes, well, that's part of the word "algebraic Poisson Lie group". Also S is an anti-Poisson map.

If $\mathfrak{g}$ is a Lie algebra, there is another natural Hopf algebra associated to $\mathfrak{g}$: the *universal enveloping algebra* $\mathcal{U}\mathfrak{g}$.

Proposition 21.1 *We saw that* $(\mathfrak{g}, \delta)$ *is a* Lie bialgebra *if and only if* $\delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ *is a one-cocycle for the Chevalley complex with coefficients in* $\bigwedge^2 \mathfrak{g}$*. The proposition is that this induces a one-cocycle* *in the Hochschild cohomology of $\mathcal{U}\mathfrak{g}$ with coefficients in $\mathcal{U}\mathfrak{g}^{\otimes 2}$.*

Recall, the *Hochschild complex*: Let A be an associative algebra, and M a module over A. Then we define

$$C^\bullet(A, M) \overset{\text{def}}{=} \bigoplus_{n\geq 0} \operatorname{Hom}_{\mathbb{C}}(A^{\otimes n}, M) \tag{21.1}$$

So all linear maps over the field; if A is infinite-dimensional, we should talk about topology on it, but we will obscure this. Anyway, suppose that $c \in C^n$; we must produce a cochain in C^{n+1}. We do it thus:

$$dc(x_1, \dots, x_{n+1}) = c(x_1x_2, x_3, \dots, x_n) - \dots + (-1)^{n-1}c(x_1, \dots, x_{n-1}x_n) + \text{next time} \tag{21.2}$$

We will do this next time; we don't want to get confused by pluses and minuses at the board. ****It should be almost that, but also with some reference to the module structure on M, I think.****

21.3 Looking forward

A *dual pair of Hopf algebras* is a pair A, B of Hopf algebras along with a nondegenerate pairing $\langle , \rangle : A \otimes B \to \mathbb{C}$ that gets along with the Hopf algebra structure:

$$\langle ab, l\rangle = \langle a \otimes b, \Delta_B(l)\rangle \tag{21.3}$$

$$\langle \Delta_A(a), l \otimes m\rangle = \langle a, lm\rangle \tag{21.4}$$

$$\langle 1_A, l\rangle = \epsilon_B(l) \tag{21.5}$$

$$\langle a, 1_B\rangle = \epsilon_A(a) \tag{21.6}$$

$$\langle S_A(a), l\rangle = \langle a, S_B(l)\rangle \tag{21.7}$$

We will use the not-ideal but at least historical notation that $\langle a\otimes b, l\otimes m\rangle = \langle a, l\rangle\langle b, m\rangle$. In general, we say that a pairing is *nondegenerate* if $\langle a, l\rangle = 0\,\forall l$ implies $a = 0$, and also on the other side.

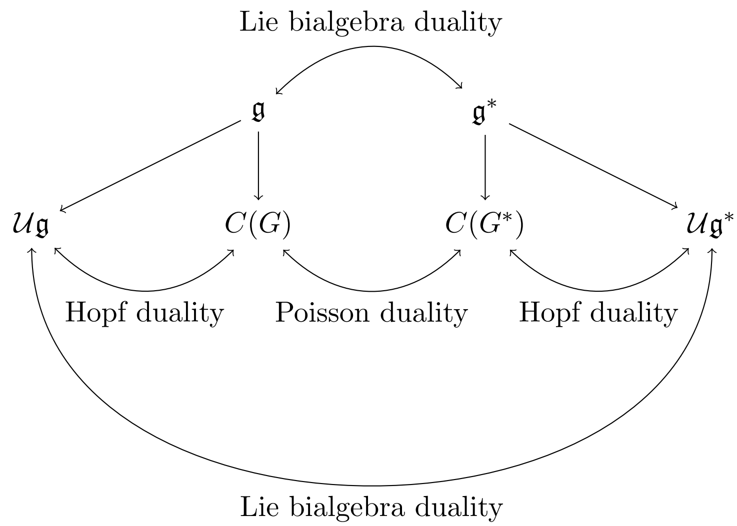
Example 21.1 If A is finite-dimensional, then there is a unique Hopf algebra $B = A^*$. ◊

Example 21.2 If $\mathfrak{g}$ is a Lie algebra, and G the corresponding algebraic Lie group, then $\mathcal{U}\mathfrak{g}$ and $C(G)$ are not dual vector spaces — the dual to an infinite-dimensional vector space is a hairy thing — but they are a dual pair of Hopf algebras. We define the pairing by $\langle 1, f\rangle \overset{\text{def}}{=} f(e)$, $\langle x, f\rangle \overset{\text{def}}{=} \frac{d}{dt}\Big|_{t=0} f(e^{tX})$ when $x \in \mathfrak{g} \hookrightarrow \mathcal{U}\mathfrak{g}$, and for a product, $\langle x_1 \dots x_n, f\rangle = \frac{d^n}{dt_1 \dots dt_n}\Big|_{t_i=0} f(e^{tX_1} \dots e^{tX_n})$. ****The opposite pairing convention above might lead to different order here.**** ◊

Now, if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, then we get two Hopf Poisson algebras $C(G)$ and $C(G^*)$, and in fact four Hopf algebras producible naturally form this data:

$$\mathcal{U}\mathfrak{g},\ \ C(G),\ \ G(G^*),\ \ \mathcal{U}\mathfrak{g}^*$$

Question from the audience: If G is algebraic, is G^* also? **Answer:** We assume that they all are: this is the case for the standard Poisson Lie structure. It is also the case in all standard interesting applications. The standard examples are (1) topological (knot) invariants, and (2) quantum integrable systems. A small detour to mathematical physics: The initial motivation for Lie to invent Lie groups and Lie algebras were the symmetries of differential equations, because Classical Mechanics was the primary game. About one hundred years later, Hopf algebras arose in the same context, to understand the symmetries in quantum field theories. This will be the 21st century challenge — constructing a mathematical theory of QFT — just like the 20th century success was the mathematics of Quantum Mathematics. There is currently only bits and pieces of a consistent mathematical theory of quantum fields. Certain models can be constructed mathematically with the above algebraic tools, whence they become representation theory of certain affine algebras. Point being, everything is algebraic in all examples.



These last dualities are $\hat{\delta} : C(G)^* \to \bigwedge^2 C(G)^*$ and $\hat{\delta} : \mathcal{U}\mathfrak{g} \to \bigwedge^2 \mathcal{U}\mathfrak{g}^{\otimes 2} \hookrightarrow \mathcal{U}\mathfrak{g}^{\otimes 2}$.

We will begin a program of *quantization*. We will deform all these Hopf algebras to non-commutative, non-cocummutative Hopf algebras. When we do, we will find that algebraically (but not topologically), $\mathcal{U}_q\mathfrak{g} \cong C_q G^*$, and this will make a dual pairing with $C_q(G) \cong U_q\mathfrak{g}^*$.

There are two ways to do this: either to deform the multiplication of the algebra, or to write the algebras in generators and relations, deform the ideals, and identify the vector spaces. In the interest of time, we will only do the second, quoting:

Theorem 21.2 (Kontsevich) *Every Poisson algebra has a formal quantization.*

There is a description of the classification of these. There is a QFT that explains these.

Next week NR will be away at a conference. Noah Snyder will substitute.

Lecture 22 March 13, 2009

22.1 Drinfeld Double of a Hopf Algebra

Let $(H, H^*, \langle\rangle)$ be a dual pair of Hopf algebras. We let H^o be the Hopf algebra H^* with the opposite comultilication: $\Delta_{H^o} = \sigma \circ \Delta_{H^*}$.

Then we define the multiplication on the vector space $H \otimes H^*$ by:

$$(a \otimes l) \cdot (b \otimes m) = \sum_{b,l} ab_{(2)} \otimes l_{(2)} m \, \langle b_{(1)}, S^{-1}(l_{(1)}) \rangle \langle b_{(3)}, l_{(3)} \rangle \tag{22.1}$$

where S is the antipode for H^*, and we introduce the notation: $\Delta_H(a) = \sum_a a_{(1)} \otimes a_{(2)}$, and $\Delta_H^{(3)}(a) = \sum_a a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$. This is well-defined by the coassociativity. Similarly for $l_{(i)}$, defined in terms of Δ_{H^o}.

Theorem 22.1 *This multiplication defines an associative unital algebra structure on $\mathcal{D}(H, H^*) \stackrel{\text{def}}{=} H \otimes H^*$ with unit $1_{\mathcal{D}} = 1_H \otimes 1_{H^*}$, such that it is also a Hopf algebra with $H \hookrightarrow \mathcal{D}(H, H^*)$ and $H^o \hookrightarrow \mathcal{D}(H, H^*)$ Hopf algebra emberrings. (So $\mathcal{D}(H, H^*) \cong H \otimes H^o$ as a coalgebra.)*

$\mathcal{D}(H, H^*)$ is called the *Drinfeld double of a Hopf algebra*.

Exercise 36 *Prove this. You will need the coassciativity, and that S is an antiautomorphism.*

Question from the audience: This has a natural symmetric bilinear form. Is there a notion in which this is invariant, like the double of a Lie bialgebra? **Answer:** You have to formulate what "invariance" means. Yes, we will do this.

A pair $(H, R \in H^{\otimes 2})$, where H is a Hopf algebra, and it's OK if R is in some natural completion of $H^{\otimes 2}$ — such a pair is a *quasitriangular Hopf algebra* if the comultipliation acts on R by:

$$(\Delta \otimes \mathrm{id})R = R_{12} R_{23} \tag{22.2}$$

$$(\mathrm{id} \otimes \Delta)R = R_{13} R_{23} \tag{22.3}$$

$$\sigma \Delta a = R \, \Delta(a) \, R^{-1} \tag{22.4}$$

So R intertwines the two multiplications. The right hand side for the first two conditions is componentwise multiplication: $R = \sum \alpha_i \otimes \beta^i$, then $R_{13} = \sum \alpha_i \otimes 1 \otimes \beta^i$, etc., and $R_{13} R_{12} = \sum_{ij} \alpha^i \alpha^j \otimes \beta^j \otimes \beta^i$.

If $\mathfrak{g}$ is a Lie algebra, then $\mathcal{U}\mathfrak{g}$ the universal enveloping algebra is a Hopf algebra. Then the comultiplication is symmetric: $\Delta x = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g} \hookrightarrow \mathcal{U}\mathfrak{g}$. Then the category $\mathcal{U}\mathfrak{g}$-mod of finite-dimensional modules over $\mathcal{U}\mathfrak{g}$ has morphisms maps $f : V \to W$ linear and intertwining the action, and it is naturally an abelian category. But moreover if $H = \mathcal{U}\mathfrak{g}$, then the comultiplication lets use define the tensor product of modules.

We will be more careful now about notation. An object is a pair (π_V, V), where V is a finite-dimensional vector space, and $\pi_V : H \to \mathrm{End}(V)$. Then $\hom\big((\pi_V, V), (\pi_W, W)\big) = \{f : V \to$

W linear s.t. $f\pi_V(a) = \pi_W(a)\, f$ for any $a \in H\}$. Now we can define:

$$(\pi_V, V) \otimes (\pi_W, W) \overset{\text{def}}{=} \big((\pi_V \otimes \pi_W) \circ \Delta, V \otimes W\big) \tag{22.5}$$

A short digression. If $H = \mathbb{K}$ (e.g. $\mathbb{K} = \mathbb{C}$), then H-mod $=$ Vect$_{\mathbb{C}}$. Then the tensor product not strict: $(V_1 \otimes V_2) \otimes V_3$ is not the same object as $V_1 \otimes (V_2 \otimes V_3)$. But there are canonical isomorphisms:

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes V_2 \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3) \tag{22.6}$$

So Vect is not a *strict monoidal category*, which is when $(V_1 \otimes V_2) \otimes V_3 = V_1 \otimes (V_2 \otimes V_3)$ on the nose. Vect is almost this: there is a canonical object $V_1 \otimes V_2 \otimes V_3$, etc. In other monoidal categories in which there is no canonical object for the tensor of n objects.

Anyway, H-mod is like Vect in this way.

Proposition 22.2 *If H is a Hopf algebra, then H-mod is an* almost strict monoidal category*, meaning that it is not strict, but for any n objects, there is a canonical unbracketed tensor product of those objects.*

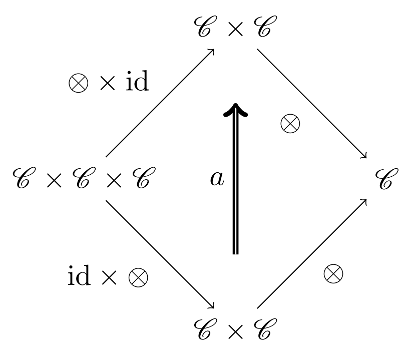
Now, if $H = \mathcal{U}\mathfrak{g}$, then $\sigma\Delta = \Delta$, so the products are isomorphic $(\pi_{V\otimes W}, V \otimes W) \xrightarrow{\sim} (\pi_{W \otimes V}, W \otimes V)$ by $S_{V,W} : v \otimes w \mapsto w \otimes v$.

Proposition 22.3 *$\mathcal{U}\mathfrak{g}$-mod is a* symmetric category*. (There exist maps $S_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$ satisfying axioms.)*

The idea is that the R in a quasitriangular Hopf algebra (H, R) will make H-mod into a braided category.

Let's recall some definitions:

- The category $\mathscr{C}$ is *monoidal* if there is a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ such that :



(22.7)

  where the diagram need not commute on the nose, but up to a particular natural isomorphism a.

  Moreover, where should also be a choice of object $\mathbb{1} \in \mathscr{C}$ with natural isomorphisms $V \otimes \mathbb{1} \underset{r_V}{\xrightarrow{\sim}} V \underset{l_V}{\xleftarrow{\sim}} \mathbb{1} \otimes V$.

Example 22.1 If H is a *bialgebra*, i.e. like a Hopf algebra except without the antipode, then H-mod is monoidal. We have to choose a unit object: pick a one-dimensional space $\mathcal{L}$, whence $\phi : \mathrm{End}_{\mathbb{C}}(\mathcal{L}) \cong \mathbb{C}$, and use the comultiplication $\epsilon : H \to \mathbb{C}$ to define the action of $H \to \mathrm{End}(\mathcal{L})$. $\Diamond$

- A monoidal category $\mathscr{C}$ is *rigid* if for any object $V \in \mathscr{C}$ there exists V^* (unique up to unique isomorphism) and maps $i_V : \mathbb{1} \to V \otimes V^*$ and $e_V : V^* \otimes V \to \mathbb{1}$, satisfying natural axioms. So $V \mapsto V^*$ is not yet a (contravariant) functor — it is once we choose V^* for each V, and different choices give isomorphic functors. ****Once we chose a functor, I expect that i and e should be dinatural transformations.**** Some references: Bakalov and Kirillov, and there are some notes (not yet a book) by Kevin Walker.

  Example 22.2 If H is a Hopf algebra, then H-mod is rigid, with $(\pi_V, V)^* \stackrel{\text{def}}{=} \big((\pi_V \circ S)^*, V^*\big)$, where on the right-hand side we mean that linear-algebraic dual (V is finite-dimensional). If you didn't have S, then you wouldn't be able to construct this homomorphism. A simple property: $\big((\pi_V, V) \otimes (\pi_W, W)\big)^* \cong (\pi_W, W)^* \otimes (\pi_V, V)^*$. $\Diamond$

Next time, we will see that if (H, R) is quasitriangular, then there will be a natural braiding $c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$, given by $c_{V,W} = S_{V,W}(\pi_V \otimes \pi_W)R$. This will be a *tensor category*.

Moreover, we will see that the Double of any Hopf algebra is quasitriangular.

Lecture 23 March 16, 2009. Guest Lecture by Noah Snyder

23.1 Quasitriangular Hopf Algebras and Braided Tensor Categories

Often when Hopf algebras are first introduced, they're seen as generalizations of "functions on a group": Δ comes from $\cdot$, ϵ from e, S from $g \mapsto g^{-1}$, and the algebra structure understands the geometry of the group.

But then there's a switcheroo: we will think about H-modules, which make a category with extra structure. We will have the following dictionary:

- $\Delta \rightsquigarrow \otimes$
- $S \rightsquigarrow *$
- $\epsilon \rightsquigarrow \mathbb{1}$
- and the algebra structure makes H-mod into an abelian category.

One can ask whether H-mod includes a natural map $\sigma_{V,W} : V \otimes W \to W \otimes V$. For $H = C(G)$, yes: the vector-space trivial "swap" map does the job. But there may be others.

Example 23.1 The category of *supervector spaces* has objects $\mathbb{Z}/2$-graded vector spaces, and braiding is $v \otimes w \mapsto (-1)^{|v|\cdot|w|} w \otimes v$, where $|v|$ is the degree of v (either 0 or 1). $\Diamond$

How would one construct a map $\sigma_{V,W}$ in general? Well, we should define $v \otimes w \mapsto \text{swap}R(v \otimes w)$, where $R \in H \otimes H$. This should be a map of modules, and so we need to demand that it intertwines the action of any $x \in H$. For this to happen, we'd need to demand:

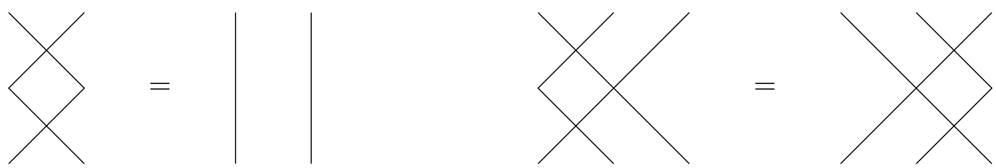
$$R\,\Delta x = \Delta^{\text{op}} x\, R \tag{23.1}$$

where $\Delta^{\text{op}} = \text{swap}\,\Delta$.

What properties do we demand of $\sigma_{V,W}$? The old fashioned answer is to satisfy the relations on the symmetric group.

We will explain this in pictures. The yoga is to draw objects as labeled strands. For example, $\uparrow V$ is an object, $\downarrow V$ is its dual, and $\curvearrowright$ is the pairing $V \otimes V^* \to \mathbb{C}$.

Anyway, so if $g \in S_n$ the symmetric group, we could demand that there is a unique and well-defined $\sigma_g : V_1 \otimes \cdots \otimes V_n \to V_{g(1)} \otimes V_{g(n)}$. I.e. any two ways to build g out of transpositions, we should get the same map. In pictures, this is equivalent to the requirement that:



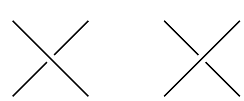
There is also another thing we might want. For example, $\sigma_{A\otimes B,C} : ABC \to CAB$ is well-defined; we should demand that it matches $ABC \stackrel{1\otimes\sigma_{B,C}}{\longrightarrow} ACB \stackrel{\sigma_{A,C}\otimes 1}{\longrightarrow} CAB$.

If any element of the symmetric group has a well-defined action on a long string of tensor products, we say the category is *symmetric*. For example, the category of supervector spaces is symmetric.

Exercise 37 *Consider the category of supervector spaces, and think of it has H-mod where $H = \mathbb{Z}/2$. Doing this is tricky: you need to consider the odd or even parts in terms of the action of the generator x. Anyway, find R. It should be something like $R = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x)$.*

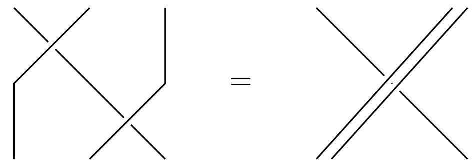
But we now introduce new hotness:

Rather than demanding that the symmetric group act on tensor products, let's only demand the braid group. This has overcrossings and undercrossings.



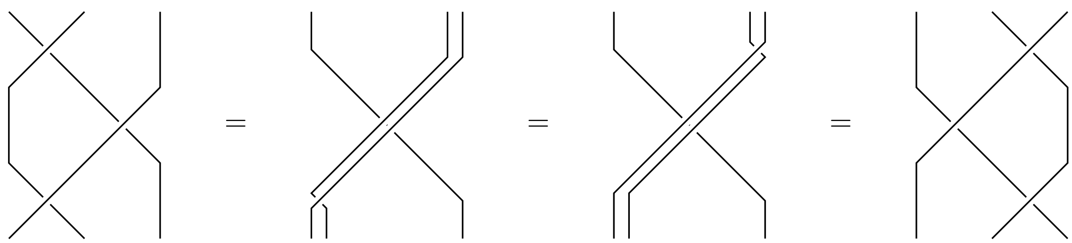
Then we will have infinitely many maps between tensor products, which will generally be different. For example, just on two elements, we have $\mathbb{Z}$ many maps. A result of Artin: the braid group can be presented by two versions, one with all overcrossings and one with all undercrossings, of Reidemeister Three.

But using naturality, there's a better description.



By the way, you read diagrams bottom to top, because then when reading diagrams top to bottom, you get the words left to right.

Ah, but naturality says exactly that you can pull things across the crossing. So for example we can prove the Braid relation:



where the first and third equalities are the relation above, and the second is naturality.

Clearly this can be made into an algebraic proof, but that would take a lot longer to write.

So, a *braiding* is a natural transformation $\sigma_{V,W} : V \otimes W \mapsto W \otimes V$ such that

$$\sigma_{A\otimes B,C} = (\sigma_{A,B} \otimes 1)(1 \otimes \sigma_{B,C}) \tag{23.2}$$

and $\sigma_{A,B\otimes C} =$ **Exercise 38**.

Let's translate this into a condition on R. The LHS of equation 23.2 mean $\mathrm{swap}_{12,3}(\Delta \otimes 1)(R)$ and the RHS means $\mathrm{swap}_{1,2}(R \otimes 1)\mathrm{swap}_{2,3}(1 \otimes R)$. This is a bit awkward because of the swap in the middle. Let's move the middle swap past $R \otimes 1$, which leaves the first part of R fixed and moves the other part past. So we have $R = \sum R_{(1)} \otimes R_{(2)}$, and we define $R_{13} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}$. Thus, the final condition, after canceling some swaps, is:

$$(\Delta \otimes 1)(R) = R_{13}R_{23} \tag{23.3}$$

There's also another condition; you can run the argument backwards to show that if R satisfies both, then it gives a braiding.

Exercise 39 *Work out the braid relation in terms of R. You will get the* Yang Baxter Relation*.*

23.2 The quantum double

In a braiding, it's easy to satisfy the desire to make $\sigma : V \otimes W \to W \otimes V$ a module map, and hard to make it satisfy the relation. But in the Double, the second part is easy, and the first hard.

You start with a Hopf algebra H, and we want to build a Hopf algebra of the form $H \otimes H^*$. I.e. we know the products on each component, and the coproduct is defined by its components, and the only thing to define is how to write x^*y as a sum of things of the form ab^*, where $a, y \in H$ and $x^*, b^* \in H^*$. Then the relation is obvious ****?****

Then we have a miracle. We can get $V \otimes W \to W \otimes V$ and that $H \otimes H^*$ is a Hopf algebra in exactly one way. It's a long calculation.

Lecture 24 March 13, 2009

****I was late, but that's OK: VS has not shown up.****

24.1 Matt describes a particular quantum group.

We pick $q \in \mathbb{C} \smallsetminus \{0\}$, such that $q^2 \neq 1$.

We define the *quantum plane* $\mathbb{C}_q^2$ to be the algebra $\langle x, y \text{ s.t. } xy = qyx \rangle$. Then normal classical matrix multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

is actually a coaction of algebras: $C(\mathbb{C}^2) \to C(M(2)) \otimes C(\mathbb{C}^2)$. What should act on the quantum plane? "Quantum matrices". So we decorate the equation with qs — $C(\mathbb{C}_q^2) \to C(M_q(2)) \otimes C(\mathbb{C}_q^2)$ — and now we want to know the relations on $C(M_q(2))$ to make everything consistent. Well, if $x' = a \otimes x + b \otimes y$ and $y' = c \otimes x + d \otimes y$, then we want $x'y' = qy'x'$. This requires that the coefficients after expanding $x'y' = qy'x'$ into xs and ys (and q-muting the ys past the xs) match:

$$ac = qca \tag{24.1}$$

$$bd = qdb \tag{24.2}$$

$$qad + bc = q(qcb + da) \tag{24.3}$$

By looking at the right action, we get more relationships. **Question from the audience:** The right action is on the dual space $C(\mathbb{C}_q^{2*})$. Is it obvious that this should be the same algebra, when you go crazy with the qs? Are you imposing the Euclidean bilinear form on $\mathbb{C}_q^2$? **Answer:** Good question. We will not answer it. We have:

$$ab = qba \tag{24.4}$$

$$cd = qdc \tag{24.5}$$

$$qad + cb = q(qbc + da) \tag{24.6}$$

We can rearrange equation 24.3 to $q(ad - da) = q^2cb - bc$, and equation 24.6 becomes $q(ad - da) = q^2bc - cb$. Subtracting and dividing gives:

$$bc = cb \tag{24.7}$$

$$ad - da = (q - q^{-1})bc \tag{24.8}$$

This approach is in Klimyk and Schmudgen, *Quantum Groups and their Representations*. The whole approach is due to Yuri Manin.

All of this is thinking of $C(M_q(2))$ as a bialgebra; there's nothing invertible yet.

Anyway, the algebra $C(M_q(2))$ we define as being the algebra generated by a, b, c, d such that equations 24.1, 24.2, 24.4, 24.5, 24.7, and 24.8 hold. The nemonic: adjacent generators in the matrix q-mute with the alphabetical order. Antidiagonal commute, and diagonal have the weird one.

One can check, moreover, that the coalgebra structure from $C(M(2))$ given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{24.9}$$

is an algebra homomorphism for $C(M_q(2))$ as well. The counit $a, d \mapsto 1$, $c, b \mapsto 0$ also works.

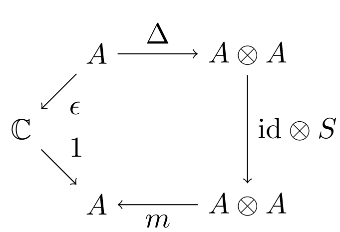
We want inverses. Thinking hard about exterior powers, one can decide that the correct "quantum determinant" is $D_q = ad - qbc$. Some facts:

- D_q is central.
- D_q is *grouplike*: $\Delta(D_q) = D_q \otimes D_q$.

These imply that the ideal I generated by $D_q - 1$ is a bialgebra ideal. We define $C(SL_q(2)) = C(M_q(2))/I$. Then on this we can define an antipode:

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -qc & a \end{pmatrix} \tag{24.10}$$

One now has to check the antipode axiom:



(24.11)

which is an exercise in multiplying matrices.

This Hopf algebra deserves to be called $C(SL_q(2))$.

****Then we left ten minutes early.****

Lecture 25 March 30, 2009

Before the break, Noah said the words "quasi-triangular Hopf algebra", and Matt introduced "quantum $SL(2)$". VS was teaching at the same time, and so was unavailable.

By the way, on NR's website, there is a reference to TQFT lectures from last year. On that website, there are notes on tensor categories, so there is some overlap. Everything related to the fact that the category of modules over a quasitriangular Hopf algebra is braided monoidal — this part is explained well there.

Let's recall NR's discussion from two weeks ago. If H is a Hopf algebra, we define its *double* to be the coalgebra $\mathcal{D}(H) \stackrel{\text{def}}{=} H \otimes H^\circ$, where H° is H^* with the opposite comultiplication. If H is infinite-dimensional, we must supply H with a dual pairing $H^*, \langle , \rangle$. The multiplication on $\mathcal{D}(H)$ is given by:

$$(a \otimes l)(b \otimes m) = \sum_{b,l} ab_{(2)} \otimes l_{(2)} m \langle b_{(1)}, S_{H^*}^{-1}(l_{(1)}) \rangle \langle b_{(3)}, l_{(3)} \rangle \tag{25.1}$$

The notation is that $a, b \in H$, $l, m \in H^\circ$, and $S_{H^*}^{-1} = S_{H^\circ}$. We also use the notation that $\delta_H^{(3)}(b) = \sum_b b_{(1)} \otimes b_{(2)} \otimes b_{(3)}$, and $\delta_{H^\circ}^{(3)}(l) = \sum_l l_{(1)} \otimes l_{(2)} \otimes l_{(3)}$. We also write $\mathcal{D}(H) = H \bowtie H^\circ$; it is a Hopf algebra.

Let A be a Hopf algebra and $R \in A^{\otimes 2}$, possibly completed to $A^{\hat{\otimes} 2}$ in some way. For example, if $A = C(M)$ for a space M, then perhaps we want to complete $A \otimes A$ to $C(M \times M)$. We will also have an example from formal power series. Anyway, the pair $(A, R \in A^{\hat{\otimes} 2})$ is a *quasitriangular Hopf algebra* if

$$(\Delta \otimes \mathrm{id})(R) = R_{13} R_{23} \tag{25.2}$$

$$(\mathrm{id} \otimes \Delta)(R) = R_{13} R_{12} \tag{25.3}$$

whence R is invertible, and we also demand that

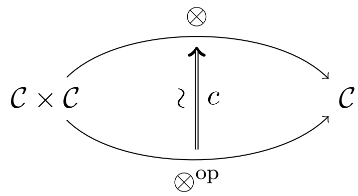
$$\Delta^{\mathrm{op}}(a) = R\, \Delta(a)\, R^{-1} \tag{25.4}$$

This definition is totally natural, given the following:

Proposition 25.1 *If (A, R) is a quasitriangular Hopf algebra, then the category $A -$ mod of finite-dimensional A-modules is a* tensor category. *I.e. it is rigid (dual objects), monoidal ($\otimes$), Abelian (a category of modules), and braided (this follows from the quasitriangular structure).*

Let us define these words better. A category $\mathcal{C}$ is *monoidal* if it is endowed with a (covariant) functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Recall, a morphism $(A, B) \to (C, D) \in \mathcal{C} \times \mathcal{C}$ is a pair $(f : A \to C, g : B \to D)$. If $(\mathcal{C}, \otimes)$ is monoidal, then we can also construct $\otimes^{\mathrm{op}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ by $(A, B) \mapsto B \otimes A$. ****Really we are using the canonical flip map $\mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$.****

So we have two functors $\otimes, \otimes^{\mathrm{op}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. When we have two functors, we can ask whether they are the same — well, not the same, because "the same" does not exist in categories, but isomorphic.



That c is an isomorphism means that for every pair of objects, there is an isomorphism $c_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$, and this system commutes with all morphisms.

So, the data $\mathcal{C}, \otimes, \mathbb{1}, a, c$, where a is the associativity (included in the word "monoidal") and c is an isomorphism $\otimes \xrightarrow{\sim} \otimes^{\mathrm{op}}$, is *braided monoidal* if c is compatible with the associativity constrain. This condition consists of two hexagon diagrams:

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{c_{A\otimes B,C}} & C \otimes (A \otimes B) & \xleftarrow{a_{C,A,B}} & (C \otimes A) \otimes B \\ \downarrow a_{A,B,C} & & & & \uparrow c_{A,C} \otimes \mathrm{id} \\ A \otimes (B \otimes C) & \xrightarrow{\mathrm{id} \otimes c_{B,C}} & A \otimes (C \otimes B) & \xrightarrow{a^{-1}_{A,C,B}} & (A \otimes C) \otimes B \end{array} \tag{25.5}$$

The other hexagon is the same, with $c_{A,B}$ replaced by $c^{-1}_{B,A}$. **Question from the audience:** So those are not always the same? **Answer:** No. For G-mod when G is a group, they are, but we will have examples where it is not, and Noah gave one: the category of tangles.

So, for example, $\mathcal{C} = A$-mod, a is trivial (well, as trivial as for Vect; so it is not completely trivial, but for any ordered collection of objects, there is a canonical tensor of the whole collection, and any bracketing is naturally isomorphic to this object), the tensor is given by

$$(\pi_V, V) \otimes (\pi_W, W) = ((\pi_V \otimes \pi_W) \circ \Delta, V \otimes W), \tag{25.6}$$

the unit $\mathbb{1}$ is the one-dimensional representation given by $\epsilon : A \to \mathbb{C}$, and the braiding is:

$$c_{V,W} = P_{V,W}(\pi_V \otimes \pi_W)(R) \tag{25.7}$$

where R is the quasitriangular structure and P is the permutation map of vector spaces.

One can check that this is an isomorphism of modules; it is equivalent to equation 25.4.

Exercise 40 *The hexagons (equation 25.5) are equivalent to equations 25.2 and 25.3.*

This continues the correspondence:

If A is ...	then A-mod is ...
an algebra	Abelian
a bialgebra	monoidal
Hopf	rigid
quasitriangular	braided

One of the main applications of all of this is the construction of topological invariants. And we certainly want a braiding, so that we can study knots.

Well, it's very nice to have a definition, but we should also have an example.

Theorem 25.2 *The double $\mathcal{D}(H)$ of any Hopf algebra with the canonical element $R = \sum_i (e_i \otimes 1) \otimes (1 \otimes e^i)$ is quasitriangular. We have picked any basis e_i of H, and its dual basis e^i of H^* defined via the pairing $\langle , \rangle : H \otimes H^* \to \mathbb{C}$. But if H is infinite-dimensional, then we have written down only a formal sum; the topology needed to define the correct completion $\mathcal{D}(H)^{\hat{\otimes} 2}$ is a sensitive question.*

We now make a logical jump, and study the double of quantized Borel in $\mathfrak{sl}_2$. Let us define this. $\mathcal{U}_h\mathfrak{b}_+$ is the algebra generated by H, E and completed over $\mathbb{C}[[h]]$ with the defining relations $[H, E] = 2E$. In other words, $\mathcal{U}_h\mathfrak{b}_+ \cong \mathcal{U}\mathfrak{b}_+[[h]]$.

Theorem 25.3 *1. The map $\Delta : \mathcal{U}_h\mathfrak{b}_+ \to \mathcal{U}_h\mathfrak{b}_+^{\otimes 2}$ defined on generators as*

$$\Delta(H) = H \otimes 1 + 1 \otimes H \tag{25.8}$$

$$\Delta(E) = E \otimes e^{hH/2} + 1 \otimes E \tag{25.9}$$

is a comultiplication for $\mathcal{U}_h\mathfrak{b}_+$.

2. $\epsilon(1) = 0$, $\epsilon(H) = \epsilon(E) = 0$ extends to a counit $\epsilon : \mathcal{U}_h : \mathfrak{b}_+ \to \mathbb{C}[[h]]$.

So $\mathcal{U}_h\mathfrak{b}_+$ is a bialgebra over the commutative ring $\mathbb{C}[[h]]$. In particular, we demand that both maps be continuous.

Exercise 41 *Find S.*

Let us know describe a dual to $\mathcal{U}_h\mathfrak{b}_+$. We define the symbol $\mathcal{U}_h\mathfrak{b}_+{}^\circ$ to be generated over $\mathbb{C}[[h]]$ by $H^\vee$ and F with the defining relation $[H^\vee, F] = -\frac{h}{2}F$. The bialgebra part is:

$$\Delta H^\vee = H^\vee \otimes 1 + 1 \otimes H^\vee \tag{25.10}$$

$$\Delta F = F \otimes 1 + e^{-2H^\vee} \otimes F \tag{25.11}$$

Well, the second term in equation 25.11 is a problem; it is an infinite series. So we need to define a topology in which $e^{H^\vee}$ can be written. We do this via a filtration on $\mathcal{U}_h\mathfrak{b}_+{}^\circ$ given by $\deg(H^\vee) \leq 1$ and $\deg(F) \leq 1$, and $\deg(ab) \leq \deg(a)\deg(b)$. If this were commutative, then we this would be the grading of the polynomial algebra, but it is noncommutative. So we define a topology in terms of the filtration, and complete to the "formal power series" ring.

Proposition 25.4 *There is a unique Hopf pairing $\langle , \rangle : \mathcal{U}_h\mathfrak{b}_+ \otimes \mathcal{U}_h\mathfrak{b}_+{}^\circ \to \mathbb{C}[[h]]$ such that $\langle H, H^\vee \rangle = 1 = \langle E, F \rangle$ and $\langle H, F \rangle = 0 = \langle E, H^\vee \rangle$. We comment that if these were commutative rings, then this would be the pairing of polynomial functions on V with polynomial functions* ****formal power series?**** *on the dual vector space.*

Next time we will construct the double, build quantized universal enveloping algebras, and connect it up with what we learned about Lie bialgebras and Poisson Lie groups.

Lecture 26 April 1, 2009

26.1 Schur-Weyl duality

Do you know about Schur-Weyl duality? No? Well, we will certainly discuss it, but we should deviate from the story of quantum groups to describe it.

Let us consider the action of GL_n on $\mathbb{C}^n$. Then GL_n acts on $(\mathbb{C}^n)^{\otimes N}$, and the symmetric group S_N also acts on $(\mathbb{C}^n)^{\otimes N}$, by permutations.

Theorem 26.1 (Schur-Weyl duality) *These actions centralize each other.*

I.e. everything that commutes with the entire GL_n action is in S_n, and vice versa.

Corollary 26.1.1 $(\mathbb{C}^n)^{\otimes N}$ *decomposes as* $\bigoplus_{\lambda \dashv N} V_\lambda^{GL_n} \otimes W_\lambda^{S_N}$, *where* λ *is a partition of* N*:* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ *with* $\lambda_i \geq 0$ *and* $\lambda_1 + \dots + \lambda_k = N$. *The sum is over partitions with* $k \leq n$.

We hope that everyone knows the representation theory of S_N, but of course it is never the case. MH described the representation of simple Lie groups.

More generally, consider the action $\Gamma \curvearrowright W$ when it is multiplicity-free. Then there are ****?**** dualities, $GL_n \times GL_m$ on the polynomials on an $n \times m$ array of variables.

Theorem 26.2 *This action is multiplicity free.*

There are names that are completely misleading. For example, "quantum groups", which are vaguely quantum but definitely not groups. Similarly, "Lie supergroups" are not groups but Hopf algebras in the category of Lie supervector spaces. Since you do not know Lie superalgebra, we will not go in this direction right now.

There is an object called the *Hecke algebra*, given by:

$$H_n(q) \overset{\text{def}}{=} \langle s_i,\, i = 1, \dots, N-1 \text{ s.t. } (s_i - q)(s_i + 1) = 0,\, s_i s_{i\pm 1} s_i = s_{i\pm 1} s_i s_{i\pm 1},\, s_i s_j = s_j s_i, |i-j| > 1 \rangle \tag{26.1}$$

It is a deformation of the symmetric group. Then there is a natural action of $H_N(q)$ on $(\mathbb{C}^n)^{\otimes N}$ and also of $\mathcal{U}_q(\mathfrak{gl}_n)$ on $(\mathbb{C}^n)^{\otimes n}$, and then we have:

Theorem 26.3 (q-Schur-Weyl duality) $H_n(q)$ *centralizes* $\mathcal{U}_q(\mathfrak{gl}_n)$ *in* $(\mathbb{C}^n)^{\otimes N}$, *and vice versa.*

More precisely, if $\pi : A \curvearrowright W$, then we define the *cetralizer* of A in W to be ****NR uses "C" but I'd prefer "Z"**** $C(A; W) = \{a : W \to W \text{ s.t. } [a, \pi(x)] = 0\, \forall x \in A\}$.

26.2 Deformations of Hopf algebras

Last time we discussed $\mathcal{U}_h\mathfrak{b}_+$ and its dual $\mathcal{U}_h\mathfrak{b}_+{}^\circ$ We briefly mentioned the words "formal deformation" last time, but now we make some definitions.

Let $(A, m, 1)$ be an associative unital algebra over $\mathbb{C}$ (or anything else); we list the algebra as a triple: a vector space, an associative multiplication, and a unit. Then $B = A[[h]]$ is a *formal (torsion free) deformation* of A if $(B, \tilde{m}, \tilde{1})$ is an associative unital algebra such that:

$$\tilde{m}(a,b) = m(a,b) + \sum_{n=1}^{\infty} h^n m^{(n)}(a,b) \tag{26.2}$$

where $m^{(n)} : A \otimes A \to A$ are extended to $B \otimes B \to B$ by h-linearity ****and continuity****; and:

$$\tilde{1} = 1 + \sum_{n=1}^{\infty} h^n 1^{(n)} \tag{26.3}$$

where 1 is the unit in A and $1^{(n)} \in A$.

Now let $(B, \tilde{m}, \tilde{1})$ and $(B, \hat{m}, \hat{1})$ be two formal deformations of $(A, m, 1)$. Then we say that these are *equivalent* and write $(B, \tilde{m}, \tilde{1}) \sim (B, \hat{m}, \hat{1})$ if there is a $\phi : B \to B$ that is $\mathbb{C}[[h]]$-linear with $\phi = \mathrm{id} + \sum_{n=1}^{\infty} h^n \phi^{(n)}$ (whence ϕ is invertible) such that $\phi \circ \tilde{m} = \hat{m} \circ (\phi \otimes \phi)$ and $\phi(\tilde{1}) = \hat{1}$.

Then there is a natural question: Given A, describe the equivalence classes of formal deformations of A. There is always, of course, the trivial deformation; there may not be any others.

So these are formal deformations, but what we really want is a family of algebras. We say that a family of algebras $\{A_h\}$ where h ranges over $h \in X$, where X is some set with a limit point 0, is a *deformation* of A if:

- We have linear isomorphisms $\phi_h : A_h \xrightarrow{\sim} A$. This can be slightly problematic: if A is an infinite-dimensional vector space, then these isomorphisms should also be continuous, in some natural topology. One way to do this is to choose bases and identify them.
- $\lim_{h \to 0} \phi_h \circ m_h \circ (\phi_h^{-1} \otimes \phi_h^{-1}) = m$. For each h, the inside is an associative multiplication $A \otimes A \to A$.
- $\lim_{h \to 0} \phi_1(1_h) = 1$.

The isomorphisms ϕ_h are part of the data; the other two conditions are properties. The meaning of $\lim_{h \to 0}$ is a bit sloppy.

But all these questions have a trivial answer. The only way we know how to construct algebras — well, there are algebras of functions, but other than that — is to take quotients of free algebras. This is how we will proceed. We will take a free algebra $F(x_1, \dots, x_n)$, and we will construct a family of ideals $I_h \subseteq F$, such that $A_0 = A$. Then we will try to define $A_h = F/I_h$, and the biggest problem is to construct the linear isomorphisms $A_h \xrightarrow{\sim} A_0$. The usual way we will do this is to construct algebras of PBW-type. For example, we may know that $A_h \cong S(x_1, \dots, x_n)$ the symmetric algebra.

This is really what we want: to have families of algebras. Not just formal deformations, which are like the formal Taylor expansions of the formal algebras. But formal deformations are easier to describe; this is the difference between micro-local analysis and global analysis.

Question from the audience: Why do we really want families of algebras **Answer:** It is a general philosophy of life. If you want to study some structures, look for stable structures that come in families. When you try to classify stable structures, you hope that there is a discrete collection of them, e.g. for simple Lie algebras, even though simplicity is an open condition. You can stratify branches of mathematics by the strength of the equivalence relations. In many branches, the goal is to classify something. If you try to classify all vector spaces, this is tedious. But up to isomorphism? Then they are classified by dimension. Similarly, if you are classifying a structure, and you are sitting at a singular point, it can be quite hard, but classifying families is easy.

Here's another reason. Well, it's another reason for someone indoctrinated by physics. The world, it turns out, is not commutative. It is at least quantum-mechanical. What we see — the commutativity of classical mechanics — is a limit of quantum mechanics as the parameter goes to 0. So the problem of quantization is that it's the wrong direction. The natural direction is the other one: to specialize to $h \to 0$. Then we get various degeneracies.

Having said this, let's forget about the real world and return to formal deformations of associative algebras.

Suppose that A is commutative, and that B is a formal deformation of A. Then the multiplication is given by:

$$\tilde{m} = m + hm^{(1)} + O(h^2) \tag{26.4}$$

We are following the previous outline, that the natural direction is not quantization by degeneration. Anyway, we define $\{a, b\} \overset{\text{def}}{=} m^{(1)}(a, b) - m^{(1)}(b, a)$.

Theorem 26.4 *$(A, m, \{,\})$ is a Poisson algebra.*

****NR: "We have a small theorem..." Theo: "I don't have a symbol for 'small theorem'." NR: "The definition of 'small theorem' is that the proof is a homework." Matt: "There are many small theorems in this class." NR: "Well, let's make this into a big theorem."****

Proof: $\{a, b\} = \frac{1}{h}[a, b] \bmod h$, and $A = B \bmod h$. So the Jacobi and Leibniz for $\{,\}$ follow from those for $[,]$. □

So, if at a special point a family of associative algebras becomes commutative, then it induces a Poisson structure. **Question from the audience:** We don't have a family. **Answer:** No, we have a formal family.

Exercise 42 *Let B be a formal deformation of A; A may not be commutative. Prove:*

1. *It induces a Poisson structure on $Z(A)$, the center of A.*
2. *This Poisson structure extends to $\{,\} : Z(A) \otimes A \to A$, which is the action of the Poisson algebra $Z(A)$ by derivations on A. In other words, $\{z, ab\} = \{z, a\}b + a\{z, b\}$.*

The theorem is a specialization of the homework, when the whole algebra is commutative.

B is a *formal deformation quantization* of the Poisson algebra $(A, m, \{,\})$ if the Poisson bracket on A agrees with the one given by the deformation. The natural question: What is the set of equivalence classes of (formal) deformation quantizations of a given Poisson algebra?

It is a complicated question. There is a complete answer in the case of algebras on a Poisson manifold. Consider $A = C^\infty(M)$ and

- $\tilde{m} = m + \sum_{n=1}^\infty h^n m^{(n)}$ is special in the sense that it is symmetric: $m^{(n)}(a, b) = (-1)^n m^{(n)}(b, a)$.
- $m^{(n)}$ are bidifferential operators on M of degree (n, n).

Theorem 26.5 (Kontesevich) *Equivalence classes of such formal deformations quantizations are in bijection with formal deformations of the Poisson structure, i.e.* $\tilde{p} = p + \sum_{n=1}^\infty h^n p^{(n)}$ *where* $p^{(n)}$ *are bivector fields and* $\tilde{p}$ *satisfies Jacobi.*

Next time we give a few more definitions, and then we will proceed with examples.

Lecture 27 April 3, 2009

Last time we define deformations of Hopf algebras. We make a quick summary: if $(A, m, \Delta, \epsilon, 1, S)$ is a Hopf algebra, then $(B, \tilde{m}, \dots)$ is a *formal deformation* of A if $B = A[[h]]$ as a vector space, and $\tilde{m} = m + \sum_{n=1}^\infty h^n m^{(n)}$, etc. The elements of the formal power series are, for example, $m^{(n)} : A \otimes A \to A$ extended to $B \otimes B \to B$ by h-linearity ****and continuity****.

Example 27.1 If A is a commutative Hopf algebra and a Poisson Hopf algebra. In other words, there is a Poisson bracket $\{,\} : A \otimes A \to A$ such that $\Delta : A \to A \otimes A$ is a morphism of Poisson algebras. Then we say that B is a *formal deformation* of A if in addition to the above we have: $m^{(1)} - m^{(1)\mathrm{op}} = \{,\}$. From the previous lecture, we saw that $m^{(1)} - m^{(1)\mathrm{op}} : A \otimes A \to A$ is always a Poisson map; now we are demanding that it be the given Poisson structure on A.

For example, let $A = C(G)$ where G is a Poisson Lie group, then any formal deformation B of $C(G)$ we will call $C_h(G)$, and call it the "quantum group" of G.

Question from the audience: There are some choices involved in constructing $C_h(G)$, so this is an imprecise notation. **Answer:** Yes, but it is the usual notation in the literature. $\Diamond$

Example 27.2 Consider now the case when A is cocommutative, i.e. $\Delta^{\mathrm{op}} = \Delta$. For example, $A = \mathcal{U}\mathfrak{g}$. Well, if $\mathfrak{g}$ is a Lie bialgebra, the Lie coalgebra lifts to a 1-cocycle on $\mathcal{U}\mathfrak{g}$. Let's describe this. We have a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, which is a 1-cocycle for the Chevalley complex for $\mathfrak{g}$ with coefficients $M = \mathfrak{g} \wedge \mathfrak{g}$. In other words, $\delta([x, y]) = [x, \delta(y)] + [\delta(x), y]$, in other words $d_{\mathrm{Ch}}\delta = 0$.

Let us recall the *Hoschild Complex* of an algebra. Let A be an associative algebra, M a module over A. We define $C_{\mathrm{Hoschild}}(A, M)$ as follows. (By the way, G. Hoschild is emeritus here, but is never around: he is very shy.) The nth part is

$$C^{(n)}(A, M) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, M) \tag{27.1}$$

and of course $C(A,M) = \bigoplus_{n\geq 0} C^{(n)}(A,M)$. The differential is: if $\alpha_n \in C^{(n)}(A,M)$, then

$$d\alpha_n(a_1,\dots,a_{n+1}) \stackrel{\text{def}}{=} \alpha_n(a_1a_2,a_3,\dots,a_{n+1}) - \alpha_n(a_1,a_2a_3,\dots,a_{n+1}) + \dots \\ \pm \left(a_1\alpha_n(a_2,a_3,\dots,a_{n+1}) - a_2\alpha_n(a_1,a_3,\dots,a_{n+1}) + \dots\right) \tag{27.2}$$

The principle is that this is exactly the what you get if you try to lift the Chevalley Complex to $\mathcal{U}\mathfrak{g}$.

So for example

$$d\alpha_1(a,b) = \alpha_1(ab) - a\alpha_1(b) + b\alpha_1(a) \tag{27.3}$$

And α_1 is a one-cocycle if $d\alpha_1 = 0$, i.e.:

$$\alpha_1(ab) = a\alpha_1(b) - b\alpha_1(a) \tag{27.4}$$

Ok, so let's let $A = \mathcal{U}\mathfrak{g}$, and by the PBW theorem we can identify this as a vector space with $S(\mathfrak{g})$. Let $a,b \in \mathfrak{g} \hookrightarrow \mathcal{U}\mathfrak{g}$. Let's let $M = A$, and $\alpha_1 : A \to A$. So we can think of $\alpha_1 : S(\mathfrak{g}) \to S(\mathfrak{g})$. And these are graded vector spaces — the multiplication respects only the filtration.

Let's change our mind now about the set-up. In particular, let's demand that M be a bimodule over A, and at least understand the one-cocycle condition to be:

$$\alpha_1(ab) = a\alpha_1(b) + \alpha_1(a)b \tag{27.5}$$

Then if α_1 is a cocycle $A \to A$, then the conditions are:

$$\alpha_1(ab+ba) = \alpha_1(ab) + \alpha_1(ba) = a\alpha_1(b) + \alpha_1(a)b + b\alpha_1(a) + \dots \tag{27.6}$$

$$\alpha_1(ab-ba) = \dots \tag{27.7}$$

Actually, this is not the story we meant to tell. It's a very nice story, but we want something else.

Let us consider a map $\alpha_1 : A \to A \otimes A$, such that A is a bialgebra, and let's ask what happens when

$$\alpha_1(ab) = \Delta a\, \alpha_1(b) + \alpha_1(a)\, \Delta b \tag{27.8}$$

Then we see that

$$\alpha_1(ab-ba) = [\Delta a, \alpha_1(b)] + [\alpha_1(a), \Delta b] \tag{27.9}$$

so α_1 restricts to a one-cocycle on $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$.

Conversely, any such one-cocycle δ, which is really just a Lie bialgebra structure, lifts to a map $\alpha_1 : \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$ by $\alpha_1(ab) = \Delta a\alpha_1(b) + \alpha_1(a)\Delta b$.

If A is a cocommutative Hopf algebra with $\alpha : A \to A \otimes A$ such that:

$$\alpha^{\text{op}} = -\alpha \tag{27.10}$$

$$\alpha(ab) = \Delta a\, \alpha(b) + \alpha(a)\, \Delta(b) \tag{27.11}$$

and also the coJacobi identity, then we say that A is a *co-Poisson Hopf algebra*. **Question from the audience:** Shouldn't there also be some compatibility between α and the multiplication? **Answer: **? Some sort of "co-Leibniz identity", and also that α is a morphism of the correct structures****

Proposition 27.1 *If A and B are a dual pair (via $\langle , \rangle$) of Hopf algebras,and $\{,\} : B \wedge B \to B$ is a Poisson structure on B, then we define $\alpha : A \to A \otimes A$ by:*

$$\langle \alpha(a), l \otimes m \rangle = \langle a, \{l, m\} \rangle \tag{27.12}$$

Then this is a co-Poisson Hopf structure.

Proof: Antisymmetry is obvious.

$$\begin{aligned}
\langle \alpha(ab), l \otimes m \rangle &= \langle ab, \{l, m\} \rangle && (27.13)\\
&= \langle a \otimes b, \Delta\{l, m\} \rangle && (27.14)\\
&= \langle a \otimes b, \{\Delta l, \Delta m\} \rangle && (27.15)\\
&= \langle a \otimes b, \sum_{l,m} \{l^{(1)}, m^{(1)}\rangle \otimes l^{(2)} m^{(2)} + \sum_{l,m} l^{(1)} m^{(1)} \otimes \{l^{(2)}, m^{(2)}\} \rangle && (27.16)\\
&= \sum_{l,m} \left(\langle a, \{l^{(1)}, m^{(1)}\} \rangle \langle b, l^{(2)} m^{(2)} \rangle + \langle a, l^{(1)} m^{(1)} \rangle \langle b, \{l^{(2)}, m^{(2)}\} \rangle \right) && (27.17)\\
&= \sum_{l,m} \left(\langle \alpha(a), l^{(1)} \otimes m^{(1)} \rangle \langle \Delta b, l^{(2)} \otimes m^{(2)} \rangle + \langle \Delta(a), l^{(1)} \otimes m^{(1)} \rangle \langle \alpha b, l^{(2)} \otimes m^{(2)} \rangle \right) && (27.18)\\
&= \sum_{l,m} \left(\langle \alpha_{13}(a) \Delta_{24}(b), l^{(1)} \otimes l^{(2)} \otimes m^{(1)} \otimes m^{(2)} \rangle + \langle \Delta_{13}(a) \alpha_{24}(b), l^{(1)} \otimes l^{(2)} \otimes m^{(1)} \otimes m^{(2)} \rangle \right) && (27.19)\\
&= \langle \alpha_{13}(a) \Delta_{24}(b) + \Delta_{13}(a) \alpha_{24}(b), \Delta(l) \otimes \Delta(m) \rangle && (27.20)\\
&= \langle m_{12} m_{34} (\alpha_{13}(a) \Delta_{24}(b)) + m_{12} m_{34} (\Delta_{13}(a) \alpha_{24}(b)), l \otimes m \rangle && (27.21)\\
&= \langle \alpha(a) \Delta(b) + \Delta(a) \alpha(b), l \otimes m \rangle && (27.22)
\end{aligned}$$

and so we have the cocycle condition. □

Ok, let's formulate the theorem:

Theorem 27.2 *Let $(A, B, \langle , \rangle)$ be a dual Hopf pairing, and let B be a Poisson Hopf algebra. Define $\alpha : A \to A \otimes A$ by $\langle \alpha(a), l \otimes m \rangle \stackrel{\text{def}}{=} \langle a, \{l, m\} \rangle$. Then α is skew-symmetric, a cocycle, and satisfies co-Jacobi. Such an algebra is a* co-Poisson Hopf algebra*.*

Theorem 27.3 *If $\mathfrak{g}$ is a Lie bialgebra, with $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g} \hookrightarrow \mathfrak{g} \otimes \mathfrak{g}$ the one-cocycle, then δ defines a a unique co-Poisson structure $\alpha : \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$ such that $\alpha|_{\mathfrak{g}} = \delta$.*

Suppose that A is co-Poisson with α the structure. Then B is its *formal deformation quantization* if $\tilde{\Delta} = \Delta + h\alpha + O(h^2)$, and other conditions.

So we have two notions: the deformation of a Poisson Hopf algebra, and of a co-Poisson Hopf algebra.

Example 27.3 If $\mathfrak{g}$ is a Lie bialgerba, $\mathcal{U}\mathfrak{g}$ is co-Poisson, and $\mathcal{U}_h\mathfrak{g}$ is a formal deformation of $\mathcal{U}\mathfrak{g}$ as a co-Poisson Hopf algebra is a *quantized universal enveloping algebra* of $\mathfrak{g}$. ◊

At the classical level we have dual pairs of Poisson and co-Poisson Hopf algebras, and these deform into dual pairs.

Next time, we will study examples: $\mathcal{U}_h\mathfrak{b}_+$ and $\mathcal{U}_h\mathfrak{b}_+{}^\circ = C_h(B_-)$.

Lecture 28 April 6, 2009

A general question is whether, given $\mathcal{U}\mathfrak{g}$, does there exist a formal deformation $\mathcal{U}_h\mathfrak{g}$ such that the multiplication and comultiplication deform:

$$\Delta_h a = \Delta a + h\Delta^{(1)}a + \dots \tag{28.1}$$
$$\text{s.t. } \Delta^{(1)} - \Delta^{(1)^{\text{op}}} = \tilde{\delta} \tag{28.2}$$

where $\tilde{\delta}$ is a one-cocycle for $\mathcal{U}\mathfrak{g}$.

Theorem 28.1 (Etingof, Kazhdan) *For any $\mathfrak{g}$ a finite-dimensional Lie bialgebra (or Kac-Moody, etc.), there exists such a $\mathcal{U}_h\mathfrak{g}$.*

Just as $\mathcal{U}\mathfrak{g}$ is dual as a Hopf algebra to $C(G)$, $\mathcal{U}_h\mathfrak{g}$ will be dual to $C_h(G)$ in the appropriate topologies.

For each simple Lie algebra $\mathfrak{g}$ with the standard Lie bialgebra structure, we can give an explicit description of $\mathcal{U}_h\mathfrak{g}$ in terms of generators and relations. We will describe this construction.

Let us begin with $\mathfrak{g} = \mathfrak{sl}_2$. Then we defined $\mathcal{U}_h\mathfrak{b}_+ = \langle H, E \text{ s.t. } [H,E] = 2E\rangle$ as an algebra over $\mathbb{C}[[h]]$, with the coalgebra structure

$$\Delta H = H \otimes 1 + 1 \otimes H \tag{28.3}$$
$$\Delta E = E \otimes e^{hH/2} + 1 \otimes E \tag{28.4}$$

Exercise 43 *Check that this defines a Hopf algebra.*

Let us check also that this in fact deforms $\mathcal{U}\mathfrak{b}_+$. We have presented the algebra in terms of generators and relations, and it's clear that in terms of the basis $\{H^n E^m\}_{n,m\geq 0}$, we have:

$$\mathcal{U}_h\mathfrak{b}_+ = \mathcal{U}\mathfrak{b}_+[[h]] \tag{28.5}$$

as a linear space, and in fact as an algebra. Let's compute the comultiplication in terms of the basis:

$$\Delta_h(H^n E^m) = (\Delta_h H)^n (\Delta_h E)^m \tag{28.6}$$
$$= (\Delta H)^n (\Delta E + h\Delta^{(1)} E + \dots)^m \tag{28.7}$$
$$= (\Delta H)^n \Big(\Delta E^m + h\big((\Delta E)^{m-1}\Delta^{(1)} E + (\Delta E)^{m-2}(\Delta^{(1)} E)(\Delta E) + \dots\big) + O(h^2)\Big) \tag{28.8}$$

where $\Delta^{(1)} E = E \otimes H/2$ is the linear term in the expansion of $E \otimes e^{hH/2}$. Thus, we compute:

$$\tilde{\delta} = \Delta^{(1)} - \Delta^{(1)\text{op}} \tag{28.9}$$
$$\Delta^{(1)}(H^n E^m) = \Delta H^n \left(\Delta E^{m-1}\Delta^{(1)} E + \dots + \Delta^{(1)} E \Delta E^{m-1}\right) \tag{28.10}$$
$$\text{Hence } \tilde{\delta}(H^n E^m) = \Delta H^n \left(\Delta E^{m-1}\delta E + \dots + \delta E \Delta E^{m-1}\right) \tag{28.11}$$

which is exactly the standard Lie bialgebra structure on $\mathfrak{b}_+$.

Anyway, recall that $\mathfrak{sl}_2 = \mathcal{D}(\mathfrak{b}_+)/I$, where $\mathcal{D}(\mathfrak{b}_+)$ is the double of the Lie bialgebra. So our strategy to compute $\mathcal{U}_h\mathfrak{sl}_2$ will be to describe it as:

$$\mathcal{U}_h\mathfrak{sl}_2 = \mathcal{D}(\mathcal{U}_h\mathfrak{b}_+)/\tilde{I} = \mathcal{U}_h\mathfrak{b}_+ \bowtie \mathcal{U}_h{\mathfrak{b}_+}^\circ/\tilde{I} \tag{28.12}$$

The full dual opposite is too big. We will define $\mathcal{U}_h{\mathfrak{b}_+}^\circ$ in terms of a dual pair $(\mathcal{U}_h\mathfrak{b}_+, \mathcal{U}_h{\mathfrak{b}_+}^\circ, \langle,\rangle)$, and see that it is a natural definition. We use:

$$\mathcal{U}_h{\mathfrak{b}_+}^\circ \stackrel{\text{def}}{=} \langle H^\vee, F \text{ s.t. } [H^\vee, F] = -\frac{h}{2}F\rangle \tag{28.13}$$

as a complete algebra over $\mathbb{C}[[h]]$, with the coproducts

$$\Delta H^\vee = H^\vee \otimes 1 + 1 \otimes H^\vee \tag{28.14}$$
$$\Delta F = F \otimes 1 + e^{-2H^\vee} \otimes F \tag{28.15}$$

As an algebra $\mathcal{U}_h{\mathfrak{b}_+}^\circ$ is a formal deformation of the commutative algebra $F(B_-)$, the formal functions on B_-, generated by $H^\vee$ and F with $\{H^\vee, F\} = -F$. We work with this to make sense of the exponent in 28.15; the algebra $F(B_-)$ is an algebra of formal power series. I.e. it is $\mathbb{C}[[H^\vee, F]]$ as an algebra, with a non-cocommutative coalgebra structure.

Indeed, we can recognize $F(B_-)$ as the algebra of matrices with coordinates

$$\begin{pmatrix} 1 & 0 \\ F & e^{-2H^\vee} \end{pmatrix} \tag{28.16}$$

Thus $F(B_-)$ is the algebra of functions on the formal neighborhood of the identity in B_-. Well, B_- should be all lower-triangular matrices with unit determinant. Fixing this is straightforward; we can use the coordinates:

$$\begin{pmatrix} e^{H^\vee} & 0 \\ e^{H^\vee}F & e^{-H^\vee} \end{pmatrix} \tag{28.17}$$

Question from the audience: What is a formal neighborhood of the identity? **Answer:** You take all analytic functions near the origin, and complete with the formal-power-series topology. This way, we don't need to worry about convergence issues and we can still do just algebra.

When we say that these coordinates explain the algebra, we mean that the comultiplication of "coordinate functions" is precisely the corresponding multiplication of matrices:

$$\begin{pmatrix} e^{H_1^\vee} & 0 \\ e^{H_1^\vee} F_1 & e^{-H_1^\vee} \end{pmatrix} \begin{pmatrix} e^{H_2^\vee} & 0 \\ e^{H_2^\vee} F_2 & e^{-H_2^\vee} \end{pmatrix} = \begin{pmatrix} e^{H_1^\vee + H_2^\vee} & 0 \\ e^{H_1^\vee + H_2^\vee}(F_1 1 + e^{-2H_1^\vee} F_2) & e^{-H_1^\vee + H_2^\vee} \end{pmatrix} \tag{28.18}$$

Theorem 28.2 *$F(B_-)$ with the Poisson bracket $\{H^\vee, F\} = -F$ is a Hopf Poisson algebra of functions on the formal neighborhood of e in the Poisson Lie group B_-.*

In particular, then, $\mathcal{U}_h\mathfrak{b}_+{}^\circ \stackrel{\text{def}}{=} F_h(B_-)$ is a formal deformation of $F(B_-)$. Incidentally, we can now choose $\mathcal{U}_h\mathfrak{b}_+{}^* = F_h(B_-)^{\text{coop}} = F_h(B_+)$, which deforms $F(B_+)$ which is dual to $\mathcal{U}\mathfrak{b}_+$.

Theorem 28.3 *There exists a unique Hopf pairing $\langle , \rangle : \mathcal{U}_h\mathfrak{b}_+ \otimes \mathcal{U}_h\mathfrak{b}_+{}^\circ \to \mathbb{C}[[h]]$ such that $\langle H, H^\vee \rangle = 1 = \langle E, F \rangle$, $\langle E, H^\vee \rangle = 0 = \langle H, F \rangle$, and such that:*

$$\langle H^n E^m, (H^\vee)^{n'} F^{m'} \rangle = \delta_{n,n'} \delta_{m,m'} n!(m)! \tag{28.19}$$

where $(m)! \stackrel{\text{def}}{=} (m)(m-1)\dots(1)$, and $(m) \stackrel{\text{def}}{=} \frac{\sinh(hm/2)}{\sinh(h/2)}$. ****I would prefer writing this as $[m]$ rather than (m).****

Question from the audience: The first part is a theorem, and the second part is a corollary, right? **Answer:** Well, but they are proved simultaneously. But yes, logically it's better to say that there is a unique Hopf pairing, and that the formula on monomials follows.

Proof: Exercise 44. □

Actually, strictly speaking this is not a dual pairing, but one with the opposite comultiplication:

$$\langle ab, l \rangle = \langle a \otimes b, \Delta^{\text{co op}}(l) \rangle \tag{28.20}$$

$$\langle S(a), l \rangle = \langle a, S^{-1}(l) \rangle \tag{28.21}$$

and the rest is the same. ****So it's the correct pairing without any crossings.****

Theorem 28.4 *$\mathcal{D}(\mathcal{U}_h\mathfrak{b}_+) \stackrel{\text{def}}{=} \mathcal{U}_h\mathfrak{b}_+ \bowtie \mathcal{U}_h\mathfrak{b}_+{}^\circ$ (which is just $\mathcal{U}_h\mathfrak{b}_+ \otimes \mathcal{U}_h\mathfrak{b}_+{}^\circ$ as coalgebras, and the natural embeddings of $\mathcal{U}_h\mathfrak{b}_+$ and $\mathcal{U}_h\mathfrak{b}_+{}^\circ$ are Hopf subalgebras) is generated by $H, E, H^\vee, F$, with defining relations:*

$$[H, H^\vee] = 0,\ [H, E] = 2E,\ [E, F] = -2F, \tag{28.22}$$

$$[H^\vee, E] = \frac{h}{2}E,\ [H^\vee, F] = -\frac{h}{2}F,\ [E, F] = e^{hH/2} - e^{-2H^\vee} \tag{28.23}$$

and the coalgebra from equations 28.3, 28.4, 28.14, and 28.15.

Theorem 28.5 *$\mathcal{D}(\mathcal{U}_h\mathfrak{b}_+)$ is a quasitriangular Hope algbera with $R = \sum e_i \otimes e^i \in \mathcal{U}_h\mathfrak{b}_+ \otimes \mathcal{U}_h\mathfrak{b}_+{}^\circ \hookrightarrow \mathcal{D} \otimes \mathcal{D}$ given by:*

$$R = \sum_{n,m\geq 0} \frac{H^n E^m \otimes H^{\vee n} F^m}{n!(m)!} = e^{H\otimes H^\vee} \sum_{m\geq 0} \frac{E^m \otimes F^m}{(m)!} \tag{28.24}$$

where $(m)!$ is as above.

The formal series $\sum_{n=0}^{\infty} \frac{x^n}{(n)!}$ was first introduced by Euler.

We will define $\tilde{I} = \langle \frac{h}{4} H - H^\vee \rangle$. And then we will claim that $\tilde{I}$ is a Hopf ideal, and we will define $\mathcal{U}_h\mathfrak{sl}_2 = \mathcal{D}(\mathcal{U}_h\mathfrak{b}_+)/\tilde{I}$.

Lecture 29 April 8, 2009

Question from the audience: Before we begin, can you say something about the motivation for the double? **Answer:** First there were groups. If $G \curvearrowright H$ by automorphisms, then we can construct $G \ltimes H$. If each acts on the other, then there is $G \ltimes H$ and $G \rtimes H$, so there must be $G \bowtie H$. This is generalized to Hopf algebras. If G acts on H, then $C(G)$ co-acts on $C(H)$. If G acts by automorphisms, then the co-action agrees with the coalgebra structure on $C(H)$. So say that A coacts on B, now Hopf algebras, by coalgebra endomorphisms.

It is a coaction, so we also want the commutativity of:

$$\begin{array}{ccc} B & \xrightarrow{\quad\alpha\quad} & A\otimes B \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \Delta_A \otimes \mathrm{id}} \\ A\otimes B & \xrightarrow{\ \mathrm{id}\otimes\alpha\ } & A\otimes A\otimes B \end{array} \tag{29.1}$$

But also we want another diagram, which we make as a homework. Also, there is a version that comes from the group algebra.

Exercise 45 *Find the Hopf algebra version of $G \ltimes H$ for $\mathbb{C}[G]$ and $C(H)$.*

This gives the *smash product* $A\#B$. There is the more complicated construction $A \bowtie H$. The best source for these is the book *Hopf Algebras* by S. Montgomery.

We now return to the quantum double construction.

Theorem 29.1 *1. There exists a unique algebra structure on the space $\mathcal{D}(A) = A \otimes A^\circ$ such that*

- *$A, A^\circ \hookrightarrow \mathcal{D}(A)$ are Hopf algebra embeddings.*
- *If $R = \sum_i e_i \otimes e^i \in A \otimes A^\circ \hookrightarrow \mathcal{D} \otimes \mathcal{D}$, where A embeds in the first copy and A° embeds in the second, then $R\Delta_{\mathcal{D}}(a) = \Delta_{\mathcal{D}}^{op}(a)R$*

2. Then R satisfies:

$$(\Delta \otimes id)(R) = R_{13}R_{23} \tag{29.2}$$

$$(id \otimes \Delta)(R) = R_{13}R_{12} \tag{29.3}$$

Drinfeld came up with the $\mathcal{D}(A)$ to generalize the double of a Lie bialgebra.

Question from the audience: To define R, are you assuming that A is finite-dimensional? **Answer:** No, but we are completing with some topology. Usually, A will not be finite-dimensional, but it will be filtered $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ such that $A_iA_j \subseteq A_{i+j}$, and the quotients A_i/A_{i+1} are finite-dimensional. Then the *associated graded space* $A^{\mathrm{gr}} \stackrel{\mathrm{def}}{=} \bigoplus_{i\geq 0} A_i/A_{i+1} \cong A$ as a filtered vector space. **Question from the audience:** The idea of R is to give a braiding on the category of finite-dimensional representations. So we need a way to make it act? **Answer:** Yes. Well, in many topological examples, R does not exist, but the category is braided anyway, because there is another algebra in which R exists.

Anyway, so we saw last time that $\mathcal{D}(\mathcal{U}_h\mathfrak{b}_+)$ contains an ideal $\tilde{I} \stackrel{\mathrm{def}}{=} \langle H^\vee - \frac{h}{4}H\rangle$, and $\tilde{I}$ is a Hopf ideal. Then we defined $\mathcal{D}(U_h\mathfrak{b}_+)/\tilde{I} \stackrel{\mathrm{def}}{=} \mathcal{U}_h\mathfrak{sl}_2$.

So $\mathcal{U}_h\mathfrak{sl}_2$ is generated by H, E, F, complete over $\mathbb{C}[[h]]$, with defining relations

$$[H, E] = 2E,\ [H, F] = -2F,\ [E, F] = \frac{\sinh(\frac{h}{2}H)}{\sinh(\frac{h}{2})} \tag{29.4}$$

It is a Hopf algebra with:

$$\Delta H = H \otimes 1 + 1 \otimes H \tag{29.5}$$

$$\Delta E = E \otimes e^{hH/2} + 1 \otimes E \tag{29.6}$$

$$\Delta F = F \otimes 1 + e^{-hH/2} \otimes F \tag{29.7}$$

Actually, it is not quite what we just said. Last time, we saw that in $\mathcal{D}(\mathcal{U}_h\mathfrak{b}_+)$, we have

$$[E, F] = e^{hH/2} - e^{-2H^\vee} \tag{29.8}$$

Then we specialize $H^\vee$ to $hH/4$, so that the commutator becomes the hyperbolic sine:

$$[E, F] = e^{hH/2} - e^{-hH/2} \tag{29.9}$$

But this begins in degree h, and we should like the $h \to 0$ limit to give $[E, F] = H$. Why is there this problem? Well, we started with $E, H \in \mathcal{U}\mathfrak{b}_+$ and deformed it; then $F, H^\vee \in \mathcal{F}(B_-)$, the formal functions on B_-, and so $F, H^\vee$ are both infinitesimal. So what we really want to do is work with E, H in $\mathcal{D}$, but the generator $F \in \mathcal{D}$ we declare divisible by h, and the corresponding generator of $\mathcal{U}_h\mathfrak{sl}_2$ will be $F' \stackrel{\mathrm{def}}{=} F/\sinh(h/2)$. Thus, let's add $'$s everywhere, so that $\mathcal{U}_h\mathfrak{sl}_2$ is generated by E', F', H' with:

$$[H', E'] = 2E',\ [H', F'] = -2F',\ [E', F'] = \frac{\sinh(\frac{h}{2}H')}{\sinh(\frac{h}{2})} \tag{29.10}$$

$$\Delta H' = H' \otimes 1 + 1 \otimes H' \tag{29.11}$$
$$\Delta E' = E' \otimes e^{hH'/2} + 1 \otimes E' \tag{29.12}$$
$$\Delta F' = F' \otimes 1 + e^{-hH'/2} \otimes F' \tag{29.13}$$

Then $\mathcal{U}_h\mathfrak{sl}_2$ is quasitriangular with

$$R = \exp\left(\frac{h}{4} H \otimes H\right) \sum_{n=0}^{\infty} \frac{\sinh(h/2)^n}{(n)!} (E')^n \otimes (F')^n \tag{29.14}$$

Exercise 46 *Clean up the $F \rightsquigarrow F'$ by thinking about filtrations on $\mathcal{D}(\mathcal{U}_h\mathfrak{b}_+)$ so that F is divisible by h.*

Now it is clear where R lives. It does not live in the algebraic tensor product, but in $\mathcal{U}_h\mathfrak{sl}_2 \hat{\otimes} \mathcal{U}_h\mathfrak{sl}_2$, where $\hat{\otimes}$ completes with the h-adic topology, i.e. $\mathcal{U}_h\mathfrak{sl}_2 \hat{\otimes} \mathcal{U}_h\mathfrak{sl}_2$ is the collection of infinite sums $\sum_{n=1}^{\infty} a_n h^n$, where $a_n \in (\mathcal{U}_h\mathfrak{sl}_2)^{\otimes 2}$

Theorem 29.2 *There exists an algebra isomorphism $\phi : \mathcal{U}_h\mathfrak{sl}_2 \cong \mathcal{U}\mathfrak{sl}_2[[h]]$ such that $\phi|_{\mathbb{C}[H'][[h]]} = id$.*

Proof: We forget about $'$s. Then we set $\phi(H) = H$, $\phi(E) = E\, f(hH)$, and $\phi(F) = g(hH)F$.

Exercise 47 *Find at least one such function — i.e. find f, g — such that $\phi(E)\phi(F) - \phi(F)\phi(E) = H$, recalling that $EF - FE = \sinh(hH/2)/\sinh(h/2)$.*

So, to find finite-dimensional representations is very easy: you represent $\mathcal{U}\mathfrak{sl}_2$, and then pull back. So finite-dimensional irreps of $\mathcal{U}_h\mathfrak{sl}_2$ are parameterized by $l \in \mathcal{Z}_{\geq 0}$, the highest weights, with

$$V_l = \mathbb{C}v_0^{(l)} \oplus \mathbb{C}v_1^{(l)} \oplus \cdots \oplus \mathbb{C}v_l^{(l)} \tag{29.15}$$

and E moves to the left, F to the right, and H diagonally. ****draw with arrows****. In particular,

$$Hv_m^{(l)} = (l - 2m)v_m^{(l)},\ Ev_m^{(l)} = v_{m-1}^{(l)},\ Fv_m^{(l)} = f(m,l)v_{m+1}^{(l)} \tag{29.16}$$

So a representation $(\pi^{(l)}, V_l)$ gives linear functions $\pi_{m,m'}^{(l)}$ on $\mathcal{U}_h\mathfrak{sl}_2$. Then we can look at the space of special linear functionals on $\mathcal{U}_h\mathfrak{sl}_2$ as

$$\mathcal{L}(\mathcal{U}_h\mathfrak{sl}_2) = \bigoplus_{l \geq 0} \bigoplus_{m,m'=0,1,\dots,l} \mathbb{C}\pi_{m,m'}^{(l)} \tag{29.17}$$

Question from the audience: What does $\pi_{m,m'}^{(l)}$ mean? **Answer:** $\pi^{(l)}$ is a map $\mathcal{U}_h\mathfrak{sl}_2 \to \mathrm{End}(V^{(l)}$. And we have chosen a basis $\{v_m^{(l)}\}$ of $V^{(l)}$, and so $\pi_{m,m'}^{(l)}$ is the matrix element: $\pi^{(l)}(a)v_m^{(l)} \stackrel{\text{def}}{=} \sum_{m'} \pi_{m,m'}^{(l)}(a)v_m'^{(l)}$.

Ok, so we can ask whether $\mathcal{L}$ is an algebra. We should define:

$$\langle \pi_{m_1,m_1'}^{(l_1)} \pi_{m_2,m_2'}^{(l_2)}, a\rangle \stackrel{\text{def}}{=} \langle \pi_{m_1,m_1'}^{(l_1)} \otimes \pi_{m_2,m_2'}^{(l_2)}, \Delta a\rangle \tag{29.18}$$

Then the question is whether the tensor product of irreps decomposes as a direct sum of irreps. We know that it does, by understanding the representation theory of $\mathfrak{sl}_2$:

$$V^{(l_1)} \otimes V^{(l_2)} \cong V^{(l_1+l_2)} \oplus \cdots \oplus V^{(|l_1-l_2|)} \tag{29.19}$$

but the isomorphism of $\mathcal{U}_h\mathfrak{sl}_2$-modules depends as a formal power series in h. See, we only have an isomorphism of algebras, not of Hopf algebras, so we know there is a splitting but not what it is. We will develop the Clebsch-Gordon coefficients with which to understand the product of these matrix elements.

We also have the multiplication $\pi^{(l)}(ab) = \pi^{(l)}(a)\pi^{(l)}(b)$, and this will give the coalgebra structure on $\mathcal{L}$. There is also an antipode, so that $\mathcal{L}$ is a Hopf algebra, and we will see next time that $\mathcal{L}$ is generated by $\pi^{(1)}_{00}$, $\pi^{(1)}_{01}$, $\pi^{(1)}_{10}$, and $\pi^{(1)}_{11}$. Matt discussed $\mathcal{U}_qSL_2$, where $q = e^h$, and these will be a, b, c, d.

Lecture 30 April 13, 2009

Last time we had a discussion of all sorts of special representations of $\mathcal{U}_h\mathfrak{sl}_2$. We have $W(\lambda, c) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}v_n^{(\lambda)}$, infinite in both directions. The Verma module $M_+^{(\lambda)} = \bigoplus_{n\geq 0} \mathbb{C}v_n^{(\lambda)}$, and its graded dual $M_+^* = \bigoplus_{n\geq 0} \mathbb{C}v_n^{(\lambda)^*}$, the lowest-weight Verma module. Also the finite-dimensional representation $V^{(l)} = \bigoplus_{n=0}^{l} \mathbb{C}v_n^{(l)}$. Here $\lambda \in \mathbb{C}$, $c \in \mathbb{C}[[h]]$, and $l \in \mathbb{Z}_{\geq 0}$, and for special c_λ we have an exact sequence $0 \leftarrow M \leftarrow W(\lambda, c_\lambda) \leftarrow M^* \leftarrow 0$, and for $\lambda = l \in \mathbb{Z}_+$, we have $0 \leftarrow V^{(l)} \leftarrow M^{(l)} \leftarrow M^{(-l-2)} \leftarrow 0$.

We now restrict our attention to finite-dimensional representations; the category is called <u>$\mathcal{U}_h\mathfrak{sl}_2$-mod</u>.

Theorem 30.1 *1. All finite-dimensional $\mathcal{U}_h\mathfrak{sl}_2$ modules are reducible.*

2. $V^{(l)} \otimes V^{(m)} \cong V^{(l+m)} \oplus \cdots \oplus V^{(|l-m|)}$.

Proof: $\mathcal{U}_h\mathfrak{sl}_2 \cong \mathcal{U}\mathfrak{sl}_2[[h]]$ as an algebra. □

Thus we define $\mathcal{L}(\mathcal{U}_h\mathfrak{sl}_2) = \bigoplus_{l\geq 0} \bigoplus_{0\leq m,m'\leq l} \mathbb{C}\pi^{(l)}_{m,m'}$. To define this, we chose a weight basis $\{v_n^{(l)}\}_{n=0}^{l}$ in $V^{(l)}$, and then for $a \in \mathcal{U}_h\mathfrak{sl}_2$ we have $\pi^{(l)}(a)v_n^{(l)} = \sum_{n'=0}^{l} \pi^{(l)}_{n,n'}(a)v_{n'}^{(l)}$, and then $\pi^{(l)}_{n,n'}$ are linear forms on $\mathcal{U}_h\mathfrak{sl}_2$.

Theorem 30.2 *$\mathcal{L}(\mathcal{U}_h\mathfrak{sl}_2)$ is a Hopf algebra with*

$$\pi^{(l_1)}_{m_1,m_1'}\pi^{(l)}_{m_2,m_2'} = \sum_{m=|l_1-l_2|}^{l_1+l_2} (\dots)\pi^{(l)}_{n,n'} \tag{30.1}$$

C.f. Kassel.

We remark that to define this multiplication we have to choose bases in $\mathrm{Hom}_{\mathcal{U}_h\mathfrak{sl}_2}(V^{(l)}, V^{(l_1)} \otimes V^{(l_2)})$ *and in* $\mathrm{Hom}_{\mathcal{U}_h\mathfrak{sl}_2}(V^{(l_1)} \otimes V^{(l_2)}, V^{(l)})$*. These are all* 0*- or* 1*-dimensional, so different bases correspond* *to choices of nonzero complex numbers. Hence different bases correspond to different but equivalent (by a linear transformation) multiplication on $\mathcal{L}(\mathcal{U}_h\mathfrak{sl}_2)$. Then $(\dots)$ above is a coefficient in this basis.*

The coalgebra is

$$\Delta\pi^{(l)}_{n,n'} = \sum_{n''=0}^{l} \pi^{(l)}_{n,n''} \otimes \pi^{(l)}_{n'',n'} \tag{30.2}$$

We then can define $C_h[SL_2] \overset{\text{def}}{=} \mathcal{L}(\mathcal{U}_h\mathfrak{sl}_2)$, which is the quantized algebra of polynomial functions on SL_2. We remark that $C_h[SL_2] = \bigoplus_{l=0}^{\infty} \mathrm{End}_{\mathbb{C}[[h]]}(V^{(l)})$ as a vector space. Also $\mathrm{End}(V^{(l)}) \cong {V^{(l)}}^* \otimes V^{(l)}$, where the right multiplicand is the $\mathbb{C}$-linear dual.

Theorem 30.3 (Peter-Weyl) *Consider $C[SL_2]$ as a module over $SL_2 \times SL_2$, where these are the multiplication on the left and on the right. Then $C[SL_2] \cong \bigoplus_{l=0}^{\infty} {V^{(l)}}^* \otimes V^{(l)}$ as a module.*

(This is an interesting theorem because "Peter" and "Weyl" are two different names, not somebody named "Peter Weyl".)

Recall that if (π, V) is a representation of a Hopf algebra, we define the *left dual representation* to be $(\pi^* \circ S, V^*)$. This is good, because $a \mapsto \pi^*$ is an antihomomorphism of algebras, so $a \mapsto \pi^*(S(a))$ is a representation. **Question from the audience:** S isn't always invertible? **Answer:** It is always invertible. It is not always an involution.

Well, anyway, so we are going in the backwards direction pedagogically. We of course have $C[SL_2] = \mathbb{C}[a,b,c,d]/\langle ad - bc - 1\rangle$, and Matt told us how to quantize this. We will see how to derive the quantized presentation.

Theorem 30.4 *Any irreducible representation occurs as a submodule of $(V^{(1)})^{\otimes N}$ for sufficiently large N.*

Corollary 30.4.1 *$\pi^{(1)}_{n,n'}$ generate $C_h[SL_2]$.*

So, we record some facts.

The category $\underline{\mathcal{U}_h\mathfrak{sl}_2\text{-mod}}$ is braided. Indeed, for any V, W,

$$c_{V,W} \overset{\text{def}}{=} P_{V,W}(\pi_v \otimes \pi_w)(R) : V \otimes W \to W \otimes V \tag{30.3}$$

where $P_{V,W}$ is the $\mathbb{C}$-linear flip map, is an isomorphism of $\mathcal{U}_h\mathfrak{sl}_2$-modules.

In particular, with $R^{(1,1)} \overset{\text{def}}{=} (\pi^{(1)} \otimes \pi^{(1)})(R)$, we see that $PR^{(1,1)} : V^{(1)} \otimes V^{(1)} \overset{\sim}{\to} V^{(1)} \otimes V^{(1)}$ commutes with $\mathcal{U}_h\mathfrak{sl}_2$. Thus:

$$PR^{(1,1)}(\pi^{(1)} \otimes \pi^{(1)})\Delta(a) = (\pi^{(1)} \otimes \pi^{(1)})\Delta(a)\, PR^{(1,1)} \tag{30.4}$$

Then equation 30.4 is a collection of relations in $C_h[SL_2]$. Because the left-hand side is just

$$\sum_{n_1',n_2'} (PR^{(1,1)})^{n_1',n_2'}_{n_1,n_2} \langle \pi^{(1)}_{n_1',n_1''} \pi^{(1)}_{n_2',n_2''}, a\rangle = \sum_{n_1',n_2'} \langle \pi^{(1)}_{n_1,n_1'} \pi^{(1)}_{n_2,n_2'}, a\rangle (PR^{(1,1)})^{n_1',n_2'}_{n_1'',n_2''} \tag{30.5}$$

since $\langle l \otimes m, \Delta a\rangle = \langle lm, a\rangle$. Then we define $\dot{\otimes}$ to be the tensor product of matrices with the product of matrix elements in the algebra. Thus we have:

$$PR^{(1,1)}(\pi^{(1,1)}\dot{\otimes}\pi^{(1,1)}) = (\pi^{(1,1)}\dot{\otimes}\pi^{(1,1)})PR^{(1,1)} \tag{30.6}$$

an identity in $\mathrm{End}(\mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^2) \otimes C_h[SL_2]$.

Ok, so what have we done. We have $\pi^{(1)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a = \pi^{(1)}_{0,0}$, etc., and $a, b, c, d \in C_h[SL_2]$. This is what happens with $\dot{\otimes}$. Then all the ns in the calculations are either 0 or 1, and we have written out the matrix products. This gives 16 identities. So $(PR^{(1,1)})^{n_1',n_2'}_{n_1,n_2} \in \mathbb{C}[[h]]$ are matrix coefficients. And we did the calculation with a arbitrary, and since the pairing is nondegenerate, we can drop a, getting equation 30.6.

Let's do an example. In $V^{(1)} \otimes V^{(1)}$ we choose an ordered basis $v_0^{(1)} \otimes v_0^{(1)}$, $v_0^{(1)} \otimes v_1^{(1)}$, $v_1^{(1)} \otimes v_0^{(1)}$, $v_1^{(1)} \otimes v_1^{(1)}$. Then in this basis let us write out $\dot{\otimes}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left[\begin{array}{cc|cc} aa & ab & ba & bb \\ ac & ad & bc & bd \\ \hline ca & cb & da & db \\ cc & cd & dc & dd \end{array}\right] \tag{30.7}$$

Question from the audience: Where is this living? **Answer:** It is a matrix, with coefficients in $C_h[SL_2]$, so it is in $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes C_h[SL_2]$.

****There is still some unhappiness.**** Let us make this absolutely clear. Let us act by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $e_1 = v_0^{(1)} \otimes v_0^{(1)}$. Then we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} e_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \dot{\otimes} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} a^2 \\ ac \\ ca \\ c^2 \end{pmatrix} \tag{30.8}$$

This is exactly the first column. It's better when things are trivial, which is why we are overdoing it.

Ok, so in this basis, let's understand the permutation matrix P. How does it act on e_1? It is $v_0 \otimes v_0$, so it acts trivially. Whereas it takes $e_2 \mapsto e_3$ Continuing, if you think about it, you get

$$P = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \tag{30.9}$$

What about $(\pi^{(1)} \otimes \pi^{(1)})(R)$? Well,

$$R = \exp\left(\frac{h}{4} H \otimes H\right) \sum_{n \geq 0}^{\infty} \frac{E^n \otimes F^n}{[n]!} \left(e^{h/2} - e^{-h/2}\right)^n \tag{30.10}$$

$$\pi^{(1)}(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \pi^{(1)}(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \pi^{(1)}(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{30.11}$$

$$(\pi^{(1)} \otimes \pi^{(1)})(R) = (\pi^{(1)} \otimes \pi^{(1)})(e^{\frac{h}{2} H \otimes H}) \left(1 + \left(e^{h/2} - e^{-h/2}\right) \pi^{(1)}(E) \otimes \pi^{(1)}(F)\right) \tag{30.12}$$

Exercise 48 *Show this is*

$$\left[\begin{array}{cc|cc} e^{h/2} & 0 & 0 & 0 \\ 0 & e^{-h/2} & e^{-h/2}(e^{h/2} - e^{-h/2}) & 0 \\ 0 & 0 & e^{-h/2} & 0 \\ 0 & 0 & 0 & e^{h/2} \end{array}\right] \tag{30.13}$$

Then equation 30.6 is just an identity of matrices. These give 16 indentities between a, b, c, d. They give all the relations in $C_h[SL_2]$ that Matt gave, except that $ad - e^{-h/2}bc = 1$. We will see next time that this comes from the embedding $V^{(0)} \hookrightarrow V^{(1)} \otimes V^{(1)}$.

Lecture 31 April 15, 2009

31.1 $C_h[SL_2]$

****We write $C[X]$ for the polynomial functions on X, and $C_h[X]$ for a formal quantization.****

Through the natural isomorphism $PR^{(1,1)} : V^{(1)} \otimes V^{(1)} \xrightarrow{\sim} V^{(1)} \otimes V^{(1)}$ of $\mathcal{U}_h\mathfrak{sl}_2$-modules, and using the embedding $V^{(0)} \hookrightarrow V^{(1)} \otimes V^{(1)}$, we want to describe the generators and relations for $C_h[SL_2]$.

Last time we introduced the matrix $\pi^{(1)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where if $x \in \mathcal{U}_h\mathfrak{sl}_2$, we defined $\pi^{(1)}(x) = \begin{pmatrix} \langle a, x\rangle & \langle b, x\rangle \\ \langle c, x\rangle & \langle d, x\rangle \end{pmatrix}$ — this defined the linear functionals a, b, c, d.

Theorem 31.1 *$C_h[SL_2]$ is generated by a, b, c, d.*

We introduce the letter $q = e^h$. Then

$$R^{(1,1)} = \left[\begin{array}{cc|cc} q^{1/2} & & & \\ & q^{-1/2} & q^{-1/2}(q - q^{-1}) & \\ \hline & & q^{-1/2} & \\ & & & q^{1/2} \end{array}\right] \tag{31.1}$$

where we have picked a basis $V^{(1)} = \mathbb{C}v_0^{(1)} \oplus \mathbb{C}v_1^{(1)}$, so that $v_0 = \begin{pmatrix}1\\0\end{pmatrix}$ and $v_1 = \begin{pmatrix}0\\1\end{pmatrix}$. We have $e_1 = v_0 \otimes v_0$, $e_2 = v_0 \otimes v_1$, $e_3 = v_1 \otimes v_0$, and $e_4 = v_1 \otimes v_1$. Then $\pi^{(1)} \otimes \pi^{(1)}$ is a 4×4 matrix with coefficients in $C_h[SL_2]$. The isomorphism $PR^{(1,1)} : V^{(1)} \otimes V^{(1)} \xrightarrow{\sim} V^{(1)} \otimes V^{(1)}$ then becomes the equation

$$PR^{(1,1)}\pi^{(1)} \otimes \pi^{(1)} = \pi^{(1)} \otimes \pi^{(1)} PR^{(1,1)} \tag{31.2}$$

Expanding these out give six relations:

$$qab = ba,\ db = qbd,\ ca = acq,\ dc = q^{?}cd,\ [a,d] = -(q-q^{-1})cb,\ cb = bd \tag{31.3}$$

When $q = 1$ these are just the commutativity relations. These relations do not define quantum SL_2, but rather the *quantum* 2×2 *matrices*, because there is no invertibility/determinant condition.

So we now consider the map $\phi : V^{(0} \to V^{(1)} \otimes V^{(1)}$, which is $\mathcal{U}_h\mathfrak{sl}_2$-linear. To define $C_h[SL_2]$ as $\bigoplus_{n\geq 0} V^{(n)^*} \otimes V^{(n)}$, we should choose a basis in each $V^{(n)}$ and in each $\mathrm{Hom}(V^{(n)} \leftrightarrows V^{(n_1)} \otimes V^{(n_2)})$. Then we can get the basis $\pi^{(n)}_{m,m'} \in V^{(n)^*} \otimes V^{(n)}$.

So let's pick the basis $\{v_0^{(0)}\}$ for $V^{(0)}$, such that $H\phi(v_0^{(0)}) = 0$, $E\phi(v_0^{(0)}) = 0$, and $F(v_0^{(0)}) = 0$. From the first equation, we see that

$$\phi(v_0^{(0)}) = \alpha v_0^{(1)} \otimes v_1^{(1)} + \beta v_1^{(1)} \otimes v_0^{(1)} \tag{31.4}$$

since $Hv_0^{(1)} = v_0^{(1)}$ and $Hv_1^{(1)} = -v_1^{(1)}$. From the second equation, and recalling that E acts on $V^{(1)} \otimes V^{(1)}$ as $E \otimes e^{hH/2} + 1 \otimes E$, we get

$$\alpha + \beta e^{h/2} = 0 \tag{31.5}$$

The F action gives the same equation. Thus, we choose

$$\phi(v_0^{(0)}) = e^{h/4} v_0^{(1)} \otimes v_1^{(1)} + e^{-h/4} v_1^{(1)} \otimes v_0^{(1)} \tag{31.6}$$

What kind of relations do these give for a, b, c, d? We demand:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \phi(v_0^{(0)}) = \phi(v_0^{(0)}) \tag{31.7}$$

which we write out in matrices:

$$\begin{bmatrix} a^2 & ab & & \\ ac & ad & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} 0 \\ e^{h/4} \\ e^{-h/4} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{h/4} \\ e^{-h/4} \\ 0 \end{bmatrix} \tag{31.8}$$

When you expand this out, you get two relations that are redundant with equation 31.3, and two which are the same:

$$ad - e^{-h/2}bc = 1 \tag{31.9}$$

Theorem 31.2 *The relations equations 31.3 and 31.9 are defining in $C_h[SL_2]$.*

Proof: By the definition and the Peter-Weyl theorem, $C_h[SL_2] \cong C[SL_2][[h]]$ as a vector space, where $C[SL_2]$ is the algebra of polynomial functions on SL_2. ****I lost some of what NR said about this**** When G is compact, $L_2(G) \cong \bigoplus_i \operatorname{End}(V_i)$, but you need some completions. Horrible things happen in the non-compact C^∞ case. For example, $C^\infty(SL_2[\mathbb{C}])$ does not have the above decomposition, because the representation theory of SL_2 includes irreducible representations infinite in both directions, and these contribute. C.f. "$SL_2(\mathbb{R})$" by S. Lang, or the yellow book by Wolfe ****?****.

So one should probe that the algebra generated by a, b, c, d with equations 31.3 and 31.9 is still isomorphic to $C[SL_2][[h]]$ as a vector space. This follows from a version of the PBW theorem.

Exercise 49 *Complete this proof. Hint: $c^n a^k d^l b^m$ is a convenient basis.* □

So this completes the following square, where by $h \to 0$ we mean the definition $\lim_{h\to 0} a \stackrel{\text{def}}{=} a \mod h$:

$$\begin{array}{ccc} \mathcal{U}_h\mathfrak{sl}_2 & \stackrel{\text{dual}}{\longleftrightarrow} & C_h[SL_2] \\ \big\downarrow h \to 0 & & \big\downarrow h \to 0 \\ \mathcal{U}\mathfrak{sl}_2 & \stackrel{\text{dual}}{\longleftrightarrow} & C[SL_2] \end{array} \tag{31.10}$$

Let us fix a linear isomorphism $C[SL_2][[h]] \cong C_h[SL_2]$, for example one can identify alphabetized monomials. Then let us understand the identity

$$R^{(1,1)}\pi_1^{(1)}\pi_2^{(1)} = \pi_2^{(1)}\pi_1^{(1)}R^{(1,1)} \tag{31.11}$$

where $\pi_1 = \pi \otimes 1$, $\pi_2 = 1 \otimes \pi$. Then we see in $\operatorname{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes C_h[SL_2]$, we have

$$R^{(1,1)} = 1 + 2h\left(\underbrace{\frac{H \otimes H}{4} + E \otimes F}_{r}\right) + O(h^2) \tag{31.12}$$

where r is the classical R-matrix. Recall that $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We see that:

$$[\pi_1^{(1)}, \pi_2^{(1)}]\left(-\frac{1}{2h}\right) = [r, \pi_1^{(1)}\pi_2^{(1)}] + O(h) \tag{31.13}$$

In this way,

$$\lim_{h\to 0}\left(-\frac{1}{2h}\right)(ab - ba) \stackrel{\text{def}}{=} \{a, b\} \tag{31.14}$$

whence

$$\{\pi_1^{(1)}, \pi_2^{(1)}\} = [r, \pi_1^{(1)}\pi_2^{(1)}] \tag{31.15}$$

Question from the audience: In the classical case, we saw that each of these objects has two different duals: the dual Hopf algebra, and the dual arising from the bialgebra/Poisson structure. But your square only has one dual? **Answer:** Well, yes, and we already have some of this, since $C_h[SL_2]$ has some form of PBW theorem, so it should be considered as a universal enveloping algebra. In fact, it will be the same as quantized universal enveloping algebra for $\mathfrak{sl}_2$-dual, but with a different topology.

31.2 q-Schur-Weyl duality

Let us consider $\mathcal{H}_N = V^{(1)} \otimes \cdots \otimes V^{(1)}$, where there are N copies. There are two natural algebras that act on this: $\mathcal{U}_h\mathfrak{sl}_2$ with the diagonal action, and the braid group $B_N = \langle s_i, 1 \leq i \leq N-1 \text{ s.t. } s_i s_j = s_j s_i \text{ for } |i-j| > 1, s_{i\pm1} s_i s_{i\pm1} = s_i s_{i\pm1} s_i \rangle$.

We recall the relation for the universal R-matrix:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \tag{31.16}$$

where $R_{12} = R \otimes 1$, etc. So we consider $S = PR^{(1,1)} e^{-h/2}$, and this then satisfies

$$(S \otimes 1)(1 \otimes S)(S \otimes 1) = (1 \otimes S)(S \otimes 1)(1 \otimes S) \tag{31.17}$$

So $\mathbb{C}[B_N]$ acts on $\mathcal{H}_n$ by $s_i \mapsto 1 \otimes \cdots \otimes S \otimes \cdots \otimes 1$, where S is in the $(i, i+1)$th spots. Let $q = e^h$, then

$$S = \begin{bmatrix} q & & & \\ & & 1 & \\ & 1 & q - q^{-1} & \\ & & & q \end{bmatrix} \tag{31.18}$$

so the eigenvalues are q with multiplicity 2 and $-q^{-1}$ with multiplicity 1, and $(S-q)(S+q^{-1}) = 0$.

Next time we will discuss this more fully, and introduce Temperly-Lieb.

Lecture 32 April 17, 2009

****I arrived five minutes late, in the middle of an historical discussion: Euler, Bernouli, et al. The question was the introduction of the letter q.****

32.1 Hecke-Iwakori algebra

We return to the discussion of the Braid group and the Hecke algebra.

The *Hecke-Iwakori algebra* is a quotient of the braid group:

$$H_n(q) \overset{\text{def}}{=} \mathbb{C}(q)[B_n]/\langle (s_i - q)(s_i + q^{-1}) \rangle \tag{32.1}$$

It is an algebra over $\mathbb{C}(q)$, the rational functions in q, where q is a formal variable. There is an isomorphism of vector spaces: $H_n(q) \cong \mathbb{C}(q)[S_n]$. One can study irreducible representations of $H_n(q)$, which are enumerated by the same data as the representations of S_n; this is a special topic.

Consider $\mathscr{H}_n = (V^{(1)})^{\otimes n}$. Then we defined

$$S = e^{-h/2} P R^{(1,1)} = \left[\begin{array}{cc|cc} e^h & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & e^h - e^{-h} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \tag{32.2}$$

This has block form:

$$S = \left[\begin{array}{c|cc|c} e^h & & & \\ \hline & 0 & 1 & \\ & 1 & e^h - e^{-h} & \\ \hline & & & 1 \end{array}\right] \tag{32.3}$$

So the eigenvalues of S are $q \overset{\text{def}}{=} e^h$ with multiplicity 2 from the top and bottom blocks, and the middle one is 2×2, and we can check that q and $-q^{-1}$. So the eigenvalues of S are q with multiplicity 3 and $-q^{-1}$ with multiplicity 1. We see that $S^t = S$, so we can write it in spectral decomposition: $SqP^{(3)}(q) - q^{-1}P^{(1)}(q)$, where $P^{(3)}$ and $P^{(1)}$ are orthogonal projections.

Therefore $(S-q)(S+q^{-1}) = 0$, and so $\mathbb{C}[B_n] \curvearrowright \mathscr{H}_n$ by $s_i \mapsto 1 \otimes \cdots \otimes S \otimes \cdots \otimes 1$, where the S is in the $(i, i+1)$ spot. Then this representation $\mathbb{C}[B_n] \to \mathrm{End}(H_n)$ factors through $H_n(q)$.

Proposition 32.1 *Irreducible representations of $H_n(q)$ are parameterized by partitions of n: $H_n(q) \cong \bigoplus_{\lambda \dashv n} \mathrm{End}(W_\lambda)$.*

Thus we can build the *Temperley-Lieb quotient* or *Temperley-Lieb algebra*, by $TL_n(q) = H_n(q)/I_n$, where $TL_n(q)|_{W^\lambda} = 0$ if λ has more than two rows. Thus I_n is the sum $\bigoplus \mathrm{End}(W_\lambda)$ where λ has more than two rows.

I_n is generated by elements $p^-_{i,i+1,i+2} = 1 - q^{-1}(s_i + s_{i+1}) + q^{-2}(s_i s_{i+1} + s_{i+1} s_i) - q^3 s_i s_{i+1} s_i$.

Question from the audience: When q is a root of unity, the isomorphism is not true. **Answer:** That's correct. For us, q is a formal variable. When q is an lth root of unity, $H_n(q)$ behaves like the representation theory of S_n in characteristic l.

Another describtion:

$$TL_n(q) = \langle e_i \text{ s.t. } e_i^2 = e_i, e_i e_{i\pm1} e_i = (q + q^{-1}) e_{i\pm1} \rangle \tag{32.4}$$

Then the homomorphism $H_n(q) \to TL_n(q)$ is $s_i \mapsto q(1 - e_i) - q^{-1} e_i$. The proof is purely algebraic.

So we have three important algebras. $\mathbb{C}[B_n] \to H_n(q) \to TL_n(q)$, and we know that $H_n(q) \to \mathrm{End}(\mathscr{H}_n)$.

Theorem 32.2 *In fact,* $TL_n(q) \to \mathrm{End}(\mathscr{H}_n)$.

Question from the audience: We defined $\mathscr{H}_n$ over formal power series in h. But now we are working with rational functions in q. **Answer:** What we do is that among the rational functions in q, some of them are regular at $q = 1$, and these can be expanded as formal power series. Conversely, we can extend the h action to a q action.

Question from the audience: We are just aiming to understand $\mathcal{U}_q\mathfrak{sl}_2$. **Answer:** Yes. For $\mathcal{U}_q\mathfrak{sl}_n$, there will not be Temperley-Lieb. **Question from the audience:** The "2" in TL is the 2 in $\mathfrak{sl}_2$? **Answer:** Yes.

Proof: We have $H_n(q) \to \mathrm{End}(\mathscr{H}_n)$ by $s_i \mapsto 1 \otimes \cdots \otimes S \otimes \cdots \otimes 1 = S_i$, and we know that $(S_i - q)(S_i + q^{-1}) = 0$, so $S_i = q(1 - P^{(1)}_{i,i+1} - q^{-1}P^{(1)}_{i,i+1}$, so this $P^{(1)}_{i,i+1}$ is a natural candidate for e_i. This just comes from $S = qP^{(3)} - q^{-1}P^{(1)}$ and $1 = P^{(3)} + P^{(1)}$.

From last time, we saw $\phi : V^{(0)} \to V^{(1)} \otimes V^{(1)}$ sends $v_0^{(0)} \mapsto \begin{pmatrix} 0 \\ e^{h/4} \\ -e^{-h/4} \\ 0 \end{pmatrix}$.

Now we do some linear algebra. The orthogonal projector onto $\phi(v_0^{(0)})$ is $P^{(1)}$:

$$P^{(1)} = \frac{1}{q^{1/2} + q^{-1/2}} \begin{pmatrix} 0 \\ e^{h/4} \\ -e^{-h/4} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & e^{h/4} & -e^{-h/4} & 0 \end{pmatrix} = \left[\begin{array}{c:c} q^{1/2} & -1 \\ \hdashline -1 & q^{-1/2} \end{array} \right] \frac{1}{q^{1/2} + q^{-1/2}} \tag{32.5}$$

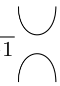
$\square$

****There is general concern about q versus $q^{1/2}$.**** We check that $v_0^{(0)}$ is the eigenvector of S. In fact, the formula should be $v_0^{(0)} = \begin{pmatrix} 0 \\ e^{h/2} \\ -e^{-h/2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ q^{1/2} \\ -q^{-1/2} \\ 0 \end{pmatrix}$. So we need to make a correction in the last lecture. Then:

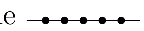
$$P^{(1)} == \left[\begin{array}{c:c} q & -1 \\ \hdashline -1 & q^{-1} \end{array} \right] \frac{1}{q + q^{-1}} \tag{32.6}$$

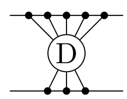
Lemma 32.3 $P^{(1)}_{i,i+1} P^{(1)}_{i+2,i+3} P^{(1)}_{i,i+1} = (q + q^{-1})^{-2} P^{(1)}_{i,i+1}$

To understand this, we write

$$P^{(1)} = \frac{1}{q+q^{-1}}$$  (32.7)

Look at the lectures on TQFT from Amsterdam on NR's website.

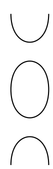
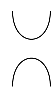
The correct way to say this. Consider the category where objects are points on a line  up to isotopy, and morphisms are diagrams that you can use to connect such points:



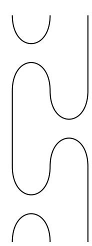
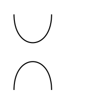
(32.8)

And the point is that there is a functor from diagrams to vector spaces.

So the way to understand the above statement is:

 $= (q+q^{-1})$  (32.9)

where the value of a loop is $q+q^{-1}$. The lemma is another diagram:

 $=$  (32.10)

****I need to write some code to draw these diagrams quickly.****

****NR said this quickly, but I didn't catch it live. To clarify, we have  $= \phi(v_0^{(0)})$ and  $= \phi(v_0^{(0)})^T$ its transpose, or perhaps the other order depending on whether you read diagrams up or down.****

Lecture 33 April 2, 2009

Last time we introduced the *Temperley-Lieb algebra.* Today we develop this further, including the graphical language for this algebra.

We first defined the category of *self-avoiding diagrams.* We need to describe: objects, morphisms, composition.

objects These are parameterized by non-negative integers. You should think of these as equivalence classes of points on a line, equivalence up to homotopy, which lets you move points around but not past each other. ****draw****

morphisms Morphisms $\mathrm{Mor}(m, n)$ are equivalence classes of self-avoiding diagrams, up to regular isotopy. There can be no intersections, and "regular isotopy" means no cusps, etc. ****draw****

composition We explain by picture: ****convention is that D_2 over D_1 is called $D_2 D_1$****. We take a geometric representative of each morphism, glue them together, and then take the equivalence class.

This is a category with many nice properties. We will actually take the $\mathbb{Z}$-linear envelop: morphisms are $\mathbb{Z}$-linear combinations of equivalence classes.

The Temperley-Lieb quotient $TL(\tau)$, where τ is a formal variable, by doing nothing with objects but we assign a simple closed loop to have the value τ (multiplication), and then we remove the loop. So now the morphisms are $\mathbb{Z}[\tau]$-linear combinations of self-avoiding diagrams with no loops.

Question from the audience: So we can have nested loops, and these are the same as disjoint loops? **Answer:** Yes, in the quotient. Not in the the original. **Question from the audience:** And that has value τ^2 or 2τ? **Answer:** τ^2.

Theorem 33.1 $\mathrm{Mor}(n, n)$ *in* $TL(\tau)$ *is isomorphic to* $TL_n(\tau)$.

Proof: We give a correct definition of the *Temperley-Lieb algebra*:

$$TL_n(\tau) \stackrel{\text{def}}{=} \langle e_i \text{ s.t. } e_i^2 = \tau e_i, e_i e_{i \pm i} e_i = e_i \rangle \tag{33.1}$$

Then the morphism from $\mathrm{Mor}(n, n) \to TL_n(\tau)$ is $E_i \mapsto e_i$, where E_i is the diagram with vertical strands for each spot $1, \ldots, n$ except $i, i+1$, and a cap-cup conbination at spot $i, i+1$. ****draw****□

Exercise 50 *Finish the proof. You check that the two algebras have the same dimension. It is a nice combinatorial exercise to compute the number of such diagrams in* $\mathrm{Mor}(n, n)$*. The word "Catalan numbers" is everywhere here. So the exercise is:*

Construct a basis of diagrams.

The category of *ribbon tangles* is a generalization of knots and braids. We really should talk about n-categories here, which are becoming very popular in every area. But we will just talk about the 1-category for now. But there are more or less one or two steps from what we are doing now to very modern results in the theory of knot invariants, the so-called "Categorification" business. See, you think of these diagrams as slices of soap bubbles in three dimensions. Imagine a movie of a plane passing through bubbles. Then at most times you get a self-avoiding diagram, and the two-morphisms are the soap bubbles. We will derive Jones' Polynomial today, and in this two-category viewpoint you can refine this, and find out that Jones is a q-Euler characteristic of the dimensions of the homologies of the Khovanov categorified story. On the representation-theory side, one can try to categorify representations. If you have an algebra over the integers and a basis in which all morphisms are integer matrices, then you can ask for a category where instead of linear maps you have functors, and the integers in the matrices are dimensions of certain spaces of morphisms. So we replace the algebra by a category and the category of modules by a two-category. We will not have time to do this.

Anyway, the category of *ribbon tangles*:

objects collections of signs $(\epsilon_1, \dots, \epsilon_k)$ where $\epsilon_i = \pm$.

morphism You put a diagram in the middle, where now the components are oriented, and agree with the signs on the boundary given by the rule **$+ =\downarrow$, $- =\uparrow$.** We mod out by the following equivalences:

- Regular homotopies just as before.
- Reidemeisters 2 and 3. **draw** We allow all possible orientations.

So these are oriented planar diagrams with an extra crossing symbol. **draw**

Why is this the category of ribbon tangles? We image a plane $\mathbb{R}^2$ with a chosen line, and points positioned irregularly with respect to the line, so that the orthogonal projects do not align. Then morphisms live in $\mathbb{R}^2 \times I$, and a geometrical braid is when $m = n$ and it is a map $\phi : I^{\sqcup n} \hookrightarrow \mathbb{R}^2 I$ where all 0s go to the bottom and all 1s go to the top, and a *braid* is an isotopy class of such an embedding. So we are taking braids, but also allowing strings to go back to the top, and to have loops.

The word "ribbon" is also called *framed*. This means that connected components are very thin ribbons. In $\mathbb{R}^3$, we can always choose the framing to be orthogonal to the ribbon. Here's a better description of a component of a framed tangle. You choose an embedding of I, and afterwords you pick a section of the normal bundle, or rather of the unit-normal sphere-bundle.

Proposition 33.2 *For any geometrical tangle, there exists a* blackboard framing*.*

We chose the two lines, and we assume that they are parallel. Then we have a projection to the plane that includes these lines. We take any tangle, deform it slightly so that its projection is regular, and then we pick the framing to be orthogonal to the plane.

Proposition 33.3 *Any framing of a tangle corresponds to a blackboard framing of some diagram of the tangle.*

Theorem 33.4 *Equivalence classes of diagrams with respect to the Reidemeister moves are in bijection with equivalence classes of framed geometrical tangles.*

Actually, let's say this this way: Let us write $\mathcal{D}$ for the category of ribbon tangles defined above, and $\mathcal{T}$ for the category with objects sequences, and morphisms homeomorphism classes of framed geometrical tangles such that the signs agree. The framing is required to be blackboard near the boundary. The composition is gluing of representatives, and then taking equivalence classes of the result.

Theorem 33.5 *$\mathcal{D}$ and $\mathcal{T}$ are equivalent.*

Question from the audience: Do we lose the information of twists? **Answer:** No. Because we do not have Reidemeister 1.

Question from the audience: I'm confused by signs. Do the signs say whether the framing is towards the viewer or away? **Answer:** No. The tangles also have a long direction, and the signs of the endpoints determine that direction.

We recall the correct definition of *equivalence of categories*. We want functors $\mathcal{D} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{T}$, so that $F \circ G$ is isomorphic as a functor to $\mathrm{id}_{\mathcal{D}}$, and $G \circ F \cong \mathrm{id}_{\mathcal{T}}$.

Exercise 51 *Construct these functors.*

Let's make $\mathcal{D}$ very precise. In addition to self-avoiding diagrams, we have two types of vertices: overcrossings and undercrossings. So a diagram is a self-avoiding four-valent directed graph with two types of vertices. ****draw****

When studying $\mathcal{T}$, the two lines are essential; without them you get much more complicated objects.

Now if you want to construct invariants of tangles, you sort of know what to do. You construct a functor. An *invariant* of ribbon (framed) tangles is a functor $\mathcal{D} \to \mathcal{C}$, where $\mathcal{C}$ is known. What is the most known category? It's the category of vector spaces. So we could do that. Next time, we can construct Jones polynomials, which is essentially a functor from $\mathcal{D}$ to $TL(\tau)$. We will use previous results about TL algebras to construct $TL(\tau) \to \underline{\mathrm{Vect}/\mathbb{Z}(\tau)}$.

Lecture 34 April 22, 2009

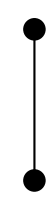
****We begin with some discussion of scheduling, voting for whether to hold a separate class.**** Next Wednesday (April 29) we will have an extra class, 4-6pm, in 939 Evans. The following week (May 6) the Representations Theory seminar will be a discussion of $\mathcal{U}_q\hat{\mathfrak{g}}$.

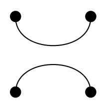
We begin by formulating an open problem in Schur-Weyl duality. This is a duality between $\mathcal{U}_h\mathfrak{sl}_n$ and $TL_n(e^h + e^{-h})$. Last time we discussed diagrams and framed tangles. If $\mathcal{T}$ is the category of framed tangles, we described the functor $\mathcal{T} \to TL(\tau)$. We did not discuss the functor $TL(\tau) \to \underline{\mathrm{Vect}}$. An outline:

- We will do this.
- We will construct the category Γ of framed tangled graphs.
- We will then define the notion of a *ribbon category* $\mathcal{C}$ and of a *ribbon Hopf algebra.*
- We will define $\Gamma(\mathcal{C})$ the category of $\mathcal{C}$-colored graphs.
- We will define the functor $\Gamma(\mathcal{C}) \to \mathcal{C}$.

- We will pose a conjecture, with input data $\mathcal{U}_h\mathfrak{g}$ and $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$ which will be reduce to Schur-Weyl when $\mathfrak{g} = \mathfrak{sl}_n$ and $V_{\lambda_i} = \mathbb{C}^n$.

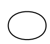
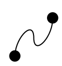
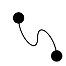
First, the functor. On objects, we define $F($ $) = (\mathbb{C}^2)^n$. On morphisms:

$$F\left(\vphantom{\Big|} \right) = \mathrm{id}_{\mathbb{C}^2} \tag{34.1}$$


$$F\left(\vphantom{\Big|} \right) = \begin{pmatrix} 0 & q^{1/2} & q^{-1/2} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ q^{1/2} \\ q^{-1/2} \\ 0 \end{pmatrix} : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C} \to \mathbb{C}^2 \otimes \mathbb{C}^2 \tag{34.2}$$


There aren't really 1/2s in the last equation: the matrix is just qs.

Theorem 34.1 *There exists a unique covariant monoidal functor $F : TL(\tau) \to \underline{\mathrm{Vect}}/\mathbb{Z}[q, q^{-1}]$ that extends F above. Here the monoidal structure on $TL(\tau)$ is given by disjoint (left-to-right) union.*

Exercise 52 *Prove the theorem. You will need to show that the above morphisms generate all morphisms, modulo*  $= \tau$ *and*  $=$  $=$  .

So, we now take any framed knot $K : S^1 \hookrightarrow \mathbb{R}^3$; this then defines a morphism $\emptyset \to \emptyset$ in $\mathcal{T}$, and hence in $TL_n(q + q^{-1})$ and thus a number in $\mathbb{Z}[q, q^{-1}]$. This number will be, up to a scalar, the Jones polynomial of framed K.

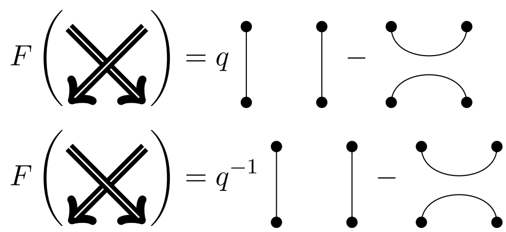
****I then missed some discussion, because I was fixing the above pictures.****

Ah, we never defined the functor from framed tangles to $TL(\tau)$.

We have $\mathcal{T}$ the category of framed tangles. Objects are sequences $(\epsilon_1, \dots, \epsilon_n)$ of signs, and morphisms are regular homotopy classes of diagrams of tangles, and module the two Reidemeister moves R2 and R3 ****draw****. We want to define a functor to TL, and we will define it, but the definition will probably be wrong.

So, objects in $TL(\tau)$ are integers (no signs), and morphisms are regular homotopy classes of non-self-intersecting diagrams with no loops. Composition is gluing and then evaluating all loops to τ.

So, we propose the functor that on objects simply forgets about the signs: $F\big((\epsilon_1, \dots, \epsilon_n)\big) = n$. On the crossings, we take:



(34.3)

(34.4)

(34.5)

On everything else (cups, caps, lines), we just forget the arrows and framings. ****the pictures need improvement****

Theorem 34.2 *The mapping F defines a unique covariant monoidal functor $\mathcal{T} \to TL(q+q^{-1})$.*

Proof: The strategy is twofold: 1. Prove that $\mathcal{T}$ is generated by the diagrams above. 2. Prove that the functor respects the relations.

Question from the audience: In $TL(q+q^{-1})$, it seems that you can only multiply by $q+q^{-1}$; it is $\mathbb{Z}[q+q^{-1}]$-linear. But we want it to be $\mathbb{Z}[q,q^{-1}]$-linear. **Answer:** Ok, so now we have extended TL, and we set $\tau = q+q^{-1}$.

Ok, so let's prove the Reidemeister moves. ****I'm not going to keep up with the drawings live. Do it yourself. Be sure to check both the down-down R2 and the down-up R2. How do we define F in a down-up crossing? We rotate the crossing by 90 degrees, and use cups and caps.**** We did R2; do R3 as **Exercise 53**. □

Thus the functor produces invariants of ****framed**** knots, and moreover given a ****framed**** tangle it produces a linear map that depends only on the topology of the tangle.

This construction depended only on the tangle. Next time, we will extend the story to framed graphs: graphs embedded in $\mathbb{R}^2 \times I$, but with a framing. So the edges are ribbons, and at any vertex the framings are all parallel. We will then construct invariants, but we will need slightly more than quasitriangularity: we will need a "ribbon" structure.

Then we will need examples. Hopf algebras were invented in the late 60s, early 70s, but there were no examples, except for Sweedler's prototype of $\mathcal{U}_q\mathfrak{b}_+$. So the subject became dormant for many years until $\mathcal{U}_q\mathfrak{g}$ was invented. Luckilly, our framed graph construction will extend to many examples.

Lecture 35 April 24, 2009

35.1 Harold: The Belavin-Drinfeld Classification

Although the classification can be extended, we restrict to the case when $\mathfrak{g}$ is a simple finite-dimensional complex Lie algebra. Then:

Theorem 35.1 *Any bialgebra structure δ on $\mathfrak{g}$ is quasitriangular.*

We will outline the proof of this. First we state two facts:

1. $H^1(\mathfrak{g}, V) = 0$ for all V.
2. $(\bigwedge^3 \mathfrak{g})^{\mathfrak{g}}$ is one-dimensional, generated by $[\Omega_{12}, \Omega_{23}]$, where Ω is the Casimir.

By the first statement, δ must be a one-coboundary, as by definition it is a one-cocycle. Recalling the Chevalley complex:

$$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \textstyle\bigwedge^2 \mathfrak{g}) \xrightarrow{d} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \textstyle\bigwedge^2 \mathfrak{g}) \to \dots \tag{35.1}$$

So there is some $\tilde{r} \in \bigwedge^2 \mathfrak{g}$ so that $\delta = d_{\tilde{r}} : x \mapsto [x \otimes 1 + 1 \otimes x, \tilde{r}]$. Then it is a general statement that δ satisfies co-Jacobi if and only if $CYB(\tilde{r}) \in (\bigwedge^3 \mathfrak{g})^{\mathfrak{g}}$, where $CYB(-)$ is the classical Yang Baxter function. We want $CYB(r) = 0$.

Let us use the second statement to write $CYB(\tilde{r})$. Then $CYB(\tilde{r}) = c[\Omega_{12}, \Omega_{23}]$. Then we define $r \stackrel{\text{def}}{=} \tilde{r} + \sqrt{c}\Omega$. Since Ω is central, it's clear that r and $\tilde{r}$ define the same δ.

Moveover, $r + r_{21} = 2\sqrt{c}\Omega$, so if $c \neq 0$, then $(\mathfrak{g}, \delta)$ is factorizable. Recall, this means that $r + r_{21}$ defines a nondegenerate form on $\mathfrak{g}^*$, i.e. an isomorphism $j : \mathfrak{g} \to \mathfrak{g}^*$.

Thus, we see that classifying factorizable structures on $\mathfrak{g}$ reduces to classifying r-matrices with $r + r_{21} = \Omega$ up to a rescaling.

Ok, so let Γ be a set of simple roots. Let $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $\tau : \Gamma_1 \to \Gamma_2$. Then $(\Gamma_1, \Gamma_2, \tau)$ is a *Belavin-Drinfeld triple* (BD triple) if:

1. τ is an orthogonal bijection.
2. $\forall \alpha \in \Gamma_1$, there exists n such that $\tau^n(\alpha) \in \Gamma_2 \setminus \Gamma_1$.

Example 35.1 In $\mathfrak{sl}_{n+1}$ (n roots), let Γ_1 be the leftmost $n-1$ roots and Γ_2 the rightmost $n-1$ roots, and let τ be the shift map once to the right. $\diamondsuit$

We remark that τ extends to $\mathbb{Z}\Gamma_1 \to \mathbb{Z}\Gamma_2$, and we get a partial order on Δ_+ where $\alpha \leq \beta$ if $\tau^n \alpha = \beta$ for some n.

Theorem 35.2 *If $(\Gamma_1, \Gamma_2, \tau)$ is a BD triple, and if $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfies:*

1. *$r_0 + r_0^{21} = \Omega_0$ (the "$\mathfrak{h}$-part" of Ω)*
2. *$(\tau\alpha \otimes id)r_0 + (id \otimes \alpha)r_0 = 0$ for all $\alpha \in \Gamma_1$.*

Then

$$r \stackrel{\text{def}}{=} r_0 + \sum_{\alpha\in\Delta_+} f_\alpha \otimes e_\alpha + \sum_{\substack{\alpha,\beta\in\Delta_+ \\ \alpha<\beta}} f_\alpha \wedge e_\beta \tag{35.2}$$

is an r-matrix with $r + r^{21} = \Omega$.

Conversely, all r matrices with $r + r^{21} = \Omega$ are of this form for some choice of $\mathfrak{h}, \Gamma, (\Gamma_1, \Gamma_2, \tau)$.

The strategy: study r by studying the induced map $f \stackrel{\text{def}}{=} r_- \circ j : \mathfrak{g} \to \mathfrak{g}$ (whence $r = (f \otimes \mathrm{id})\Omega$).

For example, $r + r_{21} = \Omega$ iff $f + f^* = \mathrm{id}$. Then $CYB(r) = 0$ iff for all $x, y \in \mathfrak{g}$ we have:

$$(f - \mathrm{id})[f(x), f(y)] = f[(f - \mathrm{id})(x), (f - \mathrm{id})(y)] \tag{35.3}$$

We write $1 = \mathrm{id}$. Let's say that both f and $f - 1$ were invertible — this can never happen. Then we write $x = (f-1)^{-1}\hat{x}$, and the same for y, and we drop the hats. Then equation 35.3 would say:

$$f(f-1)^{-1}[x, y] = [f(f-1)^{-1}x, f(f-1)^{-1}y] \tag{35.4}$$

Of course, this is nonsense. What we can define is a map θ which we will think of as $f/(f-1)$, where $\theta : \mathrm{Im}(f-1)/\ker f \to \mathrm{Im}\, f/\ker(f-1)$. To make clear that this makes sense, if $x \in \ker f$, then $(f-1)(-x) = x$, so the quotient makes sense.

We has another condition: $f + f^* = 1$. Then $\ker f = \mathrm{Im}(f-1)^\perp$, and $\ker(f-1) = (\mathrm{Im}\, f)^\perp$. Because $f(x) = 0$ iff $(f(x), y) = 0\, \forall y$ iff $(x, f^*(y)) = 0\, \forall y$ iff $(x, (1-f)(y)) = 0\, \forall y$ iff $x \perp \mathrm{Im}(f-1)$. We continue to play with the formula, discovering:

Lemma 35.3 *If $f + f^* = 1$, then equation 35.3 holds iff $\mathfrak{c}_1 \stackrel{\text{def}}{=} \mathrm{Im}(f-1)$ and $\mathfrak{c}_2 \stackrel{\text{def}}{=} \mathrm{Im}(f)$ are subalgebras and θ is an isomorphism.*

So we started out being interested in r-matrices that symmetrize to the Casimir, and now we're interested in subalgebras. We say that $\mathfrak{c}_1, \mathfrak{c}_2$ are the *Cayley transform* of f.

How does all this connect with BD triples? We remark that if $\mathfrak{g}_i$, $i = 1, 2$ is the subalgebra $\{h_\alpha, e_\alpha, f_\alpha \in \mathbb{Z}\Gamma_i\}$, then τ induces an isomorphism $\mathfrak{g}_1 \to \mathfrak{g}_2$ by $\tau(h_\alpha) = h_{\tau\alpha}$, etc. So both BD triples and r-matrices give isomorphisms of subalgebras.

Goal: construct $(\mathfrak{c}_1, \mathfrak{c}_2, \theta)$ corresponding to $(\Gamma_1, \Gamma_2, \tau)$, and figure out what r we get. Let's fix notation: each $\mathfrak{g}_i = \mathfrak{n}_i^- \oplus \mathfrak{h} \oplus \mathfrak{n}_i^+$. Then we write $\mathfrak{n}_{\smallsetminus 1}^+ \stackrel{\text{def}}{=} \langle \mathbb{C}e_\alpha \text{ s.t. } \alpha \notin \mathbb{Z}\Gamma_1 \rangle$ and $\mathfrak{n}_{\smallsetminus 2}^- \stackrel{\text{def}}{=} \langle \mathbb{C}f_\alpha \text{ s.t. } \alpha \notin \mathbb{Z}\Gamma_2 \rangle$. We mean these to be subalgebras.

Well, so we saw a rather strong condition: $\mathfrak{c}_i \supseteq \mathfrak{c}_i^\perp$. And we also want $\mathfrak{c}_i \supseteq \mathfrak{g}_i$. So, we chose $\mathfrak{c}_1 = \mathfrak{g}_1 \oplus \mathfrak{n}_{\smallsetminus 1}^+ \oplus V_1$, where $V_1 \subseteq \mathfrak{h}_1^\perp$ satisfies $V_1^\perp \subseteq (\mathfrak{h}_1^\perp \cap V_1)$. Then $\mathfrak{c}_1^\perp = \mathfrak{n}_{\smallsetminus 1}^+ \oplus V_1^\perp \subseteq \mathfrak{c}_1$, and $\mathfrak{c}_1/\mathfrak{c}_1^\perp = \mathfrak{g}_1 \oplus V_1/(V_1^\perp \cap \mathfrak{h}_1^\perp)$. Likewise, we define $\mathfrak{c}_2 = \mathfrak{g}_2 \oplus \mathfrak{n}_{\smallsetminus 2}^- \oplus V_2$.

We want f such that $\theta|_{\mathfrak{g}_1} = \tau$. Since θ respects the decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, we should hope that f does this as well: $f = f_+ + f_0 + f_-$, where $f_+ : \mathfrak{n}_+ \to \mathfrak{n}_+$, etc.

For this to work, we had better have: $\mathrm{Im}(f-1) = \mathfrak{c}_1 = \mathfrak{g}_1 \oplus \mathfrak{n}^+_{\diagdown 1} \oplus V_1$, $\mathrm{Im}\, f = \mathfrak{c}_2$, and $\ker f = c_1^\perp = \mathfrak{n}^+_{\diagdown 1} \oplus V_1/(V_1^\perp \cap \mathfrak{g}_1^\perp)$ and $\ker(f-1) = \mathfrak{c}_2^\perp$.

In particular, $\ker(f-1)$ lives only in the negative nilpotents and the Cartan. In particular, $(f_+ - 1_+) : \mathfrak{n}_+ \to \mathfrak{n}_+$ is invertible, and $\psi \overset{\text{def}}{=} f_+/(f_+ - 1_+)$ should be:

$$\psi(x) = \begin{cases} 0 & x \in \mathfrak{n}^+_{\diagdown 1} \\ \tau(x) & x \in \mathfrak{n}^+_1 \end{cases} \tag{35.5}$$

We remark that since we want $\psi - 1_+ = (f_+ - 1_+)^{-1}$, then $\psi_+ - 1_+$ is invertible if and only if $(f_+ - 1_+)$ is invertible. This gives the second condition in the definition of BD triple:

Lemma 35.4 *$\psi - 1_+$ is invertible iff $\forall \alpha \in \Gamma_1$, there exists n such that $\tau^n(\alpha) \in \Gamma_2 \diagdown \Gamma_1$.*

Proof: If the latter condition folds, then ψ is nilpotent and $(\psi - 1_+)^{-1} = -\sum_{n\geq 0} \psi^n$.

Conversely, suppose that for some $\alpha \in \Gamma_1$, $\tau^n(\alpha) \in \Gamma_1$ for all n. Since τ is a bijection and Γ_1 is finite, then eventually $\alpha = \tau^n(\alpha)$ for some n. So ψ has 1 as an eigenvalue, and so the map is not invertible. □

So one we know f_+, using $f + f^* = 1$, we can figure out f_-. Figuring out what everything is, we see that $f = f_0 - \sum_{n\geq 1} \psi^n + \mathrm{id}_- + \sum_{n\geq 1} (\psi^*)^n$, where ψ^* undoes ψ on the negative nilpotents.

Moreover, we know how to get an r-matrix from f, and using this gives equation 35.2. Most of the tools for the converse are here.

Lecture 36 April 27, 2009

36.1 Manny: Quantum GL_2

Most of this lecture is from Brown and Goodearl, *Lectures on Algebraic Quantum Groups*.

Recall that for a field k, $GL_2(k)$ is an algebraic group. This means that it's an algebraic variety, and that the group structure is compatible. In particular, it's a variety with coordinate ring — well, the coordinate ring for 2×2 matrices is the four-dimensional polynomials $k[a,b,c,d]$, and to make all our matrices invertible, we localize at $D = ad - bc$ — so the coordinate ring of $GL_2(k)$ is $k[a,b,c,d][D^{-1}]$. **NR:** So it's polynomials in five variables, with the added condition that $D = ad - bc$. **M:** Sure, that's another way to say it.

Let us recall the notation from Matt's lecture a while back. We define $\mathcal{O}_q(\mathbb{M}_2(k))$, the quantized coordinate ring of 2×2 matrices, to be the noncommutative ring generated by a,b,c,d with relations $ab = qba$, $ac = qca$, $bd = qdb$, $cd = qdc$, $bc = cb$, $ad - da = (q - q^{-1})bc$. Here we either take $q \in k^\times$ a non-zero element of the field, or we take q a formal variable and replace k with $k(q)$ the field of rational functions.

We define the *quantum determinant* to be $D_q = ad - qbc$.

Proposition 36.1 *D_q is a central element of $\mathcal{O}_q(\mathbb{M}_2(k))$.*

Proof: It's enough to check that D_q commutes with the generators. ****Manny does this for** a**.**** $\square$

NR: This fact follows from the structure of the category of modules. It follows from the functorial isomorphisms $\mathbb{1}\otimes V \cong V\otimes\mathbb{1}$. In particular, the chain $(V\otimes V^*)\otimes V \to \mathbb{1}\otimes V \xrightarrow{\sim} V\otimes 1 \to V\otimes(V^*\otimes V)$ should give the fact that D_q is central. Let us leave this as **Exercise 54**. It's a nice way to do it, because the same thing holds in GL_n, where you can also check it by hand but it becomes tedious.

M: Great. So we have a central element, and we now move to defining the coordinate ring $\mathcal{O}_q(GL_2(k))$. In noncommutative land, the noncommutative localization is doable but difficult and technical. However, localization at a *central* multiplicative subset is as easy as it gets. Take my word for it: writing things as fractions all works. So, we define $\mathcal{O}_q(GL_2(k)) \stackrel{\text{def}}{=} \mathcal{O}_q(\mathbb{M}_q(k))[D_q^{-1}]$, the localization of $\mathcal{O}_q(\mathbb{M}_q(k))$ at $\{{D_q}^n\}_{n=0}^{\infty}$.

B and G present a motto: "Quantized coordinate rings should be noetherian affine domains." "noetherian" should mean left- and right-noetherian, and "domain" should mean no zero divisors. "affine" means finitely generated algebraic.

Then $\mathcal{O}_q(\mathbb{M}_q(k))[D_q^{-1}]$ is clearly affine. Let's check that it's noetherian. This requires that we talk about "skew polynomial rings". Let R be a ring with an endomorphism $\sigma : R \to R$. A (left) σ-derivation $\delta : R \to R$ is an abelian group endomorphism satisfying:

$$\delta(xy) = \delta(x)\, y + \sigma(x)\, \delta(y) \tag{36.1}$$

If you take $\sigma = \text{id}$, you get the usual notion of derivation.

Given such σ, δ, the *skew polynomial ring* $T = R[x; \sigma, \delta]$ is defined by:

1. T is an overring of R, with $x \in T$.
2. $T = \bigoplus_{n=0}^{\infty} R \cdot x^n$ is a free left R-module (i.e. any $f \in T$ can be written uniquely as $\sum a_n x^n$)
3. $\forall r \in R$, we have:
$$xr = \sigma(r)x + \delta(r) \tag{36.2}$$

So, if you take $\sigma = \text{id}$ and $\delta = 0$, then you get the usual polynomial ring. This is some twisting.

Today we only care when σ is an automorphism.

Multiplication is determined by $x^n r$, which is a complicated formula. But an induction argument shows that $x^n r = \sigma^n(r)x^n$+lower degree in x. More generally,

$$\left(\sum_{i=0}^{m} r_i x^i\right)\left(\sum_{j=0}^{n} s_j x^j\right) = r_m \sigma^m(s_n) x^{m+n} + \text{lower degree} \tag{36.3}$$

From this we can prove:

Proposition 36.2 *Let R be a ring. Then $R[x;\sigma,\delta]$ is a domain iff: R is a domain and σ is one-to-one.*

Similarly, there is a noncommutative version of the Hilbert Basis Theorem, which says in the commutative case that if R is commutative noetherian, then so is $R[x]$.

Theorem 36.3 (Hilbert Basis) *Suppose that σ is an automorphism. Then if R is left- (right-)noetherian, then so is $R[x;\sigma,\delta]$.*

There are lots of examples showing that this fails if σ is not an automorphism.

So, if we can realize quantized coordinate rings as iterated skew polynomial rings over some noetherian domain, then we win: the quantized coordinate rings will be noetherian domains.

NR: Conceptually, the skew polynomial ring construction defines a ring structure on the tensor product. You take two algebras, R and $k[x]$, and you build a ring structure on the tensor product. We have seen this construction already in the case of Hopf algebras: the Double, the Smash Product. Then the polynomiality is asking for a filtered version of this. Are there other generalizations? **M:** Yes. There are skew group algebras. Normally in a group algebra the scalers commute with the group elements, but you can do a skew version. Also when $R = k[x]$, $\sigma = \text{id}$, and $\delta = \partial_x$, then $R[y;\sigma,\delta]$ is the differential operators in one variable. But I don't know of any presentation that puts all of these together.

Example 36.1 What about $\mathcal{O}_q(\mathbb{M}_2(k))$? Well, we build $B = k[x][y;\sigma_2][z;\sigma_3]$ by $\sigma_2 : x \mapsto q^{-1}x$ and $\sigma_3 : x \mapsto q^{-1}x, y \mapsto y$. The derivations are all 0.

Now we define $\sigma_4 : B \to B$ by $x \mapsto x$, $y \mapsto q^{-1}y$, and $z \mapsto q^{-1}z$.

At this point, we haven't used any derivations, but now we build $\delta_4 : B \to B$ by $x \mapsto (q-q^{-1})yz$, $y, z \mapsto 0$. Then the claim is that:

$$\mathcal{O}_q(\mathbb{M}_2(k)) \cong B[w;\sigma_4,\delta_4] \tag{36.4}$$

◊

Proposition 36.4 *$\mathcal{O}_q(\mathbb{M}_2(k))$ is a noetherian domain by Hilbert Basis, etc. $\mathcal{O}_q(GL_2(k))$ is qua localization of $\mathcal{O}_q(\mathbb{M}_2(k))$.*

Lastly, we suggest how this related to quantum SL_2. In the commutative case, there is an isomorphism $\mathcal{O}(GL_2(k)) \cong \mathcal{O}(SL_2(k))[z^{\pm 1}]$. It turns out that this extends to the quantum case: $\mathcal{O}_q(GL_2(k)) \cong \mathcal{O}_q(SL_2(k))[z^{\pm 1}]$. Then in fact it turns out that

$$\mathcal{O}_q(\mathbb{M}_2) \twoheadrightarrow \mathcal{O}_q(SL_2) \hookrightarrow \mathcal{O}_q(GL_2) \tag{36.5}$$

NR: There are connections to this and symplectic leaves. There are papers by the same authors. Take $M_{n\times n}(\mathbb{C})$, with the standard Poisson structure — it's the structure on GL_n, but this is algebraic, so it extends, and there there is a rather explicit coorespondence between this and that ****missed****.

36.2 NR: We give a short by useful definition

This Wednesday 4-6, everyone can make it? We're also thinking about a barbecue with the RTG seminar. This would be next Friday, at then end of next week, which is the last week of classes. There will be one more student lecture, which will concern the zoo of these algebras at roots of unity, where the representation theory is almost completely different.

A *ribbon Hopf algebra* is a refinement of the notion of a quasitriangular Hopf algebra. It is a triple (A, R, τ), where A is a Hopf algebra, $R \in A^{\otimes 2}$ invertible defining a quasitriangular structure on A, and $\tau \in A$ central and invertible satisfying:

- $\epsilon(\tau) = 1$
- $\mathcal{S}(\tau) = \tau$
- $\Delta(\tau) = (\tau \otimes \tau)(\sigma(R)R)$ ****σ now is the flip map?****

So τ is almost a "grouplike" element. A *grouplike* element in a bialgebra is g s.t. $\Delta g = g \otimes g$. It is called this because of the bialgebra structure on $\mathbb{C}[G]$.

Another definition: A ribbon braided monoidal rigid category. We have:

rigid existence of duals

monoidal $V \otimes W$ functorial

braided $c_{VW} : V \otimes W \xrightarrow{\sim} W \otimes V$ functorial with conditions.

ribbon There exist functorial isomorphisms $\tau_V : V \xrightarrow{\sim} V$ such that:

- $\tau_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$
- $\tau_{A^*} = (\tau_A)^*$
- $\tau_{A \otimes B} = \tau_A \otimes \tau_B(c_{BA}c_{AB})$

We will say *ribbon category* for a category with everything above.

For example, if (A, R, τ) is a ribbon Hopf algebra, then its category of modules is ribbon. It's rigid because A is a Hopf algebra, braided with $c_{AB} = P_{AB}(\pi_A \otimes \pi_B)R$, where P is the flip map. And we now set $\tau_A = \pi_A(\tau)$.

Next time we will see that the ribbon structure corresponds to the twist, and so ribbon categories allow us to represent the category of framed tangles. We will see that $\mathcal{U}_h\mathfrak{sl}_2$ has such a τ, and so its category provides invariants of framed graphs.

After this, we will study $\mathcal{U}_q\mathfrak{sl}_2$ over $k(q)$, and the culmination will be the structure of such algebras when q is a root of unity.

About this barbecue: May 7th starting around 4, ish?

Lecture 37 April 29, 2009

We begin by correcting the signs in the definition from last time.

A triple (A, R, τ), where A is a Hopf algebra and R is a quasitriangular structure and $\tau \in Z(A)$, is *ribbon* if: $\epsilon(\tau) = 1$, $\mathcal{S}(\tau) = \tau$, and $\Delta(\tau) = (\tau \otimes \tau)\big(\sigma(R)R\big)^{-1}$. ****Later in the lecture we add the axiom that τ be invertible.**** Then a rigid (existence of duals) braided (a crossing c_{AB}) monoidal ($\otimes$) category is *ribbon* if there is a system of functorial isomorphisms $\phi_A : A \xrightarrow{\sim} A$ such that: $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$, $\theta_{A^*} = (\theta_A)^*$, and $\theta_{A\otimes B} = (\theta_A \otimes \theta_B)(c_{BA}c_{AB})$.

Proposition 37.1 *(A, R, τ)-mod is ribbon with $\theta_A = \pi_A(\tau)$.*

Theorem 37.2 *Let (A, R) be a quasitriangular Hopf algebra. Let $R = \sum_i \alpha_i \otimes \beta_i$, and define $u = \sum_i \mathcal{S}(\beta_i)\alpha_i = m^{op}\big((id \otimes \mathcal{S})(R)\big)$. Then:*

- $\epsilon(u) = 1$
- $\mathcal{S}^2(a) = uau^{-1}$
- $\mathcal{S}(u)u = u\mathcal{S}(u) \in Z(A)$
- $\Delta u = u \otimes u\big(\sigma(R)R\big)^{-1}$

Proof: See the paper by Drinfeld "On central elements in quasitriangular Hopf algebras", or see NR's TQFT notes. □

Let use assume that there exists $b \in A$ invertible such that $\Delta b = b \otimes b$ and $\mathcal{S}^2(a) = bab^{-1}$ for every $a \in A$. Then $\epsilon(b) = 1$ and $S(b) = b^{-1}$. For example, let $A = \mathcal{U}_h\mathfrak{sl}_2$, and choose $b = e^{hH/2}$.

Proposition 37.3 *If such b exists, then (A, R, τ) is ribbon with $\tau = b^{-1}u$.*

Incidentally, in any ribbon Hopf algebra, we can choose $b = u\tau^{-1}$. **Question from the audience:** Is it obvious that τ is invertible from the axioms? **Answer:** Let's add this property.

Example 37.1 In $\mathcal{U}_h\mathfrak{sl}_2$, we have $u = \sum_i \mathcal{S}(\beta_i)\alpha_i$ and $\tau = e^{-hH/2}u \in \mathcal{U}_h\mathfrak{sl}_2$. Now, τ is central, so it must act as a scalar on any irreducible representation. So let V_λ be an irreducible $\mathcal{U}_h\mathfrak{sl}_2$-module with highest weight λ and highest-weight vector v_λ, whence $Ev_\lambda = 0$ and $Hv_\lambda = \lambda v_\lambda$.

Then $R = \exp(\frac{h}{4}H \otimes H)\sum_{n\geq 0} a_n(h)E^n \otimes F^n$. Then we compute

$$m^{\mathrm{op}}\big(\mathrm{id} \otimes \mathcal{S})(R)\big) = m^{\mathrm{op}}\left(\sum_{n,m\geq 0} \frac{(-h/2)^n}{n!} a_m(h)H^nE^m \otimes \mathcal{S}(F)^m H^n\right) \tag{37.1}$$

$$= \sum_{n,m\geq 0} \frac{(-h/2)^n}{n!} a_m(h)\, \mathcal{S}(F)^m\, H^{2n}\, E^m \tag{37.2}$$

Then uv_λ has only terms with $m = 0$, since $Ev_\lambda = 0$, and so

$$uv_\lambda = \sum_{n\geq 0} \frac{(-h/2)^n}{n!} \lambda^{2n} = e^{-h\lambda^2/2}v_\lambda \tag{37.3}$$

But this is not the action of the central element. The central element is $\tau = e^{-hH/2}u$, which acts as

$$\tau v_\lambda = e^{-\frac{h}{4}\lambda(\lambda+2)} v_\lambda \tag{37.4}$$

and this is $c_2(\lambda) = \lambda(\lambda+2)$ is the value of the second casimir for $\mathcal{U}\mathfrak{sl}_2$ on v_λ. Since τ is central, it acts by this scalar on the entire irreducible representation.

We define:

$$\tau_\lambda \overset{\text{def}}{=} e^{-\frac{h}{4}\lambda(\lambda+2)} \tag{37.5}$$

$\diamondsuit$

Corollary 37.3.1 *Let us compute*

$$(\pi_\lambda \otimes \pi_\mu)\big(\sigma(R)R\big) = \big(\pi_\lambda(\tau) \otimes \pi_\mu(\tau)\big)(\pi_\lambda \otimes \pi_\mu)\big(\Delta(\tau^{-1})\big) \tag{37.6}$$

$$= \tau_\lambda \tau_\mu (\pi_\lambda \otimes \pi_\mu)\big(\Delta(\tau^{-1})\big) \tag{37.7}$$

But we also have:

$$(\pi_\lambda \otimes \pi_\mu)(\Delta a) = \sum_{\nu=|\lambda-\mu|}^{\lambda+\mu} \pi_\nu(a) P_\nu \tag{37.8}$$

if a is central. Therefore once we know the decomposition $V_\lambda \otimes V_\mu \cong \bigoplus_\nu V_\nu$, then we have

$$(\pi_\lambda \otimes \pi_\mu)\big(\sigma(R)R\big) = \sum_{\nu=|\lambda-\mu|}^{\lambda+\mu} \tau_\lambda \tau_\mu \tau_\nu^{-1} P_\nu \tag{37.9}$$

The story for general simple Lie algebra is just the same as for $\mathfrak{sl}_2$, except that there can by multiplicity in the tensor product: $V_\lambda \otimes V_\mu = \bigoplus_\nu (V_\nu \otimes W_\nu)$. Then the formula remains the same except that P_λ is not multiplicity-free.

We are moving slowly towards the generalization of Schur-Weyl duality. In the classical case it is a duality between the action of SL_2 and the symmetric group, and in the quantum case between quantum SL_2 and the Hecke algebra.

37.1 Framed ribbon tangled graphs

A geometric standard framed tangled graph is: a pair (Γ, ϕ), where Γ is a graph with special properties, and $\phi : \Gamma \hookrightarrow \mathbb{R}^2 \times [0,1]$ is an embedding of the graph. ****There is a third bit of data, mentioned later: a framing. From a previous time, this is a choice of section of the unit normal bundle of the edges.**** We demand:

- Write $\partial\Gamma$ for the set of 1-valent vertices. Then $\phi(\partial\Gamma) \subseteq (L \times \{0\} \cup (L \times \{1\}) \subseteq \mathbb{R}^2 \times [0,1]$, where L is a chosen line in $\mathbb{R}^2$. ****There is a picture****
- Γ should have oriented edges.

- For each vertex, the set of adjacent edges is totally ordered. ****another picture****
- $\phi(\Gamma)$ is transversal to $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ at the boundary, and at every vertex.
- The framing is perpendicular to L at the boundary points and is pointing to the positive direction.
- The framing is parallel at each vertex. ****a complicated picture****

A standard framed tangled graph is the isotopy class of a geometric tangled graph. In other words, we take a geometric graph up to continuous deformations which are constant at the boundary. There are several versions: either constant at the boundary, or you could allow isotopies at the boundary, but the points and framings should not collapse.

These tangles form a natural category. The category of blah blah tangled graphs: Objects are sequences of signs $(\epsilon_1, \dots, \epsilon_n)$, and morphisms form ϵ to σ are blah tangled graphs such that the orientation at the boundary agrees with the +s and −s as in previous days. Composition is by stacking: you can always choose representatives that agree at the boundary, and then you take isotopy class. This is a standard construction in topology. It's very hard to compose geometric objects, because you need smoothness, but gluing topological objects is easy: you take a geometric representative, chosen to be smooth, glue, and then take isotopy classes.

So, let us simplify this. We have already done so in the pictures ****that I haven't drawn yet****: we replace three-dimensional objects with two-dimensional pictures of objects.

Thus, we introduce another category, the category of diagrams, and we choose it in such a way that it is equivalent to the category of tangles.

A *geometric diagram* is a regular projection of a geometric tangled graph to the plane $L \times [0, 1]$. Here "regular" means that the only singularities are double points — the crossings.

So essentially it is a graph with oriented edges with two types of vertices: special four-valent vertices called undercrossings and overcrossings, and also the inner vertices of the graph.

A *diagram* is an equivalence class of geometric diagrams with respect to regular homotopies, plus Reidemeister moves R2 and R3, and an extra move that allows you to pull vertices past crossings.

Theorem 37.4 *To be continued 4pm-6pm in 939.*

Lecture 38 April 29, 2009, extra session 4:00–6:00pm

This morning, we introduces the notion of a framed tangled graphs. It is a category $\mathcal{T}$: objects are sequences of +s and −s, and morphisms are geometric trangled graphs that look like ****picture****, with a rule how the orientation agrees with the signs at the boundary ****and rules about the framing that we described this morning****.

There is another category $\mathcal{D}$, of diagrams of framed tangled graphs. It has the same objects as above, and you should think of the morphisms as regular projections of geometric tangled graphs, with morphisms given by equivalence classes of regular isotopy and the framed Reidemeister moves: ****pictures. R2 and R3 are well-known. What I would call R$\frac{1}{2}$ is a strand with two opposite-oriented R1s that pulls through to the identity string.****

****We need also the rule that we can pull graph-vertices past crossings.****

Theorem 38.1 *The categories $\mathcal{T}$ and $\mathcal{D}$ defined above are equivalent, meaning that there are functors $F : \mathcal{T} \leftrightarrow \mathcal{D} : G$ with $F \circ G \cong id_{\mathcal{D}}$ and $G \circ F \cong id_{\mathcal{T}}$.*

These functors are trivial on objects. On morphism, F finds a representative of an isomorphism class of embedded graphs such that the projection to the plane is regular and such that the framing is always perpendicular to the plane — i.e. the representative has "blackboard framing" — and we project to get a geometric diagram, and then we take its diagram equivalence class. In the other direction, G picks out a geometric diagram from the equivalence class, and lifts it to a geometric tangle, and then takes the isotopy class of the lifting.

Let us now define *colored tangled graphs*, colored by a ribbon category $\mathcal{C}$. This is a category with objects being sequences of pairs $(A_1, \epsilon_1), \ldots, (A_n, \epsilon_n)$, where ϵ_i is a sign and $A_i \in \mathcal{C}$. The morphisms are framed tangled graphs, colored as follows: we assign an object of $\mathcal{C}$ to each edge ****agreeing with the objects on the boundary**** and we assign a morphism to each vertex. The assignment on the vertices is as follows. Remember we had a total ordering of the incoming edges for each vertex. Then we pick a morphism $g : \mathbb{1} \to \bigotimes_i B_i$, where $B_i = A_i$ if the edge is incoming, and A_i^* if the edge is outgoing.

One more remark: the framing at each vertex should be compatible with the total ordering. Here is what we mean by this. Let's take $\mathbb{R}^2$ with the standard orientation $e_1 \wedge e_2$. It you have a total ordering, then there is a unique embedding of the vertex into $\mathbb{R}^2$ such that the total ordering is counterclockwise. Then this orientation plus the framing at the vertex should be the orientation of $\mathbb{R}^3$.

Once we pick a framing, then at each vertex infinitesimally the vertex lies in a plane perpendicular to the framing. We are now demanding that the edges arrive in counterclockwise order.

We can now also define colored diagrams. Objects are $(\epsilon_1, A_i), \ldots, (\epsilon_n, A_n)$, morphisms are diagrams with edges assigned to objects. For diagrams, we require that at each vertex the edges are totally ordered, and that this total ordering agrees with the cyclic ordering in $\mathbb{R}^2$. Then we assign a morphism $g : \mathbb{1} \to A_1 \otimes A_2^* \ldots$, where as before incoming means the object that labels the strand, and outgoing means the dual.

So we have two categories. One is the three-dimensional category $\mathcal{T}(\mathcal{C})$ with colored framed tangles, and the other is a combinatoreal two-dimensional category $\mathcal{D}(\mathbb{C})$ of colored diagrams, where $\mathcal{C}$ is the ribbon category of coloring. As before, we have an equivalence of categories, and this is a small generalization of what we had before: $\mathcal{T}(\mathcal{C}) \cong \mathcal{D}(\mathcal{C})$.

We make a side remark, which is important to CFT and ****missed****. We defined these categories by saying that the endpoints of the $\mathcal{T}(\mathcal{C})$ morphisms are on special lines. There is an equivalent category — another version of $\mathcal{T}(\mathcal{C})$ — where the objects are simply enumerated points in $\mathbb{R}^2$, and the morphisms are "the same" as above, i.e. isotopy classes of tangles, where we demand that the isotopies are constant at the boundary. This is a very geometric category: it is topological in $\mathbb{R}^3$ but geometric in $\mathbb{R}^2$. Then we can generalize this category to the one where the morphisms are three-manifolds with tangles inside them, and the objects are two-manifolds with marked points. When you glue, the geometric information of the intermediate plane disappears, because you take isotopy class.

Well, now if you try to find a representation of this category, you have to ask: morphisms go where? objects go where? If you move the points around, you will need to represent $\pi_1(\mathbb{R}^2 \smallsetminus \{\text{the marked points}\})$. In any case, the mapping from this topological/geometric category to the topological one by putting the points along the line in order.

****the last two paragraphs should have the word "framed" sprinkled about****

Theorem 38.2 *$\mathcal{D}(\mathcal{C})$ is a ribbon category.*

Proof: Arnold first said: "I am more an expert in posing problems than in solving problems."

monoidal $(\epsilon_1, \ldots, \epsilon_n) \otimes (\sigma_1, \ldots, \sigma_m) = (\epsilon_1, \ldots, \epsilon_n, \sigma_1, \ldots, \sigma_m)$ and $\mathbb{1} = \emptyset$

braiding $c_{\epsilon,\sigma}$ =****crossing with top line from NE to SW****

ribbon The twist ****again NE-SW is on top, and the loop is in the east with the open ends in the NW and SW****

rigid We need to define the duals. We use $\big((\epsilon_1, A_1), \ldots, (\epsilon_n, A_n)\big)^* = \big((-\epsilon_n, A_n^*), \ldots, (-\epsilon_1, A_1^*)\big)$. Then $e : (A, \epsilon)^* \otimes (A, \epsilon) \to \mathbb{1}$ is the cap, and i is the cup. ☐

Theorem 38.3 *There is a functor $F : \mathcal{D}(\mathcal{C}) \to \mathcal{C}$. It acts on objects as $F : (\epsilon_1, A_1), \ldots, (\epsilon_n, A_n)\big) \to A_1^{\epsilon_1} \otimes \cdots \otimes A_n^{\epsilon_n}$, where $A^+ \stackrel{\text{def}}{=} A$ and $A^- \stackrel{\text{def}}{=} A^*$. We define $F(\downarrow_A) = id_A$, $F(\curvearrowright_A) = e_A : A^* \otimes A \to \mathbb{1}$, $F() = i_1 : \mathbb{1} \to A \otimes A^*$, and we define the other cups and caps using the ribbon structure. For example, $F(\curvearrowleft^A) = e_{{}^*A} \circ (id \otimes b_{{}^*A}^{-1})$. Here ${}^*A$ is the left dual of A, and A^* is the right dual. The braiding is c_{AB}* ****all these should be re-drawn. We let c_{AB} be the crossing with A on top from NE to SW, and B on bottom from NW to SE.****

The last part of the definition says how to act on morphisms from the emptyset. To the diagram with one vertex labeled by f and all the strands going straight up to the top of the page, we assign $f : \mathbb{1} \to A_1 \otimes A_2^* \to \ldots$.

The theorem is that the F given above extends uniquely to a monoidal functor $\mathcal{D}(\mathcal{C}) \to \mathcal{C}$.

By the way, if the category $\mathcal{C}$ is H-mod, where H is a Hopf algebra, then $(\pi, A)^* = (\pi(\mathcal{S}(a))^*, A^*)$, and ${}^*(\pi, A) = (\pi(\mathcal{S}^{-1}(a))^*, A^*)$, the right- and left- duals.

Proof: Look in the book by Turaev, *Quantum Topological Invariants.* ☐

So now we have a tool. We have a tool how to produce morphisms in the category $\mathcal{C}$. We have

$\mathcal{T}(\mathcal{C}) \to \mathcal{D}(\mathcal{C}) \to \mathcal{C}$, and so if we take any colored tangle, it will go to some morphism in $\mathcal{C}$, and this is an invariant of such tangles.

So this is one application. If you think about it, it produces some interesting spaces of morphisms for $\mathcal{C}$. Let's consider morphisms of $\mathcal{T}(\mathcal{C})$ that near the boundary all the strands point down, and have the same objects $(A_1, \dots, A_n)$. Then for any such tangle, $F(t) \in \mathrm{Hom}_{\mathcal{C}}(A_1 \otimes \cdots \otimes A_n, A_1 \otimes \cdots \otimes A_n)$.

Problem: Describe the image of all $F(t)$ in the algebra of endomorphisms $\mathrm{End}(A_1 \otimes \cdots \otimes A_n)$. We said "algebra": we assume that $\mathcal{C}$ is abelian, whence this is really an algebra, and the image under F is a special subalgebra. **Question from the audience:** These are tangled graphs? Then of course it's onto. **Answer:** Yes. We don't want graphs. Just tangles. Let's call the image of non-graph tangles under F by the name $\mathrm{tEnd}(A_1 \otimes \cdots \otimes A_n)$.

Example 38.1 Let $\mathcal{C} = \mathcal{U}_h\mathfrak{sl}_2$-mod, and let $A_1, \dots, A_n$ be finite-dimensional $\mathcal{U}_h\mathfrak{sl}_2$-mod.

Theorem 38.4 $\mathrm{tEnd}(A_1 \otimes \cdots \otimes A_n)$ *is the centralizer of* $\mathcal{U}_h\mathfrak{sl}_2$.

In particular, if $A_i = V = \mathbb{C}^2$, then $\mathrm{tEnd}(V^{\otimes n}) = TL_n(e^h)$. ◊

Question from the audience: So what is the statement of the problem? **Answer:** For simple $\mathfrak{g}$ a Lie algebra, and $\mathcal{C} = \mathcal{U}_h\mathfrak{g}$-mod, then $\mathrm{tEnd}(A_1 \otimes A_n)$ is the centralizer of the $\mathcal{U}_h\mathfrak{g} \curvearrowright A_1 \otimes A_n$. I think it has not been proven, but it must be true.

Let $\mathfrak{g}$ be a simple Lie algebra, and fix a borel $\mathfrak{b} \subseteq \mathfrak{g}$, and so simple roots $\Gamma \subseteq \Delta_+ \subseteq \Delta$. For convenience, let's enumerate simple roots. Then we define $\mathcal{U}_h\mathfrak{g}$ is an associative algebra over $\mathbb{C}[[h]]$ generated by H_i, E_i, F_i as i runs through Γ, with defining relations:

$$[H_i, H_j] = 0 \tag{38.1}$$
$$[H_i, E_j] = a_{ij}E_j \tag{38.2}$$
$$[H_i, F_j] = -a_{ij}F_j \tag{38.3}$$
$$[E_i, F_j] = \frac{\sinh(hd_iH_i/2)}{\sinh(hd_i/2)}\delta_{ij} \tag{38.4}$$

and also the q-deformed Serre relations: if $i \neq j$ and $a_{ij} \neq 0$, then

$$[[E_i, E_j]_{e^{ha_{ij}/2}}, E_j]_{e^{ha_{ij}/2}} \cdots = 0 \tag{38.5}$$

and similarly for h. This should be the usual Serre relations $(\mathrm{ad}_{E_j})^{1-a_{ij}}E_i = 0$. The symbol $[,]_q$ is the "q-commutator; we will take

$$[A, B]_q \stackrel{\text{def}}{=} qAB - q^{-1}BA \tag{38.6}$$

Next time we will give the correct definition.

The interesting and important fact about such algebras is that $\mathcal{U}_h\mathfrak{g}$ is a Hopf algebra with the comultiplication acting on generators by:

$$\Delta H_i = H_i \otimes 1 + 1 \otimes H_i \tag{38.7}$$
$$\Delta E_i = E_i \otimes e^{hd_i H_i/2} + 1 \otimes E_i \tag{38.8}$$
$$\Delta F_i = F_i \otimes 1 + e^{-hd_i H_i/2} \otimes F_i \tag{38.9}$$

Question from the audience: Can you say more about the q-commutators? **Answer:** We will give an example. Above are the q-Serre relations. Before Serre, one would describe $\mathfrak{g}$ by the entire root system. Chevalley noticed the Serre relations, and Serre noticed that they are enough.

Example 38.2 For $\mathcal{U}_h\mathfrak{sl}_n$, the q-Serre relations are:

$$E_i^2 E_{i\pm 1} - (e^h + e^{-h} E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 = 0 \tag{38.10}$$

which rearranges to

$$e^{h/2} E_i \left(e^{-h/2} E_i E_{i+1} - e^{h/2} E_{i+1} E_i \right) - e^{-h/2} \left(e^{-h/2} E_i E_{i+1} - e^{h/2} E_{i+1} E_i \right) E_i = 0 \tag{38.11}$$

$\diamondsuit$

So what is important about this algebra? Well, there are many important things. But one is that there is an important analogue of the Weyl group. And another is:

Theorem 38.5 *$\mathcal{U}_h\mathfrak{g}$ is quasitriangular with*

$$R = \exp\left(\frac{h}{2} \sum_{i,j=1}^{r} b_{ij} H_i \otimes H_j \right) \left(1 + \sum_{i=1}^{r} \sinh\left(\frac{hd_i}{2} \right) E_i \otimes F_i + \dots \right) \tag{38.12}$$

where r is the rank of $\mathfrak{g}$ and a_{ij} is the Cartan matrix, and $b = (da)^{-1}$, where $(da)_{ij} = d_i a_{ij}$ is the symmetrized Cartan matrix. And $\dots$ are the higher terms in E, F.

Another important fact:

Exercise 55 *$\mathcal{S}^2(a) = e^{\frac{h}{2}H_\rho} a e^{-\frac{h}{2}H_\rho}$, where H_ρ is the element of the Cartan corresponding to $\rho = \frac{1}{2}\sum_{\alpha \in \Delta_+} \alpha$.*

Corollary 38.5.1 *Let $\tau = \sum_i \mathcal{S}(\beta_i)\alpha_i e^{-\frac{h}{2}H\rho}$. Then $(\mathcal{U}_h\mathfrak{g}, R, \tau)$ is a ribbon Hopf algebra.*

Question from the audience: We motivated ribbon categories so as to compute invariants of tangles. Is there a computer somewhere that actually does this? **Answer:** I can imaging two questions in the direction you're asking. One: are these actually computable? Two: are they useful? As to One: See the homepage of D. Bar-Natan. Two: we can ask how precise are these invariants? And the answer is that they are still not very precise, that they cannot distinguish certain tangles. But an important part of these tangle invariants is that they are not just tangle invariants, but they have a long history with physics and quantum field theory.

****NR tells a story, but has declared it off the record.****

Lecture 39 May 1, 2009

Last time we started talking about $\mathcal{U}_h\mathfrak{g}$ for arbitrary simple $\mathfrak{g}$. At this case, it is better to do a slightly different version, called $\mathcal{U}_q\mathfrak{g}$. We first explain this for $\mathfrak{sl}_2$:

39.1 $\mathcal{U}_q\mathfrak{sl}_2$

This is an algebra over rational functions $\mathbb{C}(q)$, generated by $K^{\pm 1}, E, F$ with defining relations

$$KE = e^2 EK,\ KF = q^{-2}FK,\ [E,F] = \frac{K - K^{-1}}{q - q^{-1}} \tag{39.1}$$

Then the mapping $\mathcal{U}_q\mathfrak{sl}_2 \to \mathcal{U}_h\mathfrak{sl}_2[h^{-1}]$ given by $\phi(q) = e^{h/2}$, $\phi(K) = e^{hH/2}$, $\phi(E) = E$, and $\phi(F) = F$, induces an algebra homomorphism. Then we can easily guess that $\mathcal{U}_q\mathfrak{sl}_2$ is a Hopf algebra with

$$\Delta K = K \otimes K,\ \Delta E = E \otimes K + 1 \otimes E,\ \Delta F = F \otimes 1 + K^{-1} \otimes F \tag{39.2}$$

However, they are different algebras. In particular, $\mathcal{U}_q\mathfrak{sl}_2$ is not a quasitriangular Hopf algebra. The R-matrix for $\mathcal{U}_h\mathfrak{sl}_2$ does not pull back to an element in the completion of $(\mathcal{U}_q\mathfrak{sl}_2)^{\otimes 2}$. **Question from the audience:** What is the completion? **Answer:** We localize near q: add formal power series in $(q-1)$. But then $K-1$ is not close to 0. See, the element $R = \exp(\frac{h}{4}H \otimes H)(1 + \dots)$ does not come from any natural construction in $\mathcal{U}_q\mathfrak{sl}_2$. You could complete $\mathcal{U}_q\mathfrak{sl}_2$ to power series in $K-1$, but then you really would get back $\mathcal{U}_h\mathfrak{sl}_2$, and in particular you would lose representations if you do this.

There are two types of finite-dimensional representations of $\mathcal{U}_q\mathfrak{sl}_2$, called V_λ^ϵ, where $\epsilon = \pm$, and $\lambda \in \mathbb{Z}_{\geq 0}$. Then $\mathbb{C}_\lambda^\epsilon = \mathbb{C}v_0^\lambda(\epsilon) \oplus \cdots \oplus \mathbb{C}v_\lambda^\lambda(\epsilon)$. So the dimension is just the dimension of V_λ the representation of $\mathfrak{sl}_2$. The action is:

$$Kv_n^\lambda(\epsilon) = \epsilon q^{\lambda - 2n} v_n^\lambda(\epsilon) \tag{39.3}$$

$$Ev_n^\lambda(\epsilon) = \epsilon[\lambda + 1 - n]_q v_{n-1}^\lambda(\epsilon) \tag{39.4}$$

$$Fv_n^\lambda(\epsilon) = [n+1]_q v_{n+1}^\lambda(\epsilon) \tag{39.5}$$

Proposition 39.1 *Finite-dimensional $\mathcal{U}_q\mathfrak{sl}_2$ modules are completely reducible.*

In particular, one can show that $V_\lambda^\epsilon \otimes V_\mu^{\epsilon'} \cong \bigoplus_{\nu=|\lambda-\mu|}^{\lambda+\mu} V_\nu^{\epsilon\epsilon'}$.

In particular, contained within the category of $\mathcal{U}_q\mathfrak{sl}_2$ modules is the category $\underline{\mathcal{U}_\mathrm{q}\mathfrak{sl}_2 - \mathrm{mod}^+}$, of objects with $\epsilon = +$. It is a monoidal subcategory.

Moreover, $\underline{\mathcal{U}_{\mathrm{q}}\mathfrak{sl}_2 - \mathrm{mod}^+}$ is braided. This is trivial because we can use the braiding from $\mathcal{U}_h\mathfrak{sl}_2$ evaluated at q. The following is a simple exercise:

$$R = \exp(\frac{h}{4} H \otimes H) \sum_{n \geq 0} \frac{(q-q^{-1})^n}{[n]_q!} E^n \otimes F^n \tag{39.6}$$

where $q = e^{h/2}$. Then let's understand the action of R on a product of irreducibles:

$$(\pi_\lambda \otimes \pi_\mu)(R) = \exp(\frac{h}{4} \pi_\lambda(H) \otimes \pi_\mu(H)) f(q) \tag{39.7}$$

The E, F part is fine, because they act by Laurant polynomials in q. THe only problem is the other part, and it's only a problem because of the $h/4$, so we really ought to extend by scalars $\underline{\mathcal{U}_{\mathrm{q}}\mathfrak{sl}_2 - \mathrm{mod}^+} \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}[q^{\pm 1/2}]$.

A diagram commutes:

$$\begin{array}{ccc} \mathcal{U}_q\mathfrak{sl}_2 & \xrightarrow{\phi_+} & \mathcal{U}_h\mathfrak{sl}_2[h^{-1}] \\ \downarrow \pi_\lambda^+ & & \downarrow \pi_\lambda \\ \mathrm{End}(V_\lambda^+)(q) & \xrightarrow{q \mapsto e^{h/2}} & \mathrm{End}(V_\lambda)(\mathbb{C}[h^{-1}, h]] \end{array}$$

Proposition 39.2 $\underline{\mathcal{U}_{\mathrm{q}}\mathfrak{sl}_2 - \mathrm{mod}^+}$ *is braided monoidal.*

Proof: Exercise 56

Theorem 39.3 $Z(\mathcal{U}_q\mathfrak{sl}_2) = \mathbb{C}[c]$, *where* $c = EF + \dfrac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2}$.

Proof: Similar to $\mathcal{U}_h\mathfrak{sl}_2$. ☐

Let us now describe the integral forms of $\mathcal{U}_q\mathfrak{sl}_2$.

We will tautologically get rid of the denominators. We introduce the following notations:

$$\mathcal{A} = \mathbb{Z}[q,q^{-1}] [m]_q \qquad \stackrel{\mathrm{def}}{=} (q^m - q^{-m})/(q-q^{-1}) \in \mathcal{A} \tag{39.8}$$

$$[m]_q! \stackrel{\mathrm{def}}{=} [m]_q \cdots [1]_q \in \mathcal{A} \tag{39.9}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!} \in \mathcal{A} \tag{39.10}$$

$$[K;m]_q \stackrel{\mathrm{def}}{=} \frac{Kq^m - K^{-1}q^{-m}}{q-q^{-1}} \tag{39.11}$$

$$\begin{bmatrix} K;c \\ r \end{bmatrix}_q \stackrel{\mathrm{def}}{=} \prod_{s=1}^{r} \frac{[K;c+1-s]_q}{[s]_q} \tag{39.12}$$

$$\tag{39.13}$$

Lemma 39.4

$$[K;m]_q = [K;0]_q q^{-m} + [m]_q K \tag{39.14}$$
$$\Delta[K;0]_q = [K;0]_q \otimes K + K^{-1} \otimes [K;0]_q \tag{39.15}$$
$$\mathcal{S}\Big([K;0]_q\Big) = -[K;0]_q \tag{39.16}$$
$$\epsilon\Big([K;0]_q\Big) = 0 \tag{39.17}$$

Then we define $\mathcal{U}_{\mathcal{A}}\mathfrak{sl}_2$ the subalgebra of $\mathcal{U}_q\mathfrak{sl}_2$ generated over $\mathcal{A}$ by $E^{(n)} \stackrel{\text{def}}{=} \dfrac{E^n}{[n]_q!}$, $F^{(n)} \stackrel{\text{def}}{=} \dfrac{F^n}{[n]_q!}$, $\begin{bmatrix} K;c \\ r \end{bmatrix}$, and $K^{\pm 1}$.

Proposition 39.5 *$\mathcal{U}_{\mathcal{A}}\mathfrak{sl}_2$ is a Hopf algebra over A.*

Proof: We only outline the proof. You can find the details in ****?****

We have

$$\begin{bmatrix} K;r \\ c \end{bmatrix} E^{(n)} = E^{(n)} \begin{bmatrix} Kq^{2n};r \\ c \end{bmatrix} \tag{39.18}$$
$$\begin{bmatrix} K;r \\ c \end{bmatrix} F^{(n)} = \dots \tag{39.19}$$

Moverover, []s commute between themselves and K. Also, there is a long inductive proof that:

$$E^{(n)}F^{(m)} = F^{(m)}E^{(n)} + ()F^{(m-1)}E^{(n-1)} + ()F^{(m-2)}E^{(n-2)} + \dots \tag{39.20}$$

where () are counits ****?**** of [].

Exercise 57 *Find this relation.*

For example,

$$EF^{(m)} = F^{(m)}E + ()F^{(m-1)} \tag{39.21}$$

Now for the coalgebra structure. We check that $\Delta[] = \sum(\text{coeff in } \mathcal{A})[] \otimes []$. Moreover, it is a computation exercise to prove that

$$\Delta E^{(n)} = \sum_{k=0}^{n} q^{-k(n-k)} E^{(k)} \otimes K^k E^{(k)} \tag{39.22}$$

$$\Delta F^{(n)} = \sum_{k=0}^{n} q^{k(n-k)} F^{(k)} K^{-(n-k)} \otimes F^{(k)} \tag{39.23}$$

□

Did MH mention the notion of *Chevalley groups* last semester? ****I don't remember them, but it seems that the answer is "briefly".****

These are important when working over positive characteristic.

q = 1 We define $\mathcal{U}_\mathbb{Z}\mathfrak{sl}_2$ the subalgebra of $|Uu\mathfrak{sl}_2$ generated by:

$$\begin{bmatrix} H \\ n \end{bmatrix} = \frac{H(H-1)\cdots(H-n+1)}{n(n-1)\cdots 2\cdot 1}, \ E^{(n)} = \frac{E^n}{n!}, \ F^n = \frac{F^n}{n!} \tag{39.24}$$

Then $\mathcal{U}_\mathbb{Z}\mathfrak{sl}_2$ is a Hopf algebra over $\mathbb{Z}$. See, the universal enveloping algebra is bad mod p: $[H,E] = 2E$, and so $[H,E^p] = 2pE^p$. In particular, E^p is central over $\mathbb{F}_p$. But in this divided-polynomial ring, it's not too bad: the center is still reasonable.

We will see that the same happens at roots of unity. You can either do divided powers, which is what we did above, or you can work with the universal enveloping algebra, which is another integral version with completely different properties at roots of unity.

Lecture 40 May 4, 2009

Last time we described integral versions of $\mathcal{U}_q\mathfrak{sl}_2$ over $\mathbb{Z}[q,q^{-1}]$. One of these had divided powers, generated by $E^{(n)}, F^{(n)}, \begin{bmatrix} K;c \\ r \end{bmatrix}_q, K^{\pm 1}$. The other does not have divided powers, and is generated by $\bar{E} = (q-q^{-1}), \bar{F} = (q-q^{-1})F, K^{\pm 1}$.

Let us call $\mathcal{A} = \mathbb{Z}[q,q^{-1}]$, and let us call the divided-power version $\mathcal{U}_\mathcal{A}\mathfrak{sl}_2$. Then the other is $\tilde{\mathcal{U}}_\mathcal{A}\mathfrak{sl}_2$, a subring of $\mathcal{U}_q\mathfrak{sl}_2/\mathbb{C}(q)$. The only problematic relationship is $[E,F]$, and after multiplying by $(q-q^{-1})$ we have $[\bar{E},\bar{F}] = (q-q^{-1})(K-K^{-1})$, so $\tilde{\mathcal{U}}_\mathcal{A}\mathfrak{sl}_2$ really is defined over $\mathcal{A}$.

Let us turn our attention now to $\mathcal{U}_q\mathfrak{g}$ for $\mathfrak{g}$ simple. Let Γ be the set of simple roots, which we identity with the noes of the Dynkin diagram. We have a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$, and let us write a_{ij} for the Cartan matrix $(i,j \in \Gamma)$, $d_i = (\alpha_i,\alpha_i)/2$, and $q_i = q^{d_i}$, so that $q_i^{a_{ij}} = q_j^{a_{ji}}$.

Then we define $\mathcal{U}_q\mathfrak{g}$ to be generated by $K_i^{\pm 1}, E_i, F_i$ subject to

$$K_iE_j = q_i^{a_{ij}}E_jK_i, \ K_iF_j = q_i^{a_{ij}}F_jK_i, \ [E_i,F_j] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \ K_iK_j = K_jK_i \tag{40.1}$$

We also have the q-Serre relations:

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E^r = 0, \quad i \neq j \tag{40.2}$$

and the same for F. In particular, when $q = 1$, we have:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} E_i^{1-a_{ij}-k} E_j E_i^k = \sum \underbrace{[E_i, \ldots [E_i, [E_i}_{1-a_{ij} \text{ times}}, E_j] \tag{40.3}$$

This is all over the rational functions $\mathbb{C}(q)$, but it doesn't have to be $\mathbb{C}$.

Drinfeld introduced this in analogy with $\mathfrak{sl}_2$.

Theorem 40.1 *$\mathcal{U}_q\mathfrak{g}$ is a Hopf algebra with $\Delta K_i = K_i \otimes K_i$, $\Delta E_i = E_i \otimes K_i^{-1} + 1 \otimes E_i$, and $\Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i$, and with $\epsilon(E_i) = \epsilon(F_i) = 0$ and $\epsilon(K_i) = 1$.*

There is a whole collection of integral forms. In $\mathfrak{sl}_2$, you could for example take divided powers in E but not in F; then we'd get a mixture of the forms in the first paragraph. It's a perfectly valid integral form with its own interesting representation theory. The same thing happens for $\mathcal{U}_q\mathfrak{g}$: there are two extreme examples, either with all divided powers or with no divided powers. And then there are many intermediate forms. In fact, writing down the entire list probably isn't a difficult problem, but it hasn't been done. We will list the two extreme examples, both defined over $\mathcal{A} = \mathbb{Z}[q^{\pm 1}]$.

divided powers We define $\mathcal{U}_{\mathcal{A}}\mathfrak{g}$ the subalgebra of $\mathcal{U}_q\mathfrak{g}$ over $\mathcal{A}$, generated by $E+i^{(n)}$, $F_i^{(n)}$, $\begin{bmatrix} K_i; c \\ r \end{bmatrix}_q$, $K_i^{\pm 1}$. Well, actually we need divided powers of all the roots — we want $E_\alpha^{(n)}$ — but we haven't even defined E_α for α not a simple root.

We make a digression, looking for $E_\alpha^{(n)}$. To define this, we will need to understand the action of the Weyl algebra. We understand the *quantum Weyl group* thusly:

Let us define $\mathcal{B}(\mathfrak{g})$ the *braid group* for $\mathfrak{g}$.

$$\mathcal{B}(\mathfrak{g}) = \left\langle T_i : \underbrace{T_iT_jT_iT_j\dots}_{m_{ij}\text{ times}} = \underbrace{T_jT_iT_jT_i\dots}_{m_{ij}\text{ times}} \right\rangle \tag{40.4}$$

Where we write $m_{ij} = 2, 3, 4, 6$ for $a_{ij}a_{ji} = 0, 1, 2, 3$, respectively. If we also impose $T_i^2 = 1$, we get the *Tits group*, which for finite root systems is isomorphic to the Weyl group.

For example, for the groups of A-type, we get the usual braid group:

$$\mathcal{B}(A_n) = \langle T_i : T_iT_j = T_jT_i, a_{ij} \neq 0,\ \ T_iT_jT_i = T_jT_iT_j, a_{ij} = -1 \rangle \tag{40.5}$$

We make the following claim: the Weyl group $\mathcal{W}(\mathfrak{g}) = \mathcal{B}(\mathfrak{g})/\langle T_i^2 = 1\rangle$.

Let $\mathfrak{g}$ be a simple Lie algebra. We fix $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$. Then $\mathcal{W}(\mathfrak{g})$ acts by reflections on $\mathfrak{h}$. Rhetorical question: can the action of $\mathcal{W}$ on $\mathfrak{h}$ be extended canonically to all of $\mathfrak{g}$?

No. Consider $\mathfrak{sl}_2$, generated by H, E, F. Then $\mathcal{W}$ is generated by a unique reflection T, which acts by $T(H) = -H$. We could extend this to $T(E) = F$ and $T(F) = E$, but in fact we could extend it to $T(E) = \lambda F, T(F) = \lambda^{-1}E$. So there is a one-parametric family of extensions.

In fact, this is the action of $N(H) = \mathbb{Z}/2 \ltimes \mathbb{C}^\times$, the normalizer of the Cartan subalgebra in $SL_2(\mathbb{C})$. Then $SL_2(\mathbb{C})$ acts via ad^* on $\mathfrak{sl}_2$, and so $N(H) \subseteq SL_2(\mathbb{C})$ also acts naturally on $\mathfrak{sl}_2$.

Once we pick a basis, we can set $T(E) = F$ and $T(F) = E$ — for any such reflection, there is unique a basis in which it looks like this.

In any case, we pick a basis of $\mathfrak{g}$ a simple Lie algebra, and present the action with $\lambda = 1$ of $\mathcal{W}(\mathfrak{g})$ by:

$$T_i(E_i) = -F_i,\, T_i(F_i) = -E_i,\, T_i(H_j) = H_j - a_{ij}H_i, \tag{40.6}$$

$$T_iE_j = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} E_i^{(-a_{ij}-r)} E_j E_i^{(r)} \tag{40.7}$$

and similarly for F_i, where $E_i^{(n)} = E_i^n/n!$.

Any standard textbook on Lie algebras or on Chevalley groups will have this.

Theorem 40.2 (Lusztig) *The mapping T_i defined by*

$$T_i(E_i) = -F_iK_i,\, T_i(F_i) = -K_i^{-1}E_i,\, T_i(K_j) = K_jK_i^{-a_{ji}}, \tag{40.8}$$

$$T_iE_j = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} E_i^{(-a_{ij}-r)} E_j E_i^{(r)} q^{-r} \tag{40.9}$$

and similarly for F_i (with q^r), where $E_i^{(n)} = E_i^n/[n]_q!$ — This mapping extends to the action of $\mathcal{B}(\mathfrak{g})$ by automorphisms of $\mathcal{U}_q\mathfrak{g}$.

Question from the audience: The group acts on $\mathfrak{g}$, but now on $\mathcal{U}\mathfrak{g}$? **Answer:** It acts by automorophism on $\mathfrak{g}$ so extends to $\mathcal{U}\mathfrak{g}$. **Question from the audience:** But what about equation 40.7? **Answer:** This is the Serre relation; it really is in $\mathfrak{g}$, because it is primitive.

Lemma 40.3 (parametrization of positive roots) *Let w_0 be the longest element of $\mathcal{W}\mathfrak{g}$. Remember that $\mathcal{W}$ is generated by elements $s_1, \ldots, s_r$, where $s_i = [T_i]$ is the equivalence class when quotienting by $T_i^2 = 1$. Then any w can be written as $w = s_{i_1} \cdots s_{i_l}$; we say this is a* reduced expression *if l is as small as possible for that w.*

So, now we fix a reduced expression for the longest elements: $w_0 = s_{i_1} \cdots s_{i_N}$, where $N = |\Delta_+|$ is the number of positive roots.

Then the set $\{\alpha_{i_1},\, s_{i_1}(\alpha_{i_2}),\, s_{i_1}s_{i_2}(\alpha_{i_3}),\, \ldots\}$ is the set Δ_+ of positive roots.

This is a combinatorial statement, which we didn't prepare a proof of. For example, in $\mathfrak{sl}_n$, the simple roots are elements $E_{i,i+1}$. Then $s_i(\alpha_{i\pm 1}) = [E_{i,i+1}, E_{i+1,i+2}] = E_{i,i+2}$.

Then, we now define the root elements of $\mathcal{U}_q\mathfrak{g}$ by $E_{s_{i_1}\cdots s_{i_k}(\alpha_{i_{k+1}})} = T_{i_1} \cdots T_{i_k}(E_{i_{k+1}})$, and similarly for F.

In this way, we define E_α, F_α for $\alpha \in \Delta_+$.

But if you think about this definition, you realize that it is rather artificial. But there is no better definition. The definition depends on a choice of decomposition of w_0. There are many of these decompositions. For each decomposition, you have a set of positive roots. All these different definitions agree on the first element: all the definitions agree on simple roots. This is a non-obvious theorem.

The bad news is that the definition is non-canonical. The good news is that different choices of the decomposition are conjugate in $\mathcal{W}(\mathfrak{g})$, and the corresponding choices of positive roots are also conjugate. So the choice is almost canonical.

Next time we will have more discussion of this, and we will give a description of the R-matrix.

Now fix $\{E_\alpha, F_\alpha\}$. Then we define $\mathcal{U}_{\mathcal{A}}\mathfrak{g}$ to be generated by $\{E_\alpha^{(n)}, F_\alpha^{(n)}, K_i^{\pm 1}, \begin{bmatrix} K_i; c \\ r \end{bmatrix}_q\}$.

So this is one extreme of the definition. Well, this is a theorem that this definition works over $\mathcal{A}$. In any case, this is very important for QFTs, because at roots of unity the representation theory of this specific integral form gives quantum Chern-Simons theory.

One final remark. The other integral form, generated by $\bar{E}_\alpha = (q - q^{-1})E_\alpha$ and by $\bar{F}_\alpha$, and $K_i^{\pm 1}$. You can check that this is also defined over $\mathcal{A}$.

When $q = 1$ case, you can see that these are very different mod p.

Lecture 41 May 6, 2009

****We began with course evaluations.****

Last time we described two integral forms of $\mathcal{U}\mathfrak{g}$.

$\mathcal{U}_{\mathcal{A}}\mathfrak{g}$, generated by divided powers of root elements $E_\alpha^{(n)}$, $F_\alpha^{(n)}$, $K_i^{\pm 1}$, and $\begin{bmatrix} K_i; r \\ c \end{bmatrix}_q$.

$\overline{\mathcal{U}_{\mathcal{A}}\mathfrak{g}}$, generated by $\bar{E}_\alpha = (q_\alpha - q_\alpha^{-1})E_\alpha$, $\bar{F}_\alpha = (q_\alpha - q_\alpha^{-1})F_\alpha$, and $K_i^{\pm 1}$. Here q_α is defined to be $q^{(\alpha,\alpha)/2}$.

others The above are two extremes, and the others should lie between them. Classifying all integral forms has not been finished, but Noah should do it soon.

These definitions are not entirely canonical. They depend on a choice of $w_0 = s_{i_1} \cdots s_{i_N}$. There are nontrivial isomorphisms relating the different choices.

41.1 Multiplicative formula for R

Everything we do works over $\mathbb{Q}[[h]]$. We have $\mathcal{U}_h\mathfrak{sl}_2$ over $\mathbb{Q}[[h]]$. Let us define $(U_h\mathfrak{sl}_2)_W$ to be the algebra $\mathbb{Q}[[h]]$ generated by E, F, H, w, such that

$$wEw^{-1} = -e^{h/2}F,\ wFw^{-1} = -e^{-h/2}E,\ wHw^{-1} = -H,\ w^2 = \tau\varepsilon \tag{41.1}$$

where $\varepsilon^2 = 1$ (a choice of ± 1) and τ is the ribbon element defined earlier. We should write the last equation simply as $(w^2\tau^{-1})^2 = 1$.

Theorem 41.1 *The mapping $\Delta : (U_h\mathfrak{sl}_2)_W \to \left((U_h\mathfrak{sl}_2)_W\right)^{\otimes 2}$, given by the the comultiplication on $\mathcal{U}_h\mathfrak{sl}_2$ and by $\Delta w = R^{-1} w \otimes w$; and we extend $\mathcal{S} : w \mapsto w e^{hH/2}$ and $\epsilon(w) = 1$ — these are a Hopf algebra structure on $(U_h\mathfrak{sl}_2)_W$.*

Remarks:

1. G acts on $\mathfrak{sl}_2$, and so on $\mathcal{U}\mathfrak{sl}_2$ via ad_G. Then G contains the subgroup $N(H) \cong W \ltimes H$, and this also acts on $\mathcal{U}\mathfrak{sl}_2$. Then $W = N(H)/H$, and so does not act canonically. But if we choose for each $w \in W$ an element $\dot{w} \in [w] \subseteq N(H)$, where $[w]$ is the preimage of w, and so has size H — if we choose representatives, then we can make $\mathrm{ad}_{\dot{w}}(a)$, and then form $\mathcal{U}\mathfrak{sl}_2 \rtimes \mathbb{C}[N(H)]$. Then the defining relations is that $\dot{w}x\dot{w}^{-1} = \mathrm{ad}_{\dot{w}}(x)$. **Question from the audience:** How good do we need these choices to be? **Answer:** We are choosing a splitting of the semidirect product $W \ltimes H$. **Question from the audience:** Don't we have too many elements? **Answer:** Oh, yes. Any choice of splitting $\cdot : W \to N(H)$ defines a subalgebra $(\mathcal{U}\mathfrak{sl}_2, \dot{W})$ of $\mathcal{U}\mathfrak{sl}_2 \rtimes \mathbb{C}[N(H)]$. **Question from the audience:** So we are making this choice? **Answer:** Yes, the demand that $(w^2\tau^{-1})^2 = 1$ pins down the splitting. It's not that important, but it is convenient; what's important is that w^2 be central.
2. One more remark: We can add a copy of the Cartan to the quantum algebra defined above, to get $(\mathcal{U}_h\mathfrak{sl}_2)_W \rtimes \mathbb{C}[H] = (\mathcal{U}\mathfrak{sl}_2 \rtimes \mathbb{C}[N(H)])_h$. In general, one can take any algebra A and group Γ acting on it by automorphisms, and build $A \rtimes \mathbb{C}[\Gamma]$. When you do this as above for $\mathfrak{sl}_2$, it turns out that $\mathcal{U}\mathfrak{sl}_2 \rtimes \mathbb{C}[N(H)]$ is a Hopf algebra.
3. $(\mathcal{U}_h\mathfrak{sl}_2)_W$ is not canonical. It depends on the choice of splitting. $(\mathcal{U}_h\mathfrak{sl}_2)_W \rtimes \mathbb{C}[H]$ is canonical. The extra H lets us get from any choice to any other.

Ok, so now we replace $\mathfrak{sl}_2$ by $\mathfrak{g}$, and choose a splitting $w_i \mapsto \dot{w}_i$, but we will leave off the $\cdot$s. Then we define $(U_h\mathfrak{g})_W$ as the algebra generated by $\mathcal{U}_h\mathfrak{g}$ and elements w_i, where we declare that

$$w_i w_j w_i = w_i w_j w_i \qquad a_{ij} = -1 \tag{41.2}$$

$$w_i w_j w_i w_j = w_j w_i w_j w_i \qquad a_{ij} = -2 \tag{41.3}$$

$$w_i w_j w_i w_j w_i = w_j w_i w_j w_i w_j \qquad a_{ij} = -3 \tag{41.4}$$

and also that $w_i a w_i^{-1} = T_i(a)$, where T_i is the Lusztig automorphism giving the action of $B(\mathfrak{g})$.

Then the quadropole (E_i, F_i, H_i, w_i) generates a copy of $(\mathcal{U}_h\mathfrak{sl}_2)_W$ corresponding to the root i.

Theorem 41.2 *The algebra $(\mathcal{U}_h\mathfrak{g})_W$ is a Hopf algebra with $\Delta w_i = R(i)^{-1} w_i \otimes w_i$, where*

$$R(i) = \exp\left(\frac{hd_i}{4} H_i \otimes H_i\right) \sum_{n=0}^{\infty} \frac{(q_i - q_i^{-1})^n}{[n]_{q_i}!} E_i^n \otimes F_i^n \tag{41.5}$$

also $\mathcal{S}(w_i) = w_i e^{-hH_i/2}$ and $\epsilon(w_i) = 1$.

Then this is essentially $\mathcal{U}_h\mathfrak{g} \rtimes B(\mathfrak{g})$.

So $\mathfrak{sl}_2$ is remarkable because the Weyl group has only one generator, and it is the longest root and the only root. When you act by it **?**

So, let us choose a decomposition $w_0 = s_{i_1} \cdots s_{i_N}$. Then let us look at $\tilde{w}_0 = w_{i_1} \cdots w_{i_N}$. If we are lucky, maybe, we could hope that

$$\Delta \tilde{w}_0 \stackrel{?}{=} R^{-1} \tilde{w}_0 \otimes \tilde{w}_0 \tag{41.6}$$

If this is true, then we would have a wonderful product formula:

$$\Delta(w_{i_1}) \cdots \Delta(w_{i_N}) = R^{-1} w_{i_1} \cdots w_{i_N} \otimes w_{i_1} \cdots w_{i_N} \tag{41.7}$$

But the left-hand-side we can express:

$$\left(R(i_1)^{-1} w_{i_1} \otimes w_{i_1}\right) \cdots \left(R(i_N)^{-1} w_{i_N} \otimes w_{i_N}\right) \tag{41.8}$$

Then we move the factors of R past the w_is, and then cancel the w_is. Then we would see that the universal R matrix could be expressed as a product of $\mathfrak{sl}_2$-R-matrices, twisted by the w action. Remember that we said that the roots are given in terms of the decomposition by twisting the decomposition by reflections. Then if everything works we would have

$$R^{-1} = R(\alpha_{i_1})^{-1}\, R(s_{i_1}(\alpha_{i_2}))^{-1} \, \cdots \, R(s_{i_1} s_{i_2} \cdots s_{i_{N-1}}(\alpha_{i_N}))^{-1} = \overrightarrow{\prod_{\alpha \in \Delta_+}} R_(\alpha)^{-1} \tag{41.9}$$

where in the last expression the product is ordered by the decomposition of w_0.

So this would be wonderful. But we have to deal with the question mark in equation 41.6.

Well, it is false. But it is almost true. What is actually true is if we take out the part of R that goes with the Cartan.

Let us define $\tilde{R}(i) = \exp(-\frac{hd_i}{4} H_i \otimes H_i) R(i) = \sum_{n=0}^{\infty} \frac{(q_i - q_i^{-1})^n}{[n]_{q_i}!} E_i^n \otimes F_i^n$. Then we define $\breve{w}_i = e^{-\frac{hd_i}{8} H_i^2} w_i$. Then

$$\Delta \breve{w}_i = \tilde{R}(i)^{-1} \breve{w}_i \otimes \breve{w}_i \tag{41.10}$$

Look in Ch. and P., or in Korog. and Soib.

Theorem 41.3 $\Delta \breve{w}_0 = \tilde{R}^{-1} \breve{w}_0 \otimes \breve{w}_0$, *where* $\breve{w}_0 = \breve{w}_{i_1} \cdots \breve{w}_{i_N}$, *and* $R = \exp(\frac{h}{4} \sum_{ij} (B^{-1})_{ij} H_i \otimes H_j) \tilde{R}$, *with* $B_{ij} = d_i a_{ij}$ *and so that* $\tilde{R}$ *is the "nilpotent part" of* R.

Corollary 41.3.1 $R = \exp\left(\frac{h}{4} \sum_{ij} (B^{-1})_{ij} H_i \otimes H_j\right) \overrightarrow{\prod_{\alpha > 0}} \tilde{R}(\alpha)$

Here the order of the product and also the multiplicands $\tilde{R}(\alpha)$ depend on the decomposition, but the total product does not.

$$\tilde{R}(\alpha) = \sum_{n \geq 0} \frac{(q_\alpha - q_\alpha^{-1})^n}{[n]_{q_\alpha}!} E_\alpha^n \otimes F_\alpha^n \tag{41.11}$$

Let us say something about the theory over all. We have presented it without proofs, and this is fine.

1. One can do it completely algebraically: $\mathcal{U}_q\mathfrak{g} \cong \mathcal{U}_q(\mathfrak{n}_+) \otimes \mathcal{U}\mathfrak{h} \otimes \mathcal{U}_q(\mathfrak{n}_-)$, generated by E_α, $K_i^{\pm 1}$, and F_α, respectively. This works when q is generic.
2. One can specialize $q \to \epsilon$, where $\epsilon^l = 1$, i.e. ϵ is a root of unity. Then the representation is completely different, and there are two extremes. On the one hand, the representation theory of $\mathcal{U}_{\mathcal{A}}$, with divided powers, is a lot like the theory of Chevalley groups, which are $\mathcal{U}\mathfrak{g}$ over $\mathbb{F}_p$. The other, $\overline{\mathcal{U}_{\mathcal{A}}}$, has very large centers. There is some duality between these.
3. Lastly, one can chose $q \in \mathbb{C}$, and try to build a $*$-structure on the algebra $a \mapsto a^*$ that is an algebra antiautomorphism — $(ab)^* = b^*a^*$ — and is $\mathbb{C}$-antilinear — $(\lambda a)^* = \bar{\lambda} a^*$. Then you can ask for $*$-representations, i.e. representations with a $\mathbb{C}$-antilinear bilinear form, so that we have a notion of Hermetian operators: we ask for $\pi(a^*) = \pi(a)^\dagger$.

   Then such $*$ structures exists for $q \in \mathbb{R}$ and for $|q| = 1$, and these are different. They are related to Harmonic Analysis on Lie groups, but haven't been developed.

By the way. On Friday, the whether is kind of strange. We will have pizza at the pizza place on Oxford. It is called "Orso", and they have good pizza and good beer. 6pm.

Lecture 42 May 8, 2009

42.1 Dan HL: Quantum $\mathfrak{sl}_2$ at Roots of Unity

We follow Chari and Pressly, chapters 9 and 11, and also Kassel has a nice treatment.

Notation, we write $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ and $\mathcal{U} = \mathcal{U}\mathfrak{sl}_2$. Then we define $\mathcal{U}_q$ the algebra over $\mathbb{Q}(q)$ generated by $E, F, K^\pm$ with $KEK^{-1} = q^2E$, $KFK^{-1} = q^{-2}F$, and $[E,F] = (K - K^{-1})/(q - q^{-1})$. This is a Hopf algebra over $\mathbb{Q}(q)$, but the presentation does not makes sense under specializations $q \mapsto 1^{1/n}$. So we introduce other presentations that specialize better.

Let us do the "restricted" one. We define $\mathcal{U}_{q,\mathbb{Z}}^{\text{res}}$ to be the $\mathcal{A}$-subalgebra of $\mathcal{U}_q$ generated by $K^\pm$ and also the divided powers $E^{(n)}$ and $F^{(n)}$. We also have the upper- and lower-triangular parts: $\mathcal{U}_{q,\mathbb{Z}}^{\text{res}+}$ generated just by $E^{(n)}$, $\mathcal{U}_{q,\mathbb{Z}}^{\text{res}-}$ generated by $F^{(n)}$, but now the 0 piece needs all these weird brackets: $\mathcal{U}_{q,\mathbb{Z}}^{\text{res}0}$ is generated by $K^\pm$ and $\begin{bmatrix} K; c \\ r \end{bmatrix} = \prod_{s=1}^{r} \frac{Kq^{c+1-s} - K^{-1}q^{s-1-s}}{q^s - q^{-s}}$. In the full algebra, these extra weird brackets are commutators of the divided powers.

Theorem 42.1 *$\{E^{(n)}\}_{n=0}^\infty$ are a basis over $\mathcal{A}$ of $\mathcal{U}_{q,\mathbb{Z}}^{\text{res}+}$. $\{F^{(n)}\}_{n=0}^\infty$ are a basis over $\mathcal{A}$ of $\mathcal{U}_{q,\mathbb{Z}}^{\text{res}-}$. $\{\begin{bmatrix} K; 0 \\ r \end{bmatrix}, K\begin{bmatrix} K; 0 \\ r \end{bmatrix}\}_{r=1}^\infty$ are a basis for $\mathcal{U}^{\text{res}0}$. The multiplication gives $\mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+ \to \mathcal{U}$ of $\mathcal{A}$-modules.*

NR: This, and especially the last line, is the PBW theorem.

We will continue to quote facts, which are all checkable or are in Chari and Pressley.

So, $\mathcal{U}_{\mathcal{A}}^{\text{res}} \otimes_{\mathcal{A}} \mathbb{Q}(q) = \mathcal{U}_q$.

From here on out, we pick ϵ a primitive lth root of unity (l is odd).

Then we define $\mathcal{U}_{\epsilon,\mathbb{Z}}^{\text{res}} \stackrel{\text{def}}{=} \mathcal{U}_{\mathcal{A}}^{\text{res}} \otimes_{\mathcal{A}} \mathbb{Z}[\epsilon]$. And $\mathcal{U}_{\epsilon}^{\text{res}} \stackrel{\text{def}}{=} \mathcal{U}_{\epsilon,\mathbb{Z}}^{\text{res}} \otimes_{\mathbb{Z}[\epsilon]} \mathbb{Q}(\epsilon)$. This inherits a Hopf algebra structure all the way through, ad we will turn our attention to its representation theory.

Proposition 42.2 *In $\mathcal{U}_{\epsilon}^{\text{res}}$, $E^l = F^l = 0$, $K^{2l} = 1$, and K^l is central.*

Proof: $KEK^{-1} = q^2 E$. Moreover, we have $E^{(l)} = E^l/[l]_q!$, and we clear denominators: $E^l = [l]_q! E^{(l)}$. But the q-factorial $[l]_q!$ goes to 0 when $q \mapsto 1^{1/l}$. The rest of the claims are similar. □

Some things stay true in the specialization. E.g. PBW theorem. But we will come back to this.

In the rational Hopf algebra, there are relations between the divided powers. For example, $E^{(r)}E^{(s)} = \left[\frac{r+s}{r}\right]_q E^{(r+s)}$.

Let's divide r by l: $r = r_0 + lr_1$, where $0 \le r_0 < l$. Then

$$E^{(r)} = E^{(r_0)} \frac{(E^{(l)})^{r_1}}{r_1!} \tag{42.1}$$

Then $\mathcal{U}_{\epsilon}^{\text{res}}$ is generated over $\mathbb{Q}(\epsilon)$ by $K^{\pm}$, $\begin{bmatrix} K;0 \\ r \end{bmatrix}_q$ for $0 \le r < l$, and $E^{(l)}$ and $F^{(l)}$.

NR: Over $\mathbb{Q}$, isn't the bracket generated by K? **NR:** Shouldn't there be more of the Cartan?

Well, CP treat it a little differently. They say that there is a finite-dimensional Hopf algebra $\mathcal{U}_{\epsilon,\mathbb{Z}}^{\text{fin}}$ generated by $E, F, K^{\pm}$, and:

$$\mathcal{U}_{\epsilon,\mathbb{Z}}^{\text{fin}} \hookrightarrow \mathcal{U}_{\epsilon,\mathbb{Z}}^{\text{res}} \stackrel{\text{Fr}_\epsilon}{\to} \mathcal{U}_{\mathbb{Q}} \tag{42.2}$$

where the last map is a characteristic-0 analogue of the Frobenius. It takes

$$\begin{aligned} K &\mapsto 1 \\ E^{(l)} &\mapsto \begin{cases} \bar{E}^{(r/l)}, & l|r \\ 0, & \text{else} \end{cases} \\ \begin{bmatrix} K;0 \\ l \end{bmatrix} &\mapsto \bar{H} \end{aligned}$$

Then equation 42.2 is exact in the sense that the kernel is the two-sided ideal generated by the augmentation ideal of $\mathcal{U}_{\epsilon,\mathbb{Z}}^{\text{fin}}$.

In any case, PBW applies to $\mathcal{U}_{\epsilon,\mathbb{Z}}^{\text{fin}}$, so its basis is $E^{(0)}, \dots, E^{(l-1)}, F^{(0)}, \dots, F^{(l-1)}, \left[\frac{K;0}{0}\right], \dots, \left[\frac{K;0}{l-1}\right], K\left[\frac{K;0}{0}\right], \dots, K\left[\right.$

Oh, one thing. There is a rational Frobenius map. When we write $\mathcal{U}_{\mathbb{Z}}$ in equation 42.2, we mean the thing defined by Chevalley: the integral subalgebra of the thing over $\mathbb{C}$ generated by the divided powers.

So, even though the ring is not a product of the two outside guys in equation 42.2, the representation theory will still split into those two pieces.

Because $K^{2l} = 1$, in any finite-dimensional representation K acts semisimply with eigenvalues $\pm\epsilon^r$. So you use it the same way you use the Cartan element in $\mathfrak{sl}_2$ to pull apart a representation.

We say that a representation is *Type I* if $K^l = 1$ in the representation. To consider just Type I is no loss of generality. Indeed, there is a one-dimensional representation where $K^l = -1$, so tensoring with it will get the other irreducible representations.

NR: This sign is very important, and is related to the theory of Poisson Lie groups. In any Poisson Lie group, there is factorization similar to the upper-triangular and lower-triangular. But then you have to decide where to send the diagonal. The standard decision is to take the square root, but there is an ambiguity 2^{rank}, because you have to fix signs in the square root.

Sure, but to understand the representation theory, you can bounce back and forth. **NR:** Yes, there is a Galois theory.

So, a lot of the proofs are just like in the usual Lie theory. We have the following result in analogue to classical $\mathfrak{sl}_2$: $E^{(l)}$ and $F^{(l)}$ act nilpotently, and $\left[\frac{K;0}{l}\right]_\epsilon$ has integer eigenvalues. Idea: decompose $V = \bigoplus_{m_0,m_1} V_{m_0,m_1}$, where $K = \epsilon^{m_0}$ on V_{m_0,m_1} and $\left[\frac{K;0}{l}\right]_\epsilon - m_1$ is nilpotent. Then $E^{(l)}$ and $F^{(l)}$ raise and lower the indices, and if we're finite-dimensional, they must be nilpotent.

So, say $m = m_0 + lm_1$. Then define $V_m = \{v \in V \text{ s.t. } Kv = \epsilon^{m_0}v, \left[\frac{K;0}{l}\right]_\epsilon v = m_1 v\}$. Then it turns out that we can completely decompose $V = \bigoplus_{m\in\mathbb{Z}} V_m$. So this is a little stronger than before: now we're saying that the bracket is diagonal.

NR: So $E^{(l)}$, $F^{(l)}$, and the bracket are an $\mathfrak{sl}_2$-triple.

So you push through the $\mathfrak{sl}_2$ representation theory. What you find out is that for each $m \in \mathbb{Z}$, there is a unique irreducible module $V_\epsilon^{\text{res}}(m)$ with highest weight m, and each is of this form. Moreover, as $\mathcal{U}_\epsilon^{\text{res}}$ modules we have:

$$V_\epsilon^{\text{res}}(m) = V_\epsilon^{\text{res}}(m_0) \otimes V_\epsilon^{\text{res}}(lm_1) \tag{42.3}$$

Moreover, restricting $V_\epsilon^{\text{res}}(m_0)$ to $\mathcal{U}_\epsilon^{\text{fin}}$ gives an irreducible module, and $V_\epsilon^{\text{res}}(lm_1)$ is isomorphic to $V(m_1)$ under the Frobenius map.

So the representation theory is complete.

List of Homework Exercises

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