Lie Groups and Quantum Groups

Prof. Vera Serganova
notes by Theo Johnson-Freyd

UC-Berkeley Mathematics Department
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Contents

Introduction

These are notes from UC Berkeley's Math 261B, taught by Vera Serganova in the Spring of 2010. The class met three times a week — Mondays, Wednesdays, and Fridays — from 1pm to 2pm. Needless to say, the content is due to VS, and all the errors are due to me, the notetaker (Theo Johnson-Freyd). VS's website for the course is at `http://math.berkeley.edu/~serganov/261/index.html`.

This course is a continuation of the Fall 2009 course Math 261A: Lie Groups, taught by Prof. Ian Agol. IA's website for that course is `http://math.berkeley.edu/~ianagol/261A.F09/`. I don't know of notes from that course. Edited notes from previous versions of 261A are [1, 5]. Unedited notes from a previous version of 261B are [11].

As with my other course notes, I typed these mostly for my own benefit, although I do hope that they will be of use to other readers. (It was Anton's excellent notes from a variety of classes — in addition to the Lie groups notes mentioned above, he has other notes on his website — that inspired me to type my own notes, and I have borrowed from his preamble.) I apologize in advance for any errors or omissions. Places where I did not understand what was written or think that I in fact have an error will be marked ****like this****. Please e-mail me (`mailto:theojf@math.berkeley.edu`) with corrections. For the foreseeable future, these notes are available at `http://math.berkeley.edu/~theojf/QuantumGroups10.pdf`.

These notes are typeset using TEXShop Pro on a MacBook running OS 10.6; the backend is pdfLATEX. Pictures are drawn using PGF/TikZ. The raw LATEX sources are available at `http://math.berkeley.edu/~theojf/QuantumGroups10.tar.gz`. These notes were last updated May 2, 2010.

0.1 Conventions and numbering

Each lecture begins a new "section", and if a lecture breaks naturally into multiple topics, I try to reflect that with subsections. Equations, theorems, lemmas, etc., are numbered by their lecture. Theorems, lemmas, propositions, corollaries, and examples are counted with the same counter. Definitions are not marked qua definitions, but *italics* always mean that the word is being defined by that sentence. All definitions are indexed in the index. A list of all theorems, propositions, etc., is also at the end of the document. To generate these lists and to format theorems, etc., I have used the package `ntheorem`. Better referencing is done by `cleveref`.

Lecture 1 Jan 20, 2010

VS's website is `math.berkeley.edu/~serganov/`. Office hours are Monday 4-5:30 and W 11-12:30, in 709 Evans.

This course is on Lie groups and quantum groups. We will talk about:

1. Representation theory of semisimple Lie algebras:
   (a) *Algebraic Methods*: Weyl character formula, Harish-Chandra theorem, category $\mathscr{O}(BGG)$, cohomology, Zuckerman functor.
   (b) *Geometric Methods*: The ultimate goal is Beilinson-Bernstein. To move to this, we will study localization, flag varieties, nilpotent cone, Springer resolution, Borel-Weil-Bott theorem. And a little bit about the orbit method for nilpotent groups.
2. *Quantum Groups.* Hopf algebras, tensor categories, quantization. The main result is the existence of canonical bases. (We may or may not have time to talk about roots of unity.)

Question from the audience: Is there a good reference? **Answer:** We will not follow a particular book, but VS will post references on the website. And Theo is taking notes.

Some good references are [4, 10, 2, 3, 7, 8, 6].

1.1

We begin with something that you should know, but we will fix notation.

By $\mathfrak{g}$ we mean a semisimple (later, reductive) Lie algebra over $\mathbb{C}$. By G we mean a simply-connected connected group with Lie algebra $\mathfrak{g}$. There are several ways to look at this group. If you think of it as a Lie group really, then its representation theory is very rich. But usually we mean it as an *algebraic group*. Maybe you covered this in 261A: there is a unique compact simply connected real Lie group K with complexification $K_{\mathbb{C}} = G$. So we will not spend much time on compact groups, because everything follows from the complex presentations.

Let's then recall the root decomposition and Cartan subalgebra. We pick a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, which is maximal abelian consisting of semisimple elements. Then the action of $\mathfrak{h}$ on $\mathfrak{g}$ is semisimple, so we have the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

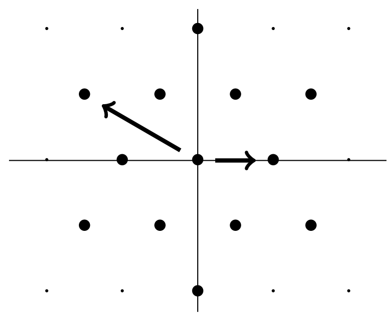
where $\Delta \subseteq \mathfrak{h}^*$ is the *root system*, and $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \text{ s.t. } [h, x] = \alpha(h)x\}$. Then $\dim \mathfrak{g}_\alpha = 1$. This follows from 261A.

Example 1.1 Let $\mathfrak{g} = \mathfrak{sl}_{n+1} \subseteq \mathfrak{gl}_{n+1}$. We take coordinates $\epsilon_0, \ldots, \epsilon_n$ given by $\epsilon_i(a_{ij}) = a_{ii}$, and $\mathfrak{h}$ is a diagraonal subalgebra. Indeed, $\Delta = \{\epsilon_i - \epsilon_j \text{ s.t. } i \neq j\}$, and $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{C}E_{ij}$. ◊

Let W be the *Weyl group*, which is a linear group acting on $\mathfrak{h}^*$, generated by all the root reflections s_α. In the above example, $W = S_{n+1}$. In general,

$$W \cong \mathcal{N}_{\mathrm{G}}(\mathfrak{h})/\mathrm{H}$$

Example 1.2 The group G_2 is an exceptional 14-dimensional Lie algebra. Its root diagram is:



◊

We recall the *triangular decomposition*, where we pick a hyperplane that does not pass through any roots, and so $\Delta = \Delta^+ \sqcup \Delta^-$, where $\Delta^+ = \{\alpha \in \Delta \text{ s.t. } (\alpha, \gamma) > 0\}$ for some generically chosen γ.

As soon as we have this decomposition, we can construct the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$. Then $\mathfrak{n}^\pm$ are nilpotent subalgebras, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ are solvable subalgebras, and we have:

Theorem 1.3 *Every maximal solvable subalgebra of $\mathfrak{g}$ is conjugate to $\mathfrak{b}$.*

We can choose a basis $\alpha_1, \ldots, \alpha_n \in \Delta$ of *simple roots* such that every positive root is:

$$\alpha = \sum m_i \alpha_i, \quad m_i \in \mathbb{Z}_{\geq 0}$$

In $\mathfrak{sl}_{n+1}$, with the standard choice $\Delta^+ = \{\epsilon_i - \epsilon_j \text{ s.t. } i > j\}$, the simple roots are $\epsilon_i - \epsilon_{i+1}$ for $i = 0, \ldots, n-1$. We define the *rank* of $\mathfrak{g}$ to be $\dim \mathfrak{h}$.

Any $\alpha \in \Delta$ gives rise to an $\mathfrak{sl}_2$-subalgebra inside $\mathfrak{g}$. How? We pick $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$, and $h_\alpha \in \mathfrak{h}$ such that $[h_\alpha, x_\alpha] = 2x_\alpha$, $[h_\alpha, y_\alpha] = -2y_\alpha$, and $[x_\alpha, y_\alpha] = h_\alpha$.

If we do this for every simple α, we get a system of generators and relations for $\mathfrak{g}$. We associate $\alpha_i \mapsto \{y_i, h_i, x_i\}$, and then consider the square matrix $C = \left(c_{ij}\right) = \left(\alpha_j(h_i)\right)$ (the h_i are *coroots*). Then we have *Chevalley's relations*:

$$[h_i, x_j] = c_{ij} x_i \tag{1.1}$$
$$[h_i, y_j] = -c_{ij} y_j \tag{1.2}$$
$$[x_i, y_j] = \delta_{ij} h_i \tag{1.3}$$

and *Serre's relations*:

$$\left(\mathrm{ad}_{x_i}\right)^{1-c_{ij}} x_j = 0 \tag{1.4}$$
$$\left(\mathrm{ad}_{y_i}\right)^{1-c_{ij}} y_j = 0 \tag{1.5}$$

Why do these hold? Does anyone know? ****no hands**** This is just $\mathfrak{sl}_2$ representation theory: x_j is the lowest vector for (x_i, h_i, y_i). A more difficult theorem, due to Serre, is that equations 1.1 to 1.5 determine $\mathfrak{g}$. We will not prove this, but perhaps will include it in the exercises.

1.2 Root and Weight lattices

The *root lattice* is Q inside $\mathfrak{h}^*$ generated by Δ. Let $(,)$ be the positive definite form induced by the Killing form. Then:

$$\alpha_i(h_j) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_j)} \tag{1.6}$$

So this justifies calling $h_j = \alpha_j^\vee$ a *coroot*. And since $(,)$ is ad-invariant, and it is in particular W-invariant.

Then the *weight lattice* is $P = \{\mu \in \mathfrak{h}^* \text{ s.t. } \mu(h_i) \in \mathbb{Z}\}$. Then $Q \subseteq P$, and in fact:

Theorem 1.4 *P/Q is isomorphic to $Z(\mathrm{G})$.*

an exercise $\left|P/Q\right| = \det(c_{ij})$

Example 1.5 What is the Weyl group of G_2? It is the Dihedral group with 12 elements. And $\det\left(\begin{smallmatrix} 2 & -3 \\ -1 & 2 \end{smallmatrix}\right) = 1$. So G_2 has trivial center. ♢

Example 1.6 In $\mathfrak{sl}(n+1)$, we have, by an easy induction:

$$\det \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{pmatrix} = n+1$$

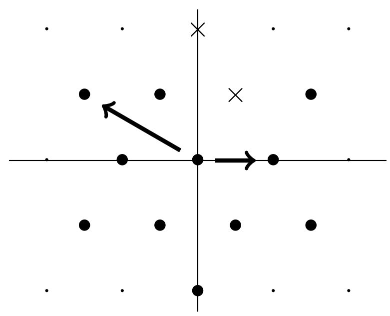
and in fact $Z(\mathrm{SL}(n+1)) = \mathbb{Z}_{n+1}$. ♢

In fact, the lattice P has a very nice geometric meaning. We have $\mathrm{H} = (\mathbb{C}^*)^{\mathrm{n}}$.

We define $P^+ = \{\lambda \in P$ s.t. $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$. Then $\alpha_1, \ldots, \alpha_n$ is the natural basis of Q, and $\omega_1, \ldots, \omega_n$, the *fundamental weights*, defined by $\omega_i(h_j) = \delta_{ij}$, are the natural basis of P.

Theorem 1.7 *Every orbit of W in P intersects P^+ at exactly one point.*

Example 1.8 Recall the root lattice of G_2 from Example 1.2. The arrows point to the simple roots, and we mark the fundamental weights with $\times$s:



◊

Finally, suppose that V is some irreducible finite-dimensional representation of $\mathfrak{g}$. Then there exists a *highest vector*: a unique up to scalar $v \in V$ with $\mathfrak{n}^+ v = 0$; then $hv = \lambda(h)v$ for $h \in \mathfrak{h}$, and so V picks out a weight $\lambda \in \mathfrak{h}^*$, the *highest weight* of V.

In fact, if V is finite-dimensional, then $\lambda \in P^+$. This is a fundamental fact, and it controls the whole representation theory, so we explain it. It comes from the $\mathfrak{sl}_2$ representation theory. Let $\mathfrak{sl}_2 = \{y, h, x\}$. The point is that $xv = 0$ and $hv = \lambda v$, so applying x, h to v doesn't do much, but we can apply y to it. Then in fact $hy^k v = (\lambda - 2k)y^k v$, so the vector remains eigen, and also by induction $xy^k v = (\lambda - k + 1)ky^{k-1}v$. But the point is that if V is finite-dimensional, then eventually $y^k v = 0$, so we must have $\lambda - k + 1 = 0$, i.e. λ is integral.

The other direction is also true:

Theorem 1.9 *There exists a bijection between P^+ and all finite-dimensional irreducible representations of G.*

Given $\lambda \in P^+$, we denote its *highest-weight representation* by $L(\lambda)$. When we do quantum groups, it will be called $L_q(\lambda)$, but it is the same λ. It is the irreducible finite-dimensional representation with weight λ.

If you want to follow this course, then everything we have said should be familiar, although maybe not all the proofs.

We now foreshadow next time. We have a representation $L(\lambda)$. We would like to know e.g. $\dim L(\lambda)$. More generally, there is a notion of *formal character*. All of you know what is the Schur Polynomial. It is a part of a more general result known as the "Weyl Character Formula". The more general statement is as follows. Let V be *semisimple* (i.e. completely reducible) over $\mathfrak{h}$. Then we can write it as $V = \bigoplus_{\mu \in P(V)} V_\mu$, where $P(V) \subseteq \mathfrak{h}^*$ (although in our case actually $P(V) \subseteq P$), and $V_\mu = \{v \in V \text{ s.t. } hv = \mu(h)\, v\}$. Then we define the following formal polynomial:

$$\operatorname{ch} V = \sum (\dim V_\mu)\, e^\mu$$

The reason this makes sense is, consider $h \in \mathfrak{h}$. Then we can exponentiate it, and:

$$\operatorname{tr}_V \exp(h) = \sum (\dim V_\mu)\, e^{\mu(h)}$$

Lecture 2 Jan 22, 2010

2.1 Weight Modules

Let $\mathfrak{g}$ be semisimple and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan. And let M be a $\mathfrak{g}$-module, possibly infinite dimensional. We write

$$M = \bigoplus_{\mu \in P(M) \subseteq \mathfrak{h}^*} M_\mu$$

where

$$M_\mu = \left\{ m \in M \text{ s.t. } hm = \mu(h)\, m \,\forall h \in \mathfrak{h} \right\}$$

and $P(M) = \{\mu \in \mathfrak{h}^* \text{ s.t. } M_\mu \neq 0\}$.

Let's say that a module is a *weight module* if $\dim M_\mu < \infty$ for each μ. Then

1. $\mathfrak{g}_\alpha M_\mu \subseteq M_{\mu+\alpha}$
2. Submodules and quotients of weight modules are weight.
3. Direct sums of weight modues are weight.
4. Every finite-dimensional module is weight.

In particular, the class of weight modules is an abelian category. **Question from the audience:** Is any extension of weight modules a weight module? **Answer:** In general, no.

We define the *character* of a weight module to be:

$$\operatorname{ch} M = \sum \dim M_\mu e^\mu$$

Then if $0 \to M \to N \to K \to 0$ is an extension of weight modules, we have:

$$\operatorname{ch} N = \operatorname{ch} M + \operatorname{ch} K$$

If V is finite-dimensional and M is weight, then $M \otimes V$ is a weight module, and:

$$\operatorname{ch}(M \otimes V) = \operatorname{ch} M \cdot \operatorname{ch} V$$

because $(M \otimes V)_\mu = \bigoplus_{\nu \in P(V)} M_{\mu-\nu} \otimes V_\nu$; the sum is finite because V is finite-dimensional.

Example 2.1 Pick $\lambda \in \mathbb{C}$ and let $\mathfrak{g} = \mathfrak{sl}_2 = \langle x, h, y \rangle$. Then the vector fields $y = \frac{\partial}{\partial t}$, $h = 2t\frac{\partial}{\partial t}$, and $x = t^2 \frac{\partial}{\partial t}$ are an $\mathfrak{sl}_2$, and we define the (formal) weight module:

$$\mathscr{F}_\lambda = t^\lambda \, \mathbb{C}[t, t^{-1}]$$

Moreover, the character is:

$$\operatorname{ch} \mathscr{F}_\lambda = \sum_{n \in \mathbb{Z}} e^{2(\lambda+n)}$$

Indeed, each weight space is spanned by the monomial $t^{2\lambda+n}$. ◊

In particular, the notion of formal character makes sense not just for finite-dimensional modules but for any weight module.

2.2 Weyl Character Formula

We introduce the weight $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Some people, e.g. Borcherds, call ρ the *Weyl vector*. Then if $\lambda \in P^+$, the Weyl formula gives the character of the irreducible finite-dimensional representation:

Theorem 2.2 (Weyl Character Formula)

$$\operatorname{ch} L(\lambda) = \frac{\displaystyle\sum_{w \in W} \operatorname{sign}(w) \, e^{w(\lambda+\rho)}}{\displaystyle\sum_{w \in W} \operatorname{sign}(w) \, e^{w(\rho)}}$$

Here $\operatorname{sign}(w) = \det w = (-1)^{\ell(w)} = \pm 1$.

an exercise $\rho(h_i) = 1$.

Example 2.3 Let $\mathfrak{g} = \mathfrak{sl}_2$; then $\lambda = n$ and $\rho(h) = 1$. Then in particular:

$$\operatorname{ch} L(n) = \frac{e^{(n+1)\rho} - e^{-(n+1)\rho}}{e^\rho - e^{-\rho}} = e^{n\rho} + e^{(n-2)\rho} + \dots + e^{-n\rho}$$

◊

Example 2.4 We now try $\mathfrak{g} = \mathfrak{gl}_n$, which is not semisimple, but it is reductive, which is a semisimple extended by a central element. We pick the usual basis $\epsilon_1, \dots, \epsilon_n$ for the Cartan, and define $z_i = e^{\epsilon_i}$. Then $\rho = z_1^{n-1} z_2^{n-2} \dots z_{n-1}$, and $W = S_n$. Let's pick $\lambda = m_1\epsilon_1 + \dots + m_n\epsilon_n$, and the condition of dominance is that $m_i - m_{i+1} \in \mathbb{Z}_{\geq 0}$.

The Weyl character formula computes a formal determinant. What we get is:

$$S_\lambda = \frac{\det \left| z_i^{m_j+n-k} \right|}{\det \left| z_i^{n-j} \right|}$$

In fact, if you know Schur polynomials, this is exactly their formula. And what is downstairs is a Vandermonde determinant:

$$\det \begin{vmatrix} z_1^{n-1} & \dots & z_n^{n-1} \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{vmatrix} = \prod_{i>j} (z_i - z_j)$$

◊

We will now talk about the algebraic proof of Theorem 2.2. Later we will give a geometric proof.

2.3 Verma module

Pick $\lambda \in \mathfrak{h}^*$, and recall $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Let's start with a one-dimensional representation of $\mathfrak{b}$, called C_λ, given by $C_\lambda = \mathbb{C}v$ with $hv = \lambda(h)v$ and $\mathfrak{n}^+ v = 0$. Now we will induce, and the induced module is called the *Verma module*. It is:

$$M(\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} C_\lambda$$

Let's understand this better. We will use the PBW theorem, which says that as a vector space $\mathcal{U}\mathfrak{g} = \mathcal{U}\mathfrak{n}^- \otimes \mathcal{U}\mathfrak{h} \otimes \mathcal{U}\mathfrak{n}^+$. Then we have an isomorphism:

$$M(\lambda) \cong \mathcal{U}\mathfrak{n}^- \otimes C_\lambda$$

In fact, the isomorphism is compatible with the $\mathcal{U}(\mathfrak{n}^- \oplus \mathfrak{h})$ action.

Let $N = |\Delta^+|$. Then for $\alpha \in \Delta^+$ pick $y_\alpha \in \mathfrak{g}_{-\alpha}$. Then $\{y_{\beta_j}\}_{j=1}^N$ is a basis for $\mathfrak{n}^-$, and so

$$\left\{ y_{\beta_1}^{s_1} \cdots y_{\beta_N}^{s_N} \otimes v \right\}$$

is a basis of $M(\lambda)$.

Then $M(\lambda)$ is a weight module, and we can calculate its character:

$$\operatorname{ch} M(\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Delta^+} \left(1 - e^{-\alpha}\right)} \tag{2.1}$$

Perhaps equation 2.1 requires some clarification. The weight of $y_{\beta_1}^{s_1} \cdots y_{\beta_N}^{s_N}$ is $\lambda - s_1\beta_1 - \dots - s_N\beta_N$. Thus:

$$\operatorname{ch} M(\lambda) = e^\lambda \left(1 + e^{-\beta_1} + e^{-2\beta_1} + \cdots\right) \cdots \left(1 + e^{-\beta_N} + e^{-2\beta_N} + \cdots\right)$$

So we can use the geometric progession formula and divide.

Moreover, $M(\lambda)$ has a unique proper maximal submodule $I(\lambda)$. To explain this, we discuss more the set of weights $P(M(\lambda))$. It is often useful to introduce a partial order on $\mathfrak{h}^*$. We say $\mu \le \lambda$ iff $\lambda - \mu = \sum_{i=1}^n m_i \alpha_i$ with all $m_i \in \mathbb{Z}_{\ge 0}$. Then $P(M(\lambda)) = P(\lambda) = \{\nu \le \lambda\}$.

Example 2.5 For $\mathfrak{sl}_2$, we have an infinite string with λ at the top, then $\lambda - 2$, etc., and y moves you down. $\diamondsuit$

Example 2.6 More generally, it is an infinite cone. When $\mathfrak{g} = \mathfrak{sl}_3$, we have ****picture****. Then it is easy to figure out the multiplicities. The boundary has weight 1, and the next layer has weight 2, then weight 3, etc. For example, for $\lambda - 2\alpha_1 - 2\alpha_2$, we can also write $2\alpha_1 + 2\alpha_2 = \alpha_1 + \alpha_2 + \alpha_3 = 2\alpha_3$. So that weight is 3. $\diamondsuit$

So, now we will show that there is a unique maximal proper submodule. Actually, what we will show, is that the sum of two proper modules is proper, and then we can just add up all the proper submodules.

So, let I_1, I_2 be two proper submodules of $M(\lambda)$. We want to show that $I_1 + I_2 \neq M(\lambda)$. But as we discussed, each of I_1, I_2 is a weight module. So $P(I_1 + I_2) = P(I_1) \cup P(I_2)$. So if $\lambda \notin P(I_a)$, then it is not in $P(I_1 + I_2)$, and $\lambda \in P(I)$ iff $I = M$, because it generates everything.

So, by quotienting out by this maximal proper submodule, we get a simple, although usually not finite-dimensional, highest weight module $L(\lambda) = M(\lambda)/I(\lambda)$. We will eventually prove, although not today, that if λ is integral and dominant, then $L(\lambda)$ is finite-dimensional; we proved the other direction last time.

Example 2.7 So let's go back to the $\mathfrak{sl}_2$ case. Recall that $hy^kv = (\lambda(h) - 2k)y^kv$ and $xy^kv - (\lambda(h) - k + 1)ky^{k-1}v$. So, if we have a submodule, it must start somewhere. So it must include some y^kv. But we want it to be proper, so we want it not to include λ. $M(\lambda)$ is not irreducible only if for some k, $(\lambda(h) - k + 1) = 0$. So λ must be integral and dominant. And if $\lambda = n$, then the vector of weight $-2 - n$ generates a proper module, so we have the following exact sequence:

$$0 \to M(-2 - n) \to M(n) \to L(n) \to 0$$

$\diamondsuit$

Example 2.8 In the $\mathfrak{sl}_3$ case the situation is more interesting, and we draw just one picture, when $\lambda = \rho$. We have two $\mathfrak{sl}_2$s, given by y_1, h_1, x_1 and y_2, h_2, x_2. But $[x_2, y_1] = 0$. So by the $\mathfrak{sl}_2$ calculation, we have $x_1y_1^2v = 0$, and by the bracket, we have $x_2y_1^2v = 0$. In fact, y_1^2v, y_2^2v generate the maximal proper, and when we take the quotient, we get the adjoint representation. The weights are found by subtraction. $\diamondsuit$

Lecture 3 Jan 25, 2010

Last time we introduced the *Verma module* $M(\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} C_\lambda$, and discussed the set of weights $P(M(\lambda)) = \{\mu \leq \lambda\} = P(\lambda)$, and we showed that the Verma module has a maximal proper submodule, and we actually considered the quotient:

$$0 \to I(\lambda) \to M(\lambda) \to L(\lambda) \to 0$$

Then we have a general remark: the highest vector of M is v, and if we have some $w \in M(\lambda)$ with $hw = \nu(h)w$ (so, some weight vector) and $\mathfrak{n}^+w = 0$, then if $\nu \neq \lambda$, then w generates a submodule. ****We draw a picture.****

Corollary 3.1 *Suppose $\lambda(h_i) = k_i \in \mathbb{Z}_{\geq 0}$, then $y_i^{k_i+1} \otimes v$ lies in a proper submodule.*

The proof is straightforward. We need to check that $x_j y_i^{k_i+1} \otimes v = 0$. But if $j \neq i$, then $[x_j, y_i] = 0$, and if $j = i$, then it is the $\mathfrak{sl}_2$ calculation that we already did.

So if $\lambda \in P^+$, and if by $\bar{v}$ we denote the image of v under $M(\lambda) \to L(\lambda)$, then a sufficiently large power of each y_i is going to kill $\bar{v}$. Therefore, the action of $y_1, \ldots, y_n \in \mathfrak{g}$ is *locally nilpotent*. I.e. for any vector, a sufficiently large power of each element kills it: $\forall u \in L(\lambda)\ \exists N(u,i)$ s.t. $y_i^{N_i} u = 0$. Why? We proved this for $u = v$, but any $u \in \mathcal{U}\mathfrak{n}^- \bar{v}$, so $u = X\bar{v}$, and $y_i X \bar{v} = (\operatorname{ad} y_i) X \bar{v} + X y_i \bar{v}$. But $\mathcal{U}\mathfrak{n}^-$ is a nilpotent algebra, so sufficiently high power of $\operatorname{ad} y_i$ kills X.

Also, $x_1, \ldots, x_n$ act locally nilpotently on $L(\lambda)$, as they did so on $M(\lambda)$. So we can exponentiate: $\exp(x_i)\exp(-y_i)\exp(x_i)$ is well-defined on $L(\lambda)$, because each $\exp(x, y)$ is, because the action is nilpotent and exp(nilpotent) truncates. On the other hand, $\exp(x_i)\exp(-y_i)\exp(x_i) \in \mathrm{SL}(2)_i$, the ith $\mathrm{SL}(2)$. Indeed, if you multiply the elements you have: $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. So this is an element of $\mathcal{N}_G(H)$.

The set of weights $P(L(\lambda))$ is W-invariant. In fact, this proves that the action respects also the multiplicities of the weights, and hence the character.

Question from the audience: Does H act trivially? What do you mean by the W action? **Answer:** We have $W \cong \mathcal{N}_G(H)/H$. But H preserves the weight spaces, so the action on the set of weights factors through W.

For example, in $\mathfrak{sl}_2$, we have only one simple reflection, and it acts by reflecting the chain of weights. ****picture****

Indeed, we see that:

$$P(L(\lambda)) \subseteq \bigcap_{w \in W} w(P(\lambda))$$

Because $P(L(\lambda)) \subseteq P(\lambda) = P(M(\lambda))$, but by W-invariance, it lies in all of them.

So this means the set of weights is inside of a convex polytope. ****Picture**** So in particular the set $P(L(\lambda))$ is finite. Thus:

Corollary 3.2 *If $\lambda \in P^+$, then $\dim L(\lambda) < \infty$.*

Question from the audience: Why the restriction on λ? **Answer:** The nilpotency argument required $\lambda \in P^+$. **Question from the audience:** $L(\lambda)$ is always irreducible. **Answer:** Yes, but finite iff $\lambda \in P^+$.

3.1 Casimir element

In our case, we have a Killing form $(,)$, which is nondegenerate. Then we choose a basis $\{e_1, \ldots, e_k\}$ of $\mathfrak{g}$, and since we have a nondegenerate form, we can construct a *dual basis* $\{e^1, \ldots, e^k\}$, by

$(e_i, e^j) = \delta_i^j$. Here $k = \dim \mathfrak{g}$. Then we define the *Casimir* element by:

$$\Omega = \sum_{i=1}^{k} e_i e^i \in \mathcal{U}\mathfrak{g}$$

In fact, $\Omega \in Z(\mathcal{U}\mathfrak{g})$, the center. This is a good an exercise, and is in the homework. The trick is that for $x \in \mathfrak{g}$, $\sum \mathrm{ad}_x(e_i e^i) = 0$, as $(,)$ is invariant. Also, Ω does not depend on a choice of $\{e_i\}$.

Thus, we can write Ω in a very special basis. We pick the basis $\{x_\alpha, y_\alpha, h_\alpha\}_{\alpha \in \Delta^+}$ with $(x_\alpha, y_\alpha) = \frac{1}{2}(h_\alpha, h_\alpha) = \frac{2}{(\alpha,\alpha)}$. Indeed, we just pick the x_αs, and then complete, using $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$. Then:

$$\Omega = \sum_{\alpha \in \Delta^+} \frac{(\alpha,\alpha)}{2} \big(x_\alpha y_\alpha + y_\alpha x_\alpha\big) + \sum_{i=1}^{n} (u_i)^2$$

Here $n = \operatorname{rank} \mathfrak{g} = \dim \mathfrak{h}$ and $\{u_i\}$ is an arbitrary orthonormal basis of $\mathfrak{h}$.

But then we write $x_\alpha y_\alpha - y_\alpha x_\alpha = h_\alpha$, so:

$$\Omega = \sum_{\alpha \in \Delta^+} (\alpha,\alpha) x_\alpha y_\alpha + \sum_{\alpha \in \Delta^+} \frac{(\alpha,\alpha)}{2} h_\alpha + \sum_{i=1}^{n} (u_i)^2$$

Moreover, $\lambda(h_\alpha) = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$, so:

$$\Omega v = \Big(\sum \frac{\alpha,\alpha}{2} \lambda(h_\alpha) + \sum \lambda(u_i)^2\Big) v = \big((\lambda,\lambda) + 2(\rho,\lambda)\big) v$$

This uses that $\mathfrak{h}$ acts diagonally and $xv = 0$. But Ω is central; so it acts as a scalar on $M(\lambda)$ (because this is generated by the eigenvector v) with eigenvalue $(\lambda + 2\rho, \lambda)$.

3.2 Category $\mathscr{O}$ (BGG category)

Here BGG=Bernstein-Gelfand-Gelfand.

We say that $M \in \mathrm{Ob}\,\mathscr{O}$ if:

1. M is a weight module.
2. M is finitely generated over $\mathcal{U}\mathfrak{g}$
3. $\mathfrak{n}^+$ acts locally nilpotently on M.

In particular, $\mathscr{O}$ is an abelian category. Also it is clear that $M(\lambda), L(\lambda) \in \mathrm{Ob}\,\mathscr{O}$. Moreover, it is clear how to describe the objects of $\mathscr{O}$. For any vector, we act on it enough by $\mathfrak{n}^+$, and end up at a highest weight vector. So actually every simple object of $\mathscr{O}$ is isomorphic to $L(\lambda)$ for some λ.

Lemma 3.3 *$M(\lambda)$ has finite Jordan-Hölder series in $\mathscr{O}$.*

Let's set $A(\lambda) = \{\nu \leq \lambda \text{ s.t. } (\nu+\rho, \nu+\rho) = (\lambda+\rho, \lambda+\rho)\}$. This is the intersection of a discrete set by a sphere, and hence is finite. I remind you that $\leq$ means less by an integer amount. But: $(\lambda+\rho, \lambda+\rho) = (\lambda+2\rho, \lambda) + (\rho, \rho)$.

If $L(\mu)$ is a simple subquotient of $M(\lambda)$, then $\mu \leq \lambda$, and the eigenvalue of the Casimir on L, M must be the same. Therefore, $(\lambda+2\rho, \lambda) = (\mu+2\rho, \mu)$, so $\mu \in A(\lambda)$. This completes the proof of lemma 3.3, by induction. In fact, probably an exercise, every object in $\mathscr{O}$ has finite Jordan-Hölder series; it is not difficult, but we do not need it.

Corollary 3.4 $\operatorname{ch} M(\lambda) = \sum_{\mu \in A(\lambda)} c_{\lambda,\mu} \operatorname{ch} L(\mu)$, *and* $c_{\lambda,\lambda} = 1$.

So the $c_{\lambda,\mu}$ are positive integer coefficients of a lower-triangular matrix with 1s on the diagonal. Therefore we can invert it, and set $a_{\lambda,\mu} = c^{-1}_{\lambda,\mu}$, and again it is lower-triangular with 1s on the diagonal. In particular:

$$\operatorname{ch} L(\lambda) = \sum_{\mu \in A(\lambda)} a_{\lambda,\mu} \operatorname{ch} M(\mu)$$

Let's now assume that $\lambda \in P^+$, and get rid of the double index: $\operatorname{ch} L(\lambda) = \sum_{\mu \in A(\lambda)} b_\mu \operatorname{ch} M(\mu)$. But we know $\operatorname{ch} M(\mu) = e^\mu / \prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})$. We multiply top and bottom by e^ρ. Then:

$$\operatorname{ch} L(\lambda) = \frac{\sum b_\mu e^{\mu+\rho}}{\prod_{\alpha \in \Delta^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right)} \tag{3.1}$$

We use that ρ is half the sum of the positive roots.

But the LHS of equation 3.1 is W-invariant, and the denominator of the RHS is skew-invariant. So the top must also be skew invariant. **Question from the audience:** Sorry, what is b_μ? **Answer:** They are unknown coefficients, but we know that $b_\lambda = 1$.

Ok, now we play a trick, knowing that $\sum_{\mu \in A(\mu)} b_\mu e^{\mu+\rho}$ must be skew invariant. So we average over the Weyl group, knowing that each orbit of the Weyl group intersects the positive chamber exactly once. We have:

$$\sum_{\mu \in A(\mu)} b_\mu e^{\mu+\rho} = \sum_{\mu \in A(\lambda) \cap P^+} b_\mu \sum_{w \in W} \operatorname{sign}(w)\, e^{w(\mu+\rho)}$$

We finish with a simple

Lemma 3.5 $A(\lambda) \cap P^+ = \{\lambda\}$

Indeed, suppose $\mu \in P^+ \cap A(\lambda)$. Then $\lambda - \mu = \sum m_i \alpha_i$ for $m_i \geq 0$. But we have $(\lambda+\rho, \lambda+\rho) = (\mu+\rho, \mu+\rho)$. Upon subtracting, we have $(\lambda+\mu+2\rho, \lambda-\mu) = 0$. On the other hand, $\lambda+\mu+2\rho$ is strictly positive, and $\lambda - \mu$ is nonnegative, so must be 0.

Next time we will discuss consequences of this formula, and formulate but not prove a result about resolutions.

Oh, we used one important fact: an exercise $\rho(h_i) = \frac{2(\rho,\alpha_i)}{(\alpha_i,\alpha_i)} = 1$.

Lecture 4 Jan 27, 2010

****VS arrives early an puts a picture on the board. Perhaps we will include it here later. It consists of a large triangular lattice, with the following labels: $\lambda = 3\omega_1 + 2\omega_2$, $\lambda+\rho = 4\omega_1+3\omega_2$, and the vertices λ, $s_1(\lambda+\rho)-\rho$, $s_2(\lambda+\rho)-\rho$, $s_1s_2(\lambda+\rho)-\rho$. And various weights drawn in.****

We will not say much about the Weyl character formula today, because we proved it last time. But we do have this formula, for dominant integral λ:

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w\in W} \operatorname{sign}(w)\, e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Delta^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right)} \tag{4.1}$$

Let's recall how much we used. We used only that the whole thing is finite and W-invariant. To get the formula we wrote at the beginning of the semester, all we need is to use the so-called "denominator formula", for which we can simply evaluate equation 4.1 at $\lambda = 0$. Then we have:

$$\prod_{\alpha\in\Delta^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right) = \sum_{w\in W} \operatorname{sign}(w)\, e^{w(\rho)} \tag{4.2}$$

In fact, this is a very important formula, especially when we move to infinite dimensions, whence it will give us combinatorial data and count ****?****. In $\mathfrak{sl}(n)$, it corresponds to Vandermonde's determinant formula.

4.1 Kostant partition function

$$\operatorname{ch} L(\lambda) = \sum_{w\in W} \operatorname{sign}(w)\ \operatorname{ch} M\big(w(\lambda+\rho)-\rho\big)$$

This is because we can take out e^ρ, which is the character of the Verma module.

So we define the *Kostant partition function* $\mathcal{P}(\gamma) = \#\big\{\{m_\alpha\} \in \mathbb{Z}_{\geq 0} \text{ s.t. } \gamma = \sum_{\alpha\in\Delta^+} m_\alpha \alpha\big\}$. I.e. it is the number of ways to write γ as a sum of positive roots. **Question from the audience:** For only positive integral γ? **Answer: **yes?****

Then we see that: $\dim M(\lambda)_\mu = \mathcal{P}(\lambda-\mu)$. We give an example in $\mathfrak{sl}(3)$. ****picture****. The dimension is 1+ the distance to the wall of the cone.

Now we can use this to calculate the dimension at weight μ in $L(\lambda)$. It's going to be the alternating sum of partition functions:

$$\mathcal{M}(\lambda,\mu) = \dim L(\lambda)_\mu = \sum_{w\in W} \operatorname{sign}(w)\, \mathcal{P}\big(w(\lambda+\rho)-\mu-\rho\big)$$

This leads to the picture we drew earlier. On the boundary, all multiplicities are 1, and they continue counting distances, because $\operatorname{ch} L(\lambda)$ is W-symmetric; it is always a hexagon (or degenerates to a triangle). See [4].

But actually, this formula is not very useful in applications, because the order of the Weyl group is very big.

In fact, there is a continuous partition function $\mathcal{P}_{\text{cont}}(\gamma) = \operatorname{Vol}\Big\{m_\alpha \in \mathbb{R}_{\geq 0} \text{ s.t. } \gamma = \sum m_\alpha \alpha\Big\}$. This function is a piecewise polynomial.

4.2 BGG resolution

We will prove this section later. But the remark is that since we have an alternating sum, it should arise from some complex. The idea is that in the category $\mathcal{O}$ we can construct a resolution in this category, in which each term is a direct sum of Verma modules:

$$0 \to M\Big(w_0(\lambda+\rho)-\rho\Big) \to \cdots \to \bigoplus_{w\in W \ell(w)=k} M\Big(w(\lambda+\rho)-\rho\Big) \to \cdots \to \bigoplus M\Big(s_i(\lambda+\rho)-\rho\Big) \to M(\lambda) \to L(\lambda) \to 0$$

Here w_0 is the longest element of W; it maps positive roots to negative roots. Also, we have $\operatorname{sign}(w) = (-1)^{\ell(w)}$, so actually the formula has some nice algebra behind it.

4.3 Dimension formula

We can write the dimension as a ratio of two products:

$$\dim L(\lambda) = \frac{\prod_{\alpha\in\Delta^+} (\lambda+\rho, \alpha)}{\prod_{\alpha\in\Delta^+} (\rho, \alpha)} \tag{4.3}$$

Example 4.1 Let $\mathfrak{g} = \mathfrak{gl}(n)$, and $\lambda = a_1\epsilon_1 + \cdots + a_n\epsilon_n$, where $a_i - a_{i+1} \in \mathbb{Z}_{\geq 0}$. Then $\rho = (n-1)\epsilon_1 + \cdots + 0\epsilon_n$. And actually the product is a double factorial. We have:

$$\frac{\prod_{i<j}(a_i - a_j + j - i)}{\prod_{i<j}(j-i)} = \frac{\cdots}{(n-1)!!}$$

$\Diamond$

We prove equation 4.3, using equation 4.2. We have $\dim L(\lambda) = \operatorname{tr} 1 = \operatorname{ch} L(\lambda)\Big|_{h=1}$. But when we try to evaluate here, we get $0/0$. It is a well-defined polynomial, but to evaluate it, we must take a limit. The point is:

$$\dim L(\lambda) = \lim_{t\to 0} \frac{\sum_{w\in W} \operatorname{sign}(w)\, e^{(w(\lambda+\rho),\rho t)}}{\sum_{w\in W} \operatorname{sign}(w)\, e^{(w(\rho),\rho t)}} \tag{4.4}$$

But equation 4.2 implies:

$$\sum_{w\in W} \operatorname{sign}(w)\, e^{(w(\rho),\mu)} = \prod_{\alpha\in\Delta^+} \left(e^{(\frac{\alpha}{2},\mu)} - e^{-(\frac{\alpha}{2},\mu)}\right)$$

So we rewrite the numerator. We have:

$$\lim_{t\to 0} \frac{\sum_{w\in W} \operatorname{sign}(w)\, e^{(w(\lambda+\rho),\rho t)}}{\sum_{w\in W} \operatorname{sign}(w)\, e^{(w(\rho),\rho t)}} = \lim_{t\to 0} \prod_{\alpha\in\Delta^+} \frac{e^{(\lambda+\rho,\frac{\alpha}{2})t} - e^{-(\lambda+\rho,\frac{\alpha}{2})t}}{e^{(\rho,\frac{\alpha}{2})t} - e^{-(\rho,\frac{\alpha}{2})t}}$$

Now we can take the limit using L'Hôspital's rule. We have: $\lim_{t\to 0} \frac{e^{at}-e^{-at}}{e^{bt}-e^{-bt}} = \frac{a}{b}$. Thus, we have:

$$\dim L(\lambda) = \prod_{\alpha\in\Delta^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}$$

This was a much more useful formula before computers. Now there is a computer program (**Question from the audience:** Which one? **Answer:** It is called "Lie". VS used it once.) that you plug in a representation and it computes things.

4.4 Application to tensor product

We have:

$$L(\lambda)\otimes L(\mu) = \sum_{\nu} \Gamma^{\nu}_{\lambda,\mu} L(\nu) \tag{4.5}$$

The question is for a formula for Γ?

Actually, you already know, from the characters of a finite group, that the characters are orthogonal. We can write the orthogonality condition here in terms of the formal character. We define the formal ring $R = \bigoplus_{\lambda\in P} \mathbb{Z}[e^{\lambda}]$. In fact, this has a W-action. So we take the invariant subring R^W.

Proposition 4.2 *The characters* $\operatorname{ch} L(\lambda)$ *form a basis in* R^W*, which is orthonormal with respect to the pairing* $(,)$ *given by:*

$$(\phi,\psi) = \textit{constant coefficient of } \frac{\mathcal{D}\bar{\mathcal{D}}}{|w|}\phi\bar{\psi}$$

where $\bar{e^{\lambda}} = e^{-\lambda}$*, so that* $\overline{\operatorname{ch} L(\lambda)} = \operatorname{ch} L(\lambda)^*$*. Also,* $\mathcal{D} = \sum \operatorname{sign}(w)\, e^{w(\rho)}$.

We describe the geometric meaning of this. Consider the compact group $K \subseteq G$, with maximal torus $T \subseteq H$. Since we didn't do it properly, think of $G = GL(n, \mathbb{C})$ and $K = U(n)$. What we do know is that every element of K is conjugate to something in T. Then the idea is that the characters are class functions:

$$\int_K \phi\,\bar{\psi}\,dg = \frac{1}{|W|}\int_T \phi\,\bar{\psi}\ \mathrm{Vol}_t\ dt$$

where Vol_t is the volume of the conjugacy class of $t \in T$ in K. The $1/|W|$ counts the redundancy of how we diagonalize unitary matrices. So the idea is that $\mathcal{D}\bar{\mathcal{D}}(e^h) = \mathrm{Vol}_t$ where $t = e^h$.

Ok, so this is why they are orthonormal. How to check that they are a basis? There is another obvious basis of R^W. Namely, each W-orbit intersects P^+ once, so for each $\lambda \in P^+$, set $E_\lambda = c_\lambda \sum_{w\in W} e^{w(\lambda)}$, where c_λ is some coefficient to get the length to be 1. But then it is obvious from the formula that

$$\mathrm{ch}\, L(\lambda) = E_\lambda + \sum_{\mu<\lambda} d_\mu E_\mu$$

But this is a lower-triangular matrix with 1s on the diagonal, so $\mathrm{ch}\, L(\lambda)$ is a basis.

Ok, but now we can calculate Γ from equation 4.5. We have:

$$\begin{aligned}
\Gamma^{\nu}_{\lambda,\mu} &= \big(\mathrm{ch}\, L(\lambda)\ \mathrm{ch}\, L(\mu), \mathrm{ch}\, L(\nu)\big) = \\
&= \text{constant coef of } \frac{1}{|W|}\frac{1}{\mathcal{D}} \sum_{w,u,v\in W} \mathrm{sign}(w)\, e^{w(\lambda+\rho)}\ \mathrm{sign}(u)\, e^{u(\mu+\rho)}\ \mathrm{sign}(v)\, e^{-v(\nu+\rho)} = \\
&= \text{const coef of} \frac{1}{\mathcal{D}} \sum_{w,\sigma\in W} \mathrm{sign}(w)\, e^{w(\lambda+\rho)}\ \mathrm{sign}(\sigma)\, e^{\sigma(\mu+\rho)-(\nu+\rho)} = \\
&= \sum_{\sigma\in W} \mathrm{sign}(\sigma)\, \mathcal{M}\big(\lambda, \nu+\rho-\sigma(\mu+\rho)\big) = \\
&= \sum_{\sigma,w\in W} \mathrm{sign}(\sigma w) \mathcal{P}\big(w(\lambda+\rho)+\sigma(\mu+\rho)-\nu-2\rho\big) \qquad (4.6)
\end{aligned}$$

where we substituted $\sigma = uv^{-1}$, and used various facts. This is the *Steinberg formula.*

Again, this is not very effective. It's much nicer to use Littlewood-Richardson rule, but that only works for $\mathfrak{gl}(n)$.

Next time, we will move to Harish-Chandra theorem, which describes the center of $\mathcal{U}\mathfrak{g}$.

Lecture 5 Jan 29, 2010

We are going to speak today about the center of universal enveloping algebra.

5.1 Center of universal enveloping algebra

We begin with some motivation: why is this important? We all know the Schur lemma, and we formulate a lemma that is a sort of infinite-dimensional version, but is less well known.

Lemma 5.1 *Let R be a countable-dimensional associative algebra over $\mathbb{C}$, and M a simple R-module. Then* $\mathrm{End}_R(M) = \mathbb{C}$.

This is well known when $\dim R < \infty$. The proof goes like this.

Proof: Any non-zero endomorphism is an isomorphism, because kernel and image are invariant subspaces. So let's pick up some endomorphism $X \neq 0$, and there are two cases: either X is algebraic over $\mathbb{C}$, or it's transcendental.

1. X is transcendental. Then $\mathbb{C}(X) \subseteq \mathrm{End}_R(M)$. But $\dim \mathbb{C}(X) = 2^{\mathbb{N}}$, because we can take $1/(x-a)$ for all $a \in \mathbb{C}$. On the other hand, $\mathrm{End}_R(M)$ is countable dimensional: if we pick up $m \in M$, $m \neq 0$, and $\phi \in \mathrm{End}_R(M)$, then ϕ is determined by $\phi(m)$, because m generates M; similarly, $\dim M$ is countable because R is countable-dimensional, and $M = Rm$. So this was impossible.
2. X is algebraic over $\mathbb{C}$. Then $p(X) = (x-\lambda_1)\dots(x-\lambda_n) = 0$, so $x = \lambda_i$ for some i. □

Question from the audience: When does this break? **Answer:** The Schur lemma does break eventually, but we do not recall a counterexample right now.

So what we will do is study irreducible representations of $\mathfrak{g}$ the Lie algebra, by studying the representation theory of $\mathcal{U}\mathfrak{g} = \mathcal{T}\mathfrak{g}/(xy - yx - [x,y])_{x,y\in\mathfrak{g}}$. Then we have:

$$\begin{array}{ccc} \mathcal{U}\mathfrak{g} & \dashrightarrow & \\ \uparrow & & \searrow \\ \mathfrak{g} & \hookrightarrow & \mathrm{End}(M) \end{array}$$

In any case, let $Z(\mathfrak{g})$ be the center of $\mathcal{U}\mathfrak{g}$ ****Z is not a functor****, and suppose that M is an irreducible $\mathfrak{g}$-module. Then for all $x \in \mathrm{Hom}(Z(\mathfrak{g}), \mathbb{C}) = \mathrm{Spec}\, Z(\mathfrak{g})$, and such that $z|_M = x(z)\,\mathrm{id}$ forall $z \in Z(\mathfrak{g})$, we have $\mathrm{Irr}\,\mathfrak{g} \to \mathrm{Spec}\, Z(\mathfrak{g})$.

Now, let's restrict attention to $\mathfrak{g}$ semisimple and $(,)$ a Killing form.

Recall that $\mathcal{U}\mathfrak{g}$ is filtered, with $\mathbb{K} \subseteq \mathcal{U}\mathfrak{g}_1 \subseteq \mathcal{U}\mathfrak{g}_2 \subseteq \dots$. Then we can construct the associated graded algebra $\mathrm{gr}\,\mathcal{U}\mathfrak{g} = \bigoplus_{i=0}^{\infty} \mathcal{U}\mathfrak{g}_{i+1}/\mathcal{U}\mathfrak{g}_i$. Then:

Theorem 5.2 (Poincarè-Birkhoff-Witt) $\mathrm{gr}\,\mathcal{U}\mathfrak{g} \cong \mathcal{S}\mathfrak{g}$

You did the proof last semester.

Ok, so the adjoint action of $\mathfrak{g}$ (or of G) on $\mathcal{U}\mathfrak{g}$ and on $\mathcal{S}\mathfrak{g}$:

Proposition 5.3 *Let* $\mathrm{char}\,\mathbb{K} = 0$. *Then* $\mathrm{gr}\,\mathcal{U}\mathfrak{g} \cong \mathcal{S}\mathfrak{g}$ *as $\mathfrak{g}$-modules with the adjoint action.*

Proof: You draw a certain triangle:

$$\begin{array}{ccc} \mathcal{S}\mathfrak{g} & \longrightarrow & \mathcal{T}\mathfrak{g} \\ & \searrow^{\sim} & \downarrow \\ & & \mathcal{U}\mathfrak{g} \end{array}$$

All the arrows are morphisms of $\mathfrak{g}$-modules, and by Theorem 5.2 the diagonal is an isomorphism.$\square$

Ok, so $Z\mathfrak{g} = \mathcal{U}\mathfrak{g}^G$ is the set of $\mathrm{ad}_{\mathfrak{g}}$-invariant elements. So we start with studying $\mathcal{S}\mathfrak{g}^G$. The killing form identifies $\mathfrak{g} \cong \mathfrak{g}^*$ as adjoint modules, so $\mathcal{S}(\mathfrak{g}^*)^G \cong \mathcal{S}\mathfrak{g}^G$. The LHS is the set of G-invariant polynomials on $\mathfrak{g}$.

Ok, so $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, and consider the restriction $r : \mathcal{S}(\mathfrak{g}^*) \to \mathcal{S}(\mathfrak{h}^*)$. How does this act on invariant polynomials?

Lemma 5.4 *1.* $\mathrm{Im}\, r \subseteq \mathcal{S}(\mathfrak{h}^*)^W$.

2. r is injective.

Proof: 1. $W \cong \mathcal{N}(\mathfrak{h})/H$, and if $f(g)$ is G-invariant, then $f(h)$ is $\mathcal{N}(\mathfrak{h})$-invariant.

2. Any semisimple element is conjugate under the adjoint action to some element of $\mathfrak{h}$. Ok, so let $\mathfrak{g}_{\mathrm{ss}}$ be the set of semisimple elements; it is dense in $\mathfrak{g}$. So if $f(\mathfrak{h}) = 0$, then $f(\mathfrak{h}_{\mathrm{ss}}) = 0$, so $f(\mathfrak{g}) = 0$. So $\ker r = 0$. $\square$

Question from the audience: What field are we over? **Answer:** We've certainly used characteristic-zero for Proposition 5.3, and we need existence of splitting form for the second part of lemma 5.4, so we probably want to be algebraically closed. Probably the first part is always true. Actually, in characteristic p, there are two ways to define the universal enveloping algebra. At some point, we will talk about this, but not now.

Theorem 5.5 *The map $r : \mathcal{S}(\mathfrak{g}^*)^G \to \mathcal{S}(\mathfrak{h}^*)^W$ is an isomorphism.*

Proof: After lemma 5.4, all we have to do is show that r is surjective. We pick fundamental weights $\omega_1, \ldots, \omega_n$, and think of them as coordinate functions on $\mathfrak{h}$. Then $\mathcal{S}(\mathfrak{h}^*)$ has a basis $\omega_1^{a_1} \cdots \omega_n^{a_n}$. Then $\mathcal{S}(\mathfrak{h}^*)^W$ is spanned by $\sum_{w \in W} w\big(\omega_1^{a_1} \cdots \omega_n^{a_n}\big)$.

Claim: $\sum_{w \in W} w(\omega_i^m)$, $m \in \mathbb{Z}_{\geq 0}$ generates the ring of invariant.

For this, we use the following *Polarization formula*: Over $x_1, \ldots, x_n$, let $\Gamma = \langle p_1, \ldots, p_{n-1} \rangle \cong \mathbb{Z}_2^{n-1}$, where $p_i(x_1, \ldots, x_n) = (x_1, \ldots, -x_i, \ldots, x_n)$. Then:

$$\sum_{\gamma \in \Gamma} \mathrm{sign}(\gamma)\, \gamma(x_1 + \cdots + x_n)^n = \text{constant} \cdot (x_1 \ldots x_n) \tag{5.1}$$

****So Γ is the group $(\mathbb{Z}/2)^{n-1}$, which acts on coordinates $\{x_1, \ldots, x_n\}$, by letting the generators p_i switch the sign of the ith variable.**** Indeed, what happens when you apply an element of the group to the LHS? You see that it's a sum of monomials of total degree n, but the only monomials that can appear must be of odd degree in each variable, by anti-symmetry.

Ok, so to prove the claim, we apply equation 5.1 to $x_i = \omega_i^{a_i}$, and thus obtain $\omega_1^{a_1} \dots \omega_n^{a_n}$.

Then to prove surjectivity, we have to show that: $\sum_{w \in W} w(\omega_i^m) \in \operatorname{Im} r$.

If V is finite-dimensional representation of $\mathfrak{g}$, $\operatorname{tr}_V(g^m) \in \mathcal{S}(\mathfrak{g}^*)^G$, since tr is ad-invariant. So if $V = L(\lambda)$ with $\lambda \in P^+$, then:

$$\operatorname{tr}_{L(\lambda)}(h^m) = \sum_{w \in W} w\Big(\lambda(h)^m\Big) + \sum_{\substack{\mu \in P^+ \\ \mu < \lambda}} d_\mu \, w\Big(\mu(h)^m\Big)$$

But the μs don't bother us: they are of lower degree, so by induction we already got them. That prove surjectivity. □

Ok, so we've got: $\mathcal{S}(\mathfrak{g}^*)^G \cong \mathcal{S}(\mathfrak{h}^*)^W$. But we are interested in the center of $\mathcal{U}\mathfrak{g}$. We will get a very similar result. **Question from the audience:** So we proved that that $Z(\mathfrak{g}) \cong \mathcal{S}(\mathfrak{h}^*)^W$ as something? **Answer:** Ah, yes, but not as rings. We proved in fact that they are isomorphic as graded vector spaces, and in fact that $\operatorname{gr} Z(\mathfrak{g}) \cong \mathcal{S}(\mathfrak{h}^*)$ as rings, but you know that you can have a commutative ring that is not isomorphic to its associated graded.

5.2 Harish-Chandra homomorphism

Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, and use PBW to write $\mathcal{U}\mathfrak{g} = \mathcal{U}\mathfrak{n} \otimes \mathcal{U}\mathfrak{h} \otimes \mathcal{U}\mathfrak{n}^+$, as vector spaces and in fact as $\mathfrak{h}$-modules. What Harish-Chandra does is to say $\mathcal{U}\mathfrak{g}^{\mathfrak{h}} = \mathcal{U}\mathfrak{h} \oplus \Big(\mathfrak{n}^-(\mathcal{U}\mathfrak{n}^-) \otimes \mathcal{U}\mathfrak{h} \otimes (\mathcal{U}\mathfrak{n}^+)\mathfrak{n}^-\Big)$, and to forget the second direct summand, so that we have $\mathcal{U}\mathfrak{g}^{\mathfrak{h}} \to \mathcal{U}\mathfrak{h}$.

Then we have a map $\theta : \mathcal{U}\mathfrak{g}^G \to \mathcal{U}\mathfrak{g} = \mathcal{S}\mathfrak{g}$, where the isomorphism is because $\mathfrak{h}$ is commutative. We claim this is a homomorphism of rings.

Example 5.6 Let $\mathfrak{g} = \mathfrak{sl}(2)$, given by x, h, y, with the Killing form $(h,h) = 8$, $(x,y) = 4$. Then $\Omega = \frac{1}{8}h^2 + \frac{1}{2}(xy + yx) = \frac{h^2}{8} + \frac{h}{4} + \frac{yx}{2}$. But when we apply the Harish-Chandra projection, we have: $\theta(\Omega) = \frac{h^2}{8} + \frac{h}{4} = \frac{1}{8}\left((h+1)^2 - 1\right)$. ◊

Lecture 6 Feb 1, 2010

Last time we defined *Harish-Chandra homomorphism*, and calculated one example. If we look at $\mathcal{U}\mathfrak{g}^{\mathfrak{h}} = \mathcal{U}\mathfrak{h} \oplus \Big(\mathfrak{n}^-(\mathcal{U}\mathfrak{n}^-) \otimes \mathcal{U}\mathfrak{h} \otimes (\mathcal{U}\mathfrak{n}^+)\mathfrak{n}^+$, which follows from PBW, and then we project onto the first part, and restrict to $Z(\mathfrak{g})$.

But $\mathcal{U}\mathfrak{h} = \mathcal{S}\mathfrak{h} = \mathbb{C}[\mathfrak{h}^*]$, the algebra of polynomial functions on $\mathfrak{h}^*$. But $\max\operatorname{Spec}\mathbb{C}[\mathfrak{h}^*] = \mathfrak{h}^*$, and so we have a dual map $\theta^* : \mathfrak{h}^* \to \operatorname{Spec}(Z(\mathfrak{g}))$. So we have $\lambda \mapsto \chi_\lambda \in \operatorname{Hom}(Z(\mathfrak{g}), \mathbb{C})$. Of course, we are in characteristic zero, but actually this works in arbitrary characteristic. **Question from the audience:** But you do need to be in a closed field. **Answer:** Yes, because we need maximal ideals.

This map has the following properties:

1. If $z \in Z(\mathfrak{g})$, then $z|_{M(\lambda)} = \chi_\lambda(z)\,\mathrm{id}$. Indeed, we have $z = \theta(z) + Y\mathfrak{n}^+$, but if v is the highest vector of $M(\lambda)$, then $\mathfrak{n}^+ v = 0$, so $zv = \theta(z)\,v$. But $\theta(z) = p(\lambda) \in \mathbb{C}[\mathfrak{h}^*]$. Then use the fact that $hv = \lambda(h)\,v$ for all $h \in \mathfrak{h}$. **Question from the audience:** What is $Y\mathfrak{n}^+$? **Answer:** Well, we mean, YX, where $X \in \mathfrak{n}^+$. This is from the far right end of the PBW statement above.

2. θ is a homomorphism of rings.

3. We now describe the image of θ. For this, we introduce an action of W on $\mathfrak{h}^*$ which is slightly different from the normal action. Instead, we call it the *shifted action*. Not to confuse it, we write it differently. It is $\lambda^w \stackrel{\mathrm{def}}{=} w(\lambda + \rho) - \rho$. Then the statement is that $\theta(Z(\mathfrak{g})) \subseteq \mathbb{C}[\mathfrak{h}^*]^{W_{\mathrm{sh}}}$.

Proof (of 3.): We will use property 1. Pick up a simple root α_i, and let $s_i \in W$ be its reflection. Then let $S_{\alpha_i} = \{\lambda \in \mathfrak{h}^* \text{ s.t. } \lambda(h_i) \in \mathbb{Z}_{\geq 0}\}$. Equivalently, $\frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}_{\geq 0}$. Well, when we studied the Verma module, we saw something like this. So:

Lemma 6.1 *Let $\lambda \in S_{\alpha_i}$ and $\mu = \lambda^{s_i}$. Then* $\mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda)) \neq 0$.

Proof (of lemma): We have $\lambda(h_i) = h_i$ and $(\lambda + \rho)(h_i) = k_i + 1$. So let $w = Y_i^{k_i+1} v$, with v the highest weight of $M(\lambda)$, and $\mathfrak{n}^+ w = 0$, and $hw = \mu(h)\,w$. Then $0 \neq \mathrm{Hom}_{\mathfrak{b}}(C_\mu, M(\lambda)) \cong \mathrm{Hom}_{\mathcal{U}\mathfrak{g}}(\mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} C_\mu, M(\lambda))$, but $\mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} C_\mu = M(\mu)$. □

So, from this, what do we have? If $\lambda \in S_{\alpha_i}$, then since the homomorphism is nontrivial the centers must act the same. So $\chi_\lambda = \chi_{\lambda^{s_i}}$. So if $f \in \mathrm{Im}\,\theta$, then $f(\lambda) = f(\lambda^{s_i})$ for all $\lambda \in S_{\alpha_i}$. But these sets are the union of countably many hyperplanes, so are dense; i.e. S_{α_i} is Zariski dense, so $f(\lambda) = f(\lambda^{s_i})$ for any $\lambda \in \mathfrak{h}^*$. This checks it on the simple reflections, and so W-invariance follows. □

Theorem 6.2 *In fact, $\theta : Z(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]^{W_{\mathrm{sh}}}$ is an isomorphism.*

Proof: The proof goes by going from filtered rings to graded rings. Suppose that A is a filtered $\mathbb{C}$-algebra, i.e. $A = \bigcup_{i=0}^\infty A_i$ with $A_i A_j \subseteq A_{i+j}$ and each A_i is a vector space; e.g. $\mathcal{U}\mathfrak{g}$. Then we do have the graded ring $\mathrm{gr}\,A = \bigoplus_i (A_i/A_{i-1})$. If we have two filtered algebras $A = \bigcup_{i=0}^\infty A_i$ and $B = \bigcup_{i=0}^\infty B_i$, and a homomrophism $\theta : A \to B$ that preserves filtrations, then we can define the map $\mathrm{gr}\,\theta : \mathrm{gr}\,A \to \mathrm{gr}\,B$. Then:

1. $\mathrm{gr}\,\theta$ is a homorphism;

2. if $\mathrm{gr}\,\theta$ is an isomorphism, then so was θ, at least when all the graded components are finite-dimensional. We remark that the converse is not true, in the following sense: You can have a filtered homomorphism $A \to B$ that is an isomorphism of algebras but not an isomorphism of filtered algebras, and it generally will not induce an isomorphism $\mathrm{gr}\,A \to \mathrm{gr}\,B$.

Neither is hard. But last time we proved that $r : \mathcal{S}(\mathfrak{g}^*)^G \to \mathcal{S}(\mathfrak{h}^*)^W$ was an isomorphism, and with the Killing form we have $\mathcal{S}(\mathfrak{g}^*)^G \cong \mathcal{S}(\mathfrak{g})^G$, and $\mathcal{S}(\mathfrak{h}^*)^W \cong \mathcal{S}(\mathfrak{h})^W$. But $\mathrm{gr}\,\mathcal{U}\mathfrak{g}^G = \mathcal{S}\mathfrak{g}^G$ and $\mathrm{gr}\,\mathcal{S}\mathfrak{h}^{W_{\mathrm{sh}}} = \mathcal{S}\mathfrak{h}^W$, and $\mathrm{gr}\,\theta = r$. ****draw a diagram**** □

Actually, there is a more general statement, called the *Duflo theorem*, that $\mathcal{S}(\mathfrak{g})^G \cong Z(\mathfrak{g})$ is an isomorphism of rings. Bar-Natan proved this provided $\mathfrak{g}$ has an invariant form $(,)$. But we do not believe that any conditions are required, but we don't remember. **Question from the audience:** Is the Duflo isomorphism canonical? **Answer:** It is clear that $\mathcal{S}\mathfrak{g}^G = \operatorname{gr} Z(\mathfrak{g})$. But even in our case we picked a ρ, which could be rotated by the Weyl group.

Theorem 6.3 *A finite group W that acts linearly on some vector space $\mathfrak{h}$, then $\mathcal{S}(\mathfrak{h})^W$ is isomorphic to a polynomial ring $\mathbb{C}[f_1, \dots, f_n]$, all homogeneous within the grading on $\mathcal{S}\mathfrak{h}$, and $n = \dim \mathfrak{h}$.*

We will not prove this. You can look it up in [13], or in VS's notes at [12].

Corollary 6.4 *$Z(\mathfrak{g})$ is isomorphic to a polynomial ring of n variables, where $n = \operatorname{rank} \mathfrak{g}$.*

The degrees of generators $m_1, \dots, m_n$ are called the *exponents* of $\mathfrak{g}$. They satisfy:

$$m_1 \cdot \dots \cdot m_n = |W| \tag{6.1}$$

$$m_1 + \dots + m_n = \frac{1}{2}(\dim \mathfrak{g} + n) \tag{6.2}$$

Question from the audience: When you say the degrees of the generators, what do you mean? **Answer:** By going to the graded. In a filtered algebra, you can always define degree by going to the graded. ****Really?****

Indeed, we have in $\mathcal{S}(\mathfrak{h})^W$, that:

$$R(t) = \sum_{k=0}^{\infty} \dim \mathcal{S}^k(\mathfrak{h})^W t^k = \prod_{i=1}^{n} \frac{1}{1 - t^{m_1}} \tag{6.3}$$

If V is a linear representation of W, then:

$$\dim V^W = \frac{1}{|W|} \sum_{w \in W} \operatorname{tr}_V w \tag{6.4}$$

It more or less follows that:

$$R(t) = \frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1 - wt)} \tag{6.5}$$

And comparing equations 6.3 and 6.5 gives:

$$\prod_{i=1}^{n} \frac{1}{1 - t^{m_1}} = \frac{1}{m_1 \dots m_n (1 - t^n)} + \frac{\sum (m_i - 1)}{2 m_1 \dots m_n (1 - t)^{n-1}} + \dots$$

whereas

$$\frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1 - wt)} = \frac{1}{|W|(1 - t)^n} + \frac{1}{|W|} \sum_{\text{reflections}} \frac{1}{2(1 - t)^{n-1}}$$

Thus, $\sum (m_i - 1) =$ number of reflections.

Example 6.5 Let's do G_2. We have $m_1 m_2 = 12$ and $m_1 + m_2 = \frac{1}{2}(14 + 2) = 8$. But we always have $m_1 = 2$, because we always have a Casimir element. So $m_2 = 6$. $\lozenge$

Example 6.6 Let's do $\mathfrak{sl}_n$. We have $m_1 \cdots m_{n-1} = n!$ and $m_1 + \cdots + m_{n-1} = \frac{n^2-1+n-1}{2} = \frac{n^2+n}{2} - 1$. You can solve this. One solution is $m_1 = 2$, $m_2 = 3$, …, $m_{n-1} = n$. One possible set of generators of $\mathcal{S}(\mathfrak{g}^*)^G$ is the traces $\operatorname{tr} X^2, \operatorname{tr} X^3, \dots, \operatorname{tr} X^n$.

Of course the set of generators is not unique. A second option is to take the characteristic polynomial $\det(X - t\mathrm{id})$ and take the coefficients. ♢

Example 6.7 Let's do B_n, C_n. These must have the same exponents, because the Weyl groups are the same. We have $B_n = \mathrm{O}(2n+1)$, and $\dim \mathfrak{g} = \frac{(2n+1)2n}{2} = (2n+1)n$. What is the order of the Weyl group? Well, $W = S_n \rtimes \mathbb{Z}_2^n$. So:

$$m_1 \cdots m_n = 2^n \, n! \tag{6.6}$$

$$m_1 + \cdots + m_n = (n+1)n \tag{6.7}$$

So the obvious solution is $2, 4, \dots, 2n$.

So, what are they? We can still take the ones from $\mathfrak{sl}_n$. But some of those vanish: the traces of odd powers of skew-symmetric matrices are 0. So we have $\operatorname{tr} X^2, \dots, \operatorname{tr} X^{2m}$. ♢

Lecture 7 Feb 3, 2010

We will first fix some notation, and then we will speak about the graded version $\mathcal{S}(\mathfrak{g}^*)^G$. This is a graded ring generated by n homogeneous invariants: $\mathcal{S}(\mathfrak{g}^*)^G = \mathbb{C}[f_1, \dots, f_n]$. We denote $\deg f_i = m_i$, ordered so that $m_1 \le m_2 \le \cdots \le m_n$.

Example 7.1 (D_n) Then $\mathfrak{g} = \mathfrak{o}(2n) = \{\text{skew-symmetric matrices of size } 2n\}$. Then $\dim \mathfrak{g} = n(2n-1)$, and $m_1 + \cdots + m_n = \frac{1}{2}(\dim \mathfrak{g} + n) = n^2$. Also, $m_1 \cdots m_n = |W| = 2^{n-1} n!$. So when you start looking for the most reasonable solution, it is $2, 4, \dots, 2(n-1)$ and one more: n, somewhere in the middle.

So, think of X as a skew-symmetric matrix. Then $f_k(X) = \operatorname{tr}(X^{2k})$ are invariant as before. But if we take $2k = 2n$, then we might as well take the determinant, but the n is actually the *Pfaffian* of X, which is a polynomial $\mathrm{Pf}(X) = \sqrt{\det X}$.

Why is $\mathrm{Pf}(X)$ a polynomial? Think of X as a matrix of some skew-symmetric form on $\mathbb{C}^{2n}$. Then by linear algebra, if X is nondegenerate, then in some basis it has a canonical form $\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. So in general, we have $X = Y^{\mathrm{t}} \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} Y$, and so $\det X = (\det Y)^2$. ♢

But in fact you can easily calculate the exponents of any semisimple Lie algbera, which we will explain now.

The first step is what's called *principal* $\mathfrak{sl}_2$ in $\mathfrak{g}$. Let's go back to the standard generators $x_1, \dots, x_n$, $h_1, \dots, h_n, y_1, \dots, y_n$. Then let $x = x_1 + \cdots + x_n$. There exists a unique $h \in \mathfrak{h}$ such that the $\alpha_i(h) = 2$ for all simple roots $\alpha_1, \dots, \alpha_n$.

So write $h = c_1 h_1 + \cdots + c_n h_n$, and then take $y = c_1 y_1 + \cdots + c_n y_n$. Then we can immediately see that $\{x, h, y\}$ is an $\mathfrak{sl}_2$ triple.

So now we will go into the Lie algebra $\mathfrak{g}$ and look at the adjoint action of this $\mathfrak{sl}_2$ on $\mathfrak{g}$. Then $\alpha(h)$ is even, so every irreducible representation appearing in $\mathfrak{g}$ has a one-dimensional 0-weight space. Also clear is that h is regular, so $\mathfrak{g}^h = \mathfrak{h}$. So that tells us that the number of $\mathfrak{sl}_2$-irreducible components is exactly n, the rank.

Example 7.2 In $\mathfrak{sl}_3$, we have $x = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$, $h = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}$, and $x = \begin{pmatrix} 0 & & \\ 2 & 0 & \\ 0 & 2 & 0 \end{pmatrix}$. So we see that $\mathfrak{sl}_3 = \mathfrak{sl}_2 \oplus V_4$. ♢

So we write $\mathfrak{g} = V_{p_1} \oplus \cdots \oplus V_{p_n}$ with $p_1 \leq \cdots \leq p_n$, where by definition $\dim V_p = p + 1$. Then $\sum_{i=1}^n (p_i + 1) = \dim \mathfrak{g}$, and so $\sum_{i=1}^n \left(\frac{p_i}{2} + 1\right) = \frac{1}{2}(\dim \mathfrak{g} + n)$. So the numbers $\frac{p_i}{2} + 1$ satisfy the same relation as the exponents, and this is no surprise: they are the same. So let's write it down:

Theorem 7.3 $m_i = \dfrac{p_i}{2} + 1$

So you know how to do it. It's a little bit of work, but you know all the weights, so you know how to decompose it.

Proof: Let $v_1, \ldots, v_n$ be the lowest weight vectors in the components. So those are all the vectors which are killed by y. So $\mathfrak{g}^y = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_n$. Let's consider a little bit bigger space $M = \mathbb{C}x \oplus \mathfrak{g}^y$. This is a linear subspace of $\mathfrak{g}$ with dimension $n + 1$. We write $\mathbb{C}$, but of course it can be any field of characteristic 0.

So $M \cong \mathbb{C}^{n+1} = \{(t_0, \ldots, t_n)\}$ the coordinates, so that any vector in M is of the form $t_0 x + t_1 v_1 + \cdots + t_n v_n$. So let $\phi : M \hookrightarrow \mathfrak{g}$ be the injection, and we are going to study the map $\phi^* : \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[M] = \mathbb{C}[t_0, t_1, \ldots, t_n]$.

Claim 1: *ϕ^* is injective.* Indeed, consider the map $\gamma : G \times M \to \mathfrak{g}$ given by first embedding and then acting: $g \times (t_0, \ldots, t_n) \mapsto (\operatorname{Ad} g)(t_0 x + t_1 v_1 + \cdots + t_n v_n)$. So we want to figure out the image of $d\gamma$ at the particular point $e \times (1, 0, \ldots, 0)$, where e is the identity element in G. Then $d\gamma|_{e,1,0,\ldots,0} : \mathfrak{g} \oplus M \to \mathfrak{g}$, and in fact $\operatorname{Im} d\gamma|_{e,1,0,\ldots,0} = \mathfrak{g}^y \oplus \operatorname{ad}_{\mathfrak{g}}(x) = \mathfrak{g}^y \oplus [x, \mathfrak{g}] = \mathfrak{g}$.

So that tells me that γ is locally surjective map. So $\operatorname{Im} \gamma$ contains an open — not Zariski, just open — subset in $\mathfrak{g}$. **Question from the audience:** So you really are working over $\mathbb{C}$? **Answer:** No, you can do this in algebraic geometry. If an algebraic map between affine spaces has surjective derivative, then the image is Zariski dense. See [4].

But on the other hand $\operatorname{Im} \gamma = G \cdot M$ is Zariski dense in $\mathfrak{g}$. So if $f \in \mathcal{S}(\mathfrak{g}^*)^G$, then $f|_M = 0$ implies $f|_{G \cdot M} = 0$ so $f = 0$.

On the other hand, if $h \in \mathfrak{sl}_2$, then ad_h acts as some diagonal matrix on M, because each v_i is an eigenvector. As a vector field, the action is:

$$\operatorname{ad}_h = 2t_0 \frac{\partial}{\partial t_0} - \sum_{i=1}^n p_i t_i \frac{\partial}{\partial t_i}$$

where $[h, v_i] = -p_i v_i$ because v_i is the lowest vector of V_{p_i}. So:

$$\mathrm{ad}_h\big(\phi^*(f)\big) = 0 \forall f \in \mathcal{S}(\mathfrak{g}^*)^G$$

And if $f_1, \ldots, f_n$ are algebraically independent, then $\phi^*(f_i), \ldots, \phi^*(f_n)$ are algebraically independent. So each must contain one t_i.

So we write:

$$\phi^*(f) = \sum c_{j_0, j_1, \ldots, j_n} t_0^{j_0} \ldots t_n^{j_n}$$

Actually, we cal also plug in $t_0 = 1$ and it is still surjective by above. Then looking at the total degrees of the weights, which must be zero by the h-invariance, then: $2j_0 = \sum_{i=1}^n p_i j_i$. And also restriction preserves degree.

So from all this, $\deg f_i \le \frac{p_i}{2} + 1$, because we have ordered everything in increasing order. Is it clear? You can figure out the details.

But $\sum \deg f_i = \sum \big(\frac{p_i}{2} + 1\big)$ is what we need, so we do have equality. □

So from this statement, which is important by itself, we need a couple of corollaries.

Corollary 7.4 *We can choose $f_1, \ldots, f_n$ so that $\phi^* f_i = t_0^{p_i/2} t_i + poly(t_0, t_1, \ldots, t_{i-1})$.*

Corollary 7.5 *The differentials $df_1, \ldots, df_n$ are linearly independent at $x = (1, 0, \ldots, 0)$.*

7.1 Nilpotent cone

We will study the geometry of the nilpotent cone. First some motivation.

Recall that our program was to study $\mathrm{Irr}\,\mathfrak{g}$. And we have a map $\mathrm{Irr}\,\mathfrak{g} \to \mathrm{Spec}\, Z(\mathfrak{g})$. Then for each $\chi \in \mathrm{Spec}\, Z(\mathfrak{g})$, we can study only those irreps that admit this character. So if $M \in \mathrm{Irr}\,\mathfrak{g}$, then $\ker \chi(M) = 0$, and so $\mathcal{U}\mathfrak{g} \ker \chi(M) = 0$.

So let's define $\mathcal{U}_\chi(\mathfrak{g}) = \mathcal{U}\mathfrak{g}/(\mathcal{U}\mathfrak{g} \ker \chi)$. So it's very important to study the structure of this algebra.

On the other hand, everything has a filtration, so one way to study such things is going to the graded. So if everything is OK, and we are going to prove that everything is OK, then $\mathrm{gr}\, \mathcal{U}_\chi \mathfrak{g} = \mathcal{S}(\mathfrak{g}^*)/\langle f_1, \ldots, f_n \rangle$. But this has a nice geometric interpretation. It is a certain affine variety inside $\mathfrak{g}$. And this is the *nilpotent cone*.

So we begin with the definition: $\mathcal{N} \overset{\mathrm{def}}{=} \{z \in \mathfrak{g} \text{ s.t. } \mathrm{ad}_z \text{ is nilpotent}\}$. Let me remind you that as $\mathfrak{g}$ is semisimple, this is equivalent to saying that z in any finite-dimensional representation is nilpotent. But then the traces of all powers are 0: for any $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ we have $\mathrm{tr}\, \pi(z)^m = 0$ for all m.

We will prove: if I is the ideal in $\mathcal{S}(\mathfrak{g}^*)^G$ generated by $f_1, \ldots, f_n$, then the nilpotent cone is exactly the set of zeros of this ideal. Later on, we will also say that the radical of I is the same as I.

Recall that an element is *regular* if its adjoint orbit has maximal dimension. So we look at:

$$\mathcal{N}_{\text{reg}} = \{z \in \mathcal{N} \text{ s.t. } \dim C_{\mathfrak{g}}(z) = n\}$$

where $C_{\mathfrak{g}}(z)$ is the centralizer of z. Then it is Zariski open set in N, and $x \in N_{\text{reg}}$.

We will prove one property:

Lemma 7.6 *If $z \in \mathcal{N}$, then $G \cdot z$, the orbit under the adjoint action, intersects $\mathfrak{n}^+$ nontrivially.*

If you think about $\mathfrak{sl}_n$ case, this is more or less trivial. It follows from Jordan form. We know that for any nilpotent, there is a flag so that it moves along the flag.

Proof: We claim that there is $u \in \mathfrak{g}$ such that $[u, z] = z$. For this, we need to show that $z \in \operatorname{Im} \operatorname{ad}_z$. But we have the Killing form, so $\operatorname{Im} \operatorname{ad} z = (\ker \operatorname{ad}_z)^{\perp}$. So we need to show that if $a \in \ker \operatorname{ad}_z$ and let $\mathfrak{g}_k = \operatorname{Im} \operatorname{ad}_z^k$. Then a commutes with ad_z, so $[a, \mathfrak{g}_k] \subseteq \mathfrak{g}_k$. So we have:

$$\operatorname{ad} z = \begin{array}{c|c|c} 0 & * & * \\ \hline 0 & 0 & * \\ \hline 0 & 0 & 0 \end{array} \qquad \operatorname{ad} a = \begin{array}{c|c|c} * & * & * \\ \hline 0 & * & * \\ \hline 0 & 0 & * \end{array}$$

But then $tr \operatorname{ad}_a \operatorname{ad}_z = 0$.

So u can be chosen semisimple: $u = u_s + u_n$, and $[u_n, z] = 0$, so $u \in \mathfrak{h}'$ for some Cartan subalgebra. So there is $g \in G$ with $\operatorname{Ad}_g(u) \in \mathfrak{h}$ and $\operatorname{Ad}_g(z) \in \mathfrak{n}^+$. Then $[\operatorname{Ad}_g(u), \operatorname{Ad}_g(z)] = \operatorname{Ad}_g(z)$.

Question from the audience: Why did $[u_n, z] = 0$? **Answer:** Because $\operatorname{ad} u_n$ is a polynomial in ad_u. □

Next time we will show that $\mathcal{U}\mathfrak{g}$ is free over its center.

And we posted new homework.

Lecture 8 Feb 5, 2010

Theorem 8.1 *Let $\mathcal{N}$ be the cone of nilpotent elements in a semisimple Lie algebra $\mathfrak{g}$.*

1. *$\mathcal{N}$ is irreducible, with $I(\mathcal{N}) = \langle f_1, \dots, f_n \rangle \stackrel{\text{def}}{=} I$, where f_i are the generators from last time.*
2. *$\mathcal{N}_{\text{reg}} = G \cdot x$, where x, h, y is the principal $\mathfrak{sl}_2$ from last time.*
3. *$\dim \mathcal{N} = \dim \mathfrak{g} - n$*

Proof: Last time we proved that $\mathcal{N} = G \cdot \mathfrak{n}^+$, by which we mean the adjoint action. So let $B = \mathcal{N}_G(\mathfrak{n}^+)$ the normalizer. Then $\dim(B \cdot x) = \dim B - n = \dim \mathfrak{n}^+$. Therefore, $\overline{B \cdot x} = \mathfrak{n}^+$, and so $\overline{G \cdot x} = \mathcal{N}$. But G is a connected group, and so $\overline{G \cdot x}$ is irreducible.

We already showed that $f_1, \dots, f_n \in I(\mathcal{N})$. And so we only need to show that $\sqrt{I} = I$. But this is follows from the fact that $df_1, \dots, df_n$ are linearly independent at x: indeed, if $\sqrt{I} \neq I$, then some df_i would have to depend on the others at some point. □

In fact, this is always true if you know a little bit about algebraic groups, and will be an exercise. The claim is that $\mathcal{N}$ has finitely many G-orbits.

Example 8.2 $\mathfrak{sl}_n$, then $\mathcal{N}$ consists of the nilpotent $n \times n$. And the G-orbit of a matrix is determined by its Jordan form. So the G-orbits are in one-to-one correspondence with ****unordered**** partitions of n. $\lozenge$

Theorem 8.3 *$\mathcal{U}\mathfrak{g}$ is free as a module over its center $Z(g)$.*

Proof: We will prove this in three steps. Our strategy will be to show in Steps 1 and 2 that $\mathcal{S}(\mathfrak{g}^*)$ is free as a $\mathcal{S}(\mathfrak{g}^*)^G$-module. Then in Step 3 we will conclude the result via the filtered-graded yoga.

Step 1 Pick up $q_1, \ldots, q_m \in \mathcal{S}(\mathfrak{g}^*)$ which are linearly independent on $G \cdot x$. Then we will show that there exists Zariski-open $U \subseteq \mathfrak{g}$, $U \ni x$ such that $q_1, \ldots, q_m$ are linearly independent on $G \cdot z$ for any $z \in U$.

This fact uses very little algebraic geometry. In fact, for every $z \in \mathfrak{g}$, define the obvious map $\phi_z : G \to G \cdot z$, $g \mapsto \mathrm{Ad}_g(z)$. Then there is a dual map. And the linear independence implies that $\operatorname{rank} \phi_x^*(q_1, \ldots, q_m) = m$. But $\operatorname{rank} \phi_z^*(q_1, \ldots, q_m) = m$ is a Zariski-open condition in z, as it is the statement that certain minors are non-zero.

Step 2 We set $\mathcal{S}(\mathfrak{g}^*) = I \oplus Y$, with Y graded. ****I is a graded subspace of $\mathcal{S}(\mathfrak{g}^*)$; so we pick some vector-space splitting.**** Then we will prove that the multiplication map $\mu : \mathcal{S}(\mathfrak{g}^*)^G \otimes Y \to \mathcal{S}(\mathfrak{g}^*)$ is an isomoprhism.

We first prove that μ is surjective. Let $p : \mathcal{S}(\mathfrak{g}^*) \to Y$ be the projection induced by the splitting. Then for $q \in \mathcal{S}^k(\mathfrak{g}^*)$, we have $q \cdot p(q) = f_1 q_1 + \cdots + f_n q_n$ with $\deg q_i < k$. So we can proceed by induction.

To prove injectivity, we argue as follows: And element of $\mathcal{S}(\mathfrak{g}^*)^G \otimes Y$ is of the form $\sum_{i=1}^m s_\otimes q_i$, with $s_i \in \mathcal{S}(\mathfrak{g}^*)^G$ and $q_i \in Y$, and we can chose it so that $q_1, \ldots, q_m$ are lienarly independent. Then suppose that $\mu\big(\sum_{i=1}^m s_\otimes q_i = \sum s_i q_i = 0$. But $q_1, \ldots, q_m$ are linearly independent on $G \cdot x$. But then they are linearly independent on $G \cdot z$ for $z \in U$, by Step 1. On the other hand, the s_i are constants on any orbit, because they are invariant. So then $s_i|_{G \cdot z} = 0$ because the q_i are linearly independent. But then $s_i = 0$, because U is Zariski-open.

Step 3 Let $\sigma : \mathcal{S}(\mathfrak{g}^*) \xrightarrow{\sim} \mathcal{S}\mathfrak{g} \hookrightarrow \mathcal{T}\mathfrak{g} \twoheadrightarrow \mathcal{U}\mathfrak{g}$, where the first map is the Killing form, the second is by symmetrization, and the last is defininining. Then as we discussed before, this is a homomorphism of $\mathfrak{g}$-modules (adjoint action).

So now set $X = \sigma(Y)$. Then take the multiplication $\tilde{\mu} : Z(\mathfrak{g}) \otimes X \to \mathcal{U}(\mathfrak{g})$. But then $\mu = \operatorname{gr} \tilde{\mu}$, and since μ is an isomorphism, so is $\tilde{\mu}$. $\square$

So, what we do is: we take a character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$. Then we are interested in $\mathcal{U}_\chi \mathfrak{g} \stackrel{\text{def}}{=} \mathcal{U}\mathfrak{g}/\big((\ker \chi)(\mathcal{U}\mathfrak{g})\big)$. As an algerba, this depends on χ. But on the other hand, their Hilbert or Poincare series are always the same. Indeed, $\operatorname{gr} \mathcal{U}_\chi(\mathfrak{g}) = \mathcal{S}(\mathfrak{g}^*)/\langle f_1, \ldots, f_n \rangle$, so this is nothing else but the ring $\mathbb{C}[\mathcal{N}]$ of regular functions on $\mathcal{N}$. That's why the nilpotent cone is so important.

In some sense, if you know about quantization, then each χ gives a deformation quantization of $\mathbb{C}[\mathcal{N}]$. We hope it's clear that $\mathbb{C}[N]$ is important, so we would like to say a lot about it. For example, G acts on $\mathbb{C}[\mathcal{N}]$, as does $\mathcal{U}_\chi\mathfrak{g}$. Then in fact these are isomorphic as modules.

By the way, when we choose the orthogonal complement Y on Step 2 above, we can make it $\mathfrak{g}$-invariant by induction on degree. If we do this, then it's trivial that $Y \cong \mathbb{C}[\mathcal{N}]$ as G-modules.

Theorem 8.4 *Y decomposes as:*

$$Y = \bigoplus_{\lambda \in P^+} L(\lambda)^{\oplus m_\lambda}$$

where the multiplicities are $m_\lambda = \dim L(\lambda)_0 = \dim L(\lambda)^{\mathfrak{h}}$.

For example, $m_\lambda \neq 0$ implies that $\lambda \in Q$. This is not surprising: only the root lattice appears as weights of $\mathcal{U}\mathfrak{g}$.

Example 8.5 When $\mathfrak{g} = \mathfrak{sl}_2$, each representation of even weight appears with multiplicity 1. ◊

So, the idea is that $G \cdot x$ is not closed. So we will deform it to $G \cdot h$, where h is from the principle $\mathfrak{sl}_2$, and hence a semisimple element, and then we will prove that we have an isomorphism $Y = \mathbb{C}[G \cdot h]$. This is the kind of thing that doesn't always work; it's rather specific to this situation.

We will first show, because we didn't do it yet:

Proposition 8.6 *If $z \in \mathfrak{g}_{\mathrm{ss}}$ (the semisimple elements), then the adjoint orbit $G \cdot z$ is closed.*

Proof: Suppose $z' \in \overline{G \cdot z}$. If $p(t)$ is the minimum polynomial for ad_z, then it also annihilates $\mathrm{ad}_{z'}$: $p(\mathrm{ad}_{z'}) = 0$. So the minimum polynomial of z' can only have smaller degree. Also, the characteristic polynomials are the same: $\det(\mathrm{ad}_z - t) = \det(\mathrm{ad}_{z'} - t)$. **Question from the audience:** Why? **Answer:** Because the characteristic polynomial is invariant, so constant on orbits, and one is in the closure of the orbit of the other.

So all multiplicities of eigenvalues are the same, and so the multiplicities of the zero eigenvalue are the same. Then $\dim \ker \mathrm{ad}_{z'} = \dim \ker \mathrm{ad}_z$. So $\dim G \cdot z = \dim G \cdot z'$, and hence $z' \in G \cdot z$. □

Ok, so now we do a nice trick.

Proposition 8.7 *Let $r : \mathcal{S}(\mathfrak{g}^*) \to \mathbb{C}[G \cdot h]$ be the restriction map. The claim is that $r : Y \to \mathbb{C}[G \cdot h]$ is an isomorphism.*

Question from the audience: This is for arbitrary semisimple h? **Answer:** No, it is for any regular h. But we will do it for the specific h in the principal $\mathfrak{sl}_2$.

Proof: Surjectivity follows from the fact that on an orbit $r(f_i)$ are constants. Injectivity is the interesting part. Remember that Y is graded; so pick up $q_1, \dots, q_m \in Y$ homogeneous and linearly independent. Then we want to show that their images $r(q_1), \dots, r(q_m)$ are also linearly independent.

So assume the opposite. Remember this notation ϕ from before: $\phi_h : G \to \mathfrak{g}^*$ birationally. Then by the assumption $\dim \phi_h^*(q_1, \dots, q_m) < m$. On the other hand, we can multiply h by any constant: because each q_i is homogeneous, we have for any $t \in \mathbb{C}^\times$, that $\dim \phi_{th}^*(q_1, \dots, q_m) < m$. On the other hand, you can check that $th + x \in G \cdot th$. So $\dim \phi^*_{th+x}(q_1, \ldots, q_m) < m$. But this rank is a semicontinuous function, so we can take $t = 0$: $\dim \phi^*_x(q_1, \ldots, q_m) < m$. But then $q_1, \ldots, q_m$ are linearly dependent on $G \cdot x$, and therefore on $\mathcal{N}$. But this is a contradiction: $Y \to \mathbb{C}[\mathcal{N}]$ is an isomorphism. □

So this was a good trick. You see what happened: you have generic orbits, which are closed, because they are maximal dimension. And then you have nongeneric orbits, but they are still in the families.

Next time we will finish Theorem 8.4, which requires that we talk a little bit about algebraic groups. Then we will talk about the symplectic structure on G-orbits in $\mathbb{C}[\mathcal{N}]$.

Lecture 9 Feb 8, 2010

We briefly recall the results from last time (Theorem 8.4). By $\mathcal{N}$ we mean the Nilpotent cone, and $\mathbb{C}[\mathcal{N}]$ its ring of regular functions. Last time, we proved that:

$$\mathbb{C}[\mathcal{N}] \cong \mathbb{C}[G \cdot h] \cong \mathcal{S}(\mathfrak{g}^*)/\langle f_1, \ldots, f_n \rangle \cong \mathcal{U}_\chi(\mathfrak{g})$$

All these are isomorphism of G-modules. **Question from the audience:** I thought the last should be the associated graded? **Answer:** No, we are just talking about G modules. As rings they are different.

We will eventually prove:

Proposition 9.1 $\mathbb{C}[G \cdot h] = \bigoplus_{\lambda \in P^+} L(\lambda) \otimes L(\lambda)^{\mathfrak{h}}$. *As G-modules, G acts on the first $L(\lambda)$, and trivially on $L(\lambda)^{\mathfrak{h}}$.*

We could talk about $\mathfrak{g}$, but we prefer G.

9.1 G as an algebraic group

Let $G \subseteq \mathrm{End}(V)$ where V is a finite-dimensional vector space. Then if G is a semisimple simply-connected Lie group over $\mathbb{C}$, then it is algebraic.

An important remark: This fails over $\mathbb{R}$. For example, $\pi_1 \mathrm{SL}(2, \mathbb{R}) = \mathbb{Z}$, which is easy to see directly. Therefore, if we take a simply-connected cover, its center is isomorphic to $\mathbb{Z}$. But this cover cannot be algebraic, as its center is not Zariski-closed.

Question from the audience: But this cover does not have a faithful finite-dimensional representation. **Answer:** That's correct. We will prove that any semisimple over $\mathbb{C}$ has a faithful representation, but we will do this when it's easy. When we say "algebraic group", we mean an affine algebraic group. Of course, elliptic curves are also algebraic groups. We will prove that if

G is an algebraic variety and a group, then there is an exact sequence $0 \to G_{\text{aff}} \to G \to A \to 0$, where G_{aff} isaffine and A is abelian. So this is all the examples.

Question from the audience: So is the failure, even over $\mathbb{R}$, to be algebraic that it doesn't have a faithful rep? **Answer:** No, certainly not in the non-semisimple case, and we think there are even semisimple counterexamples, but didn't prepare any.

So, the *regular representation* of G and algebraic group is $\mathbb{C}[G]$. It is a $G \times G$ module: if $f \in \mathbb{C}[G]$, then we set $(g_1, g_2)f|_x = f(g_1^{-1} x g_2)$. This is interesting, because each G centralizes the other.

In fact, if G is finite — and certainly finite groups are algebraic — then it is a well-known fact that

$$\mathbb{C}[G] = \bigoplus_{\text{irreps of } G} V \boxtimes V^*$$

Let's explain the symbol $\boxtimes$, which is the *exterior tensor product*. It is $M \boxtimes N = M \otimes N$ as a vector space, and if $G \curvearrowright M$ and $H \curvearrowright N$ then $G \times H \curvearrowright M \boxtimes N$.

Theorem 9.2 $\mathbb{C}[G] = \bigoplus_{\lambda \in P^+} L(\lambda) \boxtimes L(\lambda)^*$

This is a non-compact version of the *Peter-Weyl Theorem*.

To prove it we need a bit of preparation.

Recall the *maximal torus*. Let $H = (\mathcal{N}_G(\mathfrak{h}))_0$, the connected component of the normalizer in G of $\mathfrak{h}$. Then $\text{Lie}(H) = \mathfrak{h}$, and it is a torus: $H \cong \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$. Of course, we can replace $\mathbb{C}$ with any algebraically closed field of characteristic 0.

There is an exponential map: $\exp : \mathfrak{h} \to H$, which is a homomorphism of abelian group, and let $\Gamma = \ker \exp$. It is some lattice. Indeed, we have $\mathbb{C}^\times \cong \mathbb{C}/\mathbb{Z}$, or really $(2\pi i)\mathbb{Z}$.

To describe this lattice, we switch to the dual. Let $\hat{H}$ be the set of all 1-dimensional (irreducible) representations of H. Then the point is that the set of irreps of $\mathfrak{h}$ is just $\mathfrak{h}^*$, and so if $\lambda \in \mathfrak{h}^*$, we can try to exponentiate it. But in order for this to happen, we should have an integrality property. I.e. $\hat{H} \subseteq \mathfrak{h}^*$, and in fact it is:

$$\hat{H} = \{\lambda \in \mathfrak{h}^* \text{ s.t. } \langle \lambda, \gamma \rangle \in \mathbb{Z} \,\forall \gamma \in \Gamma\}$$

Moreover, $\hat{H}$ is an abelian group, because we can tensor representations.

We said already that G has a faithful representation, which could be the sum of all fundamental representations. Then $\hat{H} \cong P$, the weight lattice of $\mathfrak{g}$. **Question from the audience:** What do you mean by $\cong$? **Answer:** Of course, it is isomorphic as an abstract group, but we mean that there is a canonical isomorphism. ****They are both in $\mathfrak{h}^*$; can we write down this isomorphism?****

Then you can easily prove that any regular function on H is a finite linear combination of characters, because of Laurent polynomials. This gives:

$$\mathbb{C}[H] = \bigoplus_{\Phi \in \hat{H}} \mathbb{C}\Phi$$

Now, the torus actions on the left and on the right are more or less the same. So we identify $\hat{H} = P$, and then any element $\lambda \in P$ gives rise to a one-dimensional representation of H, which we call C_λ. So, since $C_{-\lambda} = C_\lambda^*$, we have:

$$\mathbb{C}[H] = \bigoplus_{\lambda \in P^+} C_\lambda \boxtimes C_{-\lambda}$$

****So the idea is that $C_\lambda \boxtimes C_{-\lambda} = \bigoplus_{w \in W} C_{w\lambda}$ or something...****

Now we will construct the *matrix coefficient*. For $\lambda \in P^+$, we will construct $j_\lambda : L(\lambda) \boxtimes L(\lambda)^* \to \mathbb{C}[G]$. Indeed, let $v \in L(\lambda)$ and $\varphi \in L(\lambda)^*$. Then we define:

$$j_\lambda(v \otimes \varphi)\Big|_g = \langle g^{-1}v, \phi \rangle$$

You can check directly that this is a homomorphism of $(G \times G)$-modules. It is injective by irreducibility of $L(\lambda)$s. **Question from the audience:** Why? **Answer:** The guy on the left is irreducible ****over** $G \times G$******, so the kernel is invariant, so must be 0.

Then we have:

$$\bigoplus_{\lambda \in P^+} L(\lambda) \boxtimes L(\lambda)^* \subseteq \mathbb{C}[G]$$

First, we remark that $\mathbb{C}[G]$ is indeed the direct sum of finite-dimensional representations. This for example follows from the faithfulness $G \subseteq \mathrm{End}(V)$, so $\mathbb{C}[G]$ is a quotient of $\mathbb{C}[\mathrm{End}(V)]$ by some invariant ideal, and $\mathbb{C}[\mathrm{End}(V)] = \mathcal{S}(V \otimes V^*)$. So we really cannot have an infinite-dimensional irreducible direct summand. **Question from the audience:** We're using the grading? **Answer:** No. It's always true that the quotient of a semisimple module is semisimple.

So, any finite-dimensional irrep of $G \times G$ is of the form $L(\lambda) \boxtimes L(\mu)$ for $\lambda, \mu \in P^+$. **Question from the audience:** How do we know? **Answer:** If you don't know it, do it as an exercise.

****Ah, so of course, $L(\lambda)^*$ is finite-dimensional, so not $L(-\lambda)$. Rather, $L(\lambda) \mapsto L(\lambda)^*$ is given on fundamental weights by the action of some involution of the Dynkin diagram. For many Dynkin diagrams, there are no nontrivial involutions.****

So, we are interested in $\mathrm{Hom}_G\big(L(\lambda) \boxtimes L(\mu), \mathbb{C}[G]\big)$.

We need a little preparation. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, and $N^\pm$ the group with Lie algebra $\mathfrak{n}^\pm$. Then we have a map $N^+HN^- \to G$ with Zariski-dense image. **Question from the audience:** How do we know that when you exponentiate, it becomes algebraic? **Answer:** For $N^\pm$, each matrix in $\mathfrak{n}^\pm$ is nilpotent. For H, recall that we took a normalizer and took its connected component; these are algebraic operations. The set N^+HN^- is the *big Bruhat cell*.

Now, we discussed before that $\psi \in \operatorname{Hom}_G\big(L(\lambda) \boxtimes L(\mu), \mathbb{C}[G]\big)$. Let v be a highest vector in $L(\lambda)$, and w a lowest vector in $L(\mu)$. Then $N^+v = v$ and $N^-w = w$, so let $\psi(v \otimes w) = f \in \mathbb{C}[G]$, and then we see that for any $n_\pm \in N^\pm$ and any $g \in G$ we have $f(n_+ g n_-) = f(g)$.

So, if we study the values of f on N^+HN^-, a Zariski-dense set, it is completely determined by its values on H.

There is even one more property. We know what happens to the vectors when we multiply by elements of the torus. Let $h \in \mathfrak{h}$. Then $(\exp h)v = e^{\lambda(h)}v$ and $(\exp h)w = e^{\mu'(h)}w$. **$\mu' = -\mu$?** Then $f\big((\exp h_1)h(\exp h_2)\big) = e^{\lambda(h_1)}e^{\mu'(h)}f(h)$, for $h \in H$.

So, we see that $f|_H \in C_\lambda \boxtimes C_{-\lambda}$, so $\mu' = -\lambda$, so $L(\mu) \cong L(\lambda)^*$. Weights of the dual representation are negative weights of the representation.

Therefore, we have proved:

$$\dim \operatorname{Hom}_G\big(L(\lambda) \boxtimes L(\mu), \mathbb{C}[G]\big) = \begin{cases} 1, & L(\mu) = L(\lambda)^* \\ 0, & \text{otherwise} \end{cases}$$

This proves Theorem 9.2.

Actually, everything we did works for reductive groups, but we haven't defined those.

For the last five minutes, consider the map $\xi^* : \mathbb{C}[G \cdot h] \hookrightarrow \mathbb{C}[G]$ dual to $\xi : G \twoheadrightarrow G \cdot h$. This is a homomorphism of (left) G-modules.

At this moment, we must talk about the stabilizer. We have: $\operatorname{Stab}_G(h) = H$. Why? First of all, $\mathfrak{h}$ is the centralizer of h in $\mathfrak{g}$. So $N_G(\mathfrak{h}) \subseteq \operatorname{Stab}_G(h)$. But $N_G(\mathfrak{h})/H \cong W$. And $\operatorname{Stab}_W h = \{e\}$ by regularity. This proves the claim.

Therefore in fact, we identify $G/H \cong G \cdot h$. And $\operatorname{Im} \xi^* = \{f \in \mathbb{C}[G] \text{ s.t. } f(xg) = f(x)\ \forall g \in H\}$.

So, look at $\mathbb{C}[G]^{H_{\text{right}}} = (\bigoplus L(\lambda) \boxtimes L(\lambda)^*)^{H_{\text{right}}} = \bigoplus L(\lambda) \boxtimes L(\lambda)^H$. This proves Proposition 9.1.

Question from the audience: Why can we drop the $*$? **Answer:** What is $L(\lambda)^H$. From the perspective of the weight representation, this is the weight-zero part.

Next time, we will talk about the connection with symplectic geometry and the coadjoint representation.

Lecture 10 Feb 10, 2010

Last time we had a question about algebraic groups and faithful representations. So we will begin with several general facts about algebraic groups: a few things you should know.

10.1 General facts about algebraic groups

Strangely, there is no course about algebraic groups here. As we will see in the semisimple case, compact groups and algebra groups are the same thing, but this does not cover the characteristic-p case, or even nilpotent groups. Certain things are easier to do in the framework of algebraic groups, and certain things are easier in the Lie framework.

We pick $\mathbb{K}$ algebraically closed and characteristic 0. An (affine) *algebraic group* is an algebraic variety with group structure $m : G \times G \to G$, $i : G \to G$ that are all morphisms of algebraic varieties. Then it's clear that the shift maps (left- and right-multiplication) are algebraic.

Some facts:

1. If $f : G \to H$ is a homomorphism of algebraic groups, then its image is Zariski-closed. (Henceforth, "closed" means Zariski-closed.) Why is this so? We will use the following fact from algebraic geometry. For any algebraic map of varieties $f : X, Y$, then $f(X)$ contains an open dense set inside $\overline{f(X)}$.

   So, let $U \subseteq f(G)$ open with $\overline{U} = \overline{f(G)}$. Then for $y \in \overline{f(G)}$, we have $yU \cap U$ non-empty, as it is again Zariski-dense open. But then $y \in U \cdot U$, and so $\overline{f(G)} = U \cdot U = f(G)$.

2. Let $m : G \times G \to G$ be the multiplication, and pull it back to $\Delta : \mathbb{K}[G] \to \mathbb{K}[G \times G] = \mathbb{K}[G] \otimes \mathbb{K}[G]$ via $\Delta f = \sum_{i=1}^{s} f_i \otimes f^i$ where $f(gx) = \sum_{i=1}^{s} f_i(g) f^i(x)$. But then the image of the action of the group on $\mathbb{K}[G]$ lies in the span of finitely many functions: $g \cdot f(x) \in \operatorname{span}\{f^i(x)\}$.

   So any finite-dimensional subspace $W \subseteq \mathbb{K}[G]$ (considered as a G-module with respect to left translation) is contained in some G-invariant finite-dimensional subspace.

3. But from 2. we see that if G is an algebraic group, then it has a faithful representation: pick up regular functions that separate points — you can always do this with finitely many of them — and consider the finite-dimensional invariant space containing them.

4. So, if $H \subseteq G$ is a (Zariski-)closed subgroup, and both are algebraic, then H has an ideal I_H in $\mathbb{K}[G]$. Then I_H is clearly an H-invariant subspace. So we can ask about the normalizer in G of I_H. In fact, we have:

   $$H = \{g \in G \text{ s.t. } f(gx) \in I_H \; \forall f \in I_H\}$$

   Actually, this is obvious in both directions, but in one direction it took VS some time.

   What you do is: suppose $f(e) = 0$; then $f(g) = 0$ for all $g \in H$.

5. The last fact is true for semisimple Lie algebras, but not Lie algebras in general. It is called the *Jordan-Chevalley decomposition.* If you pick any $g \in \mathrm{GL}(V)$, then we can write $g = x_s + x_n$, where x_s is semisimple and x_n is nilpotent, and $[x_s, x_n] = 0$. In fact, these conditions uniquely pick out x_s, x_n, and it turns out that there are polynomials p, q depending on g so that $x_s = p(g)$ and $x_n = q(g)$. Moreover, if $g \in \mathrm{GL}(V)$, then x_s is also invertible, although x_n never is. So we write $g = x_s(1 + x_s^{-1} x_n) = g_s g_n$. This is the *group Jordan-Chevalley*

decomposition. By the uniqueness, it doesn't matter how you embed g in $\mathrm{GL}(V)$. There is only one way to write g as a product of unipotent and semisimple elements that commute.

6. If G is an algebraic group, $g \in G$, then $g_s, g_n \in G$. This follows from fact 4. Indeed, you write $G = \{g \in \mathrm{GL}(V) \text{ s.t. } f(gx) = I_G \ \forall f \in I_G\}$. But then any polynomial of g leaves I_G invariant. **Question from the audience:** But "polynomial" doesn't make sense in the group? **Answer:** No, but it makes sense in any embedding into GL, and the answer doesn't depend on the embedding.

7. This also all works in Lie algebras, where you think in terms of the adjoint action by derivations. Then a Lie algebra of an algebraic group is closed under Jordan-Chevalley decompositions.

8. So if you can present a Lie algebra that's not closed under the JC decomposition, then it is not algebraic.

   Example 10.1 Let $G = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & e^x & xe^x \\ 0 & 0 & e^x \end{pmatrix} \text{ s.t. } x, y, z \in \mathbb{C} \right\}$. The bottom corner is $\exp\left(\begin{smallmatrix} x & x \\ 0 & x \end{smallmatrix}\right)$, so this is a closed group. And the adjoint representation is algebraic ****maybe I misheard?****, but on the other hand, you can see that $(\mathrm{Ad}_g)_s, (\mathrm{Ad}_g)_n \notin G$, so G is not an algebraic group. $\Diamond$

9. Let G be semisimple and connected $G \subseteq \mathrm{GL}(V)$. Then G is algebraic. The point is you look at the G-action in $\mathrm{End}(V)$. Then $\mathrm{End}(V) = \mathfrak{g} \oplus \mathfrak{m}$, and $[\mathfrak{g}, \mathfrak{m}] \subseteq \mathfrak{m}$, because any representation of a semisimple group is completely reducible. So take the normalizer $N_{\mathrm{GL}(V)}(\mathfrak{g})_0 = G_0 \times \left(Z_{\mathrm{GL}(V)}(\mathfrak{g})\right)$, where we took connected components. ****missed a sentence****

10. If V is any finite-dimensional representation of algebraic G, then we can construct the map $V \otimes V^* \to \mathbb{K}[G]$; last time we said that the map is injective if V is irreducible, but in fact it is always injective. The action of G is on the left on $\mathbb{K}[G]$, on V, and trivial on V^*. So we have a homomorphism of representations $V \mapsto \mathbb{K}[G] \otimes V$, which moves the action from V to $\mathbb{K}[G]$. So this is $\mathbb{K}[G] \otimes V \cong \mathbb{K}[G]^{\oplus \dim V}$.

    Question from the audience: Wait, how are you constructing the map $V \otimes V^* \to \mathbb{K}[G]$? **Answer:** You tensor with V on the right, and take the trace. **Question from the audience:** But then you're throwing something away? **Answer:** No, let's write it out. If V is finite-dimensional, then there is a canonical isomorphism $\mathrm{Hom}(A, B \otimes V) \cong \mathrm{Hom}(A \otimes V^*, B)$.

But in general, if your group is not reductive, then you can get finite-dimensional representations of arbitrary length in Jordan-Holder series.

10.2 Compact Groups

So, to make the difference with G, which is usually complex, we will denote by K a compact Lie group. Let V be a finite-dimensional complex representation of K. Then we have the operation of taking averages, which implies that V is completely reducible. In fact, V is unitary with respect to some K-invariant positive-definite Hermetian form. You do the standard thing: pick up some positive-definite Hermetian form, and then since K is compact, you can integrate. So let B be any Hermetian form; then:

$$\bar{B}(x,y) \stackrel{\text{def}}{=} \int_K B(gx, gy)\, dg$$

and if B was positive-definite, then $\bar{B}$ is K-invariant and positive-definite.

So, if K was a compact group, let's look at $\mathfrak{k} = \mathrm{Lie}(K)$ and its adjoint representation. Then it is completely reducible: $\mathfrak{k}$ splits as a direct sum of irreducible ideals. Moreover, we have $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$. Then it also is a direct sum of irreducibles. But then $\mathfrak{k}_{\mathbb{C}}$ is the direct sum of a semisimple and and abelian, so so is $\mathfrak{k}$: $\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{z}$, where $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$ is seimsimple, and $\mathfrak{z}$ is abelian. And of course, $\mathfrak{k}'_{\mathbb{C}}$ is semisimple.

So it's clear that the classification of compact groups is reduced somehow to the classification of semisimple Lie algebras.

We will not prove the following theorem, which you can easily prove from above:

Theorem 10.2 *If K is a compact connected Lie group, then $K \cong (K' \times T)/\Gamma$, where T is a torus, K' is semisimple compact, and Γ is a finite central subgroup of $K' \times T$.*

We will see very soon, but maybe not today, that if K is semisimple, and you take any connected cover, then the cover is compact.

So the classification is straightforward, and follows from the following basic question. Suppose we have a semisimple complex Lie algebra. How many ways can we get a compact group from it?

So, let $\mathfrak{k}$ be semisimple, and $\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$. The question is going back. Suppose we start with $\mathfrak{g}$: can we get $\mathfrak{k}$? The answer is "yes", and more or less in a unique way.

For this we do the following thing. Suppose that $\mathfrak{g}$ is semisimple: then it has a standard set of Chevalley generators $\{x_i, h_i, y_i \text{ s.t. } i = 1, \dots, n\}$. Then consider the skew-linear automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$. By *skew-linear*, we mean that $\sigma(\lambda x) = \bar{\lambda}\sigma(x)$, and by automorphism we want $\sigma([x,y]) = [\sigma(x), \sigma(y)]$. So it suffices to define it on generators. We will present the *Cartan involution.*

So, what it does is: $\sigma(x_i) = -y_i$, $\sigma(y_i) = -x_i$, and $\sigma(h_i) = -h_i$. This is clearly an automorphism: everything is defined over $\mathbb{Z}$, so you can definitely extend this to an involution.

So, let's take $\mathfrak{k} = \mathfrak{g}^\sigma = \{x \in \mathfrak{g} \text{ s.t. } \sigma(x) = x\}$. The point is that this is an $\mathbb{R}$-Lie subalgebra of $\mathfrak{g}$, and it's not difficult to see that $\mathfrak{g} = \mathfrak{k} \oplus \sqrt{-1}\,\mathfrak{k} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$.

Now look at the Killing form on $\mathfrak{k}$. We claim that it is negative definite. So check this, we just hve to check it on the generators. Look at how it goes on $\mathfrak{sl}_2$. Then it is:

$$\sigma : \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} -\bar{a} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}$$

So this is a Hermetian matrix. Then you look at the root spaces, which are orthogonal pretty much, and the calculation is not difficult.

So this means that if you look at $\mathrm{Ad} : K \to \mathrm{GL}(\mathfrak{k})$, then actually the image lies in the orthogonal group $\mathrm{O}(\mathfrak{k})$, and this is compact when the form is negative-definite, and the image is closed. So this means that $\mathrm{Ad}\, K$ is compact.

In fact, we will see that this construction is more or less unique.

Lecture 11 Feb 12, 2010

11.1 Unitary Representations

The plan today is to start talking about unitary representations, and to get from this some results about compact groups. We will not prove everything: it is based on some basic functional analysis

Unitary representations are very important, and for the last 50 years people have wanted to classify unitary representations of specific groups. The whole subject was started by Hermann Weyl, and is motivated by quantum mechanics. In fact, the unitary representation theory of real Lie groups is an ongoing project.

So, let V be a *Hilbert space*, i.e. a vector space over $\mathbb{C}$ with a positive-definite Hermitian form $(,)$, inducing a norm $\|v\| = \sqrt{(v,v)}$, and V is complete with respect to $\|\cdot\|$. Then $\mathrm{B}(V) = \{X \in \mathrm{End}(V) \text{ s.t. } |X| < \infty\}$, where X is operator norm, is the set of *bounded operators*. Then we also have the *unitary operators* $\mathrm{U}(V)$, the operators preserving the norm. $\mathcal{B}(V)$ is an algebra, but $\mathrm{U}(V)$ is a group.

So let G be a Lie group. A *unitary representation* of G is a homomorphism $G \to \mathrm{U}(V)$ such that (gx, y) is continuous in each variable. Actually, this is a little subtle, because we have multiple topologies, but we don't want to go into this. V is *(topologically) irreducible* if any closed invariant subspace is either 0 or V.

Lemma 11.1 (Schur) *If V is irreducible and $T \subseteq \mathrm{B}(V) \cap \mathrm{End}_G(V)$, then $T \in \mathbb{C}id$.*

Proof: Any bounded self-adjoint operator is diagonalizable. ****There are many complaints about this statement, because it is wrong, depending on what "diagonalizable" means. There is a proposed proof of the statement. The general agreement is that the statement is true but nontrivial.**** □

The point is that Hilbert spaces correctly generalize linear algebra.

Let K be a compact Lie group.

Example 11.2 $L^2(K)$ is an example of a unitary representation, where the action is $g\phi(x) = \phi(g^{-1}x)$. ◊

Example 11.3 Any finite-dimensional representation of K is unitary, by averaging to get the invariant form. $\Diamond$

In fact, for a compact group K, any continuous representation on a Hilbert space can be made into a unitary representation. But these don't give more examples:

Proposition 11.4 *Any irreducible unitary representation of K is finite-dimensional.*

Proof: Pick up $v \in V$ with $\|v\| = 1$. Then define a projection $T : V \to V$ by $T(x) = (x, v)v$. Then take the average $\bar{T} = \int g\,T\,g^{-1}\,dg$. Then T is self-adjoint and compact, so $\bar{T}$ is as well. Moreover, $(Tx, x) \geq 0$, so $(\bar{T}x, x) \geq 0$. But $\bar{T}$ is compact and self-adjoint, and so has an eigenvalue. Then $\ker(\bar{T} - \lambda\mathrm{id})$ is an invariant subspace. So $\bar{T} = \lambda\mathrm{id}$, but it is also compact, so this is only possible if $\dim V < \infty$. $\square$

This also proves:

Proposition 11.5 *Any unitary representation of K has an irreducible subrepresentation.*

So, if I pick up all irreducible subrepesentations of a unitary representation, take their sum, and then take the closure, I should get the whole thing, because otherwise I take the orthogonal complement. This holds only for compact groups: Any unitary representation is the closure of its irreducible subrepresentations.

From this, we can actually figure out that K has always a faithful finite-dimensional representation. This is very similar to the case of an algebraic group. The only change is to take $L^2(K)$. Indeed, $L^2(K)$ is clearly a faithful representation. Then pick up different irreducible subrepresentations. Suppose $V_t \subseteq L^2(K)$ is an irreducible subrepresentation, and so we have $\pi_t : K \to \mathrm{U}(V_t)$. Then $\bigcap \ker \pi_t$ is trivial. But in a compact group, any set of closed subgroups will eventually stop: we have $\ker \pi_1 \supsetneq (\ker \pi_1 \cap \ker \pi_2) \supsetneq \dots$ eventually stops at $\ker \pi_1 \cap \cdots \cap \ker \pi_s = \{1\}$. So then $V = V_1 \oplus \cdots \oplus V_s$ is a faithful finite-dimensional representation of K.

So, let's star with a compact group K. Assume it is semisimple and connected. Then it has a Lie algebra $\mathrm{Lie}(K) = \mathfrak{k}$. Denote $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$. Then we know all finite-dimensional complex irreducible representations of $\mathfrak{k}$. We have the generators $V_{\omega_1}, \dots, V_{\omega_n}$ corresponding to the fundamental weights. Then we can take $V = V_{\omega_1} \oplus \cdots \oplus V_{\omega_n}$, and construct an algebraic group $G \subseteq \mathrm{GL}(V)$. Then we take $K \subseteq G$ corresponding to $\mathfrak{k}$. **Question from the audience:** Wait, so $G = \exp \mathfrak{g}$ in GL, and how do I know it's algebraic? **Answer:** We discussed this last time. It follows from the semisimplicity of G.

Question from the audience: So the two Ks are different? **Answer:** Yes, we start with $\mathfrak{k}$, and then do above, and construct K.

Proposition 11.6 *Let $\mathfrak{k}$ be the Lie algebra of a compact semisimple group, and K constructed from $\mathfrak{k}$ as above. Then K is simply connected.*

Proof: Let $\tilde{K} \to K$ be the simply-connected cover. So $K = \tilde{K}/\Gamma$. If Γ is finite, set $K' = \tilde{K}$, and otherwise pick $\Gamma' \subsetneq \Gamma$ of finite index — it is an abelian discrete group. Then we have a finite cover $K' \to K$. So K' has a faithful representation, as it is compact, but all the faithful representations are already there, so $K' = K$.

The point is that the adjoint form of $\mathfrak{k}$ is compact, and our proof says that the simply-connected cover of a compact semisimple group is compact. $\square$

Indeed, we know that the center $Z(K) = \ker \mathrm{Ad}$. But also $Z(K) = P/Q$, the quotient of the weight lattice by the root lattice. Because inside K we have the maximal torus T, whose group of characters is P. And in the adjoint form we have $\mathrm{Ad}\, T \subseteq \mathrm{Ad}\, K$, and its characters are Q. But then the center is the quotient of one by the other.

Theorem 11.7 (Peter-Weyl) *If K is a compact group, then:*

$$L^2(K) = \overline{\bigoplus_{L(\lambda)\in \mathrm{Irr}(K)} L(\lambda) \otimes L(\lambda)^*}$$

Question from the audience: What is the bar? **Answer:** We must take closures, as opposed to the algebraic case.

We basically have this for semisimples, but actually for any compact, it is a quotient of a torus times a semisimple by a discrete group. The only thing to prove is that $\bigoplus_{\lambda\in P^+} L(\lambda) \otimes L(\lambda)^*$ is dense in $L^2(K)$. And this follows form the fact that polynomial functions are dense in L^2.

11.2 Cartan decomposiiton

Let's keep the notation of above. Then we have $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. And this we can lift to the group. $\mathfrak{k} \rightsquigarrow K \subseteq G$, and we construct a subset $M = \exp(i\mathfrak{k}) \subseteq G$. Then the claim is that the multiplication map $K \times M \to G$ is an isomorphism of real manifolds. Moreover, $\exp : i\mathfrak{k} \to M$ is an isomorphism of manifolds, so $M \cong \mathbb{R}^m$. So in particular, $\pi_1(G) = \pi_1(K)$. So if we start with simply-connected K, then G is also simply-connected. We also have $Z(G) = P/Q = Z(K)$.

So, we have $\mathrm{GL}(V) \subseteq G$. Since the representation is unitarizable, we can pick an inner product so that $K \subseteq \mathrm{U}(V) \subseteq \mathrm{GL}(V)$.

Now, let H^+ be the set of positive-definite Hermetian matrices. Then recall that the multiplication map $\mathrm{U} \times \mathrm{H}^+ \to \mathrm{GL}(V)$ is an isomorphism. Indeed, injectivity is obvious, because the intersection $\mathrm{U} \cap \mathrm{H}^+ = \{1\}$. But surjectivity is also standard. If $X \in \mathrm{GL}(V)$, then $X\bar{X}^{\mathrm{t}} \in \mathrm{H}^+$, and so we find $S \in \mathrm{H}^+$ with $S^2 = X\bar{X}^{\mathrm{t}}$. But then $XS^{-1} \subseteq \mathrm{U}(V)$. This is just linear algebra.

Lecture 12 Feb 5, 2010

First we will correct the Schur lemma, although we won't use it. We thought there was a simple proof, but the one we found depends on spectral theory.

Proposition 12.1 *If V is an irreducible unitary representation of G, then $\mathrm{B}_G(V) = \mathbb{C}$. Here $\mathrm{B}_G(V)$ is the set of bounded operators that commute with G.*

Proof: Pick $X \in \mathrm{B}_G(V)$, and think about $A = X + X^*$ and $B = (X - X^*)/i$. These are Hermetian and commute with G. Then by some functional analysis:

$$A = \int_{\operatorname{Spec} A} x \, dP(x)$$

The point is that if $E \subseteq \operatorname{Spec} A$ is a Borel subset, then $P(E)$ is a projector and commutes with A and also with G, and now the standard kernel-and-image argument works: $\ker P(E)$ is an invariant closed subspace, so $P(E) = \lambda \mathrm{id}$, and therefore A, and by the same token B, are scalars, so $X = (A + iB)/2$ is as well. □

We might talk about unitary representations later when we talk about orbit methods.

12.1 Compact Groups

We now return to our discussion of compact groups. We will prove:

Theorem 12.2 (Cartan) *Given a complex semisimple Lie algebra $\mathfrak{g}$, it has a unique up to isomorphism compact connected simply-connected Lie group K with $\mathfrak{g} \cong \operatorname{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C}$.*

We will write $\mathfrak{k} = \operatorname{Lie}(K)$.

So for each simply-connected complex group, there is exactly one compact group.

Proof: We already did the existence. Remember, we defined at some point the *Cartan involution* on $\mathfrak{g}$ by $\sigma(h_i) = -h_i$, $\sigma(x_i) = -y_i$, and $\sigma(y_i) = -x_i$, and extended by skew-linearity to an involution of $\mathfrak{g}$. We picked up $\mathfrak{k} = \mathfrak{g}^\sigma$. Then the compactness of $\mathfrak{k}$ follows from the fact that the Killing form is negative-definite ****I actually don't know why negative-definite Killing form forces a Lie algebra to be compact****.

Now, suppose that we have a Lie algebra $\mathfrak{k}$ over $\mathbb{R}$ such that $\mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}$. Now we can write $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$, and define θ to be skew-linear and the identity on $\mathfrak{k}$, so that $\mathfrak{k} = \mathfrak{g}^\theta$. This is art of a more general construction. If you know some object over an algebraically closed field, and you want to know how many objects over some subfield extend to your object, then the answer is determined by Galois cohomology. So: the possible $\mathfrak{k}$s are determined by skew-linear involutions on $\mathfrak{g}$.

But furthermore, if θ_1, θ_2 are involutions of $\mathfrak{g}$, and $\theta_1 = \phi\theta_2\phi^{-1}$ for some $\phi \in \operatorname{Aut}\mathfrak{g}$, then $\mathfrak{g}^{\theta_1} \overset{\phi}{\leftrightarrows} \mathfrak{g}^{\theta_2}$.

So we want to classify involutions up to conjugating by an automorphism.

So: since θ acts on $\mathfrak{g}$, then it also acts on $\operatorname{Aut}\mathfrak{g}$ by conjugation, and the restriction to the adjoint action is what it should be.

So, K is compact, so there is a Hermetian form on $\mathfrak{g}$ that's invariant with respect to K. **Question from the audience:** Can you just take the Killing form and skew one of the spots? **Answer:** Yes, but you must check that θ preserves the Killing form. ****If θ is a Lie algebra automorphism, then it must, right?****

So, we have: $\mathrm{GL}(\mathfrak{g}) = \mathrm{U}(\mathfrak{g}) \times \mathrm{H}^+(\mathfrak{g})$, the unitary and other parts with respect to the invariant form.

Now we look at $\mathrm{Aut}\,\mathfrak{g} \cap \mathrm{U}(\mathfrak{g}) = \{\phi \text{ s.t. } \theta\phi = \theta\phi\}$ and $\mathrm{Aut}\,\mathfrak{g} \cap \mathrm{H}^+(\mathfrak{g}) = \{\phi \text{ s.t. } \theta\phi\theta^{-1} = \phi^{-1} \text{ and positivity condition}\}$.

Question from the audience: Wait, why? **Answer:** $\theta(X) = (X^*)^{-1}$. But these coincide on $\mathfrak{k}$, and ****missed****.

We have $\mathrm{Aut}\,\mathfrak{g} \cap \mathrm{U}(\mathfrak{g}) = \mathrm{Aut}\,\mathfrak{g}^\theta$, and for notation let's write $\mathrm{Aut}\,\mathfrak{g} \cap \mathrm{H}^+(\mathfrak{g}) = \mathrm{Aut}\,\mathfrak{g}^+$.

Then we claim that $\mathrm{Aut}\,\mathfrak{g} = \mathrm{Aut}\,\mathfrak{g}^\sigma \times \mathrm{Aut}\,\mathfrak{g}^+$. The proof follows from the following:

Lemma 12.3 *Suppose that $X \in (\mathrm{Aut}\,\mathfrak{g})^+$ and $X = e^A$ for some A Hermetian, although maybe not positive-definite. Then $e^{At} \in \mathrm{Aut}\,\mathfrak{g}$ for $t \in \mathbb{R}$. So then we can define any real power X^t.*

For the proof of lemma 12.3, we choose an orthonormal basis $\{e_1, \dots, e_N\}$ of $\mathfrak{g}$, and such that $A = \lambda_i e_i$. Then we take the structure constants $[e_i, e_j] = \sum c_{ij}^k e_k$. Then e^A is an automorphism iff $\lambda_i + \lambda_j = \lambda_k$ whenever $c_{ij}^k \neq 0$. But clearly this condition is well-behaved under multiplying by a real number: e^{At} is an automorphism.

So, returning to the claim, we can take a square root. Last time, when we did the proof for the general linear group, all we needed was to take the square root. So the proof goes in exactly the same manner.

One more remark: Suppose we take θ and multiply by some automorphism $\phi \in \mathrm{Aut}\,\mathfrak{g}^+$. Then $\theta\phi = \phi^{-1/2}\theta\phi^{1/2}$.

Now we are in the following situation. We know there is this σ, and another involution θ, and they both give compact groups. Now take $\theta\sigma = \phi$. This is an automorphism, and $\sigma = \theta\phi$. So without loss of generality, by the lemma and the immediately-previous remark, we can assume without loss of generality that $\phi \in \mathrm{Aut}\,\mathfrak{g}^\theta$. But then θ commutes with ϕ, and so with σ.

Now, if we have two commuting operators in space, then they have common eigenvalues, and these two are involutions, so they have only $+$ and $-$ eigenspaces. So, we write $\mathfrak{g} = \mathfrak{g}^\sigma \oplus i\mathfrak{g}^\sigma$, and $\mathfrak{g}^\sigma$ is θ-invariant. So we have $\mathfrak{g}^\sigma = (\mathfrak{g}^\sigma)^\theta + (\mathfrak{g}^\sigma)'$, where the latter is $(\mathfrak{g}^\sigma)' = \{x \text{ s.t. } \theta x = -x\}$. But by skew-linearirty, we have $\mathfrak{g}^\theta = (\mathfrak{g}^\sigma)^\theta + i(\mathfrak{g}^\sigma)'$.

But finally, the claim is that if these give compact groups, then $(\mathfrak{g}^\sigma)' = 0$.

Indeed, if $x \in \mathfrak{g}^\sigma$ (or $\mathfrak{g}^\theta$), then ad_x is skew-Kermetian, and so ad_x is semisimple with imaginary eigenvalues. But then $i\,\mathrm{ad}_x$ is a problem. So then $(\mathfrak{g}^\sigma)' = 0$, and so $\sigma = \theta$.

Question from the audience: Where did you get the *ad*-Hermiticity? **Answer:** Because θ, σ give compact groups, and ****missed****. □

You can easily now also show:

Proposition 12.4 *Let K be semisimple. Then K is compact iff the Killing form on $\mathfrak{k}$ is negative-definite.*

****Did we already prove one direction of this? It seems we used it in the above proof.****

By the way, here's a good exercise. Take the classical series $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$, and $\mathrm{SP}(2n, \mathbb{C})$. What are their compact real forms? The answer for $\mathrm{SL}(n, \mathbb{C})$ is the special-unitary group $\mathrm{SU}(n)$. For $\mathrm{SO}(n, \mathbb{C})$, you get $\mathrm{SO}(n) = \mathrm{SO}(n, \mathbb{R})$. For $\mathrm{SP}(n, \mathbb{C})$, it is not so well known. You must take the n-dimensional space $\mathbb{H}^n$ over the quaternions. The quaternions are not commutative, but there is a well-defined notion of conjugation, and of Hermitian form. Then $\mathrm{SP}(n)$ is the group of all $\mathbb{H}$-linear operators preserving the form on $\mathbb{H}^n$.

Question from the audience: We've said that the complex representation of a compact group is the same as of the complex group. What about the real representation theory? **Answer:** It's not much harder. We didn't prepare it, but let's talk about it.

12.2 Real representation theory

Let K be a Lie group, V a finite-dimensional irreducible representation over $\mathbb{R}$. Then the Schur lemma doesn't quite work, because $\mathbb{R}$ isn't algebraically closed. But anyway $\mathrm{End}_K(V)$ must be a finite-dimensional division algebra over $\mathbb{R}$. So it must be $\mathrm{End}_K(V) = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Now, what happens when we complexify? Then in each case, it just complexifies. I.e.: let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Then $\mathrm{End}_K(V_{\mathbb{C}}) = \mathrm{End}_K(V) \otimes \mathbb{C} = \mathbb{C}$ in the first case, $\mathbb{C} \oplus \mathbb{C}$ in the second case, and $\mathrm{Mat}_2(\mathbb{C})$ in the quaternionic case.

In the first case, $V_{\mathbb{C}}$ is still irreducible. In the second, it must be the sum of two irreps: $V_{\mathbb{C}} = V_1 \oplus V_2$, and they must be different. These are called "conjugate". In the third case, it's the sum of two copies that are the same: $V_{\mathbb{C}} = V' \oplus V'$. ****So this is something like "ramification".**** So the point is: you can ask, we have a highest weight vector, and when you restrict to the real part, you get into one of these types.

You have the list: an infinite list of irreducible representations. Then some representations are fixed by this involution, $V_1 \leftrightarrow V_2$. This is related to the automorphisms of the Dynkin diagram. In A_n, there is one involution of the Dynkin diagram. You look at the weights of your representation, restrict and complexify, and then you can see what happens to it. For compact groups, there is a way to just write down some integral, and see if it's $0, 1, -1$. **Question from the audience:** I think it has something to do with the square of the character. **Answer:** Yes.

This involution is essentially passing to the dual. There are two things that can happen: either the dual of a representation is not the isomorphic to the original, in which case you're in the second situation. If it is the same, then it has a complex form, and a pairing that's either symmetric or skew-symmetric. And from this you can figure out what you have: quaternionic or real.

We will include some of this stuff in the problem sets.

12.3

Next time, we will begin looking at the coadjoint representation $\mathfrak{g}^*$. We will see that it is a Poisson manifold. In fact, we will see that the set of orbits are precisely the symplectic leaves. Then we will move to the compact case, and then coadjoint orbits are really going to be complex manifolds. They come in "flag varieties". Then we will look at flag varieties: projectivity, Bruhat decomposition, etc.

Lecture 13 Feb 19, 2010

Today we will talk a little bit about the connection between coadjoint orbits and symplectic geometry.

13.1 Poisson Algebra

A *Poisson algebra* is:

1. an associative algebra A
2. along with a Lie bracket $\{,\}$ on A
3. such that $\{a, \cdot\}$ is a derivation: $\{a, bc\} = \{a, b\}c + b\{a, c\}$

Example 13.1 Let $\mathcal{A}$ be filtered: $\mathcal{A} = \bigcup_{i=0}^{\infty}$, with filtered multiplication. And suppose that $A = \operatorname{gr}\mathcal{A}$ is commutative. Then A is naturally Poisson: you take representatives $a, b \in \mathcal{A}$ of your elements of A, take their bracket, and project, and this is well-defined.

In particular, $\mathcal{S}\mathfrak{g} = \operatorname{gr}\mathcal{U}\mathfrak{g}$ is Poisson. ♢

Now, let $\mathcal{M}$ be a manifold — analytic, smooth, whatever you want — and suppose that $C^\infty(\mathcal{M})$ is a Poisson algebra. Then actually you do know that derivations of functions are vector fields, so the Poisson structure gives you a map $C^\infty(\mathcal{M}) \to \operatorname{Vect}(\mathcal{M})$ by $f \mapsto \{f, -\} = \mathcal{D}_f$. In fact, you can show — it is not difficult — that this map must factor through the differential $d : C^\infty(\mathcal{M}) \to \Omega^1\mathcal{M}$. So actually a POisson structure on a manifold is determined by a section $\gamma \in \bigwedge^2 \mathrm{T}\mathcal{M}$. ****VS writes $\mathrm{T}\mathcal{M}$ for $\operatorname{Vect}\mathcal{M} = \Gamma(\mathrm{T}\mathcal{M})$ in the classical notation. I will do this as well.****

Question from the audience: Why must the map factor as you say? **Answer:** Because it is also a derivation in the first spot, and any derivation factors like this.

Example 13.2 $\mathfrak{g}^*$ is a Poisson manifold. The definition is as follows. Let $x \in \mathfrak{g}^*$; then $\mathrm{T}_x\mathfrak{g}^* = \mathfrak{g}^*$ naturally. If $f, g \in C^\infty(\mathfrak{g}^*)$, then $df_x, dg_x \in \mathfrak{g}$ by the identification $\mathrm{T}_x\mathfrak{g}^* = \mathfrak{g}^*$. Then we define $\{f, g\}|_x = \langle x, [df_x, df_g]\rangle$. ♢

Example 13.3 Let $\mathcal{M}$ be a symplectic manifold. I.e. it comes with a nondegenerate closed $\omega \in \Omega^2\mathcal{M}$. The nondegeneracy determines an iso $\Omega^1\mathcal{M} \to \mathrm{T}\mathcal{M}$. ♢

We will not prove the following, but it is not difficult ****It was one of my qual questions last year. I think to do the complete proof takes about 40 minutes — for my qual, I only outlined the proof. The hard part is to show that it is a manifold, and requires a little of the theory of integrable distributions.****. First: The vector field $\mathcal{D}_f = \{f, \cdot\}$ is the *Hamiltonian vector field* for the function f.

Theorem 13.4 *Let $\mathcal{M}$ be Poisson and $x \in \mathcal{M}$. Then define $\mathcal{M}_x$ to be the set of all $y \in \mathcal{M}$ which you can reach from x by Hamiltonian vector fields (maybe composing several such fields). Then $\mathcal{M}_x$ is a symplectic manifold, called the* symplectic leaf*.*

So, every Poisson manifold is a disjoint union of symplectic leaves.

Example 13.5 Let $\mathcal{M} = \mathfrak{g}^*$. Then symplectic leaves are ****connected components of**** G-orbits in $\mathcal{M}$.

Indeed, let $x \in \mathfrak{g}^*$, and then $\mathrm{T}_x(G \cdot x) \cong \mathfrak{g}/\operatorname{Stab}_{\mathfrak{g}}(x)$, where $\operatorname{Stab}_{\mathfrak{g}}(x)$ is the stabilizer of x in $\mathfrak{g}$. Then you can define the form $\omega_x(y_1, y_2) = \langle [y_1, y_2], x\rangle$. It is easy to see that the kernel of this form is the stabilizer.

Question from the audience: Why isn't the tangent space a quotient of $\mathfrak{g}^*$? **Answer:** If you have a G-action $\mathcal{M}$, then you have $\mathfrak{g} \to \operatorname{Vect} \mathcal{M}$, and for each x, a map $\mathfrak{g} \to \mathrm{T}_x(G \cdot x)$.

Indeed: $y \in \ker \omega_x$ iff$\forall z \in \mathfrak{g}$, $0 = \langle [y, z], x\rangle = \langle z, \operatorname{ad}_y^* x\rangle$, but this happens iff $\operatorname{ad}_y^* x = 0$. So ω_x is well-defined ****and nondegenerate**** on $\mathfrak{g}/\operatorname{Stab}_{\mathfrak{g}} x$. ****Why is it closed?**** ♢

Corollary 13.6 *The dimension of any coadjoint orbit is even.*

If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} \cong \mathfrak{g}^*$ as $\mathfrak{g}$-modules. ****There is a $\mathbb{C}^\times$ ambiguity in how to make this identification.****

Example 13.7 Look at the nilpotent cone $\mathcal{N}$; it has an open dense "regular" orbit $\mathcal{N}_{\mathrm{reg}}$. Then we now know that the complement $\mathcal{N} \smallsetminus \mathcal{N}_{\mathrm{reg}}$ consists of even-dimensional components. This implies that $\mathbb{C}[\mathcal{N}] = \mathbb{C}[\mathcal{N}_{\mathrm{reg}}]$. ♢

13.2 Coadjoint orbits for compact groups

We know what are compact groups, and the torus part acts trivially on the coadjoint representation. So we can assume that your group is semisimple.

Example 13.8 $K = \mathrm{U}(n)$. You can take $\mathrm{SU}(n)$ if you want. Then the coadjoint representation, which is the same as the adjoint representation, can be easily identified with Hermitian matrices. So $\mathfrak{k}^* \cong \mathfrak{k} \cong$ Hermitian matrices.

So the orbits are classified by their eigenvalues: let $x = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, with $\lambda_i \in \mathbb{R}$, and we can order them: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then to the orbit x, we associate a partition $(m_1, \dots, m_k)$ of n, by keeping track of just the multiplicities: $3, 2, 2, 1 \mapsto (1, 2, 1)$.

Then it's clear that $\operatorname{Stab}_{\mathrm{U}(n)} x \cong \mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_k)$.

Now, we want to study $K \cdot x$. It is a symplectic manifold, but it is also an almost-complex manifold, and it has a Riemannian structure, and in fact the almost-complex structure is a complex structure.

How to see this? One simple way is just to say what manifold it is. We pick up $\mathbb{C}^n$, and inside it we will pick up the flag $\mathbb{C}^n = V_k \supseteq \cdots \supseteq V_2 \supseteq V_1 \supseteq V_0 = 0$, which is a flag of a certain type: we demand that $\dim(V_i/V_{i-1}) = m_i$. So we pick up the complex manifold of such flags of this type, and these numbers in the partitions are called the"type" of the flag. So for a partition $\vec{m}$ we have the flag type $\mathrm{Fl}(m_1, \ldots, m_k)$. **Question from the audience:** This depends on the ordering of the partition? **Answer:** There is some ambiguity, yes. But at the end of the day you get isomorphic manifolds.

And $\mathrm{GL}(n, \mathbb{C})$ acts transitively on $\mathrm{Fl}(m_1, \ldots, m_k)$. But in fact $\mathrm{U}(n)$ already acts transitively on $\mathrm{Fl}(m_1, \ldots, m_k)$. This is more or less obvious from linear algebra.

And indeed, $\mathrm{Fl}(m_1, \ldots, m_k)$ is a projective algebraic variety. **Question from the audience:** I don't know what that means. **Answer:** You take an affine space, and look at a collection of homogeneous polynomials. The solution set is a closed variety, and is *projective*. Now, you have to know how to embed $\mathrm{Fl}(m_1, \ldots, m_k)$ into a big affine space. Actually, to do this, it suffices to understand the embedding of the Grassmanian $Gr(n, k) = \mathrm{Fl}(n-k, k) = \{V \subseteq \mathbb{C}^n \text{ s.t. } \dim V = k\}$. Then in fact there is the *Plücker embedding*: $Gr(n, k) \to \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ by $V = \langle e_1, \ldots, e_k \rangle \mapsto e_1 \wedge \cdots \wedge e_k$. ♢

Ok, so we did it for $K = \mathrm{U}(n)$, but actually it works for any compact (semisimple) group. What you do is you have $\mathfrak{k} \subseteq \mathfrak{g}$, and you pick $x \in \mathfrak{k}$ semisimple. Now we look at $i\,\mathrm{ad}_x$, which has all real eigenvalues. Let $\mathfrak{k}_x$ be the stabilizer, and define $\mathfrak{g}_x = \mathfrak{k}_x \otimes_{\mathbb{R}} \mathbb{C}$ the complexification. Maybe we should put ix here, but we hate it, so we multiply everything by i at the beginning. Basically this means that we're taking the stabilizer in $\mathfrak{g}$. How to classify this? We need the root data.

So take $x \in \mathfrak{h}$ where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. Then we can consider $\Delta_0 = \{\alpha \in \Delta \text{ s.t. } \alpha(x) = 0\}$. As well, let's define $\Delta^{\pm} = \{\alpha \in \Delta \text{ s.t. } \pm\alpha(x) > 0\}$.

So what we do is take a hyperplane. And if it is in generic position $\Delta_0 = 0$, but otherwise it is not.

Then you can check that $\mathfrak{g}_x = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_\alpha$. So this is a certain reductive subalgebra of $\mathfrak{g}$. And we can add to it another one which is $\mathfrak{p}_x = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$.

So for example, go back the the unitary group, and take x to be the block matrix with k 1s on the diagonal and the rest 0. Then $\mathfrak{g}_x = \mathfrak{gl}_k \oplus \mathfrak{gl}_{n-k}$, and $\mathfrak{p}_x$ is the block-upper-triangulars. On the other hand, if x is regular, then $\mathfrak{p}_x = \mathfrak{b}$ and $\mathfrak{g}_x = \mathfrak{h}$.

13.3 G/P as a projective algebraic variety

We take $G/G_x \to G/P$, because $G_x \subseteq P$. (Here P, G are the corresponding groups to above.) This is a G-equivariant map, and hence a bundle of homogeneous spaces. And the fiber is P/G_x. Let's write the nilpotent group $\mathfrak{m} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. It's nilpotent, so the exponential map is an algebraic isomorphism, and so $P/G_x = \exp \mathfrak{m} = \mathfrak{m}$. So under this identification, $G/G_x \to G/P$ is a vector bundle. And what kind of bundle is it? We claim that it is a cotangent bundle.

Why? Look at $\mathfrak{m}^- = \bigoplus_{\alpha\in\Delta^-} \mathfrak{g}_\alpha$. Then the Killing form identifies $\mathfrak{m}^+ \cong \mathfrak{m}^-$, and look at the Killing form $\mathfrak{m}^- \to \mathfrak{g}^*$. Then $\mathfrak{p} = (\mathfrak{m}^-)^\perp$. So therefore $\mathfrak{m}^- \cong (\mathfrak{g}/\mathfrak{p})^*$. This is the tangent space at a specific point, but everything is equivarient.

So actually that tells you that this bundle is nothing but the cotangent bundle. Now if you know a little bit of symplectic geometry, you know that the cotangent bundle is an example of a symplectic manifold.

Next time, we will classify compact homogeneous spaces, and we will do it in the algebraic category. It happens that G/P, G/B, etc., play extremely important roles.

Lecture 14 Feb 22, 2010

Any questions?

Question from the audience: We finished last time showing that G/Gx is a cotangent bundle. Is there a characterization of P? **Answer:** We will talk about that today. Basically the goal — maybe we won't finish it today — is to let G be an algebraic group over $\mathbb{K}$, and we assume that $\operatorname{char}\mathbb{K} = 0$ because we're not sure how the story goes otherwise, and we want to classify connected homogeneous projective G-spaces. Basically we will prove that this is always G/P.

14.1 Homogeneous spaces

If you have a pair G and algebraic group and $H \subseteq G$ a closed subgroup, then we can write down $X = G/H$. This is some space, but the question is to make it into an algebraic variety. You can prove that if G is a Lie group and H a closed subgroup, then X makes sense as a nice topological space. The problem in the algebraic case is that even if G, H are affine, then X is not affine generally. So you cannot just write down X as the spectrum of something. Instead, we will use a trick due to Chevellay.

Question from the audience: Is any closed subgroup of an algebraic group algebraic? **Answer:** When we say "closed" we mean "Zariski-closed", so yes.

Theorem 14.1 *Let $H \subseteq G$ be a closed subgroup. Then there exists a representation V of G and a line $\ell \subseteq V$ such that $H = \operatorname{Stab}_G \ell$.*

Proof: Recall that H is given via its ideal $I_H \subseteq \mathbb{K}[G]$. Then recall that $H = \mathcal{N}_G(I_H)$ the normalizer of the ideal. Then $\mathbb{K}[G]$ is Noetherian, so any ideal is finitely generated; we pick up generators $f_1, \ldots, f_n$ of I_H, and as we discussed before, there exists a finite-dimensional G-invariant subspace $\tilde{V}$ containing $f_1, \ldots, f_n$.

Then set $W = I_H \cap \tilde{V}$. Then it's an easy exercise to show that $H = \mathcal{N}_G(W)$: it contains all the generators. So actually you can say that H is the stabilizer of some space W.

If $\dim W = d$, then to get a line we can take powers. Set $V = \bigwedge^d \tilde{V}$, and $\ell = \bigwedge^d W$. □

Corollary 14.2 *$G \cdot [\ell] \cong X$ locally closed in $\mathbb{P}(V)$.*

Here $[\ell]$ is the point in $\mathbb{P}(V)$ corresponding to the line ℓ in V. Thus X is a *quasiprojective variety.*

Question from the audience: Here we're using the theorem that we had before that ****missed****. **Answer:** Yes.

Now we will discuss the special case that H is normal. Then X is a group, and the point is that it's affine algebraic:

Theorem 14.3 *If $H \subseteq G$ is closed an normal, then there exists a representation $\pi : G \to \mathrm{GL}(V)$ such that $H = \ker \pi$.*

Then clearly the image is a closed group, and it's going to be an affine group as it's in $\mathrm{GL}(V)$. **Question from the audience:** The image is closed? **Answer:** Yes, we proved that before. Any locally closed subgroup in a group is closed.

Proof: We start with V' as in the previous theorem, and choose $\ell' \subseteq V'$ a line such that $H = \mathrm{Stab}_G(\ell')$. So let's write down what this means: if $v \subseteq \ell'$, then $\forall h \in H$, $hv = \chi(h)v$. So what is this χ? It is a character of H. Because H is normal, if we take $g \in G$, then $hgv = g(g^{-1}hg)v = \chi(g^{-1}hg)gv$, and $h \mapsto \chi(g^{-1}hg)$ is another character. So the point is that G acts on the characters $\hat{H}$ by conjugation, because H is normal.

So, if $\eta \in \hat{H}$, then we set $V'_\eta = \{v \in V' \text{ s.t. } hv = \eta(h)v \, \forall h \in H\}$. So we set $W = \bigoplus_{\eta \in \hat{H}} V'_\eta$. This is G-invariant, and G permutes the V'_η. So now we construct V as:

$$V = \bigoplus \mathrm{End}_{\mathbb{K}}(V'_\eta)$$

the sum of matrix algebras. It's clear that this is a representation of G, so call it $\pi : G \to \mathrm{GL}(V)$. If you start calculating the kernel — it's an easy exercise — you find out that $\ker \pi = H$. Oh, what is this action? It's just conjugation. □

So this explains why we can quotient by any normal subgroup and get something affine.

There is something that I'm not going to prove here, that this construction does not depend on the representation. You can construct ****something**** that shows that this map $H \to \mathrm{GL}(V)$ is actually a morphism.

Proposition 14.4 *Suppose that G is abelian connected affine algebraic group. ("Algebraic" always means "affine algebraic".) Then the only projective homogeneous space is a point.*

This is just because G/H is a group, so it's affine, but also projective, and the only connected affine projective space is a point.

Example 14.5 (Warning) We know for example that we have a torus, and it is a projective as it is an elliptic curve, but it is not over an affine algebraic group. Because then you'd have to make it as $\mathbb{C}^\times/\Gamma$, but then Γ is not closed. So the action of $\mathbb{C}^\times$ on the torus is not algebraic, because the stabilizers are not Zariski-closed. $\diamondsuit$

So there are things that are quite different in the algebraic category and in the Lie category.

14.2 Solvable groups

Proposition 14.6 *If G is algebraic, then $G' = [G, G]$ is algebraic.*

In Lie groups, this is not true: you need G to be simply-connected.

Proof: Let $Y_g = gGg^{-1}G$; it is a contractible ****constructible?**** set. Then:

$$\overline{G'} = \overline{\bigcup_{g \in G} Y_g} = \overline{Y_{g_1} \cup \cdots \cup Y_{g_n}}$$

Question from the audience: Why is the union finite? **Answer:** This is an algebraic variety, and you use the Noetherian condition.

So then G' contains an open dense set in $\overline{G'}$.

Question from the audience: **missed Answer:** Don't take the union: let's take the product. Actually, we should take the product in the first one too:

$$\overline{G'} = \overline{\prod_{g \in G} Y_g} = \overline{Y_{g_1} \cdots Y_{g_n}}$$

The point is: why can you take it to be finite? The answer is that you construct a chain, and chains are only finitely long. This always causes problems. $\square$

We mention an exercise: $\operatorname{Lie} G' = [\mathfrak{g}, \mathfrak{g}]$ is G is connected.

We now make an important definition. G is *solvable* if the chain:

$$G \supseteq G' \supseteq G'' \supseteq \dots$$

stops at $G^{(n)} = \{1\}$.

Example 14.7 The fundamental example is the group $\mathrm{N}(n)$ of upper triangular matrices in $\mathrm{GL}(n)$. $\diamondsuit$

Theorem 14.8 *Let G be a connected solvable group acting on a projective variety X. Then G has a fixed point on X.*

Proof: If G is abelian, then you pick up the closed orbit, which always exists, and it must be a fixed point by Proposition 14.4.

An exercise: G connected implies that G' is connected.

Now we do induction on dimension. Let Y be the set of points fixed by G'. Then since G' is normal, Y is G-invariant. So actually the G-action factors through G/G', but this is abelian, and we are done. □

This has many nice consequences:

Theorem 14.9 (Lie-Kolchin) *If G is connected solvable and V a representation of G, then there is a full flag fixed by G. In other words, $G \hookrightarrow \mathrm{N}(V)$. More precisely, we have $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$ invariant under G.*

Proof: Take the variety of all flags. This is a projective variety, and so we use Theorem 14.8. □

Question from the audience: This is the group version of Lie's theorem? **Answer:** Yes. It's so easy, because we're in the algebraic case. It's amazing how some things become so easy in the algebraic category. The hard part is algebraic variety.

Now, let G be an affine algebraic group. A *Borel subgroup* is a maximal solvable connected closed subgroup. It is unclear that it is unique or not, but we will see soon that it is unique up to conjugation.

Theorem 14.10 *If B is a Borel subsgroup in G, then G/B is projective. Any two Borel subgroups in G are conjugate.*

Proof: First, pick up B of maximal dimension. When we say "maximal" in the definition, we mean inclusion of sets. So now we pick B of maximal dimension.

We know that there is a faithful representation $G \to \mathrm{GL}(V)$. Then we can pick a B-invariant flag $F \in \mathrm{Fl}(V)$. Now take the G-orbit of this F; then we claim that it is closed. Ah, the point is that the stabilizer of a flag is a subgroup of $\mathrm{N}(V)$, so it is solvable. So the stabilizer of F cannot be any bigger that B. Or, anyway, the connected component isn't any bigger. So $B = \mathrm{Stab}_G(F)_0$. Then $G \cdot F$ is closed, because if it is not, then we have an orbit of smaller dimension, which then has stabilizer a larger group, but we picked B of maximal dimension.

Actually, we haven't finished this yet. We will finish it next time.

The point about conjugacy: Let B' be another Borel group, then it acts on G/B, so it has a fixed point $x \in G/B$. But $\mathrm{Stab}_G\, x$ is conjugate to B, but then use maximality of B' to claim that $B' = \mathrm{Stab}_G\, x$. □

Question from the audience: Is there a good reason to worry about whether a group is disconnected? **Answer:** There's some interesting examples, especially subgroups. By itself, a disconnected group, its representations you can construct from the representations of the connected component. But it's sometimes important to consider quotients by disconnected subgroups.

Lecture 15 Feb 24, 2010

Theorem 15.1 *If B is a Borel subgroup (maximal connected solvable) then G/B is projective. Moreover, any two Borel subgroups are conjugate.*

Proof: Let V be a representation of G such that $\mathrm{Stab}_G \ell = B$ for some line $\ell \in V$. Then look at $F = \{V_n \supsetneq V_{n-1} \supsetneq \cdots \supsetneq V_2 \supsetneq V_1 = \ell \supsetneq V_0 = 0\} \in \mathrm{Fl}(V)$. Then $G \cdot F$ is closed under the assumption that $\dim B$ is maximal. Indeed: suppose $G \cdot F \subsetneq \overline{G \cdot F}$. Then $\dim(\mathrm{Stab}_G F')_0 > \dim B$, but $\mathrm{Stab}_G F'$ is solvable, as it lies in the upper-triangular matrices. But this contradicts the maximality of B. ****I don't see why we need that $\dim B$ be maximal among all Borels, just that B is not contained in a larger connected solvable.****

Ok, so suppose that B' is another Borel. Then B' has a fixed point $x \in G/B$. Then $B' \subseteq \mathrm{Stab}_G x = gBg^{-1}$, and gBg^{-1} is solvable, so by maximality of B', $B = gBg^{-1}$. □

Theorem 15.2 *Let P be a closed subgroup in G. Then G/P is projective iff P contains some Borel.*

Such a subgroup is called *parabolic*.

Proof: In one direction it should be clear: we have a map $G/B \to G/P$ if $B \subseteq P$, and the image of a projective variety is projective.

In the other direction, suppose that G/P is projective. Let B be some Borel in G. Then by the fixed point theorem, B has a fixed point $x \in G/P$. Then $\mathrm{Stab}_G x = gPg^{-1} \supseteq B$, so $P \supseteq g^{-1}Bg$. □

Question from the audience: Is there an analogue in the Lie category? Something with compact homogeneous spaces? **Answer:** More or less: we will show that they all come from coadjoint orbits. But the Lie case has more things, like the torus, which is not projective. See, there is a trick: if you have a complex group, you can still construct compact homogeneous spaces, but the action will not be algebraic. Perhaps the statement is for simply-connected things.

But the next step for us is to reduce to the semisimple case. This is useful, as it helps to understand the structure theory for algebraic groups.

So, construct $\mathrm{Nil}(G) = \left(\bigcap_{\pi \in \mathrm{Irr}(G)} \ker \pi\right)_0$. Then this is normal and unipotent ****all elements are unipotent****. Let V be a faithful representation of G, and pick a Jordan-Holder series $V \supsetneq V_1 \supsetneq \cdots \supsetneq V_k$ so that each V_i/V_{i+1} is irreducible. Then with respect to this flag, $\mathrm{Nil}(G)$ consists of upper-triangulars with 1s on the diagonal. Then in fact you can prove that $\mathrm{Nil}(G)$ is the maximal unipotent ****other adjectives****. We also have $\mathrm{nil}\,\mathfrak{g} = \mathrm{Lie}(\mathrm{Nil}\,G)$, the *nilradical* of the Lie algebra.

Lemma 15.3 *If $\mathrm{Nil}(G) = \{1\}$, then G is reductive.*

In particular, $G/\mathrm{Nil}(G)$ is reductive. We haven't defined *reductive*: we mean it's a quotient of $G_{\mathrm{ss}} \times T/$finite group, where G_{ss} is semisimple and T is a torus. **Question from the audience:** Is the converse false? **Answer:** No, it's iff. But in the other direction it's quite simple, since we have the description of all the reductive groups, so you just check.

Proof: Let V be a faithful representation of G. Then $V = V_1 \oplus \cdots \oplus V_k$, where each V_i is irreducible. We can have $V_i \cong V_j$. In any case, $\operatorname{rad}\mathfrak{g}$ acts as a scalar on each V_i; this is a standard fact. So then $\operatorname{rad}\mathfrak{g} = \mathcal{Z}(\mathfrak{g})$ — that's exactly the definition of a reductive Lie algebra. But we need to see that $\mathcal{Z}(G)_0$ is an algebraic torus. But since $\mathcal{Z}(G)_0$ acts as a scalar on each V_i, we see that it is a connected closed subgroup in $\mathbb{K}^\times \times \cdots \times \mathbb{K}^\times$. Then the end of the proof is the not-difficult exercise that any closed connected subgroup of an algebraic torus is a torus. □

So now consider the projection $p : G \to G/\operatorname{Nil}(G)$. Then consider $p^{-1}\big(\mathcal{Z}(G/\operatorname{Nil} G)_0\big)_0$. It is a maximal normal connected solvable subgroup in G. So it is an algebraic group, and it is called the *radical* of G: $\operatorname{Rad}(G)$. Of course, $\operatorname{Lie}(\operatorname{Rad} G) = \operatorname{rad}\mathfrak{g}$. And $G/\operatorname{Rad} G$ is semisimple.

But what we claim next is that any parabolic subgroup contains this radical.

Question from the audience: So this is like the Levi decomposition? **Answer:** Not exactly, as for that you need a semidirect product. We think it's true that you can write the reductive part as a semidirect product of the nilradical and a **?**. But it's not true that you can write $\mathrm{GL}(n)$ as a semidirect product: you need to take a quotient.

Anyway: every parabolic subgroup of G contains $\operatorname{Rad} G$, because $\operatorname{Rad} G$ has a fixed point on G/P, but since $\operatorname{Rad} G$ is normal it acts trivially on all of G/P, so in particular it is contained in P.

But actually, the point is you can forget about $\operatorname{Rad} G$ completely, since it doesn't contribute to the action of G on any G/P.

Corollary 15.4 *Any projective homogeneous space is isomorphic to G/P where G is semisimple is P is some parabolic subgroup.*

Question from the audience: Wait, I'm confused. I guess at some point we dropped semisimplicity as an assumption? **Answer:** That's right, we dropped it. But now we get it back.

Henceforth, we assume that G is connected. Because if it is allowed to be not connected, then you can put on any finite group, so you're stuck with the theory of finite groups.

In fact, we will give the classification of connected semisimple groups and their parabolic subgroups.

15.1 Parabolic Lie algebras

We let G be connected and semisimple.

Let P be a parabolic subgroup of G, and $\mathfrak{p} = \operatorname{Lie}(P)$. Then $\mathfrak{p} \subseteq \mathfrak{b}$ for some Borel, and they are all conjugate, so let's pick one: $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ with $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, the *standard Borel.* We also recall the generators $h_1, \dots, h_n, x_1, \dots, x_n, y_1, \dots, y_n$.

Lemma 15.5 *There exists a subset $S \subseteq \{1, \dots, n\}$ such that $\mathfrak{p}$ is generated by $\{h_1, \dots, h_n, x_1, \dots, x_n\}$ and y_j for $j \in S$.*

How do you usually use this? You draw the Dynkin diagram, and shade a few of the nodes: the unshaded nodes are S. ****picture, and an example with $\mathfrak{sl}(6)$****

Proof: This follows from a very simple fact. If $\alpha, \beta \in \Delta^+$, and $[y_\alpha, y_\beta] \in \mathfrak{p}$, then take $x_\alpha \in \mathfrak{p}$, and since y_α, x_α form an $\mathfrak{sl}(2)$-triple, then $[x_\alpha, [y_\alpha, y_\beta]] = cy_\beta$ for $c \neq 0$, so $y_\alpha, y_\beta \in \mathfrak{p}$. Then do induction. □

Lemma 15.6

1. *Let $V = L(\lambda)$. Then $\mathbb{P}(V)$ has only one closed orbit: the orbit of the line of the highest vector, which we will call ℓ_λ.*
2. *If V is not irreducible, then any closed orbit in $\mathbb{P}(V)$ is contained in $\mathbb{P}(W)$ for W some irreducible invariant subspace of V.*

Actually, this is a trivial statement: We know that the closed orbit is projective, and has a fixed line, there is only one fixed line in $L(\lambda)$. On the other hand, an invariant line in an arbitrary V is a highest weight space. The only thing that is not clear is if there are multiple isomorphic direct summands. But this part is a simple exercise.

Corollary 15.7 *P is the stabilizer of some $\ell_\lambda \subseteq L(\lambda)$.*

So this is a sort of description of P: You pick up some representation, pick some highest vector, and then you get a parabolic subgroup, and moreover any P can be obtained in this way. ****We return to the $\mathfrak{sl}(6)$ example from before.****

From the point of view of lemma 15.5, $\lambda = \sum_{i \in S} \omega_i$, where $\{\omega_i\}$ are the fundamental weights.

Example 15.8 If $P = B$ is the standard Borel, then λ is the sum of all the fundamental weights. For example, for $\mathfrak{sl}(n+1)$, this gives an embedding of the flag variety into $\mathbb{C}^{n+1} \otimes \bigwedge^2 \mathbb{C}^{n+1} \otimes \cdots \otimes \bigwedge^n \mathbb{C}^{n+1}$. ◊

Everything today was when P was connected.

Lecture 16 Feb 26, 2010

We first finish the business of the parabolic subalgebra.

Theorem 16.1 *Suppose that G is connected and semisimple.*

1. *Conjugacy classes of parabolic subgroups are in bijection with $S \subseteq \Gamma$, via $S \mapsto P_S$.*
2. *If we pick $\lambda = \sum_{i \in S} m_i \omega_i$, with $m_i > 0$, then $P_S = \mathrm{Stab}_G(\ell_\lambda)$. Notice: it does not depend on the coefficients, just that they are non-zero.*
3. *If P is parabolic, then it is connected and $\mathcal{N}_G(P) = P$.*

Example 16.2 The biggest parabolic is the whole group, whence S is empty; the smallest is the Borel, whence S is all the simple roots. ◊

Proof: We proved last time that the conjugacy class of any parabolic contains $P = \mathrm{Stab}_G(\ell_\lambda)$ for some irreducible representation $L(\lambda)$.

A remark: If $\lambda = \mu + \nu$, then $\mathrm{Stab}_G(\ell_\lambda) = \mathrm{Stab}_G(\ell_\mu) \cap \mathrm{Stab}_G(\ell_\nu)$. This is because $L(\mu) \otimes L(\nu)$ contains a unique canonical component isomorphic to $L(\lambda)$, and $\ell_\lambda = \ell_\mu \otimes \ell_\nu$. So something stabilizes ℓ_λ iff it stabilizes each of ℓ_μ, ℓ_ν.

Therefore, if $\lambda = \sum_{i \in S} m_i \omega_i$, then $\mathrm{Stab}_G(\ell_\lambda) = \bigcap_{i \in S} \mathrm{Stab}_G(\ell_{\omega_i})$. So it depends only on the *support* of λ.

So basically, that implies 1. and 2.

So now think about $\mathrm{Lie}(P_S)$. It is generated by $h_1, \ldots, h_n, x_1, \ldots, x_n, y_j$ for $j \in S$. This implies that the P_S are distinct. So to prove 3., if P_0 is a connected component of the identity, then P_0 is also parabolic, so $P_0 = \mathrm{Stab}_G(\ell_\lambda) = P$, since $\mathrm{Lie}(P_0) = \mathrm{Lie}(P)$ depends only on S.

Then the fact that it is self-normalizer also follows from this. Suppose that $P = \mathrm{Stab}_G(\ell_\lambda)$, and take $g \in \mathcal{N}_G(P)$. Then $g(\ell_\lambda)$ is fixed by P, but P has only one fixed point, because P contains B, and B has only one fixed point, so $g(\ell_\lambda) = \ell_\lambda$, so $g \in P$. This finishes 3. □

16.1 Flag manifolds for classical groups

Before we proceed, let's discuss some classical flag manifolds. Sometimes any G/P is called a *flag manifold*, and sometimes only G/B is the flag manifold and G/P are "partial flag manifolds".

Let's begin with $G = \mathrm{SL}(n)$, i.e. of type A. Then pick $k_1 < \cdots < k_s$, and pick a flag of type $\mathrm{Fl}(k_1, \ldots, k_s, n)$. These are all of them. How do you read it from the diagram?

By the way, the easiest weight. The parabolic has the Levi decomposition: $P_S = G_S \rtimes \mathrm{Nil}(P_S)$, where G_S is the reductive part. The semisimple part $(G_S)'$ of G_S can be read from the diagram, simply by deleting the marked nodes from the diagram.

Example 16.3 We take $\mathrm{SL}(7) = A_6$ and mark the third and fifth nodes. Then $(G_S)' = \mathrm{SL}(3) \times \mathrm{SL}(2) \times \mathrm{SL}(2)$, and we have the flag variet $\mathrm{Fl}(3, 5, 7)$. ◊

Now let's move to the types B_n, C_n, which are $\mathrm{SO}(2n+1)$ and $\mathrm{SP}(2n)$. Let's work over $\mathbb{C}$. Then we have representations on $\mathbb{C}^{2n+1}$ with symmetric form $(,)$ or $\mathbb{C}^{2n}$ with antisymmetric form $\langle , \rangle$.

The possible flag manifolds are isotropic submanifolds, and so never get to dimension past half the total:

$$\mathrm{OFl}(m_1, \ldots, m_s) = \{V_1 \subsetneq \cdots \subsetneq V_s \text{ s.t. } (V_i, V_i) = 0\}$$
$$\mathrm{SpFl}(m_1, \ldots, m_s) = \{V_1 \subsetneq \cdots \subsetneq V_s \text{ s.t. } \langle V_i, V_i \rangle = 0\}$$

For example, take C_n with the last node marked. This is the Lagrangian grassmanian, i.e. the set of Lagrangian subspaces in $\mathbb{C}^{2n}$.

Finally, D_n. Then $G = \mathrm{SO}(2n)$ acting on $\mathbb{C}^{2n}$, $(,)$. Then you have the same as before, but $\mathrm{OGr}(n, 2n)$, the Grassmanian of n-dimensional isotropic subspaces in $\mathbb{C}^{2n}$ has two connected components. The two components correspond to the last vertices of the Dynkin diagram.

How to see that there are two components? In $n = 2$, it's clear: there are two isotropic lines $x = iy$ and $x = -iy$. For $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ isotropic, you see that there are two projections. Take $\pi : \mathbb{C}^{2n} \to \mathbb{C}^n$, and then $\mathrm{rank}(\pi(L)) = n$ or $n-1$. The point is that the matrix of the projector is skew symmetric, and a skew symmetric matrix can have only even rank. Sorry, two irreducible components, so two connected components in the projectivisation.

For example, for the full flag variety, you have $V_1 \subsetneq \cdots \subsetneq V_{n-1}$, and then you have two choices of how to extend to the last one, V_n and V_n'. By the way, what representations are these? The spinor representations.

16.2 Bruhat decomposition

Theorem 16.4 *Let G be connected and semisimple. Then $G = \bigsqcup_{w \in W} BwB = \bigsqcup_{w \in W} N^- wB$.*

The last equality is obvious, because we can write the torus in one of these places, because $B = T \rtimes N^+$.

Then look at $N^- wB = U_w \subseteq G/B$ — think of it as a set of left cosets of B. So then we see that $G/B = \bigsqcup_{w \in W} U_w$ is a disjoint union of $|W|$ many N^- orbits.

And each orbit is very simple as a topological space. We have $U_w \cong \mathbb{C}^{\ell(w_0)-\ell(w)}$, where w_0 is the longest element of W, and $\ell(w)$ is the length of $w \in W$. We could work over any field: $\mathbb{C} \rightsquigarrow \mathbb{K}$.

Example 16.5 In $G = \mathrm{SL}(2, \mathbb{C})$, then $G/B = \mathbb{P}^1$ is the Riemann sphere. So the cells we have is the north pole U_{w_0}, which is fixed by N^-, and what is left is U_e, the *big Bruhat cell.* $\Diamond$

So you see that actually over the complex numbers, we've constructed a cell complex. And all the (real) dimensions are even, so we know the homology: $\dim H_{2i}(G/B, \mathbb{Z}) = \{w \in W \text{ s.t. } \ell(w) = i\}$. So it's clearly a very useful result.

U_w is called a *Shubert cell.*

Proof: We will do it for $G = \mathrm{GL}(n)$. Then B is upper-triangular matrices, and N^- is lower-triangulars with 1s on the diagonal. So pick up $x \in G$. Then we want to find axb, where a is lower-triangular with 1s on the diagonal and b is upper-triangular. We want $axb = w \in W$, which is to say that it is a permutation matrix. And we want to know how many ways we can do this.

But when you think in terms of matrices, this is a very easy procedure. See, multiplying on the left by a lower-triangular is some operation on the rows of the matrix, where what you can do is pick any row, and subtract from is any row above. And when you multiply on the right, you can do exactly the same thing, but with columns.

So what we're going to do is the Gaussian elimination process. So how you get your permutation matrix is the very simple thing. You look at your matrix x, and look at the very first column, and find the first non-zero element. Then by multiplying on the left, we can make 0s all below it, and on the right we can make all zeros to the right.

Then you move to the next row, and do the same thing. Then in the end, you multiply on the right by a diagonal matrix, and then you get a permutation matrix. So this gives you one of your double cosets.

But, see, the minors being non-zero are preserved by the thing. Pick up the first non-zero minor from the first column. Then find the first non-zero minor in the first two columns that contains the one you picked already. Continue. So for x, you have a unique permutation matrix you can get. That's why the double cosets do not intersect.

Finally, we want to understand U_w. Then see what you can do. Write down the permutation, and see what multiplications by a lower-triangular that don't break it. This is $N^- / \operatorname{Stab}_{N^-} w$. And $\dim = \#\{i > i_1\} + \#\{i > i_2, i \neq i_1\} + \dots$, where w is the permutation $k \mapsto i_k$. But this dimension count is $\#\{(i < j) \text{ s.t. } w(i) < w(j)\}$. This is $\frac{n(n-1)}{2} - \ell(w)$, and the $\binom{n}{2}$ is $\ell(w_0)$. □

In fact, if you think of a general proof, it may be even easier to write down, but does not explain what's going on. We will start it today, but maybe not finish it.

Proof:

1. Proof for SL(2).
2. Use the indices and the special $\mathfrak{sl}(2)$ subalgebra $\mathfrak{g}_i = \langle x_i, h_i, y_i \rangle$. These lift to algerbaic subgroups $G_i \subseteq G$, which may be the adjoint forms or may be SL(2)s, but the point is that the G_is generate G.
3. Show that $N^- WB = G$. To do this, it suffices to show that $N^- WBG_i = N^- WB$. This is a little bit of work — it is some sort of calculation, and we don't have time today. It's nothing difficult, but involves some induction. □

Next time, we will finish this, and then do the Borel-Weil-Bott theorem.

Lecture 17 March 1, 2010

Today we finish the proof from last time.

Question from the audience: Can you clarify the Bruhat decomposition? Why N^- and not N^+?
Answer: Yes. We had $N^- wV$, but $w_0(N^-)w_0^{-1} = N^+$, so that's the same. We prefer to consider the orders of N^-, because it lets us talk about highest vectors instead of lowest vectors.

We now turn to a general theorem, including some properties of the Weyl group.

1. The first thing is to show that $N^- WB = G$.

   G is generated by the little $\mathfrak{sl}(2)$s G_i corresponding to simple roots. And, as we did GL(n), it's clear that for SL(2) is works. So we just need to show that the left-hand-side is closed under multiplication by G_i. Then the point is that we have $G_i = N_i^- B_i \sqcup N_i^- r_i B_i$, because there are only these two elements $\{e, r_i\}$ of this little Weyl group — r_i is the ith simple reflection ****which we may have called s_i earlier****. But set $P_i = G_i B = G_i \rtimes \mathrm{Nil}(P_i)$, where $\mathrm{Nil}(P_i) \subseteq B$. Then wee see that $P_i = N_i^- B \sqcup N_i^- r_i B$.

Ok, so we want to check that $N^- WBG_i = N^- WB$. So we're going to multiply on the right by $g \in G_i$. Then using the previous paragraph, we see that either:

(a) $g = \exp(ty_i) K_i \exp(sx_i)$, or

(b) $g = \exp(ty_i) r_i K_i \exp(sx_i)$

where x_i, y_i are the generators of the ith $\mathfrak{sl}(2)$, s, t are numbers, and $K_i \in T \cap G_i$ is an element of the torus.

We will work out case (a), and case (b) is similar and an exercise. So, $K_i \exp(sx_i) \in B$. So it's sufficient to consider $N^- wB \exp(ty_i)$. But then $B \exp(ty_i) \in P_i$. So in particular $NwB \exp(ty_i)$ is in either $N^- w \exp(t' y_i) B$ or $N^- w \exp(t' y_i) r_i B$. Suppose for now that $w(\alpha_i) \in \Delta^+$, whence $w \exp(t' y_i) w^{-1} = \exp(t' w(y_i)) \in N^-$, and $\exp(t' y_i) r_i = r_i \exp(t'' x_i)$. So in the first case, $N^- w \exp(t' y_i) = N^- \exp(t' w(y_i)) w$.

In the second case, $w(\alpha_i) \in \Delta^-$, and so $w = \sigma r_i \tau$ with $\tau(\alpha_i) = \alpha_i$) and $\sigma(\alpha_i) \in \Delta^+$. But then $w \exp(t' y_i) = \sigma r_i \exp(t' y_i) \tau$, which is either $\sigma \exp(t'' y_i) r_i \exp(s' x_i) \tau$ or $\sigma \exp(t'' y_i) \exp(s' x_i) \tau$.

Question from the audience: I've lost the train of thought. **Answer:** We considered a few cases. We want to move one term past the Weyl group. To do this, we do a few calculations, considering a few cases, using the fact from SL(2).

Then case (b) above is similar, and is left as an exercise.

2. We want to show that $N^- wB = N^0 \sigma B$ implies that $\sigma = w$, where $\sigma, w \in W$ are arbitrary. Thus the number of double cosets is the same as the number of terms in W.

   To do this, we study N^--orbits on G/B. To do this, we will embed $G/B \subseteq \mathbb{P}(L(\lambda))$, where λ is any regular dominant weight. So look at the W-orbit of λ — the regularity condition implie that this has $|W|$ many elements.

   So now look at $U_w = N^- \ell_{w(\lambda)}$. Here a little picture might help. We do it for $\mathfrak{sl}(3)$. Our λ generates a hexagon $\{\lambda, r_1\lambda, r_2\lambda, r_1 r_2 \lambda, r_2 r_1 \lambda, w_0 \lambda\}$, where $w_0 = r_1 r_2 r_1 = r_2 r_1 r_2$. So pick $v_{w(\lambda)} \in \ell_{w(\lambda)}$. But $N^- = \exp(\mathfrak{n}^-)$, so N^- can only move us down.

   So, see, then if $U_w = U_\sigma$, then we have $\sigma(\lambda) \le w(\lambda)$ and also $w(\lambda) \le \sigma(\lambda)$. But this is only possible if $\sigma(\lambda) = w(\lambda)$, which implies that $\sigma = w$ as λ was regular.

   (In fact, you never loose the top weight when you apply N^-.)

3. The last part is that $U_w = \mathbb{K}^{\ell(w_0) - \ell(w)}$.

   We start with some extremal weight $\mu = w(\lambda)$. Then pick up the line $\ell_\mu \ni v_\mu$, and consider the map $\mathfrak{n}^- \to G/B = \mathbb{P}(G \cdot \ell_\lambda)$. Let's call this map $\overline{\exp}$, i.e. $\overline{\exp}(x) = \exp(x) v_\mu$. But x is nilpotent, so this exponential map is algebraic.

Then $\overline{\exp}^{-1}(v_\mu) = \mathrm{Stab}_{\mathfrak{n}^-}(v_\mu)$. But this subalgebra has a root decomposition: $= \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, for some $\Phi \subseteq \Delta^-$.

Question from the audience: What is this map $\overline{\exp}$? Why is it as you say? **Answer:** We have $U_\mu \cong N^-/\operatorname{Stab}_{N^-} v_\mu$, but since $\mathfrak{n}^-$ is nilpotent, we can identify this with $\mathfrak{n}^-/\operatorname{Stab}_{\mathfrak{n}^-} v_\mu$.

So we want to describe this $\Phi = \{\alpha \in \Delta^- \text{ s.t. } \mathfrak{g}_{-\alpha} v_\mu = 0\}$. Let $\alpha \in \Delta$. Then $x_\alpha v_\mu = 0$. Now let $\mu = w(\lambda)$. Then $x_{w(\alpha)} v_\lambda = 0$ iff $w(\alpha) \in \Delta^+$. So $\Phi = \{\alpha \in \Delta^- \text{ s.t. } w(\alpha) \in \Delta^+\}$, and the cardinality of this set is $\ell(w)$. So then $\dim(\mathfrak{n}^-/\operatorname{Stab}_{\mathfrak{n}^-} w_\mu) = \ell(w_0) - \ell(w)$.

This completes the proof from last time.

17.1 Bruhat order

The orbits are ordered by closure: $U_w \subseteq \overline{U_\sigma}$. We give the opposite order: $w \leq \sigma$ iff $U_\sigma \subseteq \overline{U_w}$.

For example, for $\mathfrak{sl}(3)$, we have $e \geq r_1, r_2 \geq r_1 r_2, r_2 r_1 \geq w_0$.

Now suppose that $\sigma = r_{\beta_1} \dots r_{\beta_k} w$. Then $\ell(\sigma) = k + \ell(w)$. ****I must have missed an adjective.****

We won't explore this, but it has nice combinatorics. The closures $\overline{U_\sigma}$ are *Shubert varieties*, and are singular.

By the way, everything we've done so far works for arbitrary parabolics: the cell decomposition is known for G/P as well as for G/B.

17.2 Geometric induction

We are working in the algebraic category, but we could instead work in analytic category (e.g. complex holomorphic functions), and everything works.

But, anyway, let G be an algebraic group acting on X, and suppose that $\mathcal{L} \to X$ is a vector bundle. It is a *G-vector bundle* is there is a G-action on the bundle that extends the action on X. I.e. for each $g \in G$, there should be a map of bundles $\{L \to X\} \xrightarrow{g} \{L \to X\}$ that's linear on fibers and restricts to the map $X \xrightarrow{g} X$.

We will study the case $X = G/H$ where H is a closed subgroup. Then there is a standard procedure for how you can construct G-vector bundles. Suppose we have a representation $\pi : H \to \mathrm{GL}(V)$. Then we define — it is not a fiber product — $G \times_H V = G \times V/\sim$, where the equivalence relation is that $(gh, h^{-1}v) \sim (g, v)$ for each $h \in H$, $v \in V$, and $g \in G$. Then this gives a bundle $G \times_H V \to G/H$ by forgetting the second part, and the fiber is clearly identified with V.

Now, consider the bundle $G \to G/H$. It may not be affine, but you cover it by affine subsets: $G/H = \bigcup U_i$. Then we have gluing functions $\phi_{ij} : U_i \cap U_j \to H$, and we can get a new gluing function by composing with $\pi : H \to \mathrm{GL}(V)$. Then you can check that this indeed gives you a bundle.

On the other hand, basically what we have done is to construct a functor from representations of H to G-bundles on G/H. The inverse functor is easy to construct: you pick up a point on G/H, and then the stabilizer. I.e. you pick up $x = eH \in G/H$, and then you have a natural — if you have a G-bundle — of $H = \mathrm{Stab}_G\, x$ acting linearly on the fiber over $x = eH$. This is actually an equivalence of categories.

So, let's say we have $V \to \mathcal{L}(V) = G \times_H V$. Then the next step is to consider sections $\Gamma(G/H, \mathcal{L}(V))$. It is a representation of G — it might be infinite-dimensional. The way you describe it is pretty simple. We have $G \times_H V \to G/H$. We want to construct a section $\gamma : G/H \to G \times_H V$. Then for each $g \in G$, we try to define $\gamma(g) = (g, \phi(g)$. But then we have a condition on ϕ: $\phi(gh) = h^{-1}\phi(g)$.

So, you can actually realize the sections as functions. $\Gamma(G/H, \mathcal{L}(V)) = \{\phi : G \to V \text{ s.t. } \phi(gh) = h^{-1}\phi(g)\, \forall h \in H, g \in G\}$. The action of G is actually a left action: $g\phi(x) = \phi(g^{-1}x)$.

So, see what we have done. We took a representation of H, and get a representation of G, so it is a kind of induction. **Question from the audience:** What kinds of sections? **Answer:** We are working in the algebraic category, but you can of course work in the analytic category. If you want unitary representations, take L^2-sections.**Question from the audience:** Even the smooth category? **Answer:** You can define it, but what you get is very different.

Anyway, so we have a functor Ind : H-rep $\to$ G-rep. It is an embedding, so it is exact on the left.

Incidentally, you can sort of forget about bundles. You have H-representations, G-bundles, and this is an equivalence of categories, but you also have just the straightforward induction.

Lecture 18 March 3, 2010

We begin by fixing some notation. We have $H \subseteq G$. From an H-module V we constructed a vector bundle $G \times_H V = \mathscr{L}(V)$. Then we had an *induction functor* given by:

$$\Gamma_{G/H}(V) = \Gamma\big(G/H, \mathscr{L}(V)\big) = \big\{\phi : G \to V \text{ s.t. } \phi(gh) = h^{-1}\phi(g)\, \forall g \in G, h \in H\big\}$$

Question from the audience: When you say function? **Answer:** Algebraic. But there is also a holomorphic version of the story.

18.1 Frobenius reciprocity

We will study this induction functor. As opposed to the finite case, we do not have complete reducibility. For example, the Cartan is solvable. So it's important to have the correct statement.

Theorem 18.1 *Let M be a G-module, and V an H-module. Then:*

$$\mathrm{Hom}_G\big(M, \Gamma_{G/H}(V)\big) \cong \mathrm{Hom}_H(M, V)$$

And this isomorphism is canonical.

So in this sense the induction functor is right-adjoint to the restriction functor.

Proof: You write out the definitions.

$$\mathrm{Hom}_G\big(M, \Gamma_{G/H}(V)\big) = \big\{\phi : G \to \mathrm{Hom}_{\mathbb{C}}(M, V) \text{ s.t. } \phi(g^{-1}xh) = h^{-1}\,\phi(x)\,g\ \forall x, g \in G,\, h \in H$$

So we pick $\phi \in \mathrm{Hom}_G(M, \Gamma_{G/H}(V))$, and send it to $\phi(e) \in \mathrm{Hom}_H(M, V)$. We claim this is a canonical homomorphism, because we can go back: if we have $\alpha \in \mathrm{Hom}_H(M, V)$, we can move it to $\phi_\alpha : x \mapsto \alpha x^{-1}$. (We leave for you to check if this should be x or x^{-1}. The point is that the value at any point is determined by the value at e.) □

One of the properties that is sort of important to us, which we will use later on, is:

Corollary 18.2 *If V was an injective H-module, then $\Gamma_{G/H}(V)$ is an injective G-module.*

A remark: Let G be reductive (e.g. semisimple). Then $\mathbb{C}[G] = \bigoplus L(\lambda) \boxtimes L(\lambda)^*$. Then, recalling that we have actions both on the left and on the right, then it is immediate that:

$$\Gamma_{G/H}\big(\mathbb{C}[G]\big) = \bigoplus L(\lambda) \boxtimes \big(L(\lambda)^* \otimes V\big)^H$$

and so you get that the multiplicities are:

$$= \bigoplus L(\lambda) \boxtimes \mathrm{Hom}_H\big(L(\lambda), \mathbb{C}[G]\big)$$

Let's then describe induction for the universal enveloping algebra. Let $\mathfrak{g} \subseteq \mathfrak{h}$, and V an $\mathfrak{h}$-module. Then we define:

$$\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}} V = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V \tag{18.1}$$

Notice that this is of very big dimension. Recall that many $\mathfrak{g}$ modules do not integrate to groups, and indeed this one will not, because it is too large. But there is another different, in the ordering:

$$\mathrm{Hom}_{\mathfrak{g}}\big(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(V), M\big) = \mathrm{Hom}_{\mathfrak{h}}(V, M) \tag{18.2}$$

This is a standard result, that we let you do it. ****So the point is that for universal enveloping algebra, induction is left-adjoint.****

We will talk about this a bit more later: when you move away form finite groups, you have the group algebra, and the function algebra, but also the universal enveloping algebra.

Question from the audience: So this works always? **Answer:** Yes. In equations 18.1 and 18.2, you can put any algebras $A \subseteq B$, and no extra structure on the modules.

Anyway, we want to get the ordering in the same direction as in Theorem 18.1. We can define *coinduction*:

$$\operatorname{Coind}(V) \stackrel{\text{def}}{=} \operatorname{Hom}_{\mathcal{U}\mathfrak{h}}(\mathcal{U}\mathfrak{g}, V)$$

Then we do have:

$$\operatorname{Hom}_{\mathfrak{g}}(M, \operatorname{Coind} V) = \operatorname{Hom}_{\mathfrak{h}}(M, V)$$

But this is humongous. As soon as you take the dual space to a countable-dimensional space, you get something uncountable.

So we need to cut it down. Let's define $Z(M) = \{m \in M \text{ s.t. } \dim \mathcal{U}\mathfrak{g}\, m < \infty\}$. Here M is any $\mathfrak{g}$-module. This is closely related to the *superman functor* ****?****. And for reductive groups, using this gets close to the original.

Question from the audience: How can I describe the $\mathfrak{g}$-action on the G-induction? **Answer:** Take $x \in \mathfrak{g}$. Then look at $\mathbb{K}[G] \otimes V$. Then $x(f \otimes v) = L_x(f) \otimes v + f \otimes x(v)$. The Lie algebra is always the right-translation-invariant derivations.

One more property, which is very standard, so we skip the proof:

Lemma 18.3 *If we have a chain $G \supseteq K \supseteq H$, then the composition works:*

$$\Gamma_{G/K} \circ \Gamma_{K/H} = \Gamma_{G/H}$$

Just by definitions. ****A canonical (natural in G, K, H) isomorphism of functors.****

In any case, $\Gamma_{G/H}$ is exact on the left ****being an adjoint****. But it is not exact, so we will study the derived functor. But for this, we need enough injective modules.

Proposition 18.4 *Let H be an algebraic group. We claim that $\mathbb{C}[H]$ is injective.*

Proof: In fact, $\operatorname{Hom}_H(V, \mathbb{C}[H]) \cong V^*$. How do you see this? Think about it for a moment in a different way. You have the functions in V^*. And you think about the LHS as the algebraic maps $\phi : H \to V^*$ such that $\phi(h^{-1}x) = h\phi(x)$. But such functions are completely determined by their values at e. So this is very similar to what we did previously: if I know $\phi(e)$ I know it everywhere. So we constructed a map LHS→RHS.

And now I need the inverse map, which is also very clear. If I have $\xi \in V^*$, I construct $\phi(g) = g\xi$.

Question from the audience: This is Frobenius reciprocity induced from the trivial subgroup. **Answer:** Yes, exactly.

Now, $V \mapsto V^*$ is clearly an exact functor, so then $\mathbb{C}[H]$ is injective. □

Finally, we want to show that any V can be mapped to an injective representation. But a representation is actually a map $V \to \mathbb{C}[G] \otimes V$. This is the Hopf algebra definition of a representation. ****We are using the notation $\mathbb{C}[G]$ to mean the Hopf algebra of functions on G, not the**

$\mathbb{C}$-linear combinations of elements of G.** The point is that we have:

$$\begin{array}{ccc} V & \xrightarrow{\rho} & \mathbb{C}[G]\otimes V \\ {\scriptstyle \Delta\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\rho} \\ & \mathbb{C}[G]\otimes\mathbb{C}[G]\otimes V & \end{array}$$

And the two paths are the same.

Then there is a fact. The g action on V and on $\mathbb{C}[G]\otimes V$, where on this it is $g(\phi(x)\otimes v) = \phi(g^{-1}x)\otimes v$, is intertwined by ρ.

So, every H-representation V has an injective resolutions:

$$0 \to I_0 \to I_1 \to I_2 \to \dots$$

and $H^0(I_*) = V$.

Question from the audience: Can you clarify the definition of the action? Why isn't it V^*? **Answer:** When V is finite-dimensional, it is the same, by a diagram chase. When V is infinite-dimensional, $\mathrm{Aut}(V)$ is not an algebraic group, and so you'd better take what we said as a definition.

Oh, so we define: $\Gamma^i_{G/H}(V) \overset{\text{def}}{=} R^i\Gamma_{G/H}(V)$. Then there is a fact — if you don't know what it means, we won't use it — that $\Gamma^i_{G/H}(V) = H^i(G/H, \mathscr{L}(V))$.

Also, a remark: if H is reductive, then $\Gamma^i_{G/H}(V) = 0$ for $i > 0$, by complete reducibility.

Ok, so we now use a little bit of homological algebra. If you don't know it, look it up in any book. So what we do is we take:

$$0 \to \Gamma_{G/H}(I_0) \to \Gamma_{G/H}(I_1) \to \dots$$

And then we define $\Gamma^i_{G/H} \overset{\text{def}}{=} H^i(\Gamma_{G/H}(I_\bullet))$.

Ok, so we are interested in the case $H = B$, but actually there is a certain shift for this one, so we switch to $B^- = w_0(B)$. Then $G/B^- = G/B$. And as we discussed before G-line bundles on G/B^- are in bijection with one-dimensional representations of B^-. But $B^- = T \rtimes N^-$. But on any one-dimensional representation, the nilpotent part acts trivially. So in fact they are in bijection with one-dimensional representations, i.e. characters, of T. So actually it is a remarkable fact: not only all G-modules, but B-modules, are in bijection with characters of T.

And what are the characters of T? They are the weight lattice. Now we are in some trouble: we want to denote this by P, but we want to call this also a parabolic subgroup, so we use bold: $\mathbf{P}$ is the weight lattice.

Also, the category of line bundles is a group under tensor product, and this group is isomorphic to the weight lattice:

$$\mathbf{P} \cong \mathrm{Pic}_G(G/B^-) \underset{\text{without proof}}{\overset{\text{fact}}{=}} \mathrm{Pic}(G/B^-)$$

So, if $\lambda \in \mathbf{P}$, we denote by C_λ the one-dimensional representation of B^- with character λ. **Question from the audience:** It is sort of a different λ? **Answer:** We will prefer the additive notation, so that we identify them.

So, we set $\mathcal{O}(\lambda) \stackrel{\text{def}}{=} G \times_{B^-} C_\lambda$. This is standard notation.

The motivation is from $G = \mathrm{SL}(2)$. Then $G/B^- = \mathbb{P}^1$, and $\mathcal{O}(-1)$ is the *tautolical line bundle.* When you tensor it, or take its dual, you get the other line bundles.

We found a nice proof for the following, which we will give next time:

Theorem 18.5 *Let $\lambda \in \mathbf{P}$, and recall the weight ρ. If $\lambda+\rho$ is not regular, i.e. it lies on a wall, then $H^i(G/B^-, \mathcal{O}(\lambda)) = 0$ for all $i > 0$. On the other hand, if $\lambda + \rho$ is regular, then there is a unique Weyl group element which moves it to the interior of the chamber, i.e. $w(\lambda + \rho)$ is dominant. In this case:*

$$H^i(G/B, \mathcal{O}(\lambda)) = \begin{cases} 0 & \text{if } i \neq \ell(w) \\ L(w(\lambda+\rho)-\rho) & \text{if } i = \ell(w) \end{cases}$$

So you see in the geometry we have the same shifted action as in the Weyl character formula.

Lecture 19 March 5, 2010

Recall from last time, we have $\mathcal{O}(\lambda) = G \times_{B^-} C_\lambda$, and:

Theorem 19.1 *Assume G is simply connected (there is a version without too). Let $\mu \in \mathbf{P}$. If $\mu + \rho$ is not regular, then $H^i(G/B^-, \mathcal{O}(\mu)) = 0$ for all i. If $\mu + \rho$ is regular, then there is a unique $w \in W$ with $\mu + \rho = w(\lambda + \rho)$ for $\lambda \in \mathbf{P}^+$, and in this case:*

$$H^i(G/B^-, \mathcal{O}(\mu)) = \begin{cases} 0 & i \neq \ell(w) \\ L(\lambda) & i = \ell(w) \end{cases}$$

We will prove this today.

Proof:

1. If μ is not dominant, then $\Gamma(G/B^-, \mathcal{O}(\mu)) = 0$. If μ is dominant, then $\Gamma(G/B^-, \mathcal{O}(\mu)) = L(\mu)$. Then:

$$\mathrm{Hom}_G(L(\nu), \Gamma_{G/B^-}(C_\mu)) = \begin{cases} 0 & \nu \neq \mu \\ \mathbb{C} & \nu = \mu \end{cases}$$

****I missed part of this argument****

2. Let $G = \mathrm{SL}(2)$, and pick $n \in \mathbb{Z}$. Then $I(n) = \Gamma_{B^-/T}(C_n)$. Here T is the maximal torus. Last time we proved that if we start with something injective, then we get something injective, and over the torus all one-dimensional representations are injective, and C_n is the one-dimensional on which $S^1 = T$ acts n-fold.

Then, we have $\mathfrak{b}^- = \langle h, y \rangle$, and $I(n) = \langle t^{n+2k} \text{ s.t. } k \in \mathbb{Z}_{\geq 0} \rangle$. Then h acts by $2t\frac{\partial}{\partial t}$ and y by $\frac{\partial}{\partial t}$. In the picture, we have an infinite-dimensional module, starting with degree n at the bottom, and then $n+2$, $n+4$, etc., in a chain: y decreases us by one step each time.

So what is C_n? It is the **co?**homology of:

$$0 \to I(n) \to I(n+2) \to 0$$

So, then $\mathrm{Hom}_{B^-}(L(m), I(n))$ is easy to write down. It is zero unless $m \geq n$, and thence it is just the horizontal maps on weights. So, for positive n, we have:

$$\Gamma_{G/B^-}(I(n)) = \bigoplus_{k=0}^{\infty} L(n+2k)$$

and for negative n it is:

$$\Gamma_{G/B^-}(I(n)) = \bigoplus_{k=0}^{\infty} L(-n+2k)$$

And for $n \geq 0$, we have:

$$0 \to \bigoplus_{k=0}^{\infty} L(n+2k) \xrightarrow{\partial} \bigoplus_{k=0}^{\infty} L(n+2+2k) \to 0$$

and we can calculate the cohomology. $\ker \partial = L(n)$, and all the rest maps injectively and surjectively. So, if $n \geq 0$, then:

$$H^i(G/B^-, \mathcal{O}(n)) = \begin{cases} L(n), & i = 0 \\ 0, & i = 1 \end{cases}$$

If $n < 0$, then we do not have sections, and so the map ∂ must be injective:

$$0 \to \bigoplus_{k=0}^{\infty} L(-n+2k) \xrightarrow{\partial} \bigoplus_{k=0}^{\infty} L(-n+2k+2) \to 0$$

Then the result is that, for $n < -1$:

$$H^i(G/B^-, \mathcal{O}(n)) = \begin{cases} 0, & i = 0 \\ L(-n-2), & i = 1 \end{cases}$$

The picture is symmetric around -1.

And finally:

$$H^i(G/B^-, \mathcal{O}(-1)) = 0 \text{ for } i = 0, 1$$

This gives the proof in the case of SL(2).

3. For the general case, we need some general properties of $\Gamma^i_{G/H}$, which is the derived functor of sections.

   (a) There is a long exact sequence. Suppose you have an exact sequences $0 \to A \to B \to C \to 0$ of H-modules. Then there is a long exact sequence:

   $$0 \to \Gamma^0_{G/H}(A) \to \Gamma^0_{G/H}(B) \to \Gamma^0_{G/H}(C) \to \Gamma^1_{G/H}(A) \to \Gamma^1_{G/H}(B) \to \Gamma^1_{G/H}(C) \to \dots$$

   This is general: as soon as you have a right-derived functor, you have a long exact sequence. The only thing to check is a little homology.

   (b) We said last time that $\Gamma_{G/H}(\Gamma_{H/K}(V)) = \Gamma_{G/K}(V)$ if $G \supseteq H \supseteq K$. Thus, if $\Gamma^i_{H/K}(V) = 0$ for all i, then $\Gamma^i_{G/K}(V) = 0$ for all i. Why? Because we start with injective resolution, do the induction, and if already have an exact sequence of injectives and apply the functor, we get an exact sequence, because Γ is exact on injective modules. Actually, more generally we can compute one from the other via a spectral sequence, but we don't need it.

   (c) Let M be a finite-dimensional G-module and V some H-module. Then $\Gamma^i_{G/H}(V \otimes M) \cong \Gamma^i_{G/H}(V) \otimes M$. Why is this true? First of all, the functor $\otimes M$, if M is finite-dimensional representation, moves injective modules to injective modules. So if I is injective, then $I \otimes M$ is injective. This is because $\otimes M$ has an adjoint functor $\otimes M^*$.

   And because of this, it suffices to prove the statement for Γ^0, which is just Γ. But we construct:

   $$M \otimes \Gamma_{G/H}(V) \to \Gamma_{G/H}(M \otimes V)$$

   in the way that you can. We recall the definition of induction, and then: $(m \otimes \phi)(g) \stackrel{\text{def}}{=} g^{-1}m \otimes \phi(g)$. And to construct the inverse, we do the following trick:

   $$\Gamma_{G/H}(M \otimes V) \otimes M^* \to \Gamma_{G/H}(M \otimes M^* \otimes V) \xrightarrow{\text{tr}} \Gamma_{G/H}(V)$$

   Then pulling the M^* over to the right, we get the inverse map

4. We are now ready to finish the proof of the theorem. We will prove:

   Lemma 19.2 *Let $\mu \in \mathbf{P}$, and α_i a simple root, and assume that $\mu(h_i) \geq 0$. Then let $\nu = r_i(\mu + \rho) - \rho$. Then $H^i(G/B^-, \mathcal{O}(\mu)) = H^{i+1}(G/B^-, \mathcal{O}(\nu))$.*

   Before proving this, let's explain why lemma 19.2 implies the theorem. Indeed, choose $w_0 = r_{i_1} \circ \cdots \circ r_{i_\ell}$. Then in the middle we count to $|mu$: $r_{i_p} \circ \cdots \circ r_{i_\ell}(\lambda + \rho) = \mu + \rho$. Then $\mathcal{O}(w_0(\lambda + \rho))$ has cohomology only in the highest possible degree. And then we go back, using the facts above, tracking where the cohomology goes. Oh, and we also have to show that the highest cohomology cannot be bigger than the dimension of the flag, which is obvious form the geometric picture.

   Oh, by the way, lemma 19.2 ****or maybe the theorem**** is due to Bott, and the proof we give ****or maybe lemma 19.2**** is due to Demazure.

5. **Proof (of lemma 19.2):** We will pick $G \supseteq P_i \supseteq B^-$. Here $\mathfrak{p}_i = \mathfrak{b}^- + \mathfrak{g}_{\alpha_i}$. Geometrically, what does this mean? We have $G/B^- \to G/P_i$, and the fiber is P_i/B^-. But $P_i/\operatorname{Nil}(P_i) = G_i \cdot T$, where G_i is the SL(2). So $P_i/B^- = \mathbb{P}^1$.

   So, what are the irreducible representations of P_i? They are the same as of $G_i \cdot T$. (We write $\cdot$ because it is not a direct product: they intersect.) So they are $V(\eta)$ where $\eta \in \mathbf{P}$, with one more condition: $\eta(h_i) \geq 0$. ****picture with blocks****

   So, then, first remark: $\mathcal{O}(-\rho)$ does not have cohomology. Why? Because we know this for SL(2). So $H^j(P_i/B^-, \mathcal{O}(-\rho)) = 0$. So now the trick. We pick up $C_{-\rho}$ a B^- module, and tensor it with $V(\mu+\rho)$. Then ****missed**** $V = C_{-\rho} \otimes V(\mu+\rho)$ is acyclic everywhere.

$$H^j(P_i/B^-, C_{-\rho} \otimes V(\mu+\rho)) = 0$$

   Hence the same is true for G/B^-.

   Now we look in more detail what is this module. What is $V(\mu+\rho)$? It is basically the module over SL(2) plus additional weight. So we have a three-step filtration. We have C_μ on top, and C_ν on the bottom, and there is a big piece between. What is this big piece actually? Oh, this piece is a simple thing. It is $V' = C_{-\rho} \otimes V(\mu+\rho-\alpha_i)$, because it can be obtained in the same way.

   So this gives us two exact sequences. We have:

$$0 \to C_\nu \to V \to X \to 0$$

$$0 \to V' \to X \to C_\mu \to 0$$

   for some X. And now you know what to do. You do know that V is acyclic, and V' is also acyclic. So we write down long exact sequence, but we do not have terms at V' at all. So the cohomologies match up:

$$H^i(G/B^-, \mathscr{L}(X)) \cong H^i(G/B^-, \mathcal{O}(\mu))$$

   And for the first sequence, there is a jump:

$$H^i(G/B^-, \mathscr{L}(X)) \cong H^{i+1}(G/B^-, \mathcal{O}(\nu))$$ □

So that's what we need, and we are done. Our theorem is proven, and our time is up. □

Lecture 20 March 8, 2010

Is it OK, the Borel-Weil-Bott proof?

Question from the audience: What is a good reference for the material from the last few days?
Answer: The book [7] is good, as is [4] at the end. But we don't feel comfortable following any book for this course. Perhaps we will write something.

Today, let's think philosophically about what we did last time. We gave a certain geometric construction of finite-dimensional representations: the Borel-Weil-Bott theorem allows you to realize a finite-dimensional representation of G, a semisimple group, as the sections of some line bundle. We want to push this farther. We would like to do something with infinite-dimensional representations. Thus, we are led to the following question: is it possible to get some geometric realization of the representations of the Lie algebra $\mathfrak{g}$?

The point is: if you think about the group action, it clearly acts on sections of the line bundle. And we do have $\mathcal{O}(\lambda)$, and clearly $\Gamma(\mathcal{O}(\lambda))$ is a representation $\mathfrak{g}$, as it is a representation of the group. But in fact we can go further: if we take any open set, and the sections above that open set, then we do not have a representation of the group — it would move us out of the open set — but we do have a representation of the Lie algebra. Again: if $\mathcal{U} \subseteq G/B^-$ is any open subset, then $\Gamma(U, \mathcal{O}(\lambda))$ has the natural structure of a $\mathfrak{g}$-module.

Question from the audience: What do we mean by this in the algebraic category? **Answer:** This is absolutely algebraic. We have a homomorphism $\mathfrak{g} \to \mathrm{Vect}(G/B^-)$, to the algebraic vector fields. The point is, if we have our functions, and take an open set, then restricting to the open set the space of functions enlarges.

Example 20.1 Let $G = \mathrm{SL}(2)$, and then $G/B^- = \mathbb{P}^1$, and so we take any open subset of this projective line. For example, let U be the open Schubert cell, i.e. it is the sphere without the north pole, i.e. $\mathbb{C}$. Then if we look at $\Gamma(U, \mathcal{O}(n))$, then any line bundle over $U \cong \mathbb{C}$ is trivial, so $\Gamma(U, \mathcal{O}(n)) \cong \mathbb{C}[z]$. ****Any line bundle is trivializable, but doesn't it require choices to make this?**** Then we can figure out the formulas for the Lie algebra action. We have $\mathfrak{g} = \langle y, h, x \rangle$ acting on $\mathbb{C}[z]$ by:

$$y \mapsto \frac{\partial}{\partial z} \qquad h \mapsto 2z\frac{\partial}{\partial z} - n \qquad x \mapsto -z^2\frac{\partial}{\partial z} + nz$$

So you see, these are not really acting as vector fields, because of the $-n$ there. But the action is by first-order differential operators, and the commutators are correct. ****So is there a parameterization in which the action is by derivations?**** $\Diamond$

Example 20.2 (How to realize Verma module?) We now consider generalized functions. Indeed, let's consider δ_0 the delta function with support at 0. What does this mean in this setting? We don't have any topology, so we think of it formally: $z\delta_0 = 0$. So then we freely define $\delta_0', \delta_0'', \dots$ the derivatives of z.

Let's tell this in another way. Look at the formal Weyl algebra generated by $z, \frac{\partial}{\partial z}$. Then we will construct a certain representation of this algebra. **Question from the audience:** So δ_0 is: you take $\mathbb{C}[z]$ and freely adjoint δ_0 subject to $z\delta_0 = 0$? **Answer:** Yes. Then we try to extend the representation of the Weyl algebra.

Then an easy calculation shows that $x\delta_0 = 0$, and $h\delta_0 = (-n-2)\delta_0$. And y acts freely. So what we get is the Verma module $M(-n-2)$. $\Diamond$

Now, you can get plenty of other representations. For example, remember from one of the first homeworks we can construct the weight modules. We can also remove both the north and south poles form $\mathbb{P}^1$, and then get some local system with two points removed. So you see, we can get lots of representations.

Let's now proceed with some results. First, a definition. Let X be a non-singular algebraic variety, e.g. a variety over $\mathbb{C}$ given by algebraic equations. Let $\mathcal{L}$ be a line bundle over X, and $U \subseteq X$ and affine open set. We are going to construct a certain algebra $\mathscr{D}(U, \mathcal{L})$, which can be read as the differential operators on U with coefficients in $\mathcal{L}$. It is a filtered infinite-dimensional algebra, and will live in $\mathscr{D}(U, \mathcal{L}) \subseteq \text{End}(\Gamma(U, \mathcal{L}))$.

We start with $\mathscr{D}_0(U, \mathcal{L})$: by definition, it is the linear maps along the fiber, i.e. the functions. We use the notation $\mathscr{O}(U)$, the regular functions on U. Then we define the next step inductively:

$$\mathscr{D}_i(U, \mathcal{L}) = \left\{\delta \in \text{End}(\Gamma(U, \mathcal{L})) \text{ s.t. } [\delta, \phi] \in \mathscr{D}_{i-1}(U, \mathcal{L}) \ \forall \phi \in \mathscr{D}_0(U, \mathcal{L})\right\}$$

Then $\mathscr{D}(U, \mathcal{L}) = \bigcup \mathscr{D}_i$.

For example, look at $\mathscr{D}_1(U, \mathcal{L})$. Then what we're trying to say is that locally this is vector fields and functions. I.e. let us trivialize: if we identify $\Gamma(U, \mathcal{L}) \cong \mathscr{O}(U)$, then the condition gives $\delta(f \cdot g) = \delta(f) \cdot g + f \cdot \delta(g)$. So $[\delta, -]$ is a derivation for $\delta \in \mathscr{D}_1$. If you go to the next step, you get differential operators of second, third, or any order. This definition applies to any affine variety, or actually to any commutative rings. **Question from the audience:** So this gives some sheaf of algebras. Is this different from saying that you have an action of the ordinary ring of differentional operators on the line bundle $\mathcal{L}$? **Answer:** You see, locally, your sheaf is just the ordinary differential operators, but when you start gluing together, it will be different.

Example 20.3 If $\mathcal{U} = \mathbb{C}^n$ and $\mathcal{L}$ is trivial, then $\mathscr{D}(U)$ is nothing else but the Weyl algebra. If our coordinates on $\mathbb{C}^n$ are $x_1, \ldots, x_n$, then

$$\mathscr{D}(U) = \mathcal{T}(x_1, \ldots, x_n, \partial_1, \ldots \partial_n) / \left\langle [x_i, x_j] = 0, [\partial_i, \partial_j] = 0, [\partial_i, x_j] = \delta_{ij} \right\rangle$$

The filtration is given by $\deg \partial_i = 1$, $\deg x_i = 0$. $\Diamond$

We are not very familiar with differential operators on singular manifolds. So we will continue to suppose that our variety is non-singular.

More or less by definition, if you look at it, $\text{gr}\, \mathscr{D}(U, \mathcal{L})$ is commutative. So we are in the situation where we can make the Poisson bracket.

Oh, we should say: if U is affine, then everything is good, but if not, e.g. if U is projective space, then we have to glue, and we get the sheaf. So: if X is not affine — we try to avoid terminology from algebraic variety — we can cover X by open affine subsets $X = \bigcup U_i$, so we have $\coprod \mathscr{D}(U_i, \mathcal{L})$. Then we can glue them together, and $\mathscr{D}(X, \mathcal{L})$, the things defined everywhere, is a filtered associative algebra; it is the sections of the sheaf $\mathscr{D}(-, \mathcal{L})$.

Theorem 20.4 *If X is non-singular, then $\text{gr}\, \mathscr{D}(X, \mathcal{L}) = \Gamma(X, \mathcal{S}^\bullet(TX))$. Here TX is the tangent bundle, and $\mathcal{S}^\bullet$ is the sheaf of symmetric functions. There is another way to look at this: $\Gamma(X, \mathcal{S}^\bullet(TX)) = \mathscr{O}(T^*X)$. This is a Poisson manifold, and we are asserting that this is an isomorphism of Poisson algebras.*

Question from the audience: Even if $\mathcal{L}$ has no global sections, this algebra will have lots of sections. **Answer:** That's exactly right. This is why we have to glue, and work in the sheaf language.

We will not prove Theorem 20.4, but we give some sort of explanation. It suffices to prove it in the affine case, and then go to the cover.

If $R = \mathscr{O}(X)$, and X is affine, then remembera that $\mathscr{D}_1(X)$ consists of certain ****linear, not algebraic**** endomorphisms $R \to R$. Indeed, if $\delta \in \mathscr{D}_1$, then as we explained before, $[\delta, -] : R \to R$ is a derivation. Indeed, $\mathscr{D}_1(X)/R \cong \operatorname{Der} R = \Gamma(X, TX)$. So than you can prove that if you have a derivation $\delta : R \to R$, then it factors through the differentials $d : R \to \Gamma(T^*X)$. Then we just take the contraction of the one-form with a vector field.

So the point is that if we go farther, then it is going to work in the same way. In fact, if $\delta \in \mathscr{D}_n(X)$, then $\delta \in \mathscr{D}_n(X)/\mathscr{D}_{n-1}(X)$ acts as a derivation $R \to \Gamma(X, \mathcal{S}^{n-1}(TX))$. Then exactly in the same manner as before, it factors through the de Rham differential. So what we do is first apply the differential and then contract with $\gamma \in \Gamma(X, \mathcal{S}^n(TX))$. And what you have to show is that those are all.

So, the fact is: the graded algebra does not depend on the line bundle that you have, but the Poisson bracket already does.

So, we will apply this to our situtation. Let $X = G/B^-$ the flag manifold, and $\mathcal{L} = \mathcal{O}(\lambda)$. Then you can check that we have a homomorphism $\mathfrak{g} \to \mathcal{D}^1(X, \mathcal{O}(\lambda))$. And actually in the examples we gave the formulas for SL(2). So what we're saying is that you can extend this to $\mathcal{U}\mathfrak{g} \to \mathscr{D}(X, \mathcal{O}(\lambda))$. Let's call this map θ_λ. It is a homomorphism of associative algebras. **Question from the audience:** I'm still a bit confused. **Answer:** You also have the G action on sections. So the $\mathfrak{g}$ action is not a vector field, but just a differential operator of the first order on the space of sections. **Question from the audience:** The line bundle $\mathcal{O}(\lambda)$ comes with a canonical connections. Is this the lift via the connection of the vector fields on the base? **Answer:** Yes, you can describe it that way. **Question from the audience:** How does the group action give a connection? **Answer:** We will explain this next time.

The point is, that we have a homomorphism $\theta_\lambda : \mathcal{U}\mathfrak{g} \to \mathscr{D}(X, \mathcal{O}(\lambda))$ of filtered associative algebras. We will formulate the following theorem, and next time prove it. It requires recalling some things that we did some time ago.

Recall the *Harish-Chandra homomorphism.* If you remember, this sends $\mathcal{Z}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{h}) = \operatorname{Pol}(\mathfrak{h}^*)$. Then we consider the dual map $\mathfrak{h}^* \to \operatorname{Hom}(\mathcal{Z}(\mathfrak{g}), \mathbb{C})$. Then to each weight λ we consider the central character χ_λ. Then:

Theorem 20.5 $\theta_\lambda(\ker \chi_\lambda) = 0$. *Therefore,* θ_λ *pushes through* $\mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g}/(\mathcal{U}\mathfrak{g} \ker \chi_\lambda)$. *Then* $\theta_\lambda : \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g}/(\mathcal{U}\mathfrak{g} \ker \chi_\lambda) \to \mathscr{D}(X, \mathcal{O}(\lambda))$ *is an isomorphism.*

So what does this mean? If you pick up any representation so that the center acts by a certain scalar, then this is the same as a representation of the totally geometric algebra $\mathscr{D}(X, \mathcal{O}(\lambda))$.

Question from the audience: We are sneaking up on D-modules? **Answer:** Yes, of course. Those are my favorite topic.

Our plan is: we will talk a little about this, and then after the Spring break, we will move immediately to quantum groups. So we want to do everything else before then, so we don't know how far we can go in this direction. But, yes, $\mathscr{D}(X, \mathcal{O}(\lambda))$ makes sense even when λ is not integral, when we don't have a line bundle.

Lecture 21 March 10, 2010

Last time there were some questions. So we will return to the construction from last time, before presenting a theorem.

The point is: if we have a G-action on X, and we have a G-line bundle $\mathcal{L}$, then this is more or less equivalent to saying that you have a map $\mathfrak{g} \to \mathscr{D}_1(X, \mathcal{L})$, or at least in one direction this is absolutely true. The construction is as follows. Trivialize, and think about the sections $\Gamma(X, \mathcal{L})$ as functions with values in the fiber of $\pi : \mathcal{L} \to X$. So we've trivialized, and $\pi^{-1}(x) = L$, and locally $\gamma \in \Gamma(X, \mathcal{L})$ looks like $\gamma(x) \in L$. (Actually, everything we say works for general vector bundles, but then you need matrix-valued differential operators.) So, we have G-actions on $X, \mathcal{L}$. Then the compatibility is that:

$$g^* \gamma(x) = \phi(g, x)\, \gamma(g^{-1} x) \tag{21.1}$$

And ϕ is a one-cocycle, but the point is that we have this function, and $\phi(e, x) = 1$. So now we take a curve $g(t)$ in G. Then differentiating equation 21.1 gives:

$$\frac{d}{dt} g^*(t)\, \gamma(x) \bigg|_{t=0} = \frac{d\phi(g(t), x)}{dt} \bigg|_{t=0} \gamma(x) + \frac{d}{dt} \gamma\big(g^{-1}(t) x\big) \bigg|_{t=0} \tag{21.2}$$

This is just the Leibniz rule. Now, $g'(0) = \xi \in \mathfrak{g}$, and the action is $\xi \cdot \gamma = L_\xi(\gamma) + \varphi(x, \xi)\gamma$, where $\varphi \in \mathcal{O}(X)$, and γ is a function after the trivialization. So then you have to check that:

$$\xi(\phi \cdot \gamma) = L_\xi(\phi)\, \gamma + \phi(\xi \cdot \gamma)$$

and then ξ acts as a differential operator of the first order.

So this isn't quite a connection. We know how to differentiate γ in directions along the orbits of G, by the above, but to get a connection, the G-action on X should be free and transitive. **Question from the audience:** Why free? Why not just transitive? **Answer:** Suppose we have $X = G/H$ and $x \in X$. Then we have $\mathfrak{g} \to T_x X$. But we think this should be injective and surjective to be a connection. So if the action is free and transitive, then there is an isomorphism. If the action is not free, then you have $\mathfrak{g}/\mathfrak{h} \cong T_x X$, and so you don't know how to lift this. **Question from the audience:** If you have a metric? **Answer:** Maybe. **Question from the audience:** If H is discrete? **Answer:** Then everything is OK.

Ok, so we move now to $X = G/B^-$, and $\mathcal{L} = \mathcal{O}(\lambda)$. Then by the previous remarks, we have $\mathfrak{g} \to \mathscr{D}_1(X, \mathcal{L})$, and this is a homomoprhism of Lie algebras. In fact, we have an exact sequence of Lie algebras:

$$0 \to \mathcal{O}(\lambda) \to \mathscr{D}_1(X, \mathcal{L}) \to TX \to 0$$

$**TX$ is the sections — the vector fields.** **Question from the audience:** This is the exact sequence from the first level of associated graded? **Answer:** Yes, exactly.

Today we will study $\theta_\lambda : \mathcal{U}\mathfrak{g} \to \mathscr{D}(X, \mathcal{L})$. Some time ago, recall the *Harish-Chandra homomorphism.* Recall that $\mathcal{Z}(\mathfrak{g})$ is the *center* of $\mathcal{U}(\mathfrak{g})$, and the HC map is $\mathcal{Z}(\mathfrak{g}) \to \mathcal{S}\mathfrak{h}$. Define the map $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \overset{\text{HC}}{\to} \mathcal{S}\mathfrak{h} \overset{\lambda}{\to} \mathbb{C}$.

Lemma 21.1 *If $z \in \mathcal{Z}(\mathfrak{g})$, then $\chi_\lambda(z) = \theta_\lambda(z)$.*

Proof: Pick $x \in X$ such that $\text{Stab}_G(x) = B^-$, and let $U \ni x$ be an open set. Then $\mathcal{O}(U) \supseteq I_x$ the ideal of X, and $B^-(I_x) = I_x$. Then let $\xi \in \mathfrak{b}^-$ and $\varphi \in \mathcal{O}(U)$. Then:

$$(\xi \cdot \varphi)(x) = \lambda(\xi)\, \phi(x)$$

Oh, and $\lambda : \mathfrak{b}^- \to \mathfrak{b}^-/\mathfrak{n}^-$.

Now recall that if $x \in \mathcal{Z}(\mathfrak{g})$, then $z = p(h) \mod \mathfrak{n}^-\mathcal{U}(\mathfrak{g})$ where $p(h) \in \mathcal{S}\mathfrak{h}$. This is more or less the definition: $\mathcal{U}\mathfrak{g}^{\mathfrak{h}} = \mathcal{U}\mathfrak{h} \oplus \mathfrak{n}^- \oplus \mathcal{U}\mathfrak{g} \oplus \mathfrak{n}^+$.

Ok, so now we look at $(z \cdot \varphi)(x)$. But $\mathfrak{n}^-$ kills the whole function: $\mathfrak{n}^-\mathcal{O}(U) = I_x$. Therefore, only the $p(h)$ part survives. But then we know how the h part plays a rule: $(z \cdot \varphi)(x) = \chi_\lambda(z)\phi(x)$.

So we did it at one point, but now we can do it at any point, because z is in the center. Indeed, since $z \in \mathcal{Z}(\mathfrak{g})$, z commutes with the action of G, which is transitive, so we can start with x and move it to any other point: $z \cdot g^*(\phi) = g^*(z \cdot \phi)$, □

Corollary 21.2 *θ_λ factors through $\mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g}/\mathcal{U}\mathfrak{g} \ker \chi_\lambda$.*

Theorem 21.3 *$\theta_\lambda : \mathcal{U}\mathfrak{g}/(\mathcal{U}\mathfrak{g} \ker \chi_\lambda) \to \mathscr{D}(X, \mathcal{L})$ is an isomorphism.*

The trick is to look at the associated graded algebras on both sides. Each side is naturally filtered, and θ respects the filtrations, so we will then use the standard fact that we discussed before that lets us go back.

Let's introduce some notation: $\mathcal{U}^\lambda\mathfrak{g} \overset{\text{def}}{=} \mathcal{U}\mathfrak{g}/(\mathcal{U}\mathfrak{g} \ker \chi_\lambda)$.

Lemma 21.4 *Let N be the cone of nilpotent elements in $\mathfrak{g}$. Then $\text{gr}(\mathcal{U}^\lambda\mathfrak{g}) = \mathbb{C}[N]$, the ring of regular functions on N.*

Proof: For this we have to recall what is this ring of functions. We proved that it is a polynomial algebra, and a stronger theorem that the center acts freely. So, let $\mathfrak{g} \cong \mathfrak{g}^*$ via the Killing form. Then remember what we did before: we took $I(N)$ the ideal of the cone, and then $\mathcal{S}\mathfrak{g} = I(N) \oplus Y$, where Y was a homogeneous compliment. Then we proved that $\mathcal{S}\mathfrak{g}^G \otimes Y \to \mathcal{S}\mathfrak{g}$ is an isomorphism.

Moreover, we have the natural maps $\gamma : \mathcal{S}\mathfrak{g} \to \mathcal{T}\mathfrak{g} \to \mathcal{U}\mathfrak{g}$, and so we have:

$$\begin{array}{ccc} \mathcal{S}\mathfrak{g}^G \otimes Y & \longrightarrow & \mathcal{S}\mathfrak{g} \\ \downarrow \sim & & \downarrow \sim \\ \mathcal{Z}(\mathfrak{g}) \otimes \gamma(Y) & \longrightarrow & \mathcal{U}\mathfrak{g} \end{array}$$

□

Ok, so we know $\mathrm{gr}(\mathscr{D}(X, \mathcal{L})) = \mathcal{O}(T^*X)$.

Then we have the *Springer resolution*: $X \cong \{\mathfrak{b}_x$ among Borel subalgebras in $\mathfrak{g}\}$. I.e. rather than thinking about X, think about a collection of Borels.

Now, let us describe T^*X. What is it? It is $\{(x, \xi)$ s.t. $x \in X, \xi \in (\mathfrak{g}/\mathfrak{b}_x)^*\}$, because we identify $T_xX \cong \mathfrak{g}/\mathfrak{b}_x$. But by the Killing form, $(\mathfrak{g}/\mathfrak{b}_x)^* \cong \mathfrak{n}_x$, where $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$. Why? Because $(\mathfrak{b}_x)^\perp$ with respect to the Killing form is just $\mathfrak{n}_x$.

So now, we can think about the elements of T^*X as a pair $\mathfrak{b}_x$ a Borel subalgebra and $\xi \in \mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$. In particular, ξ is nilpotent. Then the map p from the cotangent bundle to the nilpotent cone $T^*X \to N$ is just forgetting $\mathfrak{b}$: $p : (\mathfrak{b}_x, \xi) \mapsto \xi \in N$. **Question from the audience:** Wait, how can we project onto the fiber? **Answer:** We have T^*X is a subbundle of $X \times \mathfrak{g}$.

Lemma 21.5 *1. p is surjective.*

2. If $\xi \in N$ is regular, then $p^{-1}(\xi)$ is a single point.

3. $p^{-1}(\xi)$ is a connected projective variety.

4. $p^* : \mathbb{C}(N) \to \mathcal{O}(T^*N)$ is an isomorphism.

So in algebrogeometric language, p is *proper*, and is an isomorphism on open parts.

Proof: 1. we proved before. 2. Fix $\mathfrak{b}$. Pick up a regular ξ. Then ξ can be embedded in a principle $\mathrm{SL}(2) = \{\eta, h, \xi\}$. Then $C_{\mathfrak{g}}(\xi) \subseteq \mathfrak{b}$, but it can live in only one $\mathfrak{b}$. Think about it: you pick up the regular piece. And you look at a centralizer. Then what we try to say is that the centralizer of the pair $(\mathfrak{b}, \xi)$ is the same as the centralizer of ξ, and therefore there is only one $\mathfrak{b}$. The best way to see this is to pick up one particular $\mathfrak{b}$, and then construct the centralizer and see that there are only positive weights. For 3. and 4., we explained already. □

Oh, by the way, we posted new homework. So far only two people have done homework, but we hope that the new one is easy.

Then we are saying that $\mathrm{gr}\, \theta_\lambda = p^*$. Why? We have $\mathbb{C}[N] \subseteq \mathcal{S}\mathfrak{g} \to \mathcal{S}(TX)$. What does this map do? At each point $x \in X$, we want to construct a vector field. Well, we pick $\xi \in \mathfrak{g}$, and then project to $\mathfrak{g}/\mathfrak{b}_x$, and this defines $\xi \to T_xX$, and thus a vector field. But $\mathcal{S}\mathfrak{g} \to \mathcal{S}(TX)$ is p^*, and $\theta_\lambda : \mathcal{U}\mathfrak{g} \to \mathscr{D}(X, \mathcal{L})$, if you go to the graded part it is what's going on.

This proves **??**.

Lecture 22 March 12, 2010

****I was out of town. These notes are copied from the hand-written notes by Matt Tucker-Simmons. Naturally, any errors are mine, rather than his.****

Last time: $\mathscr{D}(X,\mathcal{O}(\lambda)) \cong \mathcal{U}_\lambda\mathfrak{g} = \mathcal{U}\mathfrak{g}/\big(\ker(\chi_\lambda)\mathcal{U}\mathfrak{g}\big)$, where $\lambda \in \mathbf{P}$ is an integral weight.

But: the right-hand-side can be defined for λ not necessarily integral, and the left-hand-side can be defined also: *twisted differential operators* are the sheaf of algebras locally isomorphic to the sheaf of differential operators $\mathscr{D}_1(U) = \Gamma(U,TX) \oplus \mathcal{O}(U)$, where $X = \bigcup_i U_i$ and $U = U_i \cap U_j$. We need to understand the automorphisms of $\mathscr{D}_1(U)$ for the transition maps, and they should preserve the filtration. They must be: for $v \in \Gamma(U,TX)$,

$$v \mapsto v + \langle \alpha, v\rangle$$

for some $\alpha \in \Omega^1(U)$. And we must have $d\alpha = 0$ to get the commuting relations. So the twisted differential operators tdo correspond to $H^1(X, \Omega^1_d(X))$.

For example, let $X = G/B$, and consider $\mathcal{O}_X/\mathbb{C}$:

$$0 \to \mathbb{C} \to \mathcal{O}_X \to (\Omega^1_d)_X \to 0$$

Take the long exact sequence in cohomology; we know that $\mathcal{O}_x$ has only nonzero cohomology in dimension 1. Therefore:

$$H^1(X,\Omega^1_d)_X \cong H^2(X,\mathbb{C})$$

The RHS gives Schubert cells. And if an element of the LHS gives a sheaf of twisted differential operators on a line bundle, then the corresponding class on the RHS is the *Chern class*.

Now take $\mathfrak{g}$ to be complex semisimple. Then the sheaf of functions to $\mathfrak{g}$ is $\mathcal{O}_x \otimes \mathfrak{g} = \tilde{\mathfrak{g}}$. We have $\mathfrak{g} \to TX$, and so:

$$[f \otimes \xi, g \otimes \eta] = fg \otimes [\xi,\eta] + f\, L_\xi(g) \otimes \eta - g\, L_\eta(f) \otimes \xi$$

This is a *Lie algebroid*, i.e. a sheaf of Lie algebras and $\mathcal{O}_X$-modules.

In fact, we have a more general notion. Given a map of sheaves $\alpha : \tilde{\mathfrak{g}} \to TX$ such that $[\xi,\varphi\eta] = \varphi[\xi,\eta] + L_{\alpha(\xi)}(\varphi)\eta$, then we get $\mathscr{D}(X,\mathcal{O}(\lambda))$ by taking the enveloping algebroid of …something ….

$$\ker(\alpha) = \{\varphi : X \to \mathfrak{g} \text{ s.t. } \varphi(x) \in \mathfrak{b}_x\ \forall x \in X\}$$

$$[\ker(\alpha),\ker(\alpha)] = \{\varphi : X \to \mathfrak{g} \text{ s.t. } \varphi(x) \in \mathfrak{n}_x\}$$

And $\mathfrak{b}_x/\mathfrak{n}_x \cong \mathfrak{b}_y/\mathfrak{n}_x$ for all $x, y \in X$ canonically, since the choice is up to conjugation by B. But $\mathfrak{b}_x/\mathfrak{n}_x \cong \mathfrak{h}$.

Thus, we set $\tilde{\mathcal{U}} = \mathcal{O}_X \otimes \mathcal{U}\mathfrak{g}$, and $\lambda \in \mathfrak{h}^*$ gives $\lambda : \mathfrak{b}_X \to \mathbb{C}$ for any x, since $\mathfrak{b}_x/\mathfrak{n}_x = \mathfrak{h}$. Let's denote by J_λ the ideal in $\tilde{\mathcal{U}}$ generated by $\big(\varphi(x) - \lambda(\varphi(x))\big)$. Then we define $\mathscr{D}_X^\lambda = \tilde{\mathcal{U}}/J_\lambda$ for any $\lambda \in \mathfrak{h}^*$. For $U_j \subseteq X$, we have $\mathfrak{g} \mapsto \mathscr{D}_1(U_1)$ and $\mathscr{D}_1(U_2)$, and this defines the gluing maps for the sheaf of twisted differential operators.

Lemma 22.1 *1. $\mathscr{D}_X^\lambda$ is a tdo sheaf.*

2. $\Gamma(X, \mathscr{D}_X^\lambda) \stackrel{\text{def}}{=} \mathscr{D}^\lambda(X) = \mathcal{U}_\lambda\mathfrak{g}$.

Proof: For the second statement, go to associated graded, same as before. For the first statement, calculate: for $U \subseteq X$, $\mathscr{D}_0^\lambda(U) = \mathcal{O}(U)$, and:

$$\frac{\mathscr{D}_1^\lambda(U)}{\mathscr{D}_0^\lambda(U)} = \frac{\mathcal{O}(U)\otimes\mathfrak{g}}{\ker\alpha} = \Gamma(U, TX)$$

and so we are locally isomorphic to differential operators. □

Theorem 22.2 (Beilinson-Bernstein) *Assume that λ is dominant and $\lambda+\rho$ is regular, but not necessarily integral. Then:*

- $\mathscr{D}^\lambda(X) \cong \mathcal{U}_\lambda\mathfrak{g}$*-modules.*
- $\mathscr{D}_X^\lambda$*-modules (sheaves of modules, quasicoherent as $\mathcal{O}_x$-modules)*

For $\mathcal{F} \in \mathscr{D}_X^\lambda$-mod, $\Gamma(X, \mathcal{F}) \in \mathcal{U}_\lambda\mathfrak{g}$-mod. On the other hand, if $F \in \mathscr{D}^\lambda(X) \cong \mathcal{U}_\lambda(\mathfrak{g})$-mod, put $\mathcal{F}(U) = \mathscr{D}^\lambda(U) \otimes_{\mathscr{D}^\lambda(X)} F$ as the localization functor. Then we claim that Γ and localization give an equivalence of categories.

Aside: What is quasicoherence? Take an algebraic variety X, and $\mathcal{O}_X$ the sheaf of functions. Then $\mathcal{M}$ is quasicoherent if for a small enough cover, for $V \subseteq U$, we have $\mathcal{M}(V) = \mathcal{M}(U) \times_{\mathcal{O}(U)} \mathcal{O}(V)$. For X affine, quasicoherent sheaves are the same as $\mathbb{C}[X]$-modules, e.g. sections of vector bundles.

The point is that this is not true for projective things, because usually there are too few global sections.

When λ is integral, we can see that the dominance and that $\lambda+\rho$ is regular are necessary, by the BWB theorem.

Proof: It suffices to prove the following: If $\mathcal{F}$ is a $\mathscr{D}_X^\lambda$-module, then $H^i(X, \mathcal{F}) = 0$ for $i \neq 0$, and $H^0(X, \mathcal{F}) \neq 0$ for $\mathcal{F} \neq 0$.

Let V be a finite-dimensional irrep of B, and set $\mathcal{V} = G \times_{B^-} V = \mathcal{O}_X \otimes V$. Then V has a B^--invariant filtration with 1-dimensional quotients, and this gives a filtration of $\mathcal{V}$:

$$0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_k = V$$

Here $\mathcal{V}_1 \cong \mathcal{O}(\nu)$, where ν is the lowest weight of V, and $\mathcal{V}_k/\mathcal{V}_{k-1} \cong \mathcal{O}(\mu)$, where μ is the highest weight.

Then $\mathcal{F}(\gamma) = \mathcal{O}(\gamma)\otimes\mathcal{F}$, and so consider $\mathcal{F} \to \mathcal{F}(\nu)\otimes\mathcal{V}$ and $\mathcal{F}\otimes\mathcal{V} \to \mathcal{F}(\mu)$, induces respectively by the maps $\mathcal{V} \to \mathcal{O}(\mu)$ and $\mathcal{O}(\nu) \to V$. To complete the proof we need *Serre's Theorem*: if $X \hookrightarrow \mathbb{P}^n$ and $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{F}\otimes\mathcal{O}(m)$ does not have nonzero cohomology for large enough m. □

Lecture 23 March 15, 2010

We are in the middle of the program of localization theorem. Let's recall where we are:

We have $X = G/B$ and hence an action map $\alpha : \mathfrak{g} \to TX$. Then we take $\tilde{\mathcal{U}} = \mathcal{U}\mathfrak{g} \otimes \mathcal{O}_X$, which is a trivial sheaf, and we quotient by the ideal $\mathcal{I}_\lambda = \big(\{\xi - \lambda(\xi) \text{ s.t. } \xi \in \ker \alpha\}\big)$. This gives the *sheaf of twisted differential operators* $\mathcal{D}_X^\lambda = \tilde{\mathcal{U}}/\mathcal{I}_\lambda$. Locally, it is the sheaf of differential operators. Then we showed that $\Gamma(X, \mathcal{D}_X^\lambda) = \mathcal{U}_\lambda \mathfrak{g}$.

Then our goal is to prove that if $\lambda + \rho$ is dominant and regular, then:

1. $H^i(X, \mathcal{F}) = 0$ for all $i > 0$
2. $H^0(X, \mathcal{F}) \neq 0$

where $\mathcal{F}$ is a $\mathcal{D}_X^\lambda \supseteq \mathcal{O}_X$ ****a module over this?****, and $\mathcal{F}$ must be quasicoherent over $\mathcal{O}_X$. As we explained last time, this is a sort of analog of the Borel-Weil-Bott theorem, in a much more general setting.

So, first we explain the tensor product construction. Let $\mathcal{F}$ and $\mathcal{S}$ be $\tilde{\mathcal{U}}$-modules. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{S}$ is again a $\tilde{\mathcal{U}}$-module.

Then if $\mathcal{F}$ is a $\mathcal{D}_X^\lambda$-module, then for $\mu \in \mathbf{P}$, we set $\mathcal{F}(\mu) \stackrel{\text{def}}{=} \mathcal{O}(\mu) \otimes_{\mathcal{O}} \mathcal{F}$. This is going to add weights (because we have the infinitesimal action on the fiber), as is clear from the definition, so actually this is a $\mathcal{D}_X^{\lambda+\mu}$-module.

Now let μ be dominant integral, and consider $V = L(\mu)$. Then we have the induced bundle $\mathcal{V} = G \times_B V$. And this induced bundle is a $\tilde{\mathcal{U}}$ module. Now, V has a $\mathfrak{b}^-$-invariant filtration $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_s = V$, such that the quotients are all one-dimensional $\mathfrak{b}^-$-representations. In fact, $V_i/V_{i-1} = C_{\gamma_i}$, where γ_i is a weight of V. In all this theory, this is kind of an important trick, and we almost played it when proving the BWB theorem.

Now, when we do this, we notice that γ_1 is the lowest weight of V, by construction, and the last one is the highest weight, and these are fixed — you cannot vary them.

But then, if you do this sheafification, or whatever, you actually do have an absolutely similar filtration at the level of line bundles:

$$0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_s = \mathcal{V}$$

Then $\mathcal{V}_i/\mathcal{V}_{i-1} = \mathcal{O}(\gamma_i)$.

Now we do the following thing. We take $\mathcal{F}(-\nu) \otimes_{\mathcal{O}_X} \mathcal{V}$, and we naturally have a map to this from $\mathcal{F}(-\nu) \otimes_{\mathcal{O}_X} \mathcal{V}_1$, and also a map to this one form $\mathcal{F}$. So we naturally have a map $i : \mathcal{F} \to \mathcal{F}(-\nu) \otimes_{\mathcal{O}_X} \mathcal{V}_1 \to \mathcal{F}(-\nu) \otimes_{\mathcal{O}_X} \mathcal{V}$. Is instead we take the highest term, using the map $\mathcal{V} \to \mathcal{V}_s$, we get a map $j : \mathcal{F}(-\nu) \otimes_{\mathcal{O}_X} \mathcal{V} \to \mathcal{F}(\mu)$.

Lemma 23.1 *i has a right inverse and j has a left inverse.*

So basically what we are saying is that although we have a filtration, in fact $\mathcal{F}$ splits as a direct sum.

Proof: Here is the idea. In the quotient, we have the action of $\mathcal{D}_X^{\lambda+\gamma_i}$. So then we will prove:

1. If $\gamma_i \neq \nu$ the lowest weight, then $\mathcal{U}_{\lambda-\nu+\gamma_i}\mathfrak{g} \neq \mathcal{U}_\lambda\mathfrak{g}$. Reminder: what is $\mathcal{U}_\lambda\mathfrak{g}$? It is a quotient by a central character. So the point is that we may have other central characters, but none of them is like this.
2. If $\gamma_i \neq \mu$ then highest weight, then $\mathcal{U}_{\lambda+\gamma_i}\mathfrak{g} \neq \mathcal{U}_{\lambda+\nu}\mathfrak{g}$.

So 1. is required to prove that i exists, and 2. that j exists.

But how do we prove this? We have the Harish-Chandra theorem: if two weights define the same central character, then they are on the same orbit of the shifted Weyl group.

So, let's prove 1.: We must show that $\lambda + \gamma_i - \nu \neq w(\lambda + \rho) - \rho$ for any $w \in W$. Assume the opposite. Then we do have $w(\lambda+\rho) - (\lambda+\rho) = \gamma_i - \nu$. But now, ν is the lowest weight, so $\gamma_i - \nu > 0$. But $w(\lambda+\rho) - (\lambda+\rho) \leq 0$, as it is dominant. (We did not use that it is regular: that is for 2.)

Proof of 2.: We assume $w(\lambda+\gamma_i+\rho) = \lambda+\mu+\rho$, and so $\lambda+\rho-w(\lambda+\rho) = w(\gamma_i)-\mu$. But the RHS is ≤ 0, and the LHS is > 0 because $\lambda+\rho$ is regular. □

So, the trick is, we prove it under the assumption that $\mathcal{F}$ is coherent, i.e. finitely-generated, and then use the usual trick: quasicoherent is an inductive limit of coherent, and so the coherent case suffices. We will skip this limit: it is not difficult, but you have to check various maps.

Ok, so now use that $H^i(X, \mathcal{F}(-\nu)) = 0$ for $i > 0$ if $-\nu$ is sufficiently large. This is an algebraic geometry result, because this is the Serre theorem, which says that if you pick up ν, and then multiply it by and integer, and then by the Serre theorem you get vanishing cohomology.

Then $H^i(X, \mathcal{F}(-\nu) \otimes_{\mathcal{O}_X} \mathcal{V}) = H^i(X, \mathcal{F}(-\nu)) \otimes V = 0$. But we also know that $\mathcal{F}$ is a direct summand of $\mathcal{F}(-\nu) \otimes_{\mathcal{O}_X} \mathcal{V}$, and so $H^i(X, \mathcal{F}) = 0$.

Question from the audience: Wait: ν is just an integer? How does it relate to the twist in the sheaf? **Answer:** Oh, let's explain this. Remember that $\pi : X \subseteq \mathbb{P}(L(\lambda))$, where $\lambda = -\nu$ of μ. Then we thought we explained this once: $\mathcal{O}(\lambda) = \pi^*\mathcal{O}(1)$. So we multiply λ by some integer, and then it works. See, you have different embeddings, and different line bundles. . . .

Finally, for $i = 0$ part, we do the same thing. Again, for sufficiently large μ, $H^0(X, \mathcal{F}(\mu)) \neq 0$ by the Serre theorem. Assume that $H^0(X, \mathcal{F}) = 0$. Then $H^0(X, \mathcal{F} \otimes_{\mathcal{O}_x} \mathcal{V}) = 0$ by the argument as before. But then since we have the splitting, $H^0(X, \mathcal{F}(\mu)) = 0$, because it is a direct summand, but this is a contradiction.

And so the theorem from last time is proven.

So see the condition at the top of today shows you that taking global sections is an exact functor, and the second that ****?****. So we have an equivalence of categories between ****?****

There are many applications, usually via applying the theory of D-modules to questions in representation theory. For example, we can classify unitary representations. D-modules require a course in themselves, but we will give some simply examples.

Question from the audience: Can you explain the equivalence of categories more? **Answer:** The point is that you can localize back, and generally you get something bigger. We have $\Gamma : \mathcal{D}_X^\lambda\text{-mod} \to \mathcal{U}_\lambda\text{-mod}$. We also have $L : \mathcal{U}_\lambda\text{-mod} \to \mathcal{D}_X^\lambda\text{-mod}$. **Question from the audience:** A priori: I take global sections and localize, and that's the identity on $\mathcal{U}_\lambda$-mod. But the other way is by exactness? **Answer:** So we have $\Gamma \circ L = \text{id}$, because we don't get more sections, by faithfulness. There is this general theorem — one is adjoint of another — that says that if you have two functors between two categories, and one is adjoint of another, and if they are exact and faithful, then you have an equivalence of categories. It goes like this: it should be a bijection on irreducibles. If we remember correctly the statement is:

Proposition 23.2 *If you have two adjoint functors, exact, and gives bijection on irreducibles, then they give equivalence of categories.*

So the idea is that there is a map like $0 \to \mathcal{M} \to \Gamma \circ L(\mathcal{M}) \to \mathcal{T} \to 0$. Then we do localization of this, and if we apply ΓL again, then we get something non-trivial. We'll stop; it's based on the proposition, but we cannot figure it out immediately on the board.

Question from the audience: Ok, so we proved that Γ is exact. Did we already have the bijection on irreducibles? **Answer:** This should follow from faithfulness, i.e. the H^0 part. You take a module on a sheaf, and take global sections, then the module remains (ir)reducible. And it's clear that localization never gets 0: if you localize from something non-zero, you do not get zero, but the question is whether you localize from something that does not have global sections. We will try to do this next time. Of course, this is some standard statement that works for any sheaf.

Ah, yes. One thing we know that is kind of nice:

Corollary 23.3 *If we $\mu \in \mathbf{P}^+$, then we have a functor*

$$\mathcal{D}_X^\lambda\text{-}mod \overset{\otimes\mathcal{O}(\mu)}{\longrightarrow} \mathcal{D}_X^{\lambda+\mu}\text{-}mod$$

It is an equivalence of categories. Moreover, if $\lambda+\rho$ is regular dominant, then we get the translation principle*:*

$$\mathcal{U}_\lambda\mathfrak{g}\text{-}mod \to \mathcal{U}_{\lambda+\mu}\mathfrak{g}\text{-}mod$$

Example 23.4 If λ was itself integral dominant, then we have an equivalence $\Phi : \mathcal{U}_\lambda\mathfrak{g}\text{-mod} \to \mathcal{U}_0\mathfrak{g}\text{-mod}$.

In fact, we can construct this Φ, or rather its inverse. It is $\Phi(M) = \big(M \otimes L(\lambda)\big)/(\ker \chi_\lambda)$, and the adjoint is $F(M) = \big(M \otimes L(\lambda)^*\big)/(\ker \chi_0)$. For example, the trivial module goes to a finite-dimensional module. These are *translation functors.* $\Diamond$

One example is BGG-resolution. We can prove that there exists a resolution:

$$0 \to M(-2\rho) \to \cdots \to \bigoplus_{\ell(w)=k} M(w(\rho)-\rho) \to \cdots \to M(0) \to 0$$

Then $L(0)$ is just the cohomology in the last term. And if you think about this complex, it is pretty natural, and in fact it is dual with the De Rham complex.

Indeed, you take $G/B = \mathcal{U}_0 = N^- \cdot x$, and $\operatorname{Stab}_G x = B$, and $N^- \cong \mathfrak{n}^-$. Then the De Rham complex, you get

$$\Omega^k(\mathcal{U}_0) \cong \mathcal{S}^\bullet(\mathfrak{n}^-)^* \otimes \bigwedge^k (\mathfrak{n}^-)^*$$

This is an isomorphism of vector spaces, but in fact, the differential commutes with ****?****, and so we get the one above, except in the opposite direction:

$$0 \to \Omega^0(\mathcal{U}_0) \xrightarrow{d} \Omega^1(\mathcal{U}_0) \xrightarrow{d} \dots \xrightarrow{d} \Omega^\ell(U_0) \to 0$$

This is a complex of $\mathfrak{g}$-modules. We get the cohomology of this complex for free, because it is the De Rham complex. We take its restricted dual, and we get:

$$M_k \stackrel{\text{def}}{=} \Omega^k(\mathcal{U}_0)^* = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \bigwedge^k (\mathfrak{n}^-)$$

and $\mathfrak{n}^- = \mathfrak{g}/bb$, and it's easy to prove that we do get a $\mathfrak{b}$-filtration.

Then we have:

$$0 \to M_\ell \to \cdots \to M_2 \to M_1 \to M_0 \to 0$$

The only problem is that we have too many Verma modules, so we quotient by the kernel, and $M_k/(\ker \chi_0) = \bigoplus_{\ell(w)=l} M(w(\rho)-\rho)$. The details we will put in the next exercise set. The weights you get: they are $w(\rho)-\rho = \sum_{\alpha\in\Delta^-\cap w(\Delta^+} \alpha$.

Our plan for the next two sections: we will mostly change topics, and discuss the cohomology of Lie algebras.

Lecture 24 March 17, 2010

Warning: we may have to move rooms soon.

There was a question from last time. We have a functor Γ of global section, and L the localization, and they are adjoint:

$$\operatorname{Hom}_{\mathscr{D}_X}(\mathrm{L}(F), \mathcal{F}) = \operatorname{Hom}_{\mathscr{D}(X)}(F, \Gamma(\mathcal{F}))$$

where F is a module and $\mathcal{F}$ is a sheaf, and $\mathcal{D}_X$ is the sheaf of twisted differential operators ("tdo") and $\mathcal{D}(X)$ is the sections of $\mathcal{D}_X$. **Question from the audience:** Is this the correct direction?

Answer: We think so. Localization is like taking a tensor product. If we localize at one point, we have $\mathrm{Hom}_{\mathscr{D}(U)}(\mathscr{D}(\mathcal{U}) \otimes_{\mathscr{D}(X)} F, \mathcal{F}(U)) = \mathrm{Hom}_{\mathscr{D}(X)}(F, \mathcal{F}(U))$. So from this we get:

$$\mathrm{Hom}_{\mathscr{D}_X}(\mathrm{L}F, \mathrm{L}F) \cong \mathrm{Hom}_{\mathscr{D}(X)}(F, \Gamma \mathrm{L}F)$$

and so this induces a map $F \to \Gamma \mathrm{L}F$. And we also have:

$$\mathrm{Hom}_{\mathscr{D}(X)}(\Gamma\mathcal{F}, \Gamma\mathcal{F}) \cong \mathrm{Hom}_{\mathscr{D}_X}(\mathrm{L}\Gamma\mathcal{F}, \mathcal{F})$$

and that gives a map $\mathrm{L}\Gamma\mathcal{F} \to \mathcal{F}$. So we do have: $\mathrm{L}\Gamma\mathrm{L}F \cong F$, and $\Gamma\mathrm{L}\Gamma\mathcal{F} \cong \Gamma\mathcal{F}$. In fact, we have these isomorphisms whenever we have an adjoint pair of functors.

But now we use that Γ is exact and faithful. **Question from the audience:** We proved that it does not take non-zero objects to 0. Is this the same as faithful? **Answer:** Yes if it is exact, because you just look at the kernels.

So, we want to prove that $\mathrm{L}\Gamma\mathcal{F} \to \mathcal{F}$ is an isomorphism. We do this by looking at the exact sequence

$$0 \to \mathcal{X} \to \mathrm{L}\Gamma\mathcal{F} \to \mathcal{F} \to \mathcal{Y} \to 0$$

and apply Γ:

$$0 \to \Gamma\mathcal{X} \to \Gamma\mathrm{L}\Gamma\mathcal{F} \xrightarrow{\sim} \Gamma\mathcal{F} \to \Gamma\mathcal{Y} \to 0$$

but then $\Gamma\mathcal{X} = \Gamma\mathcal{Y} = 0$, and so $\mathcal{X} = \mathcal{Y} = 0$, since we proved that non-zero objects are not sent to zero.

On the other hand, let $F \in \mathscr{D}(X)$-mod. Then we take a free resolution

$$\cdots \to A_2 \to A_1 \to A_0 \to 0$$

and apply L:

$$\cdots \to \mathrm{L}A_1 \to \mathrm{L}A_1 \to \mathrm{L}A_0 \to 0$$

and apply Γ:

$$\cdots \to \Gamma\mathrm{L}A_1 \to \Gamma\mathrm{L}A_1 \to \Gamma\mathrm{L}A_0 \to 0$$

But on free modules it is clear that $\Gamma\mathrm{L}A_i \cong A_i$, so there is no cohomology except in the last spot of the third sequence, but then since L is exact and faithful, the sequence of $\mathrm{L}A_j$ is exact, and so $\Gamma\mathrm{L}F \cong F$. **Question from the audience:** Wait, why on free things? **Answer:** Because we calculated that $\Gamma\mathrm{L}\mathscr{D}(X) = \mathscr{D}(X)$. We did this by calculating $\Gamma(\mathscr{D}(X)) = \mathcal{U}\mathfrak{g}$/something, and we already did this. So we're not sure about the general statement from last time, but it's like when you do a localization on an affine variety to get a quasicoherent sheaf: you can go back.

24.1 Cohomology of Lie algebras

This is a fairly major change of pace, although next time we will connect to BWB theorem.

We start with a definition, which requires no assumptions on the field $\mathbb{K}$ ****VS uses $\mathbb{F}$****, the modules, the dimension, etc.

Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-module. We construct the *cochain complex* via:

$$C^k(\mathfrak{g}, M) \stackrel{\text{def}}{=} \text{Hom}_{\mathbb{K}}\left(\mathfrak{g}^{\wedge k}, M\right)$$

In other words, skew-symmetric k-multi-linear functions from $\mathfrak{g}$ to M. The differential $d : C^k(\mathfrak{g}, M) \to C^{k+1}(\mathfrak{g}, M)$ is:

$$dc(g_1, \dots, g_{k+1} = \sum_{i=1}^{k+1}(-1)^{i+1} g_i c(g_1, \dots, \hat{g}_i, \dots, g_{k+1}) + \sum_{i<j}(-1)^{i+j} c([g_i, g_j], g_1, \dots, \hat{g}_i, \dots, \hat{g}_j, \dots, g_{k+1})$$

The $\hat{g}$ is standard notation, meaning that we leave that term out. We will not check that $d^2 = 0$, but it is not hard: you can almost just do it by pointing. Most parts just cancel, and there is a triple term that cancels by the Jacobi identity, and there is a term that cancels by the module axioms. So indeed we get a complex.

So then the cohomology we define in this way is the usual:

$$H^k(\mathfrak{g}, M) = \frac{\ker d \cap C^k(\mathfrak{g}, M)}{\text{Im}\, d \cap C^k(\mathfrak{g}, M)}$$

How would you come up with this complex? We will explain the origin. It can be formulated as the following proposition:

Proposition 24.1 *This cohomology is nothing else but the* Ext *functor. I.e.* $\text{Ext}^i(\mathbb{K}, M) = H^i(\mathfrak{g}, M)$.

And it is not hard to define the Ext functor because we have free resolutions using universal enveloping algebra. Actually, it follows that $\text{Ext}^i(N, M) = H^i(\mathfrak{g}, \text{Hom}_{\mathbb{K}}(N, M))$. **Question from the audience:** Ext is taken in which category? **Answer:** The category of all $\mathfrak{g}$-modules. There is a version if you restrict to some other category. **Question from the audience:** And if we look in finite-dimensional modules we get 0? **Answer:** Not exactly. First of all, that's only in the representations of a semisimple module. And H^3 still caries data ****I missed the more precise statement of when it does so****.

We also have homology. In the finite-dimensional case it is dual to this one. We have chains:

$$C_k(\mathfrak{g}, M) = \mathfrak{g}^{\wedge k} \otimes M$$
$$\partial(g_1 \wedge \dots \wedge g_k \otimes m) = \sum (-1)^{i+1} g_1 \wedge \dots \wedge \hat{g}_i \wedge \dots \wedge g_k \otimes g_i +$$
$$+ \sum (-1)^{i+j} [g_i, g_j] \wedge g_1 \wedge \dots \hat{g}_i \dots \hat{g}_j \dots \wedge g_k \otimes m$$

So then you can analogously define $H_k(\mathfrak{g}, M)$ to be the homology of this complex.

Lemma 24.2 $H_i(\mathfrak{g}, \mathcal{U}\mathfrak{g}) = \begin{cases} \mathbb{K}, & i = 0 \\ 0, & i > 0 \end{cases}$

Proof: The complex $\dots \xrightarrow{\partial} \bigwedge^k \mathfrak{g} \otimes \mathcal{U}\mathfrak{g} \xrightarrow{\partial} \dots$ has a filtration induced by the filtration on $\mathcal{U}\mathfrak{g}$ — actually, we take the filtration that is the sum: degree in $\mathcal{U}\mathfrak{g}$ plus degree in $\mathfrak{g}^{\wedge k}$. We take the graded, and then we get the *Koszul complex*:

$$\dots \to \mathfrak{g}^{\wedge k} \otimes \mathcal{S}\mathfrak{g} \to \dots$$

And for V any vector space, we know that this complex has cohomology only in degree 0. Indeed, we have two maps:

$$V^{\wedge k} \otimes V^{\vee \ell} \underset{d}{\overset{\partial}{\rightleftarrows}} V^{\wedge(k-1)} \otimes V^{\vee(\ell+1)}$$

****VS uses $\bigwedge^k V \otimes \mathcal{S}^\ell V$, which we like too; but $\bigwedge$ behaves differently in displaystyle.**** These are:

$$\partial(v_1 \wedge \dots \wedge v_k \otimes w_1 \dots w_\ell) = \sum (-1)^i w_1 \wedge \dots \hat{v}_i \dots \wedge v_k \otimes v_i w_1 \dots w_\ell$$
$$d(v_1 \wedge \dots v_k \otimes w_1 \dots w_\ell) = \sum v_1 \wedge \dots \wedge v_k \wedge w_i \otimes w_1 \dots \hat{w}_i \dots w_\ell$$

Then the point is that $d\partial + \partial d = (k+\ell)\mathrm{id}$. And d is a chain homotopy. So the Koszul complex is exact except at $k + l = 0$.

And it is a general fact: if the graded has homology only in one place, then the original complex only has homology in that same place. So the only thing left to check is that $H_0 = \mathbb{K}$, and this is because of $1 \in \mathcal{U}\mathfrak{g}$. □

There is another thing to check: that the d, ∂ are all $\mathfrak{g}$ maps. Then we see that $C_k(\mathfrak{g}, \mathcal{U}\mathfrak{g}) = \mathfrak{g}^{\wedge k} \otimes \mathcal{U}\mathfrak{g}$ gives a free resolution of $\mathbb{K}$. So now we take the complex $\mathrm{Hom}_\mathfrak{g}(\mathcal{U}\mathfrak{g} \otimes \mathfrak{g}^{\wedge k}, M)$, and compute the homology. But by Frobenius reciprocity,

$$\mathrm{Hom}_\mathfrak{g}(\mathcal{U}\mathfrak{g} \otimes \mathfrak{g}^{\wedge k}, M) \cong \mathrm{Hom}_\mathbb{K}(\mathfrak{g}^{\wedge k}, M)$$

which is just the cochain complex $C^k(\mathfrak{g}, M)$. In the above, $\mathcal{U}\mathfrak{g} \otimes \mathfrak{g}^{\wedge k}$ is a $\mathfrak{g}$-module, where $\mathfrak{g}$ acts either by left multiplication or by the adjoint action. In general, the idea is to think of C_k as induced by $\mathcal{U}\mathfrak{g} \otimes \mathfrak{g}^{\wedge k}$: we are inducing from the zero subalgebra.

Ok, so more generally, we use the associativity of tensor products to tensor again with N:

$$\mathrm{Hom}_\mathfrak{g}(\mathcal{U}\mathfrak{g} \otimes \mathfrak{g}^{\wedge k} \otimes N, M) = \mathrm{Hom}_\mathbb{K}(\mathfrak{g}^{\wedge k}, \mathrm{Hom}_\mathbb{K}(N, M))$$

So this should explain why this is the right definition.

So now we will get an interpretation of the lower-dimensional cohomology. And next time we will calculate some of the cohomology for simple Lie algebras, and we will see that the many of the structure theorems of simple Lie algebras follow from them. By the way, injective modules in $\mathfrak{g}$-rep are very complicated, but projective modules, you just take a free resolution.

Question from the audience: This may by *Priddy's resolution*? **Answer:** We haven't seen that, but maybe.

Ok, so: $H^0(\mathfrak{g}, M) = M^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(\mathbb{K}, M)$ are just the invariant.

We already explained that $H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{K}}(M,N)) = \mathrm{Ext}^1_{\mathfrak{g}}(M,N)$. We will now explain this in a different way using short exact sequences. Because what are we studying here? We are actually studying extensions $0 \to N \to X \to M \to 0$. And we would like to established an equivalence between 1-cocycles and extensions. The idea is, as a vector space $X = M \oplus N$. Then we will define a new action by $g \cdot (m,n) = (gm, c(g)m + gn)$, where c is a one-cocycle. So this defines a new action. Here $c(g) \in \mathrm{Hom}_{\mathbb{K}}(M,N)$. But we have to check that this is indeed an action:

$$g_1 \cdot g_2 \cdot (m,n) - g_2 \cdot g_1 \cdot (m,n) = [g_1, g_2] \cdot (m,n)$$

And this condition gives a condition on c, that $c([g_1, g_2]) = [g_1, c(g_2)] - [g_2, c(g_1)]$. And this is just the condition that $dc = 0$. So actually, you can calculate this precisely. If the extension splits, what does this mean? It means that there is a map $M \to X$ that commutes, and so this map is $\phi(m) = (m, \varphi(m))$, where $\varphi \in \mathrm{Hom}_{\mathbb{K}}(M,N)$, and the condition that this is a submodule is that:

$$g \cdot (m, \varphi(m)) = (gm, \varphi(gm))$$

and you write this out, what do you get? $c(g) = [g, \phi]$. And that means that c is coboundary. So indeed we see that the abelian group of exact sequences quotient by the splitting extensions gives you the first cohomology group. **Question from the audience:** What is the group operation on extensions? **Answer:** You check that if two sequences are isomorphic, then the cocycles differ by a coboundary.

Lecture 25 Feb 5, 2010

Today we will talk first about some symmetries in the Lie algebra cochain complex, and then we will prove that the cohomology of semisimples with nontrivial coefficients are trivial.

So, we have the cochain complex $C^k(\mathfrak{g}, M) \xrightarrow{d} C^{k+1}(\mathfrak{g}, M) \to \dots$, and $\mathfrak{g}$ acts in several ways. For notation, we have $x \in \mathfrak{g}$, and recall that $C^k(\mathfrak{g}) = \bigwedge^k \mathfrak{g}^* \otimes M$. So we have some actions:

1. $i_x c(g_1, \dots, g_l) = c(x, g_1, \dots, g_k)$ gives a map $i_k : C^{k+1} \to C^k$.
2. $\alpha_x = \beta_x + \mathrm{ad}_x$, where $\beta_x c(x_1, \dots, x_k) = x \cdot c(x_1, \dots, x_k)$, and

$$\mathrm{ad}_x c(x_1, \dots, x_k) = \sum_{i=1}^{r} c(x_1, \dots, \mathrm{ad}_x(x_i), \dots, x_k)$$

3. It's helpful to adopt the formulation of superalgebras. We have $d \circ i_x + i_x \circ d = \alpha_x$. This shold be familiar from the de Rham complex, where the RHS would be the Lie derivative. Then it follows that $[\alpha_x, d] = 0$. And if $dc = 0$, then $\alpha_x c = d i_x c$, and so $\ker d / \mathrm{Im}\, d$ is a trivial representation of $\mathfrak{g}$.

4. $[d, \beta_x]c(x_1, \ldots, x_k) = \sum_{i=1}^{k}(-1)^{i+1}\beta_{[x_i,x]}c(x_1, \ldots, \hat{x}_i, \ldots, x_k)$

Theorem 25.1 *Let $\mathfrak{g}$ be semisimple over characteristic 0. Let M be a nontrivial finite-dimensional irreducible representation of $\mathfrak{g}$. Then $H^i(\mathfrak{g}, M) = 0$ for all i.*

Proof: It's clear that the cohomology doesn't change when you extend fields, so we will suppose that $\mathbb{K}$ is algebraically closed, so that we can work over an orthonormal basis. Then the *Casimir* is

$$\Omega = \sum_{j=1}^{\dim \mathfrak{g}} u_j u_j$$

where $\{u_j\}$ is an orthonormal basis in $\mathfrak{g}$ with respect to the Killing form. Then let $h = \sum_j \beta_{u_j} i_{u_j}$. Then:

$$d \circ h + h \circ d = \sum_j [d, \beta_{u_j}] i_{u_j} + \sum_j \beta_{u_j} \alpha_{u_j}$$
$$\sum_j [d, \beta_{u_j}] i_{u_j} c(g_1, \ldots, g_k) = \sum_j [d, \beta_{u_j}] c(u_j, g_1, \ldots, g_k) = \sum_{i,j} (-1)^i \beta_{[u_i,u_j]} c(u_j, \ldots, \hat{g}_i, \ldots, g_k)$$
$$\sum_j \beta_{u_j} \alpha_{u_j} c(g_1, \ldots, g_k) = \sum_j \beta_{u_j} \beta_{u_j} c(q_1, \ldots, g_k) + \sum_{i,j} \beta_{u_j} c(u_j, g_1, \ldots, [u_j, g_i], \ldots, g_k)$$

If there is a sign mistake, you can fix it: we work with super algebras, and we've learned that the signs always work out.

Then we claim that $d \circ h + h \circ d = \sum_j \beta_{u_j} \beta_{u_j}$. This follows if we can show that the two sums $\sum_{ij}$ cancel. We will show that they cancel for each i when $g_i = u_p$, and then use linearity. Define the tensor c via:

$$u_j, u_p] = \sum c^q_{j,p} u_q$$

Then $c^q_{jp} = -c^q_{pj}$, and then we have:

$$\sum c^q_{jp} \beta_{u_q} (-1)^i c(u_j, \ldots, \hat{g}_i, \ldots, g_k) + \sum c^q_{jp} \beta_{u_j} c(q_1, \ldots, u_q, \ldots, g_k)$$

and it works.

We did not find a coordinate-invariant proof. ****I bet the translating all this into the Penrose notation would achieve that.****

Now, if $M \neq 0$ is irreducible, then $\Omega|_M = C\mathrm{id}$ for $C \neq 0$ (by characteristic-zero), and so h is the desired homotopy. □

Remark: $H^i(\mathfrak{g}, \mathbb{K}) = 0$ for $i = 1, 2$, but not higher. For $i = 1$, this is easy to see: $H^i(\mathfrak{g}, \mathbb{K}) = [\mathfrak{g}, \mathfrak{g}]^\perp = 0$ when $\mathfrak{g}$ is semisimple. We will explain in a moment why H^2 as well.

And this, plus the theorem, implies the Weyl theorem: any finite-dimensional representation is semisimple, i.e. completely reducible. See, we proved that $H^1(\mathfrak{g}, M) = 0$ for $M \neq \mathbb{K}$ irreducible, but then for any $0 \to K \to M \to N \to 0$ we have the long exact sequence $\to H^1(\mathfrak{g}, K) \to H^1(\mathfrak{g}, M) \to H^1(\mathfrak{g}, N) \to$. Then we do induction on the rank of M, that $H^1(\mathfrak{g}, M) = 0$ for any M. But from last time, this tells us that there are no nontrivial extensions of finite-dimensional modules: $\mathrm{Ext}^1_{\mathfrak{g}}(M, N) = H^1(\mathfrak{g}, M^* \otimes N)$ when everything is finite-dimensional.

Then the point is: whenever I look at this complex, I can pick up only the invariants. Look at $C^k(\mathfrak{g}, \mathbb{K})$. Then pick up only the invariant part: this is a new complex:

$$\cdots \to C^k(\mathfrak{g}, \mathbb{K})^{\mathfrak{g}} \to C^{k+1}(\mathfrak{g}, \mathbb{K})^{\mathfrak{g}} \to \dots$$

and since $\ker d / \operatorname{Im} d$ is invariant, and since $\mathfrak{g}$ is semisimple, then we can split each C^k, and so the cohomology of $C^{\mathfrak{g}}$ is the same as for C.

Now, $\bigwedge^2(\mathfrak{g}^*)^{\mathfrak{g}} = 0$ if $\mathfrak{g}$ is semisimple, and also $H^k(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathbb{K}) = \bigoplus H^i(\mathfrak{g}_1, \mathbb{K}) \otimes H^{k-i}(\mathfrak{g}_2, \mathbb{K})$.

Then also we should point out that $H^\bullet(\mathfrak{g}, \mathbb{K})$ is a supercommutative ring, since we can take wedge products, and $H^\bullet(\mathfrak{g}, M)$ is a module over this.

So this gives a simple proof of ****?****, and a neat interpretation of $H^2(\mathfrak{g}, M)$.

This is very similar to what we did last time. You look at exact sequences $0 \to M \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$, where M is an abelian ideal in $\tilde{\mathfrak{g}}$, and the sequence splits is $\tilde{\mathfrak{g}} = \mathfrak{g} \ni\!\!\!\!\supset M$ is a semidirect sum. ****And since M is abelian, the $\tilde{\mathfrak{g}}$ action on M factors through $\mathfrak{g}$, so M is a $\mathfrak{g}$-module.****

So, let's say that $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus M$ as a vector space, and try to define the bracket:

$$\big[(g_1, m_1), (g_2, m_2)\big] = \big([g_1, g_2], c(g_1, g_2) + g_1 m_2 - g_2 m_1\big)$$

So Jacobi identity implies that — in fact, is equivalent to — $dc = 0$. So suppose that the sequence splits as $\mathfrak{g}$-modules: $\mathfrak{g} \to \tilde{\mathfrak{g}}$, then we'd have $[(g_1, \phi(g_1)), (g_2, \phi(g_2))] = ([g_1, g_2], \phi(g_1, g_2))$, but from another point of view it would be $([g_1, g_2], g_1\phi(g_2) - g_2\phi(g_1) + c(g_1, g_2))$ at least up to signs. And so $d\phi = c$.

So this means that if $H^2(\mathfrak{g}, M) = 0$, then $0 \to M \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ always splits. So that gives you a very simple proof of the *Levi decomposition theorem*:

Theorem 25.2 *If $\mathfrak{g}$ is any finite-dimensional Lie algebra and $\mathfrak{r} \subseteq \mathfrak{g}$ is its radical, and $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{g}/\mathfrak{r}$, then we have $\mathfrak{g}_{\mathrm{ss}} \hookrightarrow \mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{g}_{\mathrm{ss}} \ni\!\!\!\!\supset \mathfrak{r}$. All such decompositions are conjugate.*

Proof: Existence: $\mathfrak{r}$ is abelian is done; if $\mathfrak{r}' \subseteq \mathfrak{r}$ so that $\mathfrak{r}/\mathfrak{r}'$ is abelian, then $0 \to \mathfrak{r}/\mathfrak{r}' \to \mathfrak{g}/\mathfrak{r}' \to \mathfrak{g}_{\mathrm{ss}} \to 0$, and we know $\mathfrak{g} = \mathfrak{g}' + \mathfrak{r}$ and $\mathfrak{g}' \cap \mathfrak{r} = \mathfrak{r}'$, and so $\mathfrak{g}' = \mathfrak{g}_{\mathrm{ss}} \ni\!\!\!\!\supset \mathfrak{r}'$ by induction.

For the conjugacy, first do when $\mathfrak{r}$ is abelian. So suppose we have $\mathfrak{g} = \mathfrak{g}_{\mathrm{ss}} \oplus \mathfrak{r} = \mathfrak{g}'_{\mathrm{ss}} \oplus \mathfrak{r}$, we have $\psi : \mathfrak{g}_{\mathrm{ss}} \hookrightarrow \mathfrak{g}$, and $\psi(\mathfrak{g}_{\mathrm{ss}})\mathfrak{g}'_{\mathrm{ss}}$. Then $\mathfrak{g}'_{\mathrm{ss}} = \{(g, \psi(g)) \text{ s.t. } g \in \mathfrak{g}_{\mathrm{ss}} \text{ and } \phi(g) \in \mathfrak{r}\}$; then $d\phi = 0$ so $H^1(\mathfrak{g}_{\mathrm{ss}}, \mathfrak{r}) = 0$, and so $\phi(g) = gm$ and so $\exp(\mathrm{ad}_m)\mathfrak{g} = \mathfrak{g}'$.

And for the last part you do induction which we will skip. □

Ok, so how to calculate this cohomology. We will ssume that $\mathfrak{g}$ is the Lie algebra of a compact group — nothing depends on the field, so we might as well work over $\mathbb{R}$, and we know that each semisimple Lie algebra, you complexify and then take the compact form: $\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{su}(2)$.

Now look at $C(\mathfrak{g}, \mathbb{R})$. You can identify this with the complex of right-invariant differential forms on the group: $C(\mathfrak{g}, \mathbb{R}) \xrightarrow{\nu} \Omega_{\text{r-inv}}(G)$. **Question from the audience:** Why right versus left? **Answer:** It doesn't really matter, of course. The point is that if $g \in \mathfrak{g}$, then $L_g(\mu(c)) = \mu(\alpha_g(c))$. So in fact, in this case, when the group is compact, the Lie algebra cohomology is the same as the cohomology of G as a manifold, because it is the same as the right-invariant cohomology:

$$H^i(g, \mathbb{R}) = H^i_{\text{dR}}(G, \mathbb{R})$$

if G is compact.

And this can be calculated. We will say a few words about $\mathrm{U}(n)$ (or $\mathfrak{gl}(n, \mathbb{R})$):

Proposition 25.3 *If $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}), \mathfrak{gl}(n, \mathbb{C}), \mathfrak{u}(n)$, then $H^\bullet(\mathfrak{g}, \mathbb{K})$ is a free grassman algbera (super-commutative ring) with generators in degrees $1, 3, 5, \dots, 2n-1$.*

Why $2n-1$? If you look at $\mathrm{U}(1)$, this is just the circle. In $\mathrm{U}(2)$, $\mathrm{SU}(2)$ is the three-dimensional sphere, and we know its cohomology. And more generally, we have bundles: $\mathrm{U}(n) \xrightarrow{\mathrm{U}(n-1)} S^{2n-1}$. And so you can calculate cohomology inductively, using the long exact sequence.

$$H^\bullet_{\text{dR}}(\mathrm{U}(n-1)) \rightleftarrows H^\bullet_{\text{dR}}(\mathrm{U}(n)) \leftarrow H^\bullet_{\text{dR}}(S^{2n-1})$$

So then you can prove that $H^\bullet(\mathrm{U}(n)) = H^\bullet(\mathrm{U}(n-1)) + \omega H^\bullet(\mathrm{U}(n-1))$. Here ω is the unique up to proportionality invariant form on S^{2n-1}.

Next time we will formulate one more thing about BWB theorem, and then start quantum groups.

Lecture 26 March 29, 2010

26.1 Kostant theorem

We are in the following situation: suppose that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is semisimple over $\mathbb{C}$, and that M is a finite-dimensional $\mathfrak{g}$-module. We are interested in describing $H^i(\mathfrak{n}^+, M)$. Then you look at the chain complex $\bigwedge^i(\mathfrak{n}^+)^* \otimes M$, and there is an $\mathfrak{h}$ action (on both pieces, either the adjoint action of the $\mathfrak{h} \hookrightarrow \mathfrak{g}$-action) that commutes with the action of the differential. So H^i is an $\mathfrak{h}$-module.

If $\lambda \in \mathfrak{h}^*$, then by C_λ we denote the one-dimensional $\mathfrak{h}$-module with weight λ.

Theorem 26.1 $H^i(\mathfrak{n}^+, L(\lambda)) = \bigoplus_{\ell(w)=i} C_{w(\lambda+\rho)-\rho}$

We will give two proofs.

Proof (1): The first proof is based on BGG resolution, which is a pretty strong result in itself.

Recall that the Killing form identifies $(\mathfrak{n}^+)^* \cong \mathfrak{n}^-$. Then it's not too difficult to see that the *homology* $H_i(\mathfrak{n}^-, L(\lambda))$ is the same as the *cohomology* $H^i(\mathfrak{n}^+, L(\lambda))$. Then the BGG resolution tells us that we have:

$$0 \to M_\ell \to M_{\ell-1} \to \cdots \to M_0 \to 0$$

a resolution of $L(\lambda)$, where $M_i = \bigoplus_{\ell(w)=i} M(w(\lambda+\rho)-\rho)$.

And what is $M(\mu)$? As an $\mathfrak{n}^-$ module, it is $M(\mu) \cong \mathcal{U}\mathfrak{n}^- \otimes C_\mu$. Actually, this is an isomorphism of $\mathfrak{h} \oplus \mathfrak{n}^-$ modules. And the point is that it is free over $\mathfrak{n}^-$, and we proved that a free module has homology only in the first term. So:

$$H_i(\mathfrak{n}^-, M(\mu)) = \begin{cases} 0 & i > 0 \\ C_\mu & i = 0 \end{cases}$$

Moreover, we have $H_i(\mathfrak{n}^-, L(\lambda)) = H_0(\mathfrak{n}^-, M_i)$, and the result follows. □

Proof (2): The second proof uses the Borel-Weil-Bott theorem.

We look at a certain category, which we denote by the category of $(\mathfrak{b}, H)$-mod. What is this category? In general, these are *Harish-Chandra modules*. In our case, we will write is explicitly. The objects are $\mathfrak{b}$-modules that are locally nilpotent over $\mathfrak{n}^+$ and semisimple over $\mathfrak{h}$ with weights in $\mathbf{P}$ the weight lattice of $\mathfrak{g}$.

So if we would like to study the extensions, we have:

$$H^i(\mathfrak{n}^+, L(\lambda))_\mu = \mathrm{Ext}^i_{(\mathfrak{b},H)}(C_\mu, L(\lambda))$$

where the left-hand side is the weight-μ space, and the right-hand side is computed in this category $(\mathfrak{b}, H)$-mod. Indeed, you take a projective resolution, and take its semisimple part, and its still projective ****I think this is what VS said, but I might have misheard****.

On the other hand, we have an equivalence of categories B-mod $\xrightarrow{\sim} (\mathfrak{b}, H)$-mod. In one direction it's clear how to go — every B-module if $(\mathfrak{b}, H)$ — and in the other direction we have exponentiation. And so we have:

$$H^i(\mathfrak{n}^+, L(\lambda))_\mu = \mathrm{Ext}^i_B(C_\mu, L(\lambda)) = \mathrm{Ext}^i_B(L(\lambda)^*, C_{-\mu})$$

So we pick up an injective resolution of $C_{-\mu}$:

$$0 \to I_0 \to I_1 \to \cdots \to I_\ell \to 0$$

And so from our point of view, how do you compute? You take the Hom functor, and look at $\mathrm{Hom}_B(L(\lambda)^*, I_i)$. But on the other hand, by Frobenius reciprocity, this is the same as:

$$= \mathrm{Hom}_G(L(\lambda)^*, \mathrm{Ind}_B^G(M_i)) = \mathrm{Hom}_G\big(L(\lambda)^*, H^i(G/B, \mathcal{O}(-\mu))\big)$$

And on the other hand, $\mathrm{Hom}_B(L(\lambda)^*, I_i) = \mathrm{Ext}^i(\mathfrak{n}^+, L(\lambda))_\mu$. Incidentally, the above isn't quite in the form that we have used, and there is always the business of twists. But in any case, if you dualize:

$$\mathrm{Hom}_G\big(L(\lambda)^*, H^i(G/B, \mathcal{O}(-\mu))\big) = \mathrm{Hom}_G\big(L(\lambda), H^i(G/B^-, \mathcal{O}(\mu))\big)$$

And if you look at it, this gives the theorem. □

Or alternately, this gives the BWB theorem as a corollary of the Kostant theorem: originally Kostant proved his theorem using spectral sequences. On the other hand, the existence of a BGG resolution is stronger, because cohomology doesn't know everything.

Our final remark is that one can prove the Weyl character formula using the Kostant theorem. It is not the quickest way to prove it, but it is not very difficult.

26.2 Quantum Groups

We will spend the rest of today and the next lecture discussion Hopf algebras and duality between them, because quantum groups are nothing else but certain deformations of Hopf algebras.

The most remarkable thing is that most of these results can be done for quantum groups, if you do them appropriately.

We begin by fixing some notation. Fix a field $\mathbb{K}$, and later on we will have restrictions on characteristic, but not right now. A *Hopf algebra* over $\mathbb{K}$ has in fact two structures: it is an associative algebra and coalgebra, satisfying certain axioms that all can be drawn ****if you type fast enough****. We have

$$m : A \otimes A \to A \qquad \text{multiplication} \Delta : A \to A \otimes A \qquad \text{comultiplication}$$

Such that:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}} & A \otimes A \\ \downarrow{\scriptstyle \mathrm{id} \otimes m} & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \mathrm{id} \otimes \Delta} \\ A \otimes A & \xrightarrow{\Delta \otimes \mathrm{id}} & A \otimes A \otimes A \end{array}$$

The first diagram is the associativity you are used to. The second is its dual. We also demand that Δ be a homomorphism of algebras, and also that we have an antipode and unit and counit:

$$\begin{array}{cccc} & A & \xrightarrow{\Delta} & A \otimes A \\ \swarrow{\scriptstyle \epsilon} & & & \\ \mathbb{K} & & & {\scriptstyle \mathrm{id} \otimes S} \downarrow\downarrow {\scriptstyle S \otimes \mathrm{id}} \\ \searrow{\scriptstyle 1} & & & \\ & A & \xleftarrow{m} & A \otimes A \end{array}$$

Example 26.2 If G is a finite group, and A is its group algebra, then we set $\Delta g = g \otimes g$, $S(g) = g^{-1}$, and $\epsilon(g) = 1$. $\Diamond$

Example 26.3 If A is a finite-dimensional Hopf algebra, then so is A^* ****up to choosing some left-right conventions****. $\Diamond$

Example 26.4 The dual of the group algebra is $\mathbb{K}[G] = \{f : G \to \mathbb{K}\}$. Then $\Delta f(x, y) = f(xy)$, $S(f)(x) = f(x^{-1})$, and $\epsilon(f) = f(e)$. $\Diamond$

Example 26.5 When we have an algebraic group, we do not have the group algebra. But we do have its dual $\mathbb{K}[G]$. We must have the group algebraic, so that the comultiplication is well-defined: in the algebraic category, $\mathbb{K}[G \times G] = \mathbb{K}[G] \otimes \mathbb{K}[G]$ by definition. $\Diamond$

Example 26.6 When $G = \mathbb{K} = \mathbb{A}^1$, we have $\mathbb{K}[G] = \mathbb{K}[x]$, and $\Delta x = x \otimes 1 + 1 \otimes x$, $S(x) = -x$, and $\epsilon(x) = 0$. When $G = \mathbb{K}^*$, then $\mathbb{K}[G] = \mathbb{K}[t, t^{-1}]$, $\Delta t = t \otimes t$, $S(t) = t^{-1}$, and $\epsilon(t) = 1$. $\Diamond$

In fact, we have a theorem that every algebraic group is a Zariski-closed subgroup of $\mathrm{GL}(n)$. So our next example is a universal example:

Example 26.7 $G = \mathrm{GL}(n)$. $\mathbb{K}[G] = \mathbb{K}[x_{ij}]$, localized at $\det x$. Then $\Delta x_{ij} = \sum_k x_{ik} \otimes x_{kj}$, $S(x_{ij}) = (x^{-1})_{ij}$, and $\epsilon(x_{ij}) = \delta_{ij}$. $\Diamond$

Changing gears, our next example is the universal enveloping algebra:

Example 26.8 Let $\mathfrak{g}$ be a Lie algebra. Then $\mathcal{U}\mathfrak{g}$ is a Hopf algebra with $\Delta x = x \otimes 1 + 1 \otimes x$, $S(x) = -x$, and $\epsilon(x) = 0$ for $x \in \mathfrak{g}$. This requires a bit of checking to see that it is a Hopf algebra. What you have to check is that Δ extends to a homomorphism. The easiest way to do it is to look at the tensor algebra $\mathcal{T}\mathfrak{g}$, which is a free algebra, so there is nothing to check, and then quotient by J the ideal generate by elements of the form $x \otimes y - y \otimes x - [x, y]$. So all it amounts to is a certain calculation, that $\Delta([x, y]) = [x, y] \otimes 1 + 1 \otimes [x, y]$, which follows from an easy calculation. $\Diamond$

In general, $x \in A$ is *primitive* if $\Delta x = x \otimes 1 + 1 \otimes x$. The set of primitive elements in a Hopf algebra is a Lie algebra.

There is a general fact: $\mathbb{K}[G]$ and $\mathcal{U}\mathfrak{g}$ are "dual". To make this precise, we must introduce the Hopf algebra oflocal distributions supported at $e \in G$. We denote is $\mathscr{S}_G$, defined as follows. Take the ideal $I_e \subseteq \mathbb{K}[G]$, which is the maximal ideal of e. Then we say that:

$$\mathscr{S}_G = \left\{\alpha : \mathbb{K}[G] \to \mathbb{K} \text{ linear such that } \alpha(I_e^N) = 0 \text{ for sufficiently large } N\right\}$$

We set $\deg \alpha$ to be the minimal n such that $\alpha(I_e^{n+1}) = 0$. This gives a natural filtration on $\mathscr{S}_G$.

We now must define the multiplication and comultiplication on $\mathscr{S}_G$. To do this, we use the pairing $\langle , \rangle : \mathscr{S}_G \times \mathbb{K}[G] \to \mathbb{K}$. Then:

$$\langle \Delta_{\mathscr{S}}(\alpha), f \otimes g \rangle \stackrel{\text{def}}{=} \alpha(fg)$$
$$\langle m_{\mathscr{S}}(\alpha \otimes \beta), f \rangle \stackrel{\text{def}}{=} \langle \alpha \otimes \beta, \Delta f \rangle$$

Question from the audience: Is it obvious that $\Delta f \in \mathscr{S} \otimes \mathscr{S}$, and not in some completion? **Answer:** Not yet, but it will follow from the filtration.

We denote the multiplication by *convolution*: $m_{\mathscr{S}}(\alpha \otimes \beta) = \alpha * \beta$.

Proposition 26.9 *1. $\mathscr{S}(G)$ is a filtered associative algebra, i.e. $\mathscr{S}(G)_m * \mathscr{S}(G)_n \subseteq \mathscr{S}(G)_{m+n}$*

2. $\mathscr{S}(G)_0 \cong \mathbb{K} \cdot \mathrm{ev}_e$.

3. $\mathrm{T}_e(G) = \mathfrak{g} = \{\alpha \in \mathscr{S}(G)_1 \text{ s.t. } \alpha(1) = 0\}$. *These are precisely the maps* $\mathbb{K}[G] \to \mathbb{K}$ *such that* $\alpha(fg) = \alpha(f)\, g(e) + f(e)\, \alpha(g)$.

4. $\mathfrak{g}$ *is the Lie algebra* $[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha$.

We have to check certain things, but the point is that we have a natural homomorphism $\mathcal{U}\mathfrak{g} \to \mathscr{S}(G)$.

Theorem 26.10 *If* $\operatorname{char} \mathbb{K} = 0$*, then* $\mathcal{U}\mathfrak{g} \to \mathscr{S}(G)$ *is an isomorphism of Hopf algebras.*

In fact, this justifies the Hopf structure on $\mathcal{U}\mathfrak{g}$. Next time we will define quantum GL(2) as both a group algebra and a DJ algebra.

Lecture 27 March 31, 2010

We have posted a new homework set.

Today we will prove a statement about the duality between universal enveloping algebra and the coordinate function algebra.

Recall that we defined $\mathscr{S}(G)$ the algebra of distributions supported at e. We defined it by dualization, and gave it a filtration. We need to show that it is actually a filtered algebra.

Lemma 27.1

$\mathscr{S}(G)_0 = \mathbb{K}\delta_e$

$\mathscr{S}(G)_m * \mathscr{S}(G)_n \subseteq \mathscr{S}(G)_{m+n}$ *and* $\Delta(\mathscr{S}(G)_n) \subseteq \sum \mathscr{S}(G)_i \otimes \mathscr{S}(G)_{n-i}$

$\mathrm{T}_e G = \mathfrak{g} = \{\alpha \in \mathscr{S}(G)_1 \text{ s.t. } \alpha(1) = 0\}$ *and* $\Delta\alpha = \alpha \otimes 1 + 1 \otimes \alpha$.

Proof: 1. is obvious. For 2., we see had the ideal I_e, and it is actually a Hopf ideal — $\Delta I_e \subseteq I_e \otimes \mathbb{K}[G] + \mathbb{K}[G] \otimes I_e$, because $e \cdot e = e$. And so $\Delta I_e^n \subseteq \sum I_e^k \otimes I_e^{n-k}$, and using duality gives 2. For 3., we see that $\alpha\big((f - f(e))(g - g(e))\big) = 0$, and since α kills constants, using the Leibniz identity we actually have a derivation: $\alpha(fg) = \alpha(f)g(e) + f(e)\alpha(g)$, which is one of the definitions of the tangent space, and this also implies the final statement. $\square$

Now we will define the convolution with an arbitrary function. If you've worked with generalized functions, you know that in general you can convolve them with ordinary functions. We define: $\alpha * f \overset{\text{def}}{=} (\mathrm{id} \otimes \alpha)(\Delta f)$, for any $f \in \mathbb{K}[G]$ and $\alpha \in \mathscr{S}(G)$. And so if $\alpha \in \mathfrak{g}$, what we have defined is a left-invariant vector field $\alpha*$ on G. Now you can check the following property: $(\alpha*\beta)*f = \alpha*(\beta*f)$. So what we are actually saying is that $\mathscr{S}(G)$ act on G as left-invariant operators. **Question from the audience:** Remind me what is $\alpha * \beta$? **Answer:** It is $(\alpha * \beta)(f) = \langle \alpha \otimes \beta, \Delta f \rangle$. **Question from the audience:** And the reason that this is left-invariant is because we put id on the left? **Answer:** Yes, I think it is left-invariant. But this depends on what you call left and right.

To do these calculations, we introduce *Sweedler notation*, where you have $\Delta f = \sum_i f_i^1 \otimes f_i^2$, and in the Sweedler notation you simply ignore the summation over indices, writing this instead as $f^1 \otimes f^2$. ****I've seen this more commonly written as $\Delta f = \sum f_{(1)} \otimes f_{(2)}$.**** Then you see that:

$$\begin{aligned}
(\alpha * \beta) * f &= f^1 (\alpha * \beta)(f^2) \\
&= f^1 \langle \alpha \otimes \beta, \Delta f^2 \rangle \\
&= (\mathrm{id} \otimes \alpha \otimes \beta)(\mathrm{id} \otimes \Delta(\Delta f)) \\
\alpha * (\beta * f) &= \alpha * (f^1 \beta(f^2)) \\
&= \beta(f^2) \langle \mathrm{id} \otimes \alpha, \Delta f^1 \rangle \\
&= (\mathrm{id} \otimes \alpha \otimes \beta)(\Delta \otimes \mathrm{id}(\Delta f))
\end{aligned}$$

And the statement follows from the coassociativity of Δ.

So, we have $\mathfrak{g} \to \mathrm{Vect}^G G$ the left-invariant vector fields, and so $\mathcal{U}\mathfrak{g} \to \mathscr{S}(G)$. But if you take graded, you see that $\mathrm{gr}\,\mathcal{U}\mathfrak{g} \to \mathrm{gr}\,\mathscr{S}(G)$ is actually an isomorphism of graded algebras, because $\mathrm{gr}\,\mathscr{S}(G) = \mathcal{S}(\mathrm{T}G)^G$, and $\mathrm{T}G = \mathfrak{g} \times G$ is trivial. Here we use that $\mathscr{S}(G)$ is generated by $\mathrm{Vect}^G G$, and this uses characteristic-zero: in characteristic-p this is not true. So we see:

Theorem 27.2 $\mathcal{U}\mathfrak{g} \cong \mathscr{S}(G)$ *if* $\mathrm{char}\,\mathbb{K} = 0$.

On the other hand, in characteristic p, the point is that $(\partial/\partial x)^p$. So the point is to use divided powers.

Historically, the first thing that was defined was quantum SL(2) for the group, and then the deformation of universal enveloping algebra was constructed. VS was very young when she attended Gelfand seminar and Drinfeld gave a talk on quantum SL(2).

27.1 $\mathrm{SL}_q(2)$ and $\mathrm{GL}_q(2)$

The idea is to define these as algebras of linear transformations of quantum plane. So how to define quantum plane? We set $\mathbb{K}_q[x,y] = \mathbb{K}\langle x,y\rangle/(yx - qxy)$. Later on, we will assume that $q^2 \neq \pm 1$, where $q \in \mathbb{K}^\times$.

Now, it's not hard to see that $\{x^k y^l\}$ form a basis of $\mathbb{K}_q[x,y]$, but we do not have the same notion of homogeneity any more.

So before we say more, let's define this as a functor. Any algebraic variety is a functor from rings to sets. So what we will define is:

$$\mathbb{A}_q^2(R) = \{(x,y) \in R \times R \text{ s.t. } yx = qxy\}$$

Sometimes it's easier to work in this language: you check something by checking it on any ring R, and it is functorial.

The other thing we will now define are the *q-binomial coefficients*. First we introduce the *q-number*: $(n)_q = 1 + \cdots + q^{n-1} = \frac{q^n-1}{q-1}$. Then the *q-factorial*: $(n)_q! = (1)_q \cdots (n)_q$. Then finally: $\binom{n}{k}_q = \frac{(n)_q!}{(k)_q!\,(n-k)_q!}$ In particular, you can check by induction that:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}$$

You always have to think about xy versus yx, and q versus q^{-1}.

Ok, so now we will talk about matrices. The idea is that we do not deform the ****co?****multiplication — it is already noncommutative. But the idea is that we should declare that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x'' & y'' \end{pmatrix}$$

are automorphisms of $\mathbb{K}_q[x,y]$.

So this gives relations on a, b, c, d.

The point is that we want $\mathbb{K}_q[x,y]$ to be a comodule over the Hopf algebra $\mathrm{GL}_q(2)$. **Question from the audience:** So we are working out the endomorphisms of $\mathbb{A}_q^2$? **Answer:** Yes, exactly.

Ok, so we have:

$$(cx + dy(ax + by) = q(ax + by)(cx + dy)$$

and so $ca = qac$, $db = qbd$, $ba = qab$, and you also do the other one, so you get at the end of the day also $dc = qcd$, and a few more. All together:

$$ba = qab \quad ca = qac \quad db = qbd \quad dc = qcd \quad cb = bc \quad ad - da = (q^{-1} - q)bc \tag{27.1}$$

Then we define the bialgebra of *quantum* 2×2 *matrices* to be $M_q(2) = \mathbb{K}\langle a, b, c, d\rangle /$the relations in equation 27.1. The *comultiplication* is defined in the natural way via the multiplication of matrices:

$$\Delta(a) = a \otimes a + b \otimes c \quad \Delta(b) = a \otimes b + b \otimes d \quad \Delta(c) = c \otimes a + d \otimes c \quad \Delta(d) = c \otimes b + d \otimes d \tag{27.2}$$

and the counit $\epsilon(a) = \epsilon(d) = 1$ and $\epsilon(b) = \epsilon(c) = 0$, and you extend these to algebra homomorphisms. To check that these really work, it is easy to work with the functor of points, and then you just check with regular matrices. The point is that when you multiply matrices, you actually just multiply row-by-row.

Now, $M_q(2)$ is not a Hopf algebra, because there is no antipode: we do not have inverse matrices. So we need to see if there is such a thing as "quantum determinant".

Lemma 27.3 *Define* $\det_q = ad - q^{-1}bc = da - qbc \in M_q(2)$. *It is a multiplicative function from the functor-of-points perspective, and lies in the center of* $M_q(2)$.

We don't know any conceptual way to see that $\det_q$ is central, but you can do it simply by computing. But there is a conceptual way to see why it exists: you take the exterior algebra. We define:

$$\Lambda_q(\xi,\eta) = \mathbb{K}\langle \xi,\eta\rangle / \big(\eta\xi + q\xi\eta, \xi^2, \eta^2\big)$$

If you now work with the functor of points, you see that $m(\xi\eta) = \det_q m\, \xi\eta$. So $\Lambda_q(\xi,\eta)$ has a natural structure of a comodule over $M_q(2)$.

So in addition, $\det_q$ is *grouplike*: $\Delta(\det_q) = \det_q \otimes \det_q$. This follows from $\det_q m_1 \cdot \det_q m_2 = \det_q(m_1 m_2)$, where m_1, m_2 are R-points, i.e. $m_i \in M_q(2,R)$.

****There is an important question about representability and functors and modules, but I missed most of it, sorry.** Question from the audience:** We had a bialgebra of "quantum matrices". Were we guaranteed in advance that there is a maximal Hopf quotient of that? **Answer:** This is the question of representability? **Question from the audience:** Well, the group $\mathrm{GL}_q(2)$ should be the biggest Hopf algebra quotient of $M_q(2)$? **Answer:** Ah, yes. Incidentally, there is a general description of SO_q, SP_q, etc. We don't know of a description of the exceptional quantum groups, although there is are quantized enveloping algebras.

Now it is pretty clear how to define the groups:

$$\mathrm{SL}_q(2) \stackrel{\text{def}}{=} M_q(2)/(\det_q - 1)$$

The only thing that you have to check is that $(\det_q - 1)$ is a Hopf ideal, i.e. that $\Delta(\det_q - 1) = (\det_q - 1) \otimes \det_q + \det_q \otimes (\det_q - 1)$, which follows from the fact that $\det_q$ is grouplike. Then we can define the antipode, because we know the formula for the inverse matrix:

$$\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

Then we see a new feature. In the usual case, the antipode is an involution, but here it is not any more. In fact, S^2 is conjugation by $\left(\begin{smallmatrix} q & 0 \\ 0 & q^{-1} \end{smallmatrix}\right)$.

For GL:

$$\mathrm{GL}_q(2) = M_q(2) \otimes \mathbb{K}[t]/\big(t \det_q - 1\big) \quad \text{where } \Delta t = t \otimes t$$

Then the antipode is:

$$\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = \det_q^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

Then there is some checking, but in fact these are both Hopf algebras. Moreover, $\mathbb{K}_q[x,y]$ is a comodule over $M_q(2)$.

Next time, we will construct $\mathcal{U}_q\mathfrak{sl}(2)$, and see that it is dual to $\mathrm{SL}_q(2)$.

Friday we are still here. On Monday we move rooms.

Lecture 28 April 2, 2010

****I arrive late, having missed the generators-and-relations presentation of $\mathcal{U}_q\mathfrak{sl}(2)$.****

We define the *Verma module* of $\mathcal{U}_q\mathfrak{sl}(2)$ as follows. We declare that it contains v_0 a "highest weight vector", and pick λ; then we declare that $M(\lambda) = \text{span}\langle v_i \text{ s.t. } i \geq 0\rangle$, with action $Kv_0 = \lambda v_0$, $Ev_0 = 0$, and $Fv_n = v_{n+1}$. Then we claim that this implies that $Kv_n = KF^n v_0 = q^{-2n}\lambda v_0$ and

$$Ev_n = [n]\frac{q^{-(n-1)}\lambda - q^{n-1}\lambda^{-1}}{q - q^{-1}} v_{n-1} \tag{28.1}$$

We claim that:

Lemma 28.1 *1. $M(\lambda)$ is a $U_q\mathfrak{sl}(2)$ module.*

2. $\{F^p K^l E^q\}$ is a basis for $U_q\mathfrak{sl}(2)$. ****q is an unfortunate choice.****

Proof: 1. is a calculation, as is the check that $\{F^p K^l E^q\}$ span, by using the commutation relations. For linear independence, we suppose that $0 \sum F^p c_{p,q}(K) E^q = A$ for $c_{p,q} \in \mathbb{K}[K^{\pm 1}]$. Then $Av_0 = 0$ implies that $c_{p,0}(K) = 0$, and $Av_1 = 0$ implies that $c_{p,1}(K) = 0$, and so on. □

Lemma 28.2 *$U_q\mathfrak{sl}(2)$ is a Noetherian algebra with no zero divisors.*

Proof: The lack of zero divisors is by induction. You know that when you have a Noetherian ring, and add polynomials, you get something Noetherian again. And $\mathbb{K}[K^{\pm 1}]$ is clearly Noetherian. But in general there is a notion of *skew extension* or *Ore extensions*, which says that if A is both left- and right-Noetherian, then $A\langle t\rangle / \big(at = t\sigma(a) + \delta(a)\big)$ is too, where $\sigma : A \to A$ is an endomorphism ****and $\delta : A \to A$ is a derivation twisted by σ, I think****. □

Now, how to you see that $\mathcal{U}_q\mathfrak{sl}(2)$ is a deformation of the usual $\mathcal{U}\mathfrak{sl}(2)$? Informally, you set $q = e^t$ and $K = e^{tH/2}$, and take the limit as $t \to 0$. More formally:

28.1 Classical limit (when $q = 1$)

When you cannot specialize at a point, the trick is to add a new element so that you can. You add a new element $H = \frac{K - K^{-1}}{q - q^{-1}}$. Then consider the algebra generated by K, E, F, H satisfying the relations above and: $[H, E] = q(EK + K^{-1}E)$ and $[H, F] = -q^{-1}(FK + KF)$ ****dropped a $^{-1}$?****. Then at $q = 1$, you get clearly $\mathcal{U}\mathfrak{sl}(2) = \mathcal{U}_1\mathfrak{sl}(2)/(K - 1)$.

28.2 Finite-dimensional irreducible representations of $\mathcal{U}_q\mathfrak{sl}(2)$

It all goes pretty well and pretty similarly to the classical case until q is a root of unity.

Notice that the K action is diagonalizable, and that if $Kv = \lambda v$, then $KEv = \lambda q^2 Ev$ and $KFv = \lambda q^{-2} Fv$.

So if V is finite-dimensional, then it is semisimple over $\mathbb{K}[K^{\pm 1}]$.

And so if q is not a root of unity, then there exists a vector v such that $Ev = 0$. And so we can conclude that our module is a quotient of some $M(\lambda)$, if it is irreducible.

So how do we check what are the conditions on λ so that our module is finite-dimensional? It must be that some power of F kills the highest vector. Suppose that $F^{n+1}v_0 = 0$ but $F^n v_0 \neq 0$. Then recall equation 28.1: then $\lambda = \pm q^n$.

So all irreducible finite-dimensional representations are $V_{n,\epsilon}$, where $\epsilon = \pm 1$ and $n \in \mathbb{Z}_{\geq 0}$.

If you remember, at some point we constructed a principle $\mathfrak{sl}(2)$ subalgebra of $\mathfrak{sl}(n)$. And here the formulas are just like the usual actions if you use not regular numbers but quantum numbers. We rescale $w_n = v_n/[n]!$. Then you can write E, F, K as matrices:

$$E = \begin{pmatrix} 0 & [n] & & & \\ & 0 & [n-1] & & \\ & & \ddots & \ddots & \\ & & & 0 & [1] \\ & & & & 0 \end{pmatrix} \qquad F = \epsilon \begin{pmatrix} 0 & & & & \\ [1] & 0 & & & \\ & [2] & 0 & & \\ & & \ddots & \ddots & \\ & & & [n] & 0 \end{pmatrix} \qquad K = \epsilon \begin{pmatrix} q^n & & & & \\ & q^{n-2} & & & \\ & & q^{n-4} & & \\ & & & \ddots & \\ & & & & q^{-n} \end{pmatrix}$$

Ok, back to $\mathfrak{sl}(2)$. Recall that we have an action on $\mathbb{K}[x, y]$ by $E = x\frac{\partial}{\partial y}$ and $F = y\frac{\partial}{\partial x}$ and $H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

Similarly, $\mathcal{U}_q\mathfrak{sl}(2)$ actions on $\mathbb{K}_q[x, y]$ via:

$$\delta_x f(x, y) = \frac{f(qx, y) - f(q^{-1}x, y)}{qx - q^{-1}x} \quad F = \delta_x y \quad E = x\delta_y \quad K(x) = qx \quad K(y) = q^{-1}y$$

See, we do not want to quantize the Hopf structure on the torus. Classically, we have $\mathrm{SL}_2(\mathbb{K})$, and $\exp \mathfrak{h} = T$. The group algebra is $\mathbb{K}[T] = \mathbb{K}[K, K^{-1}] \subseteq \mathbb{K}[\mathrm{SL}_2(K)]$. And when we quantize, we do not deform this torus: it is rigid.

We would like now to describe the center. First of all, we construct the quadratic Casimir element. We set:

$$C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}$$

Then it is a little work so check that it is central. The hard part is E, F; it commutes with K almost automatically.

Proposition 28.3 $\mathcal{Z}(\mathcal{U}_q\mathfrak{sl}(2)) = \mathbb{K}[C_q]$.

Proof: Let V_0 be the centralize of K. For $\mathfrak{sl}(2)$, we use the fact that $V_0 = \mathbb{K}[K^{\pm 1}, C_q] = \mathbb{K}[K^{\pm 1}, EF]$. To continue further, we proceed exactly as for $\mathfrak{sl}(2)$, by constructing the Harish-Chandra isomorphism. For $u \in V_0$, it can be written in the form:

$$u = \sum_{i=0}^{N} F^i P_i(K) E^i$$

where P_i is a Laurent polynomial. Then by definition the *Harish-Chandra homomorphism* is $\theta(u) = P_0(K)$. Then:

1. If $u \in \mathcal{Z}$, then $u|_{M(\lambda)} = P_0(\lambda)$, because everything else kills the highest vector.

2. We leave the following statement as an exercise: $\theta : \mathcal{Z} \to \mathbb{K}[K^{\pm 1}]$ has trivial kernel.

   Now pick up all finite-dimensional representation and take the direct sum: then you get a faithful representation. So if something lies in the kernel, then it acts by zero on all finite-dimensional representations, and therefore it is zero. Or take all Verma modules instead of all finite-dimensional modules.

3. Now, if $\lambda = q^n \epsilon$, then we have an exact sequence

   $$0 \to M(\lambda q^{-2(n+1)}) \to M(\lambda) \to V_{n,\epsilon} \to 0$$

   So let $f \in \operatorname{Im} \theta$. Then $f(q^{-1}\lambda) = f(q^{-1}\lambda^{-1})$. Because this identity is true for any **?** and so is true for any λ, since q is not a root of unity.

   And so $f(q^{-1}K)$ is invariant under $K \leftrightarrow K^{-1}$, and so it is of the form $g(K + K^{-1})$.

   But this is a polynomial in the Casimir, since $\theta(\mathcal{Z}) = \mathbb{K}[qK + q^{-1}K]$. This is just the shift by ρ written multiplicatively. So $\mathcal{Z} = \mathbb{K}(C_q)$. □

We move rooms next time.

Lecture 29 April 5, 2010

We continue to study $\mathcal{U}_q\mathfrak{sl}(2)$. So far we have only discussed it as an associative algebra; today we describe its Hopf structure. Recall that we have generators K, E, F. We define the comultiplication by:

$$\Delta K = K \otimes K \tag{29.1}$$
$$\Delta E = 1 \otimes E + E \otimes K \tag{29.2}$$
$$\Delta F = K^{-1} \otimes F + F \otimes 1 \tag{29.3}$$

Then equation 29.1 tells you that K is a grouplike element.

Then we extend Δ to all of $\mathcal{U}_q\mathfrak{sl}(2)$ by demanding that it be a homomorphism $\mathcal{U}_q\mathfrak{sl}(2) \to \mathcal{U}_q\mathfrak{sl}(2)^{\otimes 2}$. It is rather easy to check that Δ does in fact extend to such a map, because we need only check:

$$KFK^{-1} = q^{-2}F \tag{29.4}$$
$$KEK^{-1} = q^2 \tag{29.5}$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}} \tag{29.6}$$

Then equations 29.4 and 29.5 are immediate, because we have a grading. For equation 29.6, we check that:

$$[\Delta E, \Delta F] = F \otimes E - F \otimes E + EK^{-1} \otimes KF - K^{-1}E \otimes FK + \\ + K^{-1} \otimes \frac{K - K^{-1}}{q - q^{-1}} + \frac{K - K^{-1}}{q - q^{-1}} \otimes K = \Delta \frac{K - K^{-1}}{q - q^{-1}} \quad (29.7)$$

Then as soon as we know that Δ is a homomorphism of algebras, to check coassociativity it is sufficient to check it only on generators. We leave this as an exercise. So $\mathcal{U}_q\mathfrak{sl}(2)$ is a bialgebra.

Now we check that we have a counit and an antipode. The counit is $\epsilon(K) = 1$, $\epsilon(E) = 0 = \epsilon(F)$. The antipode is:

$$S(E) = -EK^{-1} \qquad S(F) = -KF \qquad S(K) = K^{-1} \quad (29.8)$$

Then how do we extend it to all elements? It is fairly easy to see that S extends to an antiautomorphism for $\mathcal{U}_q\mathfrak{sl}(2)$ — an *antiautomorphism* should satisfy $S(u_1u_2) = S(u_2)\,S(u_1)$.

We remark that the antipode is not an involution. Rather, $S^2(E) = KEK^{-1}$, $S^2(F) = KFK^{-1}$, and $S^2(K) = K$, so in fact for all $u \in \mathcal{U}_q\mathfrak{sl}(2)$, we have $S^2(u) = KuK^{-1}$.

Question from the audience: Can you mention which book uses these conventions? **Answer:** [9], and [10]. I should say more. The choice of Δ is not unique. But it is unique up to a twist. See, for any $\phi \in \mathbb{K}[K^{\pm 1}]^{\otimes 2}$ invertible, then you can consider $\Delta' = \phi \circ \Delta$, where $\phi\circ$ is the adjoint (i.e. conjugation) action of ϕ on $\mathcal{U}_q\mathfrak{sl}(2)^{\otimes 2}$. These are *gauge transformations*, and up to a gauge transformation, Δ is determined.

So, recall that as soon as you have a Hopf algebra, its category of representations is automatically a tensor category. If V, W are representations, then the action on $V \otimes W$ is given by composing with Δ. Also, if V is a finite-dimensional representation of a $\mathcal{U}_q\mathfrak{sl}(2)$, and $\varphi \in V^*$, then we define $u\varphi$ for $u \in \mathcal{U}_q\mathfrak{sl}(2)$ by $\langle u\varphi, v\rangle = \langle \phi, S(u)v\rangle$. More generally, for homomorphisms, we have: if $A \in \mathrm{Hom}_{\mathbb{K}}(V, W)$, then

$$uA(v) = \sum u_1\big(A(S(u_2)v)\big)$$

in the Sweedler notation.

Theorem 29.1 *If q s not a root of unity, then the category of finite-dimensional $\mathcal{U}_q\mathfrak{sl}(2)$ modules is semisimple.*

All the irreducible modules are $V_{n,\epsilon}$ for $n \in \mathbb{Z}_{\geq 0}$ and $\epsilon = \pm 1$ from last time.

Proof: The idea is to use the Casimir. Let's recall how it goes for the semisimple Lie algebras. We consider sequences

$$0 \to V \to M \to W \to 0$$

1. Any such sequence splits if W is a trivial $\mathcal{U}_q\mathfrak{sl}(2)$ module, i.e. $\dim W = 1$ and $uw = \epsilon(u)w$. This corresponds to $\epsilon = 1$, $n = 0$ case. For this, it suffices to prove for an irreducible V, and then use the length.

Now, recall we have a Casimir C_q, with

$$C_q|_{V_{n,\epsilon}} = \epsilon \frac{q^{n+1} + q^{-n-1}}{(q-q^{-1})^2} \text{id}$$

****We should distinguish ϵ, ε for ± 1 from the counit. But oh, well.****

If q is not a root of unity, then $\epsilon q^{n+1} + \epsilon q^{-n-1} = q + q^{-1}$ if and only if $\epsilon = 1$ and $n = 0$.

So now consider $0 \to W \to M \to W \to 0$. Then $\dim M = 2$, and $K = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ if the sequence does not split, and so by the relations $E|_M = F|_M = 0$, by the K-grading, and so $0 = [E, F] \neq \frac{K-K^{-1}}{q-q^{-1}}$.

2. For the next step, we do it for arbitrary irreducible V, W. Then the general argument gives the general case. We consider the sequence of homs:

$$0 \to \text{Hom}_{\mathbb{K}}(V, W) \to \text{Hom}_{\mathbb{K}}(M, W) \to \text{Hom}_{\mathbb{K}}(W, W) \to 0$$

This is still a sequence of finite-dimensional $\mathcal{U}_q\mathfrak{sl}(2)$ modules, since we have the Hopf structure. Then pick up $\mathbb{K}\text{id} \subseteq \text{Hom}(W, W)$. Then we have

$$u\text{id}(w) = \sum u_1 S(u_2) w = \epsilon(u) w$$

So $\mathbb{K}\text{id}$ is the trivial representation

Now, since we have exact sequences, we can take N to be the preimage of $\mathbb{K}\text{id}$:

$$0 \to \text{Hom}_{\mathbb{K}}(V, W) \to N \to \mathbb{K}\text{id} \to 0$$

And so by step 1, this sequence splits, and so we can find $\gamma : \mathbb{K}\text{id} \to N$ a $\mathcal{U}_q\mathfrak{sl}(2)$ module. Let $\pi = \gamma(\text{id}) \in N$. Then $\mathbb{K}\pi$ is a trivial submodule of N, and so $K \cdot \pi = K\pi K^{-1} = \pi$. Then since $\Delta E = 1 \otimes E + E \otimes K$ and $S(E) = -EK^{-1}$, we have $E \cdot \pi = -\pi E K^{-1} E \pi K^{-1}$, but this must be 0. We can do something similar with F. So π commutes with all elements of $\mathcal{U}_q\mathfrak{sl}(2)$. So $\pi : M \to W$ is a $\mathcal{U}_q\mathfrak{sl}(2)$ homomorphism, and $\ker \pi$ is an invariant subspace complementary to $V \subseteq M$. ****VS wrote $W \subseteq M$ here, but that doesn't make any sense.**** □

There are many reasons that this argument fails when q is a root of unity.

We will conclude today by discussing the duality between $\mathcal{U}_q\mathfrak{sl}(2)$ and $\text{SL}_q(2)$.

First, if A, B are Hopf algebras, we say that they are *in duality* if there is a nondegenerate pairing between them — a bilinear map $\langle , \rangle : A \times B \to \mathbb{K}$ — and all the operations must be dual, one to another. For example, $\langle \Delta a, b_1 \otimes b_2 = \langle a, b_1 b_2 \rangle$ and $\langle a_1 \times a_2, \Delta b \rangle = \langle a_1 a_2, b \rangle$. And there should be something good about antipode and counit as well. For example, $\langle 1, b \rangle = \epsilon(b)$, and $\langle S(a), b \rangle = \langle a, S(b) \rangle$.

Now, recall that $\text{GL}_q(2)$ was defined by

$$ba = qab \quad ca = qac \quad bc = cb \quad db = qbd \quad dc = qcd \quad ad - da = (q^{-1} - q)bc \tag{29.9}$$

and that the determinant is invertible?, where we think of a, b, c, d as the coordinates of the (quantum) matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Now, there is a two-dimensional representation of $\mathcal{U}_q\mathfrak{sl}(2)$ in which $E \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $F \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $K \mapsto \left(\begin{smallmatrix} q & 0 \\ 0 & q^{-1} \end{smallmatrix}\right)$. Then the main word is to say that if $u \in \mathcal{U}_q\mathfrak{sl}(2)$ and we define the coordinates A, B, C, D by $u \mapsto \left(\begin{smallmatrix} A(u) & B(u) \\ C(u) & D(u) \end{smallmatrix}\right)$. So we check that A, B, C, D satisfy equation 29.9, and then we have a map $\mathrm{GL}_q(2) \to \mathcal{U}_q^*\mathfrak{sl}(2)$, and it is already a homomorphism of coalgebras.

Lecture 30 April 7, 2010

Today we complete the proposition from last time:

Proposition 30.1 *$\mathcal{U}_q\mathfrak{sl}(2)$ is dual to $\mathrm{SL}_q(2)$.*

Proof: The main idea was to construct the duality by considering the standard representation of $\mathcal{U}_q\mathfrak{sl}(2)$ by 2×2 matrices. Recall, we had:

$$\mathrm{SL}_q(2) = \frac{\mathrm{Mat}_q(2)}{(\det_q - 1)}$$

where $\mathrm{Mat}_q(2)$ is the space of quantum matrices, thought of as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with relations:

$$ba = qab \quad ca = qac \quad bc = cb \quad db = qbd \quad dc = qcd \quad ad - da = (q^{-1} - q)bc \tag{30.1}$$

Then the standard representation is that $E \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $F \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $K \mapsto \left(\begin{smallmatrix} q & 0 \\ 0 & q^{-1} \end{smallmatrix}\right)$. Then what we do is define $u \mapsto \left(\begin{smallmatrix} A(u) & B(u) \\ C(u) & D(u) \end{smallmatrix}\right)$. Then we have to check that $\{A, B, C, D\}$ satisfy equation 30.1.

Well, let $x, y \in \{A, B, C, D\}$. Then we have to check $(xy)(u) = \sum x(u_1)y(u_2)$, which is the Sweedler notation $\Delta u = \sum u_1 \otimes u_2$. We will check this on a basis, and extend by linearity. The trick is that $x(E^2) = x(F^2) = 0$ in our representation. So we need only check $\{E^i F^j K^l\}$ for $i, j \leq 2$.

Recall that $\Delta E = (E \otimes K + 1 \otimes E)$ and $\Delta F = (K^{-1} \otimes F + F \otimes 1)$ and $\Delta K^l = K^l \otimes K^l$.

We make a big table:

	K^l	FK^l	F^2K^l	EK^l	E^2K^l	EFK^l	E^2FK^l	EF^2K^l	$E^2F^2K^l$
BA							β		
AB							$q^{-1}\beta$		
CA		q^{2l}						αq^{2l-1}	
AC		q^{2l-1}						αq^{2l-2}	
DA	1					q			
AD	1					q^{-1}			
BC						1			
CB						1			
DB					q^{-2l}				
BD					q^{-2l-1}				
DC									
CD									

and the rest is 0, where α, β are constants, approximately $\beta = 1 + q^{-2}$ or so.

So, this proves that we have a homomorphism of algebras $\mathrm{Mat}_q(2) \to \mathcal{U}_q\mathfrak{sl}(2)^*$. On the other hand, we have a homomorphism of algebras $\mathcal{U}_q\mathfrak{sl}(2) \to \mathrm{Mat}_q(2)^*$ given by the fact that we have a representation. And therefore the map is a morphism of bialgebras, and so we have constructed our pairing $\mathrm{Mat}_q(2) \otimes \mathcal{U}_q\mathfrak{sl}(2) \to \mathbb{K}$, and it is a pairing of bialgebras.

However, this pairing has kernel, because you can check that $\langle \det_q, u\rangle = \epsilon(u)$, for $u \in \mathcal{U}_q\mathfrak{sl}(2)$. And therefore, in fact the map $\mathrm{Mat}_q(2) \to \mathcal{U}_q\mathfrak{sl}(2)^*$ factors through $\mathrm{SL}(2)$.

Now, is it clear how to check that $\langle \det_q, u\rangle = \epsilon(u)$? We have to check it only on generators, because $\det_q$ is a grouplike element, and ϵ is multiplicative.

So we have to check now only that the antipodes are well-behaved. **Question from the audience:** Do we have to check that there is not more kernel? **Answer:** Ah, that's true, and of course there is something to check, because it is not true when q is a root of unity. We will save it as homework: you can probably do it by thinking about all finite-dimensional representations.

Ah, yes, the antipode. We have to check that if $x \in \mathrm{SL}_q(2)$ and $u \in \mathcal{U}_q\mathfrak{sl}(2)$, then $\langle S(x), u\rangle = \langle x, S(u)\rangle$. Now, since S is an antipode, it is sufficient to check this for any x but only generators $u = E, F, K$. What we play is that S is an antiautomorphism of $\mathcal{U}_q\mathfrak{sl}(2)$: if we know it for u, v, then we say that $\langle x, S(uv) = \langle x, S(v)\, S(u)\rangle = \sum \langle x_1, S(v)\rangle\langle x_2, S(u)\rangle = \sum \langle S(x_1), v\rangle\langle S(x_2), u\rangle = \sum \langle S(x_2), u\rangle\langle S(x_1), v\rangle = \langle \Delta S(x), u \otimes v\rangle = \langle S(x), uv\rangle$.

Question from the audience: Wait, do I ever need to check this? **Answer:** Well, not if the Hopf algebras are finite-dimensional, but the argument is not obvious for infinite-dimensional representations.

Anyway, then you check the claim for $u = E, F, K$, which is a calculation left to the reader. $\square$

30.1 If q is a root of unity (degree l)

We will be interested actually in the number $d = \begin{cases} l, & l \text{ odd} \\ l/2, & l \text{ even} \end{cases}$. Because we are interested when $q^{2d} = 1$. The interesting feature is that the center is too large: it is not generated just by the Casimir.

Lemma 30.2 *When $q^{2d} = 1$, then K^d, E^d, F^d lie in the center of $\mathcal{U}_q\mathfrak{sl}(2)$.*

Proof: $K^d E K^{-d} = q^{2d} E$, and similarly for F. Also:

$$[E, F^m] = [m] \frac{q^{1-m}K - q^{m-1}K^{-1}}{q - q^{-1}} F^{m-1}$$

and when $m = d$, then $[m] = \frac{q^m - q^{-m}}{q - q^{-1}} = 0$. □

Corollary 30.3 *If V is an irreducible representation of $\mathcal{U}_q\mathfrak{sl}(2)$, then K^d, E^d, F^d act by scalar operators.*

And from this we will see that the dimension cannot be larger than d, and generically must be exactly d.

Proposition 30.4 $\dim V \leq 0$. *Moreover, if any of the following conditions — $F^d \neq 0$, $E^d \neq 0$, $K^d \neq \pm 1$ — then* $\dim V = d$.

Proof: We start as before: we pick up some eigenvector $v_0 \in V$ such that $Kv_0 = \lambda v_0$. Then if $v_i = F^i v_0$, then $K v_i = \lambda q^{-2i} v_i$. But notice that if we come to $i = d$, it will repeat itself: rather than being in a line, it will be in a circle.

So first assume that $F^d = a$. Then it's clear that the space spanned by $v_0, \ldots, v_{d-1}$ is invariant under the action of K and F. To see that it is also invariant with respect to E, we can use the Casimir, which is also in the center. Recall that $C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2}$. So if we would like to apply $EF^i v_0 = EF \cdot F^{i-1} v_0$, but since we know that C_q acts as c, we can get rid of EF. So the dimension is not more than d, and if F^d is not zero, then the dimension is d. A similar calculation works for E.

And finally, if $E^d = 0$, then we have a highest weight vector, so we are in a situation we understand. Indeed, we have $Ev_0 = 0$, and so $EF^m v_0 = [m] \frac{q^{1-m}\lambda - q^{m-1}\lambda}{q - q^{-1}} F^{m-1} v_0$, and if $\dim V < d$, then this coefficient must be zero. So then λ must be acting as a power of q. And therefore $K^d = \pm 1$. □

So, we see, only for very special scalars for E^d, F^d, K^d do we have interesting representations. In fact, there are very nice formulas, which we will put on the next homework.

The point is that we always have a map $\gamma : \operatorname{Irr} \mathcal{U}_q\mathfrak{sl}(2) \to \operatorname{Spec} Z$, where $Z = \mathcal{Z}(\mathcal{U}_q\mathfrak{sl}(2))$. Or maybe $\operatorname{Specm} Z$, the set of maximal ideas. You have Schur's lemma, so you look with which scalars each element acts. Here the map is: for $\mu : Z \to \mathbb{K}$, we set $\gamma^{-1}(\mu) = \{V \in \operatorname{Irr} U_q\mathfrak{sl}(2) \text{ s.t. } z|_v = \mu(z)\, \forall z \in Z\}$.

Theorem 30.5 *If $\mu(E^d) \neq 0$ or $\mu(F^d) \neq 0$, then $\gamma^{-1}(\mu)$ is just one representation of dimension d.*

Proof: Because $\dim \mathcal{U}_q\mathfrak{sl}(2)/(\ker\mu) \leq d^2$, because E^iK^j for $0 \leq i,j < d$ span the quotient. This is a simple calculation that we know: there is an irreducible representation of degree d, so by density theorem the image is the full matrix algebra, so is the full dimension. □

So, the most interesting representation theory is of the *small quantum group*, which is when we quotient $\mathcal{U}_q\mathfrak{sl}(2)$ by $E^d = F^d = 0$ and $K^d = 1$. This has many representations, and it is not a semisimple algebra. Next time we move to higher rank.

Lecture 31 April 9, 2010

We've decided to follow [7] for a while. Also, we still don't know how to show that the maps $\mathcal{U}_q\mathfrak{sl}(2) \to \mathrm{SL}_q(2)^*$ and $\mathrm{SL}_q(2) \to \mathcal{U}_q\mathfrak{sl}(2)$ are injections — i.e. how to show that the Hopf pairing from last time is nondegenerate. Presumably the proof looks inside the two Hopf algebras and shows that they have no Hopf ideals.

Today, we show how to go from $\mathfrak{g}$ a semisimple Lie algebra over $\mathbb{C}$ to $\mathcal{U}_q\mathfrak{g}$. The motivation for the construction is from the theory of Kac-Moody algebras, so we will study that analogue first:

31.1 Kac-Moody Lie algebras

Let Π be a set of "indices" or "simple roots" and $a_{\alpha,\beta}$ a matrix with $\alpha,\beta \in \Pi$. In general, all you want from this matrix a is to have $a_{\alpha\alpha} = 2$, $a_{\alpha\beta} \in \mathbb{Z}_{\leq 0}$ if $\alpha \neq \beta$, and for many applications the symmetrizability: there is some diagonal matrix d with $a_{\alpha\beta} \cdot d_\beta$ symmetric ****no sum****.

We will construct $\mathfrak{h}$ so that $\dim \mathfrak{h} = 2\Pi - \mathrm{rank}(a_{\alpha\beta})$. Usually we will have a nondegenerate, so that $\dim \mathfrak{h} = \Pi$.

Then we will enumerate the coroots via $\{h_\alpha\}_{\alpha\in\Pi}$, and we will set $\alpha(h_\beta) = a_{\beta\alpha}$, so that we have defined $\alpha \in \mathfrak{h}^*$ the roots.

Then we construct a Lie algebra $\tilde{\mathfrak{g}}$ in the most natural way. First, you say $\mathfrak{h} \subseteq \tilde{\mathfrak{g}}$ is an abelian subalgebra $[\mathfrak{h},\mathfrak{h}] = 0$, and also you introduce generators $\{x_\alpha, y_\alpha\}$ for each $\alpha \in \Pi$ and declare $[h_\alpha, x_\beta] = \beta(h_\alpha)x_\beta$, $[h_\alpha, y_\beta] = -\beta(h_\alpha)y_\beta$, and $[x_\alpha, y_\beta] = \delta_{\alpha\beta}h_\beta$. Then it's clear that $\{y_\alpha, h_\alpha, x_\alpha\}$ is an $\mathfrak{sl}(2)$-subalgebra.

So, and we will do this carefully in the quantum case, there is a triangular decomposition: $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}^+$, where $\tilde{\mathfrak{n}}^+$ is generated by x_α, $\tilde{\mathfrak{n}}^-$ is generated by the y_αs, $\tilde{\mathfrak{n}}^\pm$ are free Lie algebras on $\alpha \in \Pi$.

Question from the audience: How all the rest of $\mathfrak{h}$ comes in? **Answer:** We chose linearly independent h_αs, and made the declarations above. There might be more in $\mathfrak{h}$. But let's just assume that a is nondegenerate.

Now, how do the Serre relations come in? We are looking for a maximal ideal $I \subseteq \tilde{\mathfrak{g}}$ such that $I \cap \mathfrak{h} = 0$.

Now, there is a very simple way to construct some elements of the idea. In particular, consider for $\alpha \neq \beta$ the elements

$$u^+_{\alpha\beta} = (\mathrm{ad}\, x_\alpha)^{1-a_{\alpha\beta}} x_\beta \text{ and } u^-_{\alpha\beta} = (\mathrm{ad}\, y_\alpha)^{1-a_{\alpha\beta}} y_\beta$$

Then they are both in I. Indeed, this follows from the $\mathfrak{sl}(2)$ corresponding to $\langle x_\alpha, h_\alpha, y_\alpha \rangle$. Then you pick up x_β, and it will be a lowest vector in its $\mathfrak{sl}(2)_\alpha$ representation, and y_β is a highest weight vector. So then a simple calculation with $\mathfrak{sl}(2)$ shows that $[y_\gamma, u^+_{\alpha\beta}] = 0 = [x_\gamma, u^-_{\alpha\beta}]$. When $\gamma \neq \alpha, \beta$, this follows from the commutation relations with $\mathfrak{h}$, and for $\gamma = \alpha, \beta$ it is from $\mathfrak{sl}(2)$.

Ok, so take the root lattice Q, and you see that $I^+ = \langle u^+_{\alpha\beta} \rangle$, and then it's pretty clear that $P(I^+) \geq P(u^+_{\alpha\beta})$, where by P we mean the weights. So $I^+ \subseteq \tilde{\mathfrak{n}}^+$, and similarly for $-$, and we quotient by I, and $\tilde{\mathfrak{g}}/I$ is the *Kac-Moody algebra* associated to a.

In fact, it's a non-obvious result that there are not more relations, because we said we want the maximal ideal. But in fact:

Theorem 31.1 *If $a_{\alpha\beta}$ is symmetrizable, then the maximal ideal I is generated by $u^\pm_{\alpha\beta}$.*

We think this is still true if a is not symmetrizable, but it was proven much later. ****I missed who proved what.****

Now, we'd rather write down the Serre relations as follows. Let $r = 1 - a_{\alpha\beta}$. Then we have:

$$\sum_{s=0}^{r} (-1)^2 \binom{r}{s} x_\alpha^{r-s} x_\beta x_\alpha^s = u^+_{\alpha\beta}$$

Then this has a chance to be quantized, because we have the quantum binomial coefficients.

Recall: a is symmetrizable and positive-definite if and only if $\mathfrak{g} = \tilde{\mathfrak{g}}/I$ is finite-dimensional. But the point is that many things — the Weyl group, for example — survive moving to Kac-Moody algebras.

So, anyway, we will move back to the finite-dimensional case, but you will see that most of the construction works for KM algebras in general.

31.2 Quantum enveloping algebras for finite-dimensional semisimple Lie algebras

Fix $\mathfrak{g}$ a finite-dimensional semisimple Lie algebra over $\mathbb{C}$, with Δ a root system, W the Weyl group, Q the lattice generated by W. Then we pick $(,)$ to be W-invariant, and we will normalize it as [7] does — we think [7] is following [10] here — by letting Π be the set of simple roots and $(\alpha, \alpha) = 2$ for α a short root.

Question from the audience: When $\mathfrak{g}$ is semisimple but not simple, you do this for each component of the Dynkin diagram? **Answer:** Yes. Or we can restrict our attention to simple Lie algebras: $\mathcal{U}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = \mathcal{U}\mathfrak{g}_1 \otimes \mathcal{U}\mathfrak{g}_2$ canonically, and we think this still holds for $\mathcal{U}_q$.

Ok, so let's declare $d_\alpha = (\alpha, \alpha)/2$, and $q_\alpha = q^{d_\alpha}$. Then we define $\mathcal{U}_q\mathfrak{g}$ as follows. We pick generators $\{K_\alpha^\pm, E_\alpha, F_\alpha\}$ with:

$$K_\alpha K_\alpha^{-1} = 1 \qquad K_\alpha K_\beta = K_\beta K_\alpha \tag{31.1}$$

$$K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} E_\beta \qquad K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)} F_\beta \tag{31.2}$$

$$E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \tag{31.3}$$

Then equations 31.1 to 31.3 define the algebra $\tilde{\mathcal{U}}_q\mathfrak{g} = \tilde{U}$.

Then the point is that for $\alpha \in \Pi$, we have a homomorphism $\mathcal{U}_{q_\alpha}\mathfrak{sl}(2) \to \tilde{\mathcal{U}}_q\mathfrak{g}$.

Moreover, if $\lambda \in Q$, let $K_\lambda = \prod K_\alpha^{m_\alpha}$ where $\lambda = \sum m_\alpha \alpha$. Then we have: $K_\lambda E_\beta K_\lambda^{-1} = q^{(\lambda,\beta)} E_\beta$ and similarly for F. This defines a Q-grading on $\tilde{U}$. We set $\tilde{U}_\mu = \{u \in U \text{ s.t. } K_\lambda u K_\lambda^{-1} = q^{(\lambda,\mu)} u\}$.

Lemma 31.2 *$\tilde{U}$ has a unique Hopf algebra structure with the following properties:*

- $\Delta E_\alpha = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha$
- $\Delta F_\alpha = 1 \otimes F_\alpha + F_\alpha \otimes K_\alpha^{-1}$
- $\Delta K_\alpha = K_\alpha \otimes K_\alpha$
- $\epsilon(E_\alpha) = \epsilon(F_\alpha) = 0$ *and* $\epsilon(K_\alpha) = 1$
- $S(E_\alpha) = -K_\alpha^{-1} E_\alpha$ *and* $S(F_\alpha) = -F_\alpha K_\alpha$ *and* $S(K_\alpha) = K_\alpha^{-1}$.

Proof: Uniqueness is obvious, because theses are generators, so we need only check that Δ respects the relations. Except for one relation, everything is already checked in the $\mathfrak{sl}(2)$ case. So we need only check that $[\Delta E_\alpha, \Delta F_\beta] = 0$, which is straightforward. □

We remark that S is an antiautomorphism of $\tilde{U}$ and that $S^2(u) = K_{2\rho}^{-1} u K_{2\rho}$, where ρ is the sum of simple roots, so 2ρ is half the sum of the positive roots.

Now, notice that if q is a root of unity, then E^d, F^d are central, and otherwise the construction is just like in $\mathfrak{sl}(2)$ case.

Ok, so now we will introduce the *Serre elements*, exactly as in the KM case, but with q-numbers. As before, we set $r = 1 - a_{\alpha\beta}$, and define:

$$u_{\alpha\beta}^+ = \sum_{s=0}^{r} (-1)^s \binom{r}{s}_{q^\alpha} E_\alpha^{r-s} F_\beta E_\alpha^s$$

****The binomial coefficient should be in square brackets.****

Lemma 31.3 *The element* $u^+_{\alpha\beta}$ *is* quasiprimitive*, i.e. it is primitive up to a torus part:*

$$\Delta u^+_{\alpha\beta} = u^+_{\alpha\beta} \otimes 1 + K^r_\alpha K_\beta \otimes u^+_{\alpha\beta}$$

Then antipode acts as $S(u^+_{\alpha\beta}) = -K^r_\alpha K^{-1}_\beta u^+_{\alpha\beta}$.

There's a similar statement for $u^-_{\alpha\beta}$. We will not prove lemma 31.3, because it is an intricate calculation, but you can do it for yourself or read it.

Corollary 31.4 *Let* I *be the ideal in* $\tilde{U} = \tilde{\mathcal{U}}_q\mathfrak{g}$ *generated by all* $u^\pm_{\alpha\beta}$*. Then* $\Delta I \subseteq \tilde{U} \otimes I + I \otimes \tilde{U}$*, and* $S(I) \subseteq I$.

Therefore we can take the quotient. We define the *Drinfeld-Jimbo quantum universal enveloping algebra* to be $\mathcal{U}_q\mathfrak{g} = \tilde{U}/I$.

We will stop, but quickly mention what else should be done, but it is technical:

1. Triangular decomposition.
2. If q is not a root of unity, classify all finite-dimensional representations.
3. R-matrix.
4. The dual Hopf algebra.

Lecture 32 April 12, 2010

We continue today with basic facts about quantized universal enveloping algebra. Recall that we defined a very large algebra $\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_q\mathfrak{g}$, and then we constructed the *Serre elements*

$$u^+_{\alpha\beta} = \sum_{s=0}^{r} (-1)^s \begin{bmatrix} r \\ s \end{bmatrix}_\alpha E^{r-s}_\alpha E_\beta E^s_\alpha \tag{32.1}$$

$$u^-_{\alpha\beta} = \sum_{s=0}^{r} (-1)^s \begin{bmatrix} r \\ s \end{bmatrix}_\alpha F^{r-s}_\alpha F_\beta F^s_\alpha \tag{32.2}$$

where $r = 1 - a_{\alpha\beta}$. And if I is the ideal generated by $u^\pm_{\alpha\beta}$, then I is a Hopf ideal, and so $\mathcal{U} = \mathcal{U}_q\mathfrak{g} = \tilde{\mathcal{U}}/I$ is a Hopf algebra.

To do highest weight theory, we need a bit more. Today we will describe this story, in analogy with the theory of Kac-Moody algebra.

Recall that we have $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}^+$, where $\tilde{\mathfrak{n}}^\pm$ are free Lie algebras.

Now, let's take $\beta_1, \ldots, \beta_k \in \Pi$, pick an ordering $I = \beta_1, \ldots, \beta_k$, then we define $E^I = E_{\beta_1} \ldots E_{\beta_k}$, and $F^I = F_{\beta_1} \ldots F_{\beta_k}$. Then if $\mu \in Q$ is the weight of $I = \beta_1 + \cdots + \beta_k$, then it's clear that $E^I \in \tilde{\mathcal{U}}_\mu$.

Lemma 32.1 *As I, J run through sequences of roots and λ runs through the root lattice Q, then $\{F^I K_\lambda E^J\}$ is a basis for $\tilde{\mathcal{U}}$.*

Proof: By the commutation relations it is clear that the proposed basis spans. More difficult is to prove the linear independence. We do this by constructing certain modules. Before we do this, we report a straightforward calculation by induction:

$$\Delta(E^I) = \sum c^I_{A,B}(q)\, E^A\, K_{\mathrm{wt}\, B} \otimes E^B \tag{32.3}$$

$$\Delta(F^I) = \sum c^I_{A,B}(q^{-1})\, F^A \otimes K^{-1}_{\mathrm{wt}\, A} F^B \tag{32.4}$$

Here $c^I_{A,B}$ are certain polynomials in q, and A, B are complementary subsequences of I.

Now, if we pick any function $c : \Pi \to \mathbb{K}$, then we will construct a "Verma module" M_c, with a basis given by the v_I the set of sequences. The action is:

$$F_\alpha \cdot v_I = v_{\alpha,I} \qquad\qquad K_\alpha \cdot v_I = c_\alpha q^{-(\alpha, \mathrm{wt}\, I)} v_I \tag{32.5}$$

So, see, we try to make this into a weight module, hence the second relation, and the Fs act freely as if it's a tensor algebra. Then the last relation raises degree:

$$E_\alpha \cdot v_I = \sum_j \frac{c_\alpha q^{(-\alpha,\mu_j)} - c_\alpha^{-1} q^{(\alpha,\mu_j)}}{q_\alpha - q_\alpha^{-1}} v_{I \smallsetminus \beta_j} \tag{32.6}$$

Here $I = \beta_1, \ldots, \beta_k$, and $\mu_j = \beta_{j+1} + \cdots + \beta_k$.

This is a rather natural construction, but just like in the PBW theorem, you have to work a bit with the induction to prove that M_c is a $\tilde{\mathcal{U}}$ modules.

We picked up the proof from [7]. He uses the following trick: consider the dual module M_c', which is M_c^ω, i.e. the same module twisted by the Cartan involution. Do you remember what is the *Cartan involution*? It is the automorphism of $\tilde{\mathcal{U}}$ given by $\omega(E_\alpha) = F_\alpha$, $\omega(F_\alpha) = E_\alpha$, and $\omega(K_\alpha) = K_\alpha^{-1}$. The point is that then you can check things only for the positive part, and do the negative part by Cartan involution.

Let's denote the basis of M_c^ω by w_I.

Anyway, so now we will prove the theorem, provided the Verma module exists. Suppose that $\sum a_{I\lambda J} F^I K_\lambda E^J = 0$. Then pick I_0 maximal such that $a_{I_0 \lambda J} \neq 0$. Also, pick up M_1, which is the M_c above with c the constant function 1. And tensor it with M_c^ω, i.e. act by $A = \sum a_{I\lambda J} F^I K_\lambda E^J$ on $M_1 \otimes M_c^\omega$. Then we have:

$$0 = \sum a_{I\lambda J} F^I K_\lambda E^J (v_\emptyset \otimes w_\emptyset) \tag{32.7}$$

$$= \sum a_{I\lambda J} F^I K_\lambda \sum_{A,B} c^J_{A,B}(q) \big(E^A K_{\mathrm{wt}\, B} v_\emptyset \otimes w_B\big) \tag{32.8}$$

$$= \sum a_{I\lambda J} F^I K_\lambda (v_\emptyset \otimes w_J) \tag{32.9}$$

$$= \sum a_{I\lambda J} q^{(-\lambda, \mathrm{wt}\, J)} c_\lambda F^I (v_\emptyset \otimes w_J) \tag{32.10}$$

$$= \sum a_{I\lambda J} q^{(-\lambda, \mathrm{wt}\, J)} c_\lambda \sum c^I_{C,D}(q^{-1})\, v_C \otimes K^{-1}_{\mathrm{wt}\, C} F^D w_J \tag{32.11}$$

In equation 32.8, the only term that does not kill it is when $A = \emptyset$. In equation 32.11, we look at only the v_{I_0} part: by maximality, it must vanish. I.e.:

$$\sum_{\lambda} a_{I_0 \lambda J} q^{-(\lambda, \mathrm{wt}\, J)} c_{\lambda - \mathrm{wt}\, I_0} q^{-(\lambda, \mathrm{wt}\, J) + (\mathrm{wt}\, I_0, \mathrm{wt}\, J)} v_{I_0} \otimes w_J = 0 \tag{32.12}$$

Because all other terms will be less than the I_0 term. And so:

$$\sum_{I_0 \lambda J} q^{(-\lambda, \mathrm{wt}\, J)} c_{\lambda} = 0 \tag{32.13}$$

But this must be true for any c, and so this proves the theorem when $a_{I_0 \lambda J} = 0$ if $|\mathbb{K}| = \infty$, i.e. if $\mathrm{char}\, \mathbb{K} = 0$. □

Corollary 32.2 *There is an isomorphism of vector spaces $\tilde{U} = \tilde{U}^- \otimes \tilde{U}_0 \otimes \tilde{U}^+$, where $\tilde{U}_0$ is generated by K_λ, and $\tilde{U}^{+(-)}$ is generated by E_α (resp. F_α).*

At the end of the day, we want:

Proposition 32.3 $U = U^- \otimes U_0 \otimes U^+$.

This should follow from the statement for $\tilde{U}$. Here $U_0 = \tilde{U}_0$ and $U^\pm$ are the parts of U generated by F, E respectively. This follows from:

Lemma 32.4 $[F_\gamma, u^+_{\alpha\beta}] = 0 = [E_\gamma, u^-_{\alpha\beta}]$ *for all* α, β, γ.

Because then

$$I = \tilde{U}^- \tilde{U}_0 \langle u^+_{\alpha\beta} \rangle + \langle u^-_{\alpha\beta} \rangle \tilde{U}_0 \tilde{U}^+ \tag{32.14}$$

where $\langle u^\pm_{\alpha\beta} \rangle$ are ideals in $\tilde{U}^\pm$. Lemma 32.4 shows that the RHS of equation 32.14 is an ideal, and hence we have the equality. Why does equation 32.14 imply the claim?

We have the following commuting square, where the horizontal maps are the multiplication maps:

$$\begin{array}{ccc} U^- \otimes U_0 \otimes U^+ & \longrightarrow & U \\ \Big\uparrow & & \Big\uparrow \ker = I \\ \tilde{U}^- \otimes \tilde{U}_0 \otimes \tilde{U}^+ & \xrightarrow{\sim} & \tilde{U} \end{array}$$

So then the end is just linear algebra: we have $I = I^- + I^+$, and so $U = \tilde{U}^-/I^- \otimes U_0 \otimes \tilde{U}^+/I^+$, which does it.

Proof (of lemma 32.4): The point is that for any Hopf algebra we have the *adjoint representation*:

$$\mathrm{ad}(x) y = \sum x_1^i \, y \, S(x_2^i)$$

where $\Delta x = \sum x_1^i \otimes x_2^i$. Then we have:

$$\mathrm{ad}(E_\alpha)u = E_\alpha u - K_\alpha u K_\alpha^{-1} E_\alpha \tag{32.15}$$

$$\mathrm{ad}(F_\alpha)u = \big(F_\alpha u - u F_\alpha\big) K_\alpha \tag{32.16}$$

$$\mathrm{ad}(K_\alpha)u = K_\alpha u K_\alpha^{-1} \tag{32.17}$$

So then we see that $u^+_{\alpha\beta} = \mathrm{ad}(E^r_\alpha)E_\beta$ and $u^-_{\alpha\beta} = \mathrm{ad}(F^r_\alpha)F_\beta$, exactly as in the classical case.

But then recall that if $\gamma \neq \alpha, \beta$, $[F_\gamma, E_\alpha] = 0 = [F_\gamma, E_\beta]$, and so lemma 32.4 follows.

Now, if $\gamma = \alpha$, then we use $\mathcal{U}_{q_\alpha}\mathfrak{sl}(2)$. We have already discussed the following identity:

$$F_\alpha E^r_\alpha = [r]_\alpha E^{r-1}_\alpha \frac{q_\alpha^{r-1} K_\alpha^{-1} - q_\alpha^{1-r} F_\alpha}{q_\alpha - q_\alpha^{-1}} \tag{32.18}$$

where $r = 1 - \frac{r(\alpha,\beta)}{(\alpha,\beta)}$ **?**. So then along with $[F_\alpha, F_\beta] = 0$ **?**, we have:

$$\mathrm{ad}(F_\alpha)\,\mathrm{ad}(E^r_\alpha)E_\beta = 0 \tag{32.19}$$

And the case when $\gamma = \beta$ is an easier direct calculation, which we will put in the homework. □

Next time we discuss the category of representations.

Lecture 33 April 14, 2010

Today we briefly describe the category of finite-dimensional $U = \mathcal{U}_q\mathfrak{g}$ modules when q is not a root of unity. In fact, we will really assume that q is transcendental. Of course, we suppose that $\mathrm{char}\,\mathbb{K} = 0$, but in fact we assume that $\mathbb{K} = \mathbb{C}$.

So, the main thing is that like in the case of $\mathcal{U}_q\mathfrak{sl}(2)$ when we had an extra sign, here we again has a bit more combinatorial data.

From the classical theory, we have Q the *root lattice* and we define the *weight lattice* $P = \{\mu \in \sum_{\alpha\in\Pi} m_\alpha \alpha \text{ s.t. } \frac{2(\mu,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}\}$. Now, let $\sigma : Q \to \{\pm 1\} = \mathbb{Z}/2$ be a morphism of abelian groups — clearly this is the same as a function $\Pi \to \mathbb{Z}/2$.

1. Let M be a finite-dimensional $\mathcal{U}_q\mathfrak{g}$-module. Then we can define $M = \bigoplus M_{\lambda,\sigma}$, where

   $$M_{\lambda,\sigma} = \{m \in M \text{ s.t. } K_\alpha m = \sigma(\alpha)\, q^{(\lambda,\alpha)} m \;\forall \alpha \in \Pi\}$$

   In fact, we can do this from the $\mathfrak{sl}(2)$ theory. For each $\alpha \in \Pi$, we have $\mathcal{U}_{q_\alpha}\mathfrak{sl}(2) \hookrightarrow \mathcal{U}_q\mathfrak{g}$, where $q_\alpha = q^{(\alpha,\alpha)/2}$. Then for each K_α, we have $M = \bigoplus V_{n,\epsilon}(\alpha)$, and M is semisimple over $\mathcal{U}^\circ = \mathbb{K}[K_\mu]$. Then for m an eigenvector of $\mathcal{U}^\circ$, we have $K_\alpha m = \pm q_\alpha^{n_\alpha}$ for some $n_\alpha \in \mathbb{Z}$. And of course we can simultaneously diagonalize the K_α actions, and so from the $\mathfrak{sl}(2)$ decomposigion we do have $M = \bigoplus M_{\lambda,\sigma}$, where $\sigma(\alpha) = \pm$ depending on the sign in $K_\alpha m = \pm q_\alpha^{n_\alpha}$.

Moreover, by $q_\alpha^{n_\alpha} = q^{n_\alpha\,(\alpha,\alpha)/2}$, we see that the λ in $M = \bigoplus M_{\lambda,\sigma}$ is in P.

So that gives the *weight decomposition of modules.*

2. The next thing is that we have the *root decomposision* of $U = \bigoplus_{\mu\in Q} U_\mu$. Then it's clear that $U_\mu M_{\lambda,\sigma} \subseteq M_{\lambda+\mu,\sigma}$. In fact, in the usual way, the weights are multiplicative (although in P, Q we use additive characters):

$$(M \otimes N)_{\lambda+\lambda',\sigma\sigma'} \supseteq M_{\lambda,\sigma} \otimes N_{\lambda',\sigma'}$$

and $(M^*)_{\lambda,\sigma} = (M_{-\lambda,\sigma})^*$.

Then usually people try to get rid of the σ as soon as they can. We set $M_\sigma = \bigoplus M_{\lambda,\sigma}$ for each σ,, and clearly $M = \bigoplus M_\sigma$. Then if we let $\mathscr{F}$ be the full category of finite-dimensional U-modules, then we have $\mathscr{F} = \bigoplus \mathscr{F}_\sigma$. Moreover, for each σ, you have a trivial representation twisted by σ. And so you can go between the different subcategories: $\mathscr{F}_\sigma \xrightarrow{\otimes C_\sigma} \mathscr{F}_1$.

How is this C_σ? We set $\sigma(K_\alpha) = \sigma(\alpha)K_\alpha$ and $\sigma(E_\alpha) = \sigma(F_\alpha) = 0$. And so C_σ is a one-dimensional representation with character σ, and $\mathscr{F}_\sigma \leftrightarrow \mathscr{F}_1$ is an equivalence of categories.

We say that the *Type 1* modules are those in $\mathscr{F}_1 \subseteq \mathscr{F}$.

So, so far the only placed we used anything about q is the semisimplicity over $\mathfrak{sl}(2)$. We will at some point prove the semisimplicity for arbitrary $\mathfrak{g}$, and today we will do it for q transcendental. We were very lucky for $\mathfrak{sl}(2)$, because we constructed the Casimir by hand. In principle, that proof could be translated to the general case.

3. Ok, so if $M \in \mathscr{F}_1$, then there exists $v \in M_\lambda$ for some λ such that $E_\alpha v = 0$. Indeed, what you do is look at all weights — all the λs that occur with nonzero multiplicty — and we denote this set $P(M)$. Then we have $E_\alpha M_\mu \subseteq M_{\mu+\alpha}$. And exactly as in the classical case, we have a partial ordering $\mu \leq \lambda$, $\mu, \lambda \in P$, when $\lambda - \mu \in Q^+$. And so then we can just find a weight $\lambda \in P(M)$ that is not less than any other vector in $P(M)$ — it is *weakly maximal.* Then we have the claim automatically.

4. And so we can now again build the Verma module. For each $\lambda \in P$, we define $M(\lambda)$ via:

$$M(\lambda) = U \otimes_{U^0 U^+} C_\lambda$$

where in C_λ, the U^0s act by weight λ and the U^+ act by zero. Then we leave it as an exercise that $M(\lambda)$ has a unique simple quotient. You do this exactly as in the Lie algebra case: it has a unique maximal submodule. This quotient is called $L(\lambda)$.

Then it is clear that every irreducible finite-dimensional module ****of type 1**** is isomorphic to some $L(\lambda)$, but not every $L(\lambda)$ is finite-dimensional.

The next thing we would like to say is that if $\dim L(\lambda) < \infty$, then $\lambda \in P^+$, where $P^+ = \{\lambda \in P \text{ s.t. } \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}_{\geq 0}\ \forall \alpha \in \Pi\}$. Now why is this true? It's true because of the $\mathfrak{sl}(2)$ theory. We must have $q^{(\lambda,\alpha)} = q_\alpha^{n_\alpha}$, where $n_\alpha \in \mathbb{Z}_{\geq 0}$.

The slightly harder part is to prove that this is if and only if. The argument goes in the following way. We first construct some finite-dimensional quotient, which is like a maximal finite-dimensional quotient, but in general, eg. in the characteristic-p case or in super algebras, it is not simple. In many representation theories, as soon as you don't have semisimplicity, very probably this guy is not simple. For example, in the characteristic-p case, the dual to this quotient has a nice geometric structure coming from Borel-Weil-Bott theory.

In any case, let's construct it:

5. For each $\beta \in \Pi$, we define $m(\beta) = \frac{2(\lambda,\beta)}{(\beta,\beta)} \geq 0$. Then we define a homomoprhism

$$\varphi_\beta : M\big(\lambda - (m(\beta)+1)\beta\big) \to M(\lambda$$

How is this homomorphism constructed? If v is the highest vector on the LHS and w is the highest in $M(\lambda)$, then we will send $v \mapsto F_\beta^{m(\beta)+1} w$. So this has the required weight, and the only thing we have to check is that it is killed by all the E_αs. But if $\alpha \neq \beta$, this is trivial, because they commute, and if $\alpha = \beta$, this is an $\mathfrak{sl}(2)$ calculation that we already did.

6. So, assume that $\lambda \in P^+$. Then we define $\tilde{L}(\lambda) = M(\lambda)/\sum \mathrm{Im}(\varphi_\beta)$. We will prove two things about it. First, look at the set of weights $P(\tilde{L}(\lambda))$. It is invariant under the W-action, where W is the Weyl group of our root system. From this it follows more or less immediately that $\dim \tilde{L}(\lambda) < \infty$. Because it's the same argument: every $\mu \in P(\tilde{L}(\lambda))$ is $\leq \lambda$, and also by W-invariance $w(\mu) \leq \lambda$. So you pick up λ, consider the orbit under W; its convex hull is a polytope and all weights are inside it. ****And the multiplicities are all finite because it is a quotient of $M(\lambda)$.****

So why the W-invariance? It suffices to show that the set of weights is invariant under each simple reflection. But this is an $\mathfrak{sl}(2)$ statement. As soon as we can show that M is a direct sum of finite-diemnsional $\mathcal{U}_{q_\alpha}\mathfrak{sl}(2)$ representations, we are done. And this is equivalent to showing that the generators E_α, F_α act locally nilpotently. ****I think we want to write $\tilde{L}(\lambda)$ as a direct sum, not M.****

Claim: E_α, F_α act locally nilpotently on $\tilde{L}(\lambda)$.

Proof: For E_α, we have $E_\alpha^N M_\mu \subseteq M_{\mu+N\alpha}$, and so for sufficiently large N, we win.

And for F_α — that follows from Serre's relation. So far, everything could have happened with $\tilde{U}$, but actually here we need some sort of nilpotency of an adjoint action. Indeed, we see that $F_\alpha^N F_\beta$, if $r = 1 - a_{\alpha\beta}$, then for $N > r$ we have $F_\alpha^N F_\beta \in \sum_{j=0}^{r-1} \mathbb{K} F_\alpha^j F_\beta F_\alpha^{N-j}$. Why? When $N = r$, this is the Serre relation, and then just do induction — you multiply on the left by F_α, then apply the Serre relation to the F^r part to push it through.

Ok, so if v is a highest vector, then $\tilde{L}(\lambda)$ is spanned as a vector space by $F_{\beta_1} \cdots F_{\beta_k} v$. By the way, this is not absolutely trivial — it follows from the triangular decomposition, without which it would be not true. And so what we want, we want to show that if we pick up F_α^N for N sufficiently large, it kills this. So we use the calculation from the previous paragraph.

We go by induction on k. For $k=0$ it is the definition of $\tilde{L}(\lambda)$. And then what we do is try to move F^N to the right, and we use the calculation. □

So there are a few things that are left. We look at all finite-dimensional modules, because we would like to construct the dual algebra to $\mathcal{U}_q\mathfrak{g}$.

Lemma 33.1 *Let $u \in U$. If $u \in \operatorname{Ann} \mathscr{F}_1$, then $u = 0$.*

Proof: Let $\tilde{L}(\lambda)$ be as above — it is finite-dimensional. Then consider also the highest-weight module $\tilde{L}(\lambda)^\omega$, where ω is the Cartan involution from last time. Then we have

$$u|_{\tilde{L}(\lambda)\otimes\tilde{L}(\mu)^\omega} = 0$$

and so

$$u(v_\lambda \otimes w_\mu) = 0, \tag{33.1}$$

and this is true for all dominant λ, μ.

Now, choose a basis $\{x_1\}$ in U^+, and $\{y_j\}$ a basis for U^-, and all we want from this basis is that it be homogeneous: $y_j \in U_{\mu_j}$ for some μ for each j. Then let's write $u = \sum a_{i,\nu,i} y_j K_\nu x_i$. Then as in the previous class, let's assume that $\operatorname{wt}(x_{i_0})$ is maximal such that $a_{j,\nu,i_0} \neq 0$ for some j, ν. Then we apply this u to equation 33.1, and we end up with, for fixed i_0,

$$\sum a_{j,\nu,i_0} q^{(\operatorname{wt}(x_{i_0})+\lambda-\mu,\nu)} = 0 \tag{33.2}$$

Anyway, you do this calculation exactly as we did last time, and there is something you have to do to use the fact that the decreasing operators act freely. So to do this, we take λ, μ large enough. See, if (λ, α) is very large, then we might need to use that condition. But anyway, we end up getting equation 33.2 for all $\lambda, \mu \in P^+$ that are sufficiently large.

But by a change, we can write this as $\sum b_{j,\nu,i_0} q^{(\lambda-\mu,\nu)} = 0$. But then we use a theorem by Artin that distinct characters of an (abelian) group are linearly independent. So that applies that the as are 0.

Question from the audience: What on q did we use here? **Answer:** Only that it is not a root of unity. If it is not, then the different characters $q^{(-,\nu)}$ are all honestly distinct. □

Next time we will prove that $\tilde{L}(\lambda) = L(\lambda)$, and this requires the transcendence of q.

Lecture 34 April 16, 2010

Recall that for each weight λ we have two modules: $L(\lambda)$ is the unique simple quotient of $M(\lambda)$, and $\tilde{L}(\lambda)$ is the quotient of $M(\lambda)$ by $F_\alpha^{m(\alpha)+1} v$, i.e. it is the maximal finite-dimensional quotient of $M(\lambda)$. We will prove:

Theorem 34.1 *If q is transcendental over $\mathbb{Q}$, then $L(\lambda) = \tilde{L}(\lambda)$, and $\operatorname{ch} L(\lambda)$ is given by the Weyl character formula.*

Recall that the weight-space decomposition defines the *character* $\operatorname{ch} V = \sum_{\mu \in P(V)} \dim V_\mu e^\mu$. In fact, Theorem 34.1 is true provided q is not a root of unity, but we will not do that part.

Proof: We let q be a polynomial variable, writing it differently from $q \in \mathbb{C}$ the constant — we think of q as q.

Suppose that $\mathbb{K} = \mathbb{Q}(q)$, and define A to be the subalgebra $\mathbb{Q}[q, q^{-1}] \subseteq \mathbb{K}$. Then let V be any quotient of $\tilde{L}(\lambda)$ — we are interested in $V = L(\lambda)$ and $V = \tilde{L}(\lambda)$. Then V is spanned by $F^I v_\lambda$ for finitely many I — only finitely many $F^I v_\lambda$ are non-zero. Let V_A be the A submodule generated by $F^I v_\lambda$.

Since A is a principle ideal domain, V_A is a free finitely-generated A module. The following is also easy to check: if we pick up $E_\alpha, F_\alpha, K_\alpha$, and also $[K_\alpha, n] \overset{\text{def}}{=} \frac{K_\alpha q^n - q^{-n} K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$ ****not a commutator; notation from [7]****, then V_A is invariant under all these operators. The most difficult part of checking this is the E_α part, and for this it suffices to do an $\mathfrak{sl}(2)$ calculation.

Now, what happens if we compute $V_A \otimes_A \mathbb{K}$? Then of course we get V back. And in fact this is even true on each weight space: $V_{A,\mu} \otimes_A \mathbb{K} = V_\mu$.

Now we will do the following trick: we will specialize q to 1. We set $\bar{V} = V_A \otimes_A \mathbb{C}$, where A acts on $\mathbb{C}$ by $q = 1$. Then it is just a $\mathbb{C}$ vector space. We write $e_\alpha, f_\alpha, k_\alpha, h_\alpha$ the operators on $\bar{V}$ corresponding to $E_\alpha, F_\alpha, K_\alpha$, and $[K_\alpha, 0]$. Then $k_\alpha = 1$, and

$$h_\alpha v_\mu = \frac{q^{(\mu,\alpha)} - q^{-(\mu,\alpha)}}{q^{(\alpha,\alpha)/2} - q^{-(\alpha,\alpha)/2}} v_\mu = \frac{2(\mu,\alpha)}{(\alpha,\alpha)} v_\mu$$

Then $h_\alpha, e_\alpha, f_\alpha$ also satisfy the classical Serre's relations for $\mathfrak{g}$, because we take the quantum Serre relations and set $q \mapsto 1$. Moreover $[h_\alpha, e_\beta] = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} e_\beta$.

But then we see that $\bar{V}$ is a representation of $\mathfrak{g}$, and so $\overline{\tilde{L}(\lambda)} = \overline{L(\lambda)}$ because they are both the irreducible representation of $\mathfrak{g}$ with height λ. But this implies that $\dim \tilde{L}(\lambda)_\mu = \dim L(\lambda)_\mu$. Therefore they coincide: the result is proven for $\mathbb{K} = \mathbb{Q}(q)$.

Finally, let $\mathbb{K}$ be arbitrary ****containing** $\mathbb{Q}(q)$******. Then we can construct the modules $L(\lambda)^{\mathbb{Q}(q)}$ and $\tilde{L}(\lambda)^{\mathbb{Q}(q)}$. Then if q is transcendental over $\mathbb{Q}$, the proof above shows that these modules are the same. Now we do the base extension $V^{\mathbb{Q}(q)} \mapsto V^{\mathbb{Q}(q)} \otimes_{\mathbb{Q}(q)} \mathbb{K}$, where we have $\mathbb{Q}(q) \subseteq \mathbb{K}$ a subfield. But the point is that when you do the base extension, things play well: suppose that $\mathbb{F} \subseteq \mathbb{E}$ is a field extension and V is a simple module over an algebra $R_\mathbb{F}$ over $\mathbb{F}$, and that $\operatorname{End}_{R_\mathbb{F}}(V) = \mathbb{F}$. Then $V \otimes_\mathbb{F} \mathbb{E}$ is still irreducible. So therefore $L(\lambda)^{\mathbb{Q}(q)} \otimes_{\mathbb{Q}(q)} \mathbb{K}$ is an irreducible module, and hence $L(\lambda)^\mathbb{K}$, but $\tilde{L}(\lambda)^{\mathbb{Q}(q)} \otimes_{\mathbb{Q}(q)} \mathbb{K} = \tilde{L}(\lambda)^\mathbb{K}$. □

Corollary 34.2 *If q is transcendental, then the category $\mathscr{F}$ of finite-dimensional U-modules is semisimple.*

Proof: It is sufficient to consider the category $\mathscr{F}_1$ of modules of type 1. In fact, it is sufficient to show that there is no nontrivial extension of two irreducible modules $L(\lambda)$ and $L(\mu)$. So suppose that we have an exact sequence

$$0 \to L(\mu) \to M \to L(\lambda) \to 0$$

and we want to show that it splits. There are $3 + \epsilon$ cases:

1. $\mu = \lambda$. Then $\dim M_\lambda = 2$, but the module is semisimple over U^0, so $M_\lambda = \mathbb{K}v_\lambda \oplus \mathbb{K}v'_\lambda$, where v_λ is the highest weight vector of $L(\mu) = L(\lambda) \hookrightarrow M$, so build the submodule in M_λ generated by v'_λ, and it must be isomorphic to $L(\lambda)$, and so we have the splitting.
2. $\mu < \lambda$, then $\dim M_\lambda = 1$, and in fact we do understand that if $M \neq L(\lambda) \oplus L(\mu)$, then M_λ must generated M. Hence M is a finite-dimensional quotient of $M(\lambda)$, so it must be a quotient of $\tilde{L}(\lambda) = L(\lambda)$, and we get a contradiction.
3. If $\lambda < \mu$, then we go to the dual modules, and reduce to the previous case. We consider

$$0 \to \big(L(\lambda)^*\big)^\omega \to \big(M^*\big)^\omega \to \big(L(\mu)^*\big)^\omega \to 0$$

   The $*$ switches the subs and quotients, but also switches the weight spaces, and the ωs just switch the weight spaces, and so $\big(L(\lambda)^*\big)^\omega = L(\lambda)$.
4. If λ, μ are incomparable, then λ is not a weight of $L(\mu)$, but it is a weight of M, and so $UM_\lambda \cap L(\mu) = \{0\}$ by reducibility over U^0, and hence $M = L(\mu) \oplus UM_\lambda$ is the desired splitting. □

This is a standard style of argument by the way. It works for category $\mathcal{O}$, for Kac-Moody algebras, etc.

Recall that we had an isomorphism of vector spaces $U = U^- \otimes U^0 \otimes U^+$. Then we also had the weight decomposition:

$$U = \bigoplus_{\mu,\nu \in Q^+} U^-_{-\mu} \otimes U^0 \otimes U^+_\nu$$

Corollary 34.3 $\dim U^+_\mu = \dim U^-_{-\mu}$ *by the Cartan involution. Moreover, when q is transcendental:*

$$\dim U^+_\mu = \mathcal{P}(\mu) = \Big|\{\mu = \sum m_\beta \beta \text{ s.t. } m_\beta \in \mathbb{Z}_{\geq 0} \text{ and } \beta \text{ ranges over all of } \Delta^+\}\Big|$$

In the classical case, we have $U = \mathcal{U}\mathfrak{g}$, and Corollary 34.3 follows from the PBW decomposition.

Proof: Let λ be much bigger than μ. Then we know that $L(\lambda) = \tilde{L}(\lambda)$, and moreover $L(\lambda)_{\lambda-\mu} = U^-_{-\mu} v_\lambda$. The idea is that we start at λ, and begin quotienting, but if λ is large enough, then $\lambda - \mu$ is relatively close to λ, and in particular is not quotiented by anything.

But $\dim L(\lambda)_{\lambda-\mu}$ is given by the Weyl character formula, and it is $\mathcal{P}(\mu)$ by the classical theory. □

Finally, we look at the center and the Harish-Chandra formula.

We have $U = \bigoplus U_\mu$, and in particular we have $U_0 \supseteq U^0 = \mathbb{K}[K_\mu]_{\mu\in Q}$. Let Z be the center of U. Then U_0 is precisely the centralizer of U^0, and so $Z \subseteq U_0$. Moreover, if $u \in U_0$, then by the triangular decomposition $u = \sum_{\mathrm{wt}\, I = -\,\mathrm{wt}\, J} a_{I\mu J} F^I K_\mu E^J$. Let $\pi(\mu) = \sum a_{\emptyset\mu\emptyset} K_\mu$. Then this defines a map $U_0 \to U^0$, and this restricts to $\pi : Z \to U^0$, the *Harish-Chandra homomorphism.* If you think about it, this exactly corresponds to what we did in the classical case.

Lemma 34.4 *1. If $z \in Z$, then $z|_{M(\lambda)} = \lambda(\pi(z))id$.*

2. $\pi : Z \to U^0$ is a homomorphism of rings.

What do we mean $\lambda(\pi(z))$? We have $\lambda \in P$, and so we define $\lambda : U^0 \to \mathbb{K}$ by $\lambda(K_\mu) = q^{(\lambda,\mu)}$.

Proof: 2. is immediate from 1. For 1., since z is central, it suffices to look at the highest vector, and $zv_\lambda = \sum a_{\emptyset\mu\emptyset} q^{\lambda,\mu} v_\lambda$. □

Lemma 34.5 *π is injective.*

Proof: Suppose that $\pi(z) = 0$. Then $z|_{L(\lambda)} = 0$, so by semisimplicity $z = 0$ on any finite-dimensional module, but we proved that the annihilator of all finite-dimensional modules is 0. □

Lecture 35 April 19, 2010

Today we will try to prove the analog of Harish-Chandra theorem. We recall what we finished last time: We constructed a map $\pi : Z \to U^0$, and we know that if $z \in Z$, then $z|_{M(\lambda)} = \lambda(\pi(z))\mathrm{id}$. We also know that π is injective. We want to describe the image of the map. As in the classical case, we will get invariant polynomials, under the shifted Weyl group action.

For this, then, it is convenient to consider the shifted map $\gamma_{-\rho} \circ \pi$, where for $\gamma \in P$, we define $\gamma_\nu(K_\mu) = q^{(\nu,\mu)} K_\mu$.

Lemma 35.1 $\gamma_{-\rho} \circ \pi(Z) \subseteq (U^0)^W$.

Proof: We use the same idea as in the classical case. We have already constructed a nontrivial map $\phi_\alpha : M(s_\alpha(\lambda+\rho)-\rho) \to M(\lambda)$. Then:

$$\lambda(\pi(z)) = \big(s_\alpha(\lambda+\rho)-\rho\big)\pi(z) \tag{35.1}$$

$$(\lambda+\rho)\big(\gamma_{-\rho}\circ\pi(z)\big) = s_\alpha(\lambda+\rho)\big(\gamma_{-\rho}\circ\pi(z)\big) \tag{35.2}$$

$$\gamma_{-\rho}\circ\pi(z) = \sum a_\mu K_\mu \tag{35.3}$$

$$s_\alpha\big(\gamma_{-\rho}\circ\pi(z)\big) = \sum b_\mu K_\mu \tag{35.4}$$

$$\sum a_\mu q^{(\lambda+\rho,\mu)} = \sum b_\mu q^{(\lambda+\rho,mu} \tag{35.5}$$

where the last identity is for all $\lambda \in P^+$. But then $a_\mu = b_\mu$. □

Lemma 35.2 *Define* $U^0_{\mathrm{ev}} = \left\{\langle K_\mu\rangle \text{ s.t. } \mu \in 2P\cap Q\right\}$. *Then* $\gamma_{-\rho}\circ\pi(Z) \subseteq \left(U^0_{\mathrm{ev}}\right)^W$.

Proof: Let $\sigma : Q \to \mathbb{Z}/2$. Then recall that we defined a one-dimensional representation $\tilde{\sigma} : U \to \mathbb{K}$ by $\tilde{\sigma}(E_\alpha) = \tilde{\sigma}(F_\alpha) = 0$ and $\tilde{\sigma}(K_\mu) = \sigma_\mu K_\mu$. Then it is clear that $\tilde{\sigma}$ reserves the center, and it commutes with the Harish-Chandra map: $\gamma_{-\rho}\circ\pi\circ\tilde{\sigma} = \tilde{\sigma}\circ\gamma_{-\rho}\circ\pi$. So let $\gamma_{-\rho}\circ\pi(z) = \sum\sum a_\mu K_\mu$, and then $\gamma_{-\rho}\pi(\tilde{\sigma}(z)) = \sum a_\mu\sigma(\mu)K_\mu$. But both of these sums must be W-invariant, and so that implies that $a_\mu = a_{w\mu}$, and also $a_\mu\sigma(\mu) = \sigma(w\mu)a_{w\mu}$. So if we take w to be a reflection s_α, then $s_\alpha(\mu) = \mu - m_\mu(\alpha)\alpha$, where $m_\mu(\alpha) = \frac{2(\mu,\alpha)}{(\alpha,\alpha)}$. So then it's clear that we must have $m_\mu(\alpha) \in 2\mathbb{Z}$, and so $\mu \in 2P$. □

Our aim is the following theorem:

Theorem 35.3 $\gamma_{-\rho}\circ\pi : Z \to \left(U^0_{\mathrm{ev}}\right)^W$ *is an isomorphism.*

To do this, our first goal is to construct an invariant form on U. Recall that $U^{\geq 0} = U^0U^+$ and $U^{\leq 0} = U^-U^0$ are Hopf subalgebras. We will first construct a homomoprhism $U^{\leq 0} \to (U^{\geq 0})^*$ of associative algebras.

Recall that in our notation every element of $U^{\geq 0}$ can be written as $E^I K_\mu$. So we will define the image f_α, k_λ of F_α, K_λ under the above map by:

$$f_\alpha(E^I K_\mu) = \begin{cases} 0 & \text{if } I \neq \alpha \\ \frac{-1}{q_\alpha - q_\alpha^{-1}} & \text{if } I = \alpha \end{cases} \tag{35.6}$$

$$k_\lambda(E^i K_\mu) = \begin{cases} 0 & \text{if } I \neq \emptyset \\ q^{-(\lambda,\mu)} & \text{if } I = \emptyset \end{cases} \tag{35.7}$$

It is then not very difficult to check that this is well-defined as a map $\tilde{U}^{\leq 0} \to (U^{\geq 0})^*$: you must only check that $k_\lambda f_\alpha k_{-\lambda} = q^{-(\lambda,\alpha)}f_\alpha$ and that $k_\lambda k_\mu = k_{\lambda+\mu}$.

So we have obtained a pairing $\tilde{U}^{\leq 0}\otimes U^{\geq 0} \to \mathbb{K}$, which satisfies

$$\langle y_1y_2, x\rangle = \langle y_1\otimes y_2, \Delta x\rangle \tag{35.8}$$

$$\langle y, x_1x_2\rangle = \langle \Delta y, x_2\otimes x_1\rangle \tag{35.9}$$

Notice that in equation 35.9, we have switched the order. To check equation 35.9, it is sufficient to check when y is a generator, and then use equation 35.8, which is by construction.

Now we would like to remove the $\sim$. To do this, we must check that for all $x \in U^{\geq 0}$:

$$\langle u^-_{\alpha\beta}, x\rangle = 0 \tag{35.10}$$

And to check equation 35.10, we recall that $\Delta u^-_{\alpha\beta} = u^-\alpha\beta\otimes K_\alpha^{-r}K^{-1}\beta + 1\otimes u^-_{\alpha\beta}$, and then use the multiplicativity to reduce to the case when x is a generator.

So these are all required calculations, but we skip them.

So therefore we have: $\langle,\rangle : U^{\leq 0}\otimes U^{\geq 0} \to \mathbb{K}$.

Lemma 35.4 *This $\langle , \rangle$ is nondegenerate.*

Proof: The maps behave nicely with respect to weights — everything is graded by the action of the torus. So we will show that $U^- - \mu \otimes U^+_\mu \to \mathbb{K}$ is nondegenerate. These are finite-dimensional spaces, and by the Cartan involution that switches them they are the same dimension. Fix $y \in U^-_\mu$. So it suffices to show: if $\langle y, x\rangle = 0$ for every $x \in U^+_\mu$, then $y = 0$.

We will do this inductively on the order of the weight μ. It is a trivial check on simple roots. So, suppose $\langle y, x\rangle = 0$ for every x. Then we do also have

$$\langle y, E_\alpha x\rangle = \langle y, xE_\alpha\rangle = 0 \tag{35.11}$$

for every $x \in U^+_{\mu-\alpha}$. But one can prove by induction that we have something like this:

$$\Delta y = y \otimes K_\mu^{-1} + \sum_{\alpha\in\Pi} r_\alpha(y) \otimes F_\alpha K_{\mu-\alpha}^{-1} + \cdots = 1 \otimes y + \sum_{\alpha\in\Pi} F_\alpha \otimes r'_\alpha(y) K_\alpha^{-1} + \ldots \tag{35.12}$$

So then from equations 35.11, 35.12, 35.8, and 35.9 we get:

$$\langle r_\alpha(y), x\rangle = \langle r'_\alpha(y), x\rangle = 0 \tag{35.13}$$

Then by the inductive assumption, $r_\alpha(y) = r'_\alpha(y) = 0$.

We leave the following calculation as an exercise:

$$E_\alpha y - yE_\alpha = \frac{K_\alpha r_\alpha(y) - r'_\alpha(y)K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \tag{35.14}$$

So then we see that $yE_\alpha = E_\alpha y$. We claim that this implies $y = 0$. Indeed, look at $L(\lambda)$ with sufficiently large highest weight. Then look at yv_λ: if $y \neq 0$, then $yv_\lambda \neq 0$ by the Character formula. But then yv_λ generated a proper submodule in $L(\lambda)$ by the commutativity, and this is a contradiction.

(Here in principle we used that q is transcendental, as we only proved the description of $L(\lambda)$ in that case. But in fact our description holds more generally when q is not a root of unity.) □

Notice that in the classical case we never have a Hopf pairing like $\langle , \rangle : U^{\leq 0} \otimes U^{\geq 0} \to \mathbb{K}$, because both are cocommutative but neither is commutative. On the other hand, just thinking of them as algebras and then constructing such a pairing from the Killing form leads to Drinfeld's *double construction*, which is a way to define $\mathcal{U}_q\mathfrak{g}$.

In fact, now we will define a map on the whole space using the Killing form. Suppose we have $y \in U^-_{-\nu}$, $y' \in U^-_{-\nu'}$, $x \in U^+_\mu$, $x' \in U^+_{\mu'}$. Then we define the pairing $(,)$ by:

$$\left(yK_\nu K_\lambda x, y'K_{\nu'}K_{\lambda'}x'\right) \stackrel{\text{def}}{=} \langle y', x\rangle\langle y, x'\rangle q^{(2\rho,\nu)} q^{-(\lambda,\lambda')/2} \tag{35.15}$$

Here λ, λ' are another pair of roots. We have assumed that $q^{1/2} \in \mathbb{K}$.

Then equation 35.15 is non-zero only if $\mu = \nu'$ and $\mu' = \nu$. Then it is clear that equation 35.15 defines actually a pairing by linearity on all of $U = \bigoplus U^-_{-\eta} \otimes U^0 \otimes U^+_\xi$.

Lemma 35.5 $\big(\mathrm{ad}(u)a, b\big) = \big(a, \mathrm{ad}\, S(u)\, b\big)$

We will not check this, but the hint is that it is sufficient to check for generators $u = E_\alpha, F_\alpha, K_\alpha$.

Question from the audience: I still don't understand why you have written equation 35.15 as you have, with the K_λs, etc. **Answer:** Multiplicitation by K_λ is clearly invertible, so it doesn't matter. We chose this one to make the exponents on the right-hand side nice.

The pairing in equation 35.15 is not symmetric, but it almost is. We have $(a, b) = q^{(2\rho, \mu-\nu)}(b, a)$, where $a \in U_\nu^- U^0 U_\mu^+$ and $b \in U_\mu^- U^0 U_\nu^+$.

Now we are ready to construct the *matrix coefficients*. Let M be a finite-dimensional U-module, $f \in M^*$, $m \in M$, and $c_{f,m}(u) \stackrel{\text{def}}{=} \langle f, um\rangle$.

Lemma 35.6 *Suppose that* $2P(M) \subseteq Q$*. Then there exists a unique* $u \in U$ *such that* $(u, a) = c_{f,m}(a)\,\forall a \in U$*.*

Proof: Uniqueness is trivial, because the pairing is nondegenerate; the important part is existence. And for this it is sufficient to check for irreducible M, supposing semisimplicity, and its sufficient to check for f, m weight vectors, because then we just do the summing. So assume that $f \in M_\gamma^*$ and $m \in M_\delta$.

Now suppose that $a = yx$, where $y \in U_{-\mu}^-$ and $x \in U^+$. Then in order for $c_{f,m}(a) \neq 0$, we must have $\delta + \nu - \mu + \gamma = 0$. Now we pick up a specific weight $\eta = -2(\delta + \nu)$, and the condition on M implies that $\eta \in Q$. Then by nondegeneracy of the pairing, we do have a $u_0 \in U_{-\nu}^- K_\eta U_\mu^+$ such that $c_{f,m}(yx) = (u_0, yx)$.

Now, actually, we can put any weight in place of K_η. The trick is to check that it is consistent.

$$\langle f, yk_\lambda xm\rangle = q^{(\delta+\nu,\lambda)}\langle f, yxm\rangle \tag{35.16}$$

$$(u_0, yK_\lambda x) = q^{-(\eta,\lambda)/2}(u_0, yx) \tag{35.17}$$

And in order to be consistent, we must have $(\delta + \nu, \lambda) = -(\eta, \lambda)/2$, which we do have.

So what have we done? For any weights μ, ν, we have constructed $(u_0^{\mu,\nu}, a) = c_{f,m}(a)$ for all $a \in U_{-\mu}^- U^0 U_\nu^+$.

But the point is that for any finite-dimensional module, the set of weights for which $c_{f,m}(a) \neq 0$ is finite. So just take $u = \sum u_0^{\mu\nu}$, and we are done and out of time. $\square$

Next time we will prove Theorem 35.3.

Lecture 36 May 21, 2010

Today we prove the *Harish-Chandra theorem*:

Theorem 36.1 $\gamma_{-\rho} \circ \pi : Z \to \big(U_{\text{ev}}^0\big)^W$ *is an isomorphism*

Proof: We already know that $\gamma_{-\rho} \circ \pi$ is injective. Recall how we proved the surgectivity in the classical case: we studied invariant polynomials. We will do something similar this time, and the shift by ρ will be very natural.

Last time, we saw that matrix coefficients come from the universal enveloping algebra: if $\lambda \in P^+$ such that $2\lambda \in Q$, then there exists a unique z_λ such that $(u, z_\lambda) = \mathrm{tr}_{L(\lambda)}(uK_{-2\rho})$. We claim now that $z_\lambda \in Z$. Indeed, consider the following diagram:

$$\begin{array}{ccc} U & \longrightarrow & \mathrm{End}(L(\lambda)) \\ & \searrow & \downarrow {\scriptstyle \phi \mapsto \mathrm{tr}(K_{-2\rho}\phi)} \\ & & \mathbb{K} \end{array}$$

Here U is a U-module under the adjoint action, so that poth arrows are U-module homomorphisms. We will show that the above diagram commutes.

So, let $u \in U$, and $\Delta u = \sum u_i^1 \otimes u_i^2$. Let $\phi \in \mathrm{End}\, L(\lambda)$. Then we want to show:

$$\mathrm{tr}\left(\sum u_i^1\, \phi\, S(u_i^2)\, K_{2\rho}^{-1}\right) = \epsilon(u)\ \mathrm{tr}\left(\phi K_{2\rho}^{-1}\right) \tag{36.1}$$

As usual, it suffices to check on generators. For K_α it is easy. So let $u = E_\alpha$. Then $E_\alpha(\phi) = E_\alpha\phi - K_\alpha\phi K_\alpha^{-1}E_\alpha$. We want to show $\mathrm{tr}\Big(E_\alpha(\phi)K_{2\rho}^{-1}\Big) = 0$. But $(2\rho, \alpha) = (\alpha, \alpha)$, and so:

$$\mathrm{tr}\left(E_\alpha\phi K_{2\rho}^{-1} - q^{(\alpha,\alpha)}K_\alpha\phi E_\alpha K_\alpha^{-1}K_{2\rho}^{-1}\right) = \mathrm{tr}\left(E_\alpha\phi K_{2\rho}^{-1} - K_\alpha\phi K_{2\rho}^{-1}E_\alpha K_\alpha^{-1}\right) = 0 \tag{36.2}$$

So this means that the shifted trace is the correct notion of trace in the quantum case, because the usual trace is not a homomorphism of U-modules, but this shows that the shifted trace is.

So, we have:

$$\epsilon(u)\,(x, z_\lambda) = \Big(\mathrm{ad}(u)\,x, z_\lambda\Big) = \Big(x, \mathrm{ad}\, S(u)\, z_\lambda\Big) \tag{36.3}$$

$$\mathrm{ad}\, S(u)\, z_\lambda = \epsilon(u)\, z_\lambda = \epsilon\Big(S(u)\Big)\, z_\lambda \tag{36.4}$$

$$\mathrm{ad}(u)\, z_\lambda = \epsilon(u)\, z_\lambda \tag{36.5}$$

and this happens if and only if $z_\lambda \in Z$.

So, for each $\lambda \in P^+$ such that $2\lambda \in Q$, we construct such a $z_\lambda \in Z$.

Pick up now an arbitrary torus element $K_\mu \in U^0$. Then we can use the above identity to evaluate **?** under the homomorphism:

$$\Big(K_\mu, z_\lambda\Big) = \Big(K_\mu, \phi(z_\lambda)\Big) = \mathrm{tr}_{L(\lambda)}\, K_{\mu-2\rho} = \sum_{\nu \in P(L(\lambda))} \dim L(\lambda)_\nu\, q^{(\nu,\mu-2\rho)} \tag{36.6}$$

And we do know $(K_\nu, K_\mu) = q^{-(\nu,\mu)/2}$. So let $\pi(z_\lambda) = \sum a_\nu K_\nu$ and we have:

$$\Big(K_\mu, \pi(z_\lambda)\Big) = \sum a_\nu q^{(-\nu,\mu)/2} = \sum \dim L(\lambda)_{\nu'}\, q^{(\nu',\mu-2\rho)} \tag{36.7}$$

But equation 36.7 works for any **?** and so we use Artin's independence of characters, and, recalling the shift, we have:

$$\pi(z_\lambda) = \sum \dim L(\lambda)_{-\nu/2}\, q^{(\nu,\rho)}\, K_\nu \tag{36.8}$$

and if we do the shift we get right of the q, and so:

$$\gamma_{-\rho} \circ \pi(z) = \sum \dim L(\lambda)_{-\nu/2}\, K_\nu = \frac{1}{\operatorname{Stab}_W \lambda} \sum_{w \in W} K_{-w(\lambda)} + \sum_{\mu \lneq \lambda} c_\mu K_{-\mu} \tag{36.9}$$

for some coefficients c_μ. So equation 36.9 is some lower-triangular property. So we can start with 0 and proceed by induction.

So from all this it follows that $\gamma_{-\rho} \circ \pi(z_\lambda)$ spans $\left(U^0_{\text{ev}}\right)^W$, and hence we have surjectivity. □

We think the most remarkable part is that the 2ρ appears quite naturally when we do the quantized trace. We wonder if a similar proof exists in the classical case.

36.1 R-matrices and the group algebra

Now we plan to talk a little bit about R-matrix and the group algebra.

We keep our assumption that q is transcendental. Then we know that our category $\mathscr{F}$ of finite-dimensional $\mathcal{U}_q\mathfrak{g}$ modules is semisimple. So in principle we can pick up two modules, take their tensor product, and decompose it, and it is pretty clear that the decomposition will be the same as in the classical case. But we know that the $L(\lambda)$s over $\mathcal{U}_q\mathfrak{g}$ have the same characters as over $\mathcal{U}\mathfrak{g}$. So this implies that $M_1 \otimes M_2 \cong M_2 \otimes M_1$, and that the decomposition of $M_1 \otimes M_2$ into irreducibles is isomorphic to the classical case. In the classical case, the isomorphism $M_1 \otimes M_2 \cong M_2 \otimes M_1$ is trivial, because if the Hopf algebra is cocommutative then $P : m_1 \otimes m_2 \mapsto m_2 \otimes m_1$ is a homomorphism of algebras. But in the quantum case it is not, so we look for a homomorphism $R : M_1 \otimes M_2 \xrightarrow{\sim} M_2 \otimes M_1$, which clearly depends on the modules.

We will pick $R = P \circ \Theta^f$, where $\Theta^f \in \operatorname{End}_{\mathbb{K}}(M_1 \otimes M_2)$. How to construct Θ^f? It will be some twist of some other element Θ. We start with constructing this other Θ: it will be some formal sum of tensor products in U. In fact, we will have $\Theta \in U^- \hat{\otimes} U^+$, where $\hat{\otimes}$ is some completed tensor product, and it won't matter which because on any given finite-dimensional module the sum will truncate.

So, let $\mu \in Q^+$. Last time we discussed a pairing $\langle , \rangle : U^-_{-\mu} \otimes U^+_\mu \to \mathbb{K}$. So choose a basis $\{y_i^\mu\}$ for $U^-_{-\mu}$ and its dual basis $\{x_i^\mu\}$ for U^+_μ. Then consider the formal expression:

$$\Theta = \sum_\mu \sum_i y_i^\mu \otimes x_i^\mu \tag{36.10}$$

Then we claim that only for finitely many μs does the right-hand side of equation 36.10 act non-trivially. **Question from the audience:** So there is no U^0 component? **Answer:** No. It is going to be later on, in the form of this f.

So what is f? We take $f : P \times P \to \mathbb{K}$, such that — we never remember signs, but think this is right —

$$f(\lambda + \nu, \mu) = q^{-(\nu,\mu)}\, f(\lambda, \mu) \tag{36.11}$$

$$f(\lambda, \mu + \nu) = q^{-(\nu,\lambda)}\, f(\lambda, \nu) \tag{36.12}$$

Then equations 36.11 and 36.12 do not uniquely determine f. We could take $f(\lambda, \mu) = q^{-(\lambda,\mu)}$, but this sometimes requires $\sqrt{q}$ to exists.

Then pick some f, and construct $\tilde{f} : M_1 \otimes M_2 \to M_1 \otimes M_2$ by

$$\tilde{f}(m_1 \otimes m_2) = f(\lambda, \mu)\, m_1 \otimes m_2 \quad m_1 \in (M_1)_\lambda,\ m_2 \in (M_2)_\mu \tag{36.13}$$

Then we set $\Theta^f = \Theta \circ \tilde{f}$.

Proposition 36.2 *$R = P \circ \Theta^f : M_1 \otimes M_2 \to M_2 \otimes M_1$ is an isomorphism of U-modules.*

Proof: As usual, we will skip some calculations.

$$\begin{array}{ccc} M_1 \otimes M_2 & \xrightarrow{\Delta(u)} & M_1 \otimes M_2 \\ \downarrow{\scriptstyle \Theta^f} & & \downarrow{\scriptstyle \Theta^f} \\ M_1 \otimes M_2 & \xrightarrow{\Delta^{\mathrm{op}}(u)} & M_1 \otimes M_2 \end{array}$$

We claim that $\tau : U \to U$ given by $\tau(E_\alpha) = E_\alpha$, $\tau(F_\alpha) = F_\alpha$, and $\tau(K_\alpha) = K_\alpha^{-1}$ is a homomorphism $U \to U^{\mathrm{op}}$. I.e. we define $\Delta^\tau(u) = \tau\Delta\tau(u)$, and we claim that $(\Delta u)\Theta = \Theta\Delta^\tau(u)$.

Let's prove this claim. It is sufficient to check on generators, whence:

$$(K_\lambda \otimes K_\lambda)\Theta_\mu = \Theta_\mu(K_\lambda \otimes K_\lambda) \tag{36.14}$$

$$(E_\alpha \otimes 1)\Theta_\mu + (K_\alpha \otimes E_\mu)\Theta_{\mu-\alpha} = \Theta_\mu(E_\alpha \otimes 1) + \Theta_{\mu-\alpha}(K_\alpha^{-1} \otimes E_\alpha) \tag{36.15}$$

$$(1 \otimes F_\alpha)\Theta_\mu + (F_\alpha \otimes K_\alpha^{-1})\Theta_{\mu-\alpha} = \Theta_\mu(1 \otimes F_\alpha) + \Theta_{\mu-\alpha}(F_\alpha \otimes K_\alpha) \tag{36.16}$$

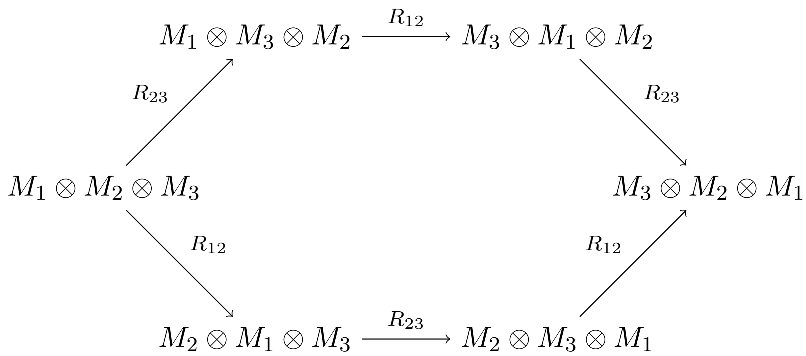
You can check this: it is like checking that the Casimir is central if you do it in coordinates.

We also clim that $\Delta^\tau(u)\tilde{f} = \tilde{f}\Delta^{\mathrm{op}}(u)$ on $M_1 \otimes M_2$. This is very simple: you check on weight vectors and when u is a generator.

Question from the audience: What is Θ_μ? **Answer:** It is a piece. $\Theta = \sum_\mu \sum_i \ldots$, so Θ_μ is the μ piece. **Question from the audience:** So what is the equality $(\Delta u)\Theta = \Theta\Delta^\tau(u)$? Is it in a completion? **Answer:** We'd rather not introduce a completion. We assert it is true on any finite-dimensional module. It ammounts to the calculation above. $\square$

So, we have constructed the *R-matrix*. It isn't quite unique: it depends on a choice of $f : P \times P \to \mathbb{K}$ satisfying equations 36.11 and 36.12.

So what can you do? You can study the braid relations. Say you have three modules M_1, M_2, M_3. Then there are two ways to get an isomorphism $M_1 \otimes M_2 \otimes M_3 \cong M_3 \otimes M_2 \otimes M_1$:



And in fact it doesn't matter which way you go: $R_{23} \circ R_{12} \circ R_{23} = R_{12} \circ R_{23} \circ R_{12}$. In fact, if you have more modules, $M_1 \otimes M_2 \otimes M_3 \otimes M_4$, then of course $R_{12} \circ R_{34} = R_{34} \circ R_{12}$. So the point is that you have a braid group action, but the weird thing: $R_{12}^2 \neq \mathrm{id}$.

Lecture 37 April 23, 2010

We begin by recalling some material from last time, and then restrict our attention to $V = V_{1,1}$ the standard representation of $\mathcal{U}_q\mathfrak{sl}(2)$. Then look at $V \otimes V$, with the basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$. Then we will write down an isomorphism $R : V \otimes V \to V \otimes V$ of $\mathcal{U}_q\mathfrak{sl}(2)$ modules in this basis as the matrix:

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q^{-1} - q & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}$$

Then $R_{i,i+1} = \mathrm{id} \otimes \cdots \otimes R \otimes \cdots \otimes \mathrm{id}$ gives an action of the braid group on $V^{\otimes n}$. Note that in the specialization $q \mapsto 1$, R is the usual flip map.

Now recall the classical situation. Then the symmetric group S_n acts on $V^{\otimes n}$, and the action commutes with the $\mathfrak{sl}(2)$-action. Indeed, S_n is the centralizer of $\mathcal{U}\mathfrak{sl}(2)$, and in fact this works for any $\mathfrak{sl}(m)$. This gives *Schur-Weyl duality*: we can decompose $V^{\otimes n} = \bigoplus V_\nu \otimes W_\nu$, where ν ranges over Young diagrams with n boxes and not more than m rows, W_ν is the corresponding representation of S_n, and V_ν is the corresponding representation of $\mathfrak{sl}(m)$. The point is that S_n acts only on W_ν and $\mathfrak{sl}(m)$ acts only on V_ν — since we have a *dual pair* S_n and $\mathfrak{sl}(m)$ (each centralizes the other), actually $\mathbb{K}[S_n] \otimes \mathcal{U}\mathfrak{sl}(m)$ acts on $V^{\otimes n}$, and the decomposition is as modules of these. Notice that the highest weights of V_ν can be read from the difference of the lengths of the rows.

Now, the point is that S_n has its usual presentation

$$\langle \sigma_1, \dots, \sigma_{n-1} \text{ s.t. } \sigma_i\sigma_j = \sigma_j\sigma_i,\, j \neq i \pm 1\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma i + 1\ \sigma_i^2 = 1\rangle \tag{37.1}$$

We will see that something similar happens in the quantum case.

Now, the Rs satisfy $R^2 = (q^{-1} - q)R + 1$, and so we define $\bar{\sigma}_i = qR^{-1}_{i,i+1}$. Then these satisfy the *braid relations*, which are the first two relations of equation 37.1, and also

$$\left(\bar{\sigma}_i - q^2\right)\left(\bar{\sigma}_i + 1\right) = 0 \tag{37.2}$$

Then we define the *Hecke algebra* $\mathscr{H}_q \overset{\text{def}}{=} \mathbb{K}[\bar{\sigma}_1, \dots, \bar{\sigma}_{n-1}]$/braid relations and equation 37.2. It is straightforward to check that $\mathscr{H}_q$ and $\mathbb{K}[S_n]$ have the same dimension.

Theorem 37.1 *$\mathcal{U}_q\mathfrak{sl}(m)$ and $\mathscr{H}_q$ form a dual pair in $V^{\otimes n}$, i.e. each centralizes the other.*

37.1 Quantum function algberas

Now we let $\mathfrak{g}$ be a semisple Lie algebra. We define $\mathbb{K}_q[G]$ to be the Hopf algebra of matrix coefficients $c_{f,m}$ for all finite-dimensional representations of $\mathcal{U}_q\mathfrak{g}$. If you want a non-simply-connected one, you take the corresponding lattice and the subcategory.

Now, how to know that $\mathbb{K}_q[G]$ is closed under multiplication? Let M_1, M_2 be finite-dimensional representations of $\mathbb{U}_q\mathfrak{g}$, and look at $M_1 \otimes M_2$. Then for $f_i \in M_i^*$ and $m_i \in M_i$, and for $\Delta u = \sum u_j^1 \otimes u_j^2$, then:

$$c_{f_1\otimes f_2, m_1\otimes m_2}(u) = \left\langle f_1 \otimes f_2, \sum u_j^1 m_1 \otimes u_j^2 m_2 \right\rangle = \sum \left\langle f_1, u_j^1 m_1 \right\rangle \left\langle f_2, u_j^2 m_2 \right\rangle = \\ = \sum c_{f_1,m_1}(u_j^1)\, c_{f_2,m_2}(u_j^2) \tag{37.3}$$

So this tells you that $c_{f_1,m_1}c_{f_2,m_2} = c_{f_1\otimes f_2, m_1\otimes m_2}$, and the multiplication in $\mathbb{K}_q[G]$ corresponds to the comultiplication in $\mathcal{U}_q\mathfrak{g}$.

So let M_1, M_2 be finite-dimensional representations of $\mathcal{U}_q\mathfrak{g}$, with bases $\{e_1^1, \dots, e_n^1\}$ and $\{e_1^2, ;e_m^2\}$. Then we define the coefficients $R^{pq}ij$ by $R(e_i^1 \otimes e_j^2) = \sum R_{ij}^{pq}\, e_p^2 \otimes e_q^1$, where R is from yesterday. Then for the matrix coefficients c_{ij}^1, c_{ij}^2 corresponding to our basis, we see that:

$$\sum_{pq} R_{pq}^{rs} c_{qi}^1 c_{pj}^2 = \sum_{pq} R_{ij}^{pq} c_{rp}^2 c_{sq}^1 \tag{37.4}$$

Question from the audience: Remind me what all these things are? **Answer:** R_{pq}^{rs} are complex numbers, and c_{ij}^a are complex-valued functions of u, i.e. elements of $\mathbb{K}_q[G]$.

Now, what we do know about $\mathcal{U}_q\mathfrak{sl}(2)$ is that the representation theory is generated by the two-dimensional defining representation, and so the algebra $\mathbb{K}_q[\mathrm{SL}(2)]$ of matrix coefficients is generated by the matrix coefficients for this defining rep V. I.e. we know generators for $\mathbb{K}_q[\mathrm{SL}(2)]$. Indeed, picking our basis, we have the generators $c_{11}, c_{12}, c_{21}, c_{22}$, and we know the R-matrix is $R^{11}_{11} = R^{22}_{22} = q^{-1}$, $R^{21}_{12} = R^{12}_{21} = 1$, and $R^{21}_{21} = q^{-1} - q$, and all the rest are 0. And so by just applying equation 37.4, we get almost all the relations for $\mathcal{K}_q[\mathrm{SL}(2)]$. Indeed, the only one we miss is the determinant.

So how to get the determinant relation that $\det_q = c_{11}c_{22} - qc_{21}c_{12} = 1$? It is that inside $V \otimes V$ there is a trivial representation $\mathbb{K}(e_1 \otimes e_2 - qe_2 \otimes e_1)$, which is the quantum analog of $V \otimes V = \mathcal{S}^2 V \oplus \Lambda^2 V$. Indeed, we see that:

$$(c_{11}e_1 + c_{12}e_2) \otimes (c_{21}e_1 + c_{22}e_1) - q(c_{21}e_1 + c_{22}e_2)(c_{11}e_1 + c_{12}e_2) = \\ = (c_{11}c_{22} - qc_{21}c_{12})e_1 \otimes e_2 + (c_{12}c_{21} - qc_{22}c_{11})e_2 \otimes e_1 \quad (37.5)$$

And so wee see that $c_{11}c_{22} - qc_{21}c_{12} = 1$ in our new definition of $\mathbb{K}_q[\mathrm{SL}(2)]$. We still do not know why these conditions (the ones from long ago) are all the relations.

Lecture 38 April 26, 2010

38.1 Kashiwara crystal bases

We restrict our attention to the case when q is transcendental over $\mathbb{Q}$. Actually, we will work over the transcendental extension $\mathbb{K} = \mathbb{Q}(q)$. Inside this we have a ring $A \subseteq \mathbb{Q}(q)$, which consists of all fractions $\frac{f(q)}{g(q)}$, where $f, g \in \mathbb{Q}[q]$ are polynomials and $g(0) \neq 0$. A is a local ring with unique maximal ideal (q), and $\mathbb{K}$ is the field of fractions of A.

So, let M be a finite-dimensional $\mathcal{U}_q\mathfrak{g}$ module, and pick $\alpha \in \Pi$. Let $U^\alpha = \mathcal{U}_{q_\alpha}\mathfrak{sl}(2)$; this is an $\mathfrak{sl}(2)$ inside $\mathcal{U}_q\mathfrak{g}$. Then over U^α we completely understand the structure of M. We assume it is type-1. Then we have $M = \bigoplus L(n_i)$, where each $L(n_i)$ is a chain of length n_i.

Then we need the divided powers $F^{(j)}_\alpha = F^j_\alpha/[j]_{q_\alpha}!$. Then we see that for each $\alpha \in \Pi$, any $x \in M$ has a unique decomposition as $x = \sum F^{(j)}_\alpha x_j$, where $E_\alpha x_j = 0$ for each j. It's clear that this decomposition doesn't depend on anything — no basis. If x_j is a weight vector of weight μ, then $x_j \in M_{\mu + j\alpha}$.

Then we define the *Kashiwara operators*:

$$\tilde{F}_\alpha x = \sum F^{(j+1)}_\alpha x_j \quad (38.1)$$
$$\tilde{E}_\alpha x = \sum F^{(j-1)}_\alpha x_j \quad (38.2)$$

Then it is clear that $\tilde{F}_\alpha \tilde{E}_\alpha = x - x_0$, where x_0 is the only vector killed by $\tilde{E}_\alpha$, and $\tilde{E}_\alpha \tilde{F}_\alpha x = x - F^{(r)}_\alpha x_r$. Here $r = (\mu, \alpha^\vee)$, and $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

Then we will now define a lattice over A, which is invariant under these Kashiwara operators. In particular, we say that an A-module $\tilde{M}$ in M is an *admissible lattice* if:

1. $\tilde{M} \otimes_A \mathbb{K} = M$
2. $\tilde{M} = \bigoplus \tilde{M}_\lambda$, where $\tilde{M}_\lambda = \tilde{M} \cap M_\lambda$
3. $\tilde{M}$ is invariant with respect to $\tilde{E}_\alpha, \tilde{F}_\alpha$.

We remark that these operators play well under homomorphisms. If $\phi : M \to N$ is a homomorphism of U-modules, then $\tilde{F}_\alpha, \tilde{E}_\alpha$ commute with ϕ.

Therefore, given $\phi : M \xrightarrow{\sim} N$, we see that $\phi(\tilde{M})$ is admissible iff $\tilde{M}$ is admissible. Moreover, $M_1 \oplus M_2 \subseteq \tilde{M}_1 \oplus \tilde{M}_2$ is admissible iff each $\tilde{M}_i$ is admissible.

So, we can consider $\tilde{M}/q\tilde{M}$, and since $\times q : M \to M$ is a U-isomorphism, then $\tilde{E}_\alpha, \tilde{F}_\alpha$ are well-defined on $\tilde{M}/q\tilde{M}$.

So, we define a *crystal base* of M to be a pair: an admissible lattice $\tilde{M}$ and a basis B of $\tilde{M}/q\tilde{M}$ over A. We require the following conditions:

1. $B = \bigcup_{\lambda \in P(M)} B_\lambda$, where $B_\lambda = B \cap (\tilde{M}_\lambda / q\tilde{M}_\lambda)$.
2. $\tilde{F}_\alpha B \supseteq B \cup \{0\}$ and $\tilde{E}_\alpha B \supseteq B \cup \{0\}$, for each $\alpha \in \Pi$.
3. If $b_1, b_2 \in B$, then $b_2 = \tilde{E}_\alpha b_1$ iff $\tilde{F}_\alpha b_2 = b_1$.

So the idea is that to describe the action of $\tilde{E}, \tilde{F}$ in a crystal base, you need only to use a *crystal graph*, which explains how $\tilde{F}$ acts. The vertices for the graph are the elements of B, and we have a directed edge colored by the root α for the $\tilde{F}_\alpha$ action: $b_1 \xrightarrow{\alpha} b_2$ iff $\tilde{F}_\alpha b_1 = b_2$. So for example, for $\mathfrak{sl}(2)$ we have only one color, and the irreducible graphs look like chains $\bullet \to \bullet \to \cdots \to \bullet$.

We now spend some time proving that these graphs are essentially unique.

Theorem 38.1 *Let $v_\lambda \in L(\lambda)$ be a highest weight vector. Pick up all possible non-zero guys of the form $\tilde{F}_{\alpha_1} \cdots \tilde{F}_{\alpha_r} v_\lambda$. Then let $\tilde{L}(\lambda)$ be spanned by these $\{\tilde{F}_{\alpha_1} \cdots \tilde{F}_{\alpha_r} v_\lambda\}$, and let $B(\lambda) = \{\tilde{F}_{\alpha_1} \cdots \tilde{F}_{\alpha_r} \bar{v}_\lambda \in \tilde{M}/q\tilde{M}$ s.t. $\tilde{F}_{\alpha_1} \cdots \tilde{F}_{\alpha_r} \bar{v}_\lambda \neq 0\}$, where we write $\bar{v}_\lambda$ to remember that this is already an element of the quotient. Then $(\tilde{L}(\lambda), B(\lambda))$ is a crystal base for $L(\lambda)$.*

Question from the audience: In the classical case, I could start with a highest weight vector and try to make a graph? **Answer:** Yes, you can try, but it is not going to work: the idea is that as soon as you introduce the parameter q, then you can look at leading terms, etc. — in $\tilde{M}$, q acts as 0. Whereas in the classical case, $F_1 F_2 v_\lambda$ and $F_2 F_1 v_\lambda$ may have dependency. Here, they are either the same, or one is 0.

The idea is that we will construct certain representations for which we can prove Theorem 38.1 by hand, and then also prove that crystal bases play well with tensor products and decomposition into sums of irreducibles. We say that λ is *nice* if Theorem 38.1 holds for λ.

We begin with some examples: Theorem 38.1 is true for $\mathfrak{sl}(2)$, more or less by definition. On the other hand, we can take M, and pick any $\mathfrak{sl}(2)$, and if we prove that for any $\mathfrak{sl}(2)$ we have a crystal base, then it is a crystal base for the whole thing, because if you check the properties 1–3 above for each α, then you are done.

Example 38.2 (Minuscule representation) Classically, the idea is that the weights are all on one orbit: $L(\lambda)$ is *minuscule* if $P(L(\lambda)) = W\lambda$. For example, if you pick up $\mathfrak{sl}(n)$, then every fundamental representation is minuscule. In general, the number of minuscule representations is exactly the number of elements in the quotient P/Q. Because what you do is pick up a class in P/Q, and then pick a minimal dominant weight in that class, and that gives you all the minuscule representations. So we say that a minimal dominant weight in $\mu + Q$ is a *minuscule weight.*

So, let $\mu \in P(\lambda)$ and α any simple root. Then $(\mu, \alpha^\vee) = 0$ or ± 1. The idea is that in a minuscule representation, there is no problem to choose a basis, because the eigenbasis is unique up to proportionality if we require that W act well. So we construct the following basis: $\{x_\mu\}$, with $x_\lambda = v_\lambda$, and

$$E_\alpha x_\mu = \begin{cases} x_{\mu+\alpha}, & (\mu, \alpha^\vee) = -1 \\ 0, & (\mu, \alpha^\vee) = 0, 1 \end{cases} \tag{38.3}$$

$$F_\alpha x_\mu = \begin{cases} x_{\mu-\alpha}, & (\mu, \alpha^\vee) = 1 \\ 0, & (\mu, \alpha^\vee) = 0, 11 \end{cases} \tag{38.4}$$

So we have $\tilde{E}_\alpha = E_\alpha$ and $\tilde{F}_\alpha = F_\alpha$, and we claim that $\sum A x_\mu$ is an admissible lattice and $\{\bar{x}_\mu\}$ is a crystal base.

For example, for the standard representation of $\mathfrak{sl}(3)$, the picture is $\bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet$. $\Diamond$

Then the idea is that we will start with the minuscule representations, and start tensoring them. This is good enough for $\mathfrak{sl}(n)$, because the fundamental representations include everything. But in general it is not enough: there are no nontrivial minuscule representations for E_8, G_2.

Example 38.3 Suppose that λ is the dominant short root. Then $P(L(\lambda)) = W\lambda \cup \{0\}$. If all the roots are the same length, this is the adjoint representation. We define Δ_{sh} to be the set of all short roots, and Π_{sh} to be the set of simple short roots. Then we will construct our basis to be:

$$\{x_\beta \text{ s.t. } \beta \in \Delta_{\text{sh}}\} \cup \{h_\alpha \text{ s.t. } \alpha \in \Pi_{\text{sh}}\}$$

For example, for $\mathfrak{sl}(n)$, this is clearly a basis for the adjoint representation. In general, we have the relations:

$$E_\alpha x_\alpha = 0 \qquad F_\alpha x_\alpha = h_\alpha \qquad F_\alpha^{(2)} x_\alpha = x_{-\alpha} \qquad F_\alpha^{(3)} x_\alpha = 0 \tag{38.5}$$

and if $\beta \neq \pm\alpha$, then $(\beta, \alpha^\vee) = 0, \pm 1$, and so $E_\alpha x_\beta, F_\alpha x_\beta$ are the same as in the minuscule case:

$$E_\alpha \left(h_\beta + \frac{(\beta, \alpha^\vee)}{[2]_{q_\alpha}} h_\alpha \right) = 0 \qquad F_\alpha \left(h_\beta + \frac{(\beta, \alpha^\vee)}{[2]_{q_\alpha}} h_\alpha \right) = 0 \tag{38.6}$$

For the second one, you start with $h_\alpha = F_\alpha x_\alpha$, and work with it: $E_\alpha h_\alpha = E_\alpha F_\alpha x_\alpha = \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} x_\alpha = (q_\alpha + q_\alpha^{-1})^{-1} x_\alpha$, because $K_\alpha x_\alpha = q_\alpha^2 x_\alpha$. And $E_\alpha h_\beta = E_\alpha F_\beta x_\beta = F_\beta E_\alpha x_\beta$, and remember that $E_\alpha x_\beta = 0$ if $(\alpha, \beta) = 0$ or $x_{\beta+\alpha}$ otherwise, and then when we apply F_β, we subtract β, and so $F_\beta E_\alpha x_\beta = x_\alpha$ or 0.

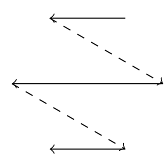
Then the claim is that from this, we get something nice: by a little calculation, $\frac{(\beta,\alpha^\vee)}{[2]_\alpha} = \frac{(\beta,\alpha^\vee)}{q_\alpha^2+1} q_\alpha \in qA$, and so:

$$0 = \tilde{E}_\alpha \left(\bar{h}_\beta + \frac{(\beta,\alpha^\vee)}{[2]_{q_\alpha}} \bar{h}_\alpha \right) = \tilde{E}_\alpha \bar{h}_\beta \tag{38.7}$$

So we claim that $\sum_{\alpha\in\Pi_{\rm sh}} A h_\alpha + \sum_{\mu\in\Delta_{\rm sh}} x_\mu$ is an admissible lattice, and $\{\bar{x}_\mu, \bar{h}_\alpha\}$ is a crystal base.

Oh, notice that here we see the difference between B_n and C_n. In B_n, we get the standard representation, and for C_n the second exterior power of the standard representation. $\lozenge$

Question from the audience: Can you draw the graph for the seven-dimensional representation of G_2, please? **Answer:** Sure:



Lecture 39 April 28, 2010

We continue to work with crystal bases, trying to prove Theorem 38.1. There are certain things we have to check.

Lemma 39.1 *Let $\tilde{M}$ be an admissible lattice, and suppose we have a weight vector $x = \sum F_\alpha^{(j)} x_j$. If $x \in \tilde{M}_\mu$, then $x_j \in \tilde{M}$ for all j. Moreover, if $\tilde{E}_\alpha x \in q\tilde{M}$, then $x_j \in q\tilde{M}$ for all $j > 0$.*

So, what are we saying? Each of these canonical elements — the E_α highest vectors — we are saying that they are in $\tilde{M}$, so that everything is involutive.

Proof: $x - \tilde{F}_\alpha \tilde{E}_\alpha x = x_0 \in \tilde{M}$, and so we do now the increasing operator: $\tilde{E}_\alpha x = \sum F_\alpha^{(j-1)} x_j \in \tilde{M}$, and so $x_1 \in \tilde{M}$, and you proceed by induction.

For the second statement, the lattice $q\tilde{M}$ is admissible, so $\tilde{E}_\alpha x = \sum F_\alpha^{(j-1)} \tilde{E}_\alpha x_j$, so from this we claim that we have $\tilde{E}_\alpha x_j \in q\tilde{M}$, but this is just for $j > 0$, and so $x_j \in q\tilde{M}$ and we are done. $\square$

Now we look at a description of highest weight vectors.

Basically, we are interested in the quotient $\tilde{M}/q\tilde{M}$. Suppose that $S \subseteq \tilde{M}/q\tilde{M}$, and we define $\mathrm{HW}(S) \stackrel{\rm def}{=} \{x \in S \text{ s.t. } \tilde{E}_\alpha x = 0 \ \forall \alpha \in \Pi\}$.

Lemma 39.2 *Let $(\tilde{M}, B)$ be a crystal base. Then:*

1. *Any $b \in B$ can be obtained by the α operators from some highest weight, i.e. it can be written as*

$$b = \tilde{F}_{\alpha_1} \cdots \tilde{F}_{\alpha_r} b'$$

for some $b' \in \mathrm{HW}(B)$.

2. *For each $\lambda \in P(M)$, the corresponding weight space* $\mathrm{HW}(\tilde{M}_\lambda / q\tilde{M}_\lambda)$ *is generated as a vector space by* $\mathrm{HW}(B_\lambda)$*. For this, remember that* $B = \bigsqcup B_\lambda$ *by definition.*

3. *If* $\mathrm{HW}(B_\lambda) \neq \emptyset$*, then λ is dominant.*

Proof: 1. We can always find some sequence so that $b' = \tilde{E}_{\alpha_r} \cdots \tilde{E}_{\alpha_1} b \in \mathrm{HW}(B)$. And on B, $\tilde{E}_\alpha$ and $\tilde{F}_\alpha$ are sort of inverses: if they are both not zero, they are. So $b = \tilde{F}_{\alpha_1} \cdots \tilde{F}_{\alpha_r} b'$.

2. Suppose that $x \in \mathrm{HW}(\tilde{M}_\lambda / q\tilde{M}_\lambda)$. Then $x = \sum c_b c$. Now apply $\tilde{E}_\alpha$. Some of the bs are killed, but those that are not killed are distinct. So throw away the zero ones: $\tilde{E}_\alpha x = \sum c_b \tilde{E}_\alpha b$, and the non-zero $\tilde{E}_\alpha b$ are nonzero and so linearly independent. But therefore $c_b = 0$ for these.

3. Finally, the condition that λ is dominant is just a condition on $\mathfrak{sl}(2)$, so what we have to do is check that $(\lambda, \alpha^\vee) \geq 0 \ \forall \alpha \in \Pi$, but we know that this is true for $\mathfrak{sl}(2)$. □

Now, recall from last time that λ is *nice* if $L(\lambda)$ has a crystal base $(\tilde{L}(\lambda), B(\lambda))$, where what we do is pick up a highest weight vector v_λ, and define $\tilde{L}(\lambda) = \sum A\tilde{F}_{\alpha_1} \cdots \tilde{F}_{\alpha_r} v_\lambda$, and $B(\lambda) = \{\tilde{F}_{\alpha_1} \cdots \tilde{F}_{\alpha_r} v_\lambda\}$. Last time we showed that all weights for $\mathfrak{sl}(2)$ and all small weights for $\mathfrak{g}$ are nice.

Proposition 39.3 *If λ is nice, then* $\mathrm{HW}(B) = B_\lambda$*. Moreover, if $(\tilde{L}, B)$ is some other crystal base, then there exists $a \in \mathbb{K}$ such that $\tilde{L} = a\tilde{L}(\lambda)$ and $B = aB(\lambda)$.*

Proof: Let us prove first the first sentence. See, if you start with the highest weight vector, you can get all vectors, but by lemma 39.2.2, we can go back. Now for the second sentence, we know that $B_\lambda = \{\overline{av_\lambda}\}$. And $\tilde{L} \ni av_\lambda$. Because it is obvious that we must have a unique element in $\tilde{L}$ of weight λ. So if λ is nice, then by definition any other element of the base can be obtained by applying the operators, and since the lattice is invariant under **?**, then $\tilde{L} = \sum A\tilde{F}_{\alpha_1} \cdots \tilde{F}_{\alpha_r} \overline{av_\lambda}$. □

Question from the audience: Can you remind me what $\mathbb{K}$ is? **Answer:** q is transcendental and $\mathbb{K} = \mathbb{Q}(q)$.

So what can go wrong? We start with the highest vector, and then hope that we can get everywhere, and if we can then the crystal base is unique up to endomorphisms. In fact, Proposition 39.3 applies to any representation whose irreducible components are nice. See, from the beginning, the construction of the base is not unique, because you have to pick a highest weight vector, but up to rescaling the highest weight vector it is unique.

Theorem 39.4 *Let M be a finite-dimensional representation of U such that $M = \bigoplus L(\lambda_i)$ and all λ_i are nice. Suppose that $(\tilde{M}, B)$ is a crystal base for M. Then there exists $\phi_i : L(\lambda_i) \to M$ such that $\tilde{M} = \bigoplus \phi_i\big(\tilde{L}(\lambda_i)\big)$ and $B = \bigsqcup \phi_i\big(B(\lambda_i)\big)$.*

Proof: We use induction on the number of components. You order the components so that λ_1 is not less than λ_2 and so on in the standard order: i.e. we let λ_1 be weakly maximal. Then we do know that B_{λ_1} contains only highest weight vectors: $\mathrm{HW}(B_{\lambda_1}) = B_{\lambda_1}$. Now pick up $b_1 \in B_{\lambda_1}$ and start generating. Then $b_1 = \overline{v_{\lambda_1}}$ for some highest weight vector with highest weight λ_1, and a choice of such a vector determines $\phi_1 : L(\lambda_1) \to M$. But then we do know that $\{\tilde{F}_{\alpha_r} \cdots \tilde{F}_{\alpha_1} b_1\} \subseteq B$, and by uniqueness theorem this is $\phi_1(B(\lambda_1)$. So we take $B \smallsetminus \phi_1(B(\lambda_1))$, and take the sublattice generated by it. We take the natural map $\tilde{M} \to \tilde{M}/q\tilde{M}$, and let $\tilde{N}$ be the preimage of the sublattice generated by $b \in B \setminus B(\lambda_1)$. Then you can proceed by induction. □

So for $\mathfrak{sl}(2)$ we know everything: every irrep is nice, so every rep is.

Now we come to the interesting part, which is to study the behavior of crystal bases under tensor products.

Theorem 39.5 *Let $(\tilde{M}_1, B_1)$ and $(\tilde{M}_2, B_2)$ be the crystal bases for M_1, M_2. Then $\big(\tilde{M}_1 \otimes \tilde{M}_2, B_1 \otimes B_2\big)$ is a crystal base for $M_1 \otimes M_2$.*

But here in fact this is not the usual tensor product: $M_1 \otimes M_2$ is the Kashiwara tensor product, in which we take not the usual comultiplication but the twisted comultiplaction

$$\Delta' = (\omega\tau \otimes \omega\tau)\Delta(\omega\tau)$$

in which ω is the Cartan involution and τ is a certain antiautomorphism introduced earlier. In formulas:

$$\Delta'(E_\alpha) = E_\alpha \otimes K_\alpha^{-1} + 1 \otimes E_\alpha \tag{39.1}$$
$$\Delta'(F_\alpha) = F_\alpha \otimes 1 + K_\alpha \otimes F_\alpha \tag{39.2}$$
$$\Delta'(K_\mu) = K_\mu \otimes K_\mu \tag{39.3}$$

and in fact we would like formulas for the divided powers

$$\Delta'(E_\alpha^{(r)}) = \sum_{i=0}^{r} q_\alpha^{-i(r-i)} E_\alpha^{(i)} \otimes E_\alpha^{(r-i)} K_\alpha^{-i} \tag{39.4}$$
$$\Delta'(F_\alpha^{(r)}) = \sum_{i=0}^{r} q_\alpha^{-i(r-i)} F_\alpha^{(i-r)} K_\alpha^{i} \otimes F_\alpha^{(i)} \tag{39.5}$$

Question from the audience: So why don't I just use the original tensor product, and a different definition of crystal base? We started with a definition that was asymmetric between E_α and F_α.
Answer: Well, then you would have to work with lowest vectors rather than highest vectors, and probably with q^{-1} rather than q. But we will do it the way Kashiwara did it.

All of equations 39.1 to 39.5 are easily checked by induction.

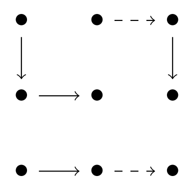
In any case, if $b \in B_1$ or B_2, define $e_\alpha(b) \stackrel{\text{def}}{=} \max\{r \text{ s.t. } \tilde{F}_\alpha^r b \neq 0\}$, and $f_\alpha(b) = \max\{r \text{ s.t. } \tilde{E}_\alpha^r b \neq 0\}$. Then the theorem not only tells you that you have a crystal base, but it also tells you the recipe by how the operators act. They act by simple formulas:

$$\tilde{F}_\alpha(b \otimes b') = \begin{cases} \tilde{F}_\alpha b \otimes b', & \text{if } f_\alpha(b) > e_\alpha(b') \\ b \otimes \tilde{F}_\alpha b', & \text{if } f_\alpha(b) \leq e_\alpha(b') \end{cases} \tag{39.6}$$

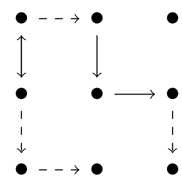
$$\tilde{E}_\alpha(b \otimes b') = \begin{cases} \tilde{F}_\alpha b \otimes b', & \text{if } f_\alpha(b) \geq e_\alpha(b') \\ b \otimes \tilde{F}_\alpha b', & \text{if } f_\alpha(b) < e_\alpha(b') \end{cases} \tag{39.7}$$

Question from the audience: Should we switch the definitions of e and f? Shouldn't e go with E? **Answer:** We think this is correct. We match our notation to [7].

Example 39.6 Let $\mathfrak{g} = \mathfrak{sl}(3)$. If you take the standard representation V and multiply it by itself — the crystal basis for V is terribly simple $\bullet \longrightarrow \bullet \dashrightarrow \bullet$. Then for $V \otimes V$ we get:



whereas for $V \otimes V^*$ you get:



These show hands-on the decompositions $V^{\otimes 2} = \mathcal{S}^2(V) \oplus \bigwedge^2(V)$ and $V \otimes V^* = \mathfrak{sl}(3) \oplus 1$. $\diamondsuit$

Finally, let's mention that the proof can be reduced to the $\mathfrak{sl}(2)$ case. Because $(\tilde{M}, B)$ is a crystal base iff it is for each U^α. And so it suffices to consider $L(m) \otimes L(n)$. And this case can be restricted further, to the case of $L(1) \otimes L(n)$, because what you do is you realize $L(m+1)$ as a direct summand of $L(m) \otimes L(1)$.

Next time is our last day; we will try to finish the proof.

Lecture 40 April 30, 2010

Today we prove the theorem we formulated last time:

Theorem 40.1 *Let $(\tilde{M}_1, B_1)$, $(\tilde{M}_2, B_2)$ be crystal bases. Then $(\tilde{M}_1 \otimes \tilde{M}_2, B_1 \otimes B_2)$ is a crystal base for $M_1 \otimes M_2$, where the tensor product is given by the actions:*

$$\tilde{F}_\alpha(b_1 \otimes b_2) = \begin{cases} \tilde{F}_\alpha b_1 \otimes b_2, & f_\alpha(b_1) > e_\alpha(b_2) \\ b_1 \otimes \tilde{F}_\alpha b_2, & f_\alpha(b_1) \le e_\alpha(b_2) \end{cases} \tag{40.1}$$

$$\tilde{E}_\alpha(b_1 \otimes b_2) = \begin{cases} \tilde{F}_\alpha b_1 \otimes b_2, & f_\alpha(b_1) \ge e_\alpha(b_2) \\ b_1 \otimes \tilde{F}_\alpha b_2, & f_\alpha(b_1) < e_\alpha(b_2) \end{cases} \tag{40.2}$$

Proof: So in principle it is sufficient to prove it in $\mathfrak{sl}(2)$, because we have $\mathcal{U}_{q_\alpha}\mathfrak{sl}(2) \hookrightarrow \mathcal{U}_q\mathfrak{g}$, and q is transcendental. We will begin with the case when $M_1 = L(1)$ and $M_2 = L(m)$, and take $x \in L(1)$ and $y \in L(m)$ highest vectors. Then we have $M_1 \otimes M_2 = L(m+1) \oplus L(m-1)$, where the highest vectors are $z_0 = x \otimes y$ and $z_1 = x \otimes Fy - q^m[m]Fx \otimes y$ — the latter is a simple calculation. Moreover, $q^m[m] \in qA$.

Second, using the formulas equations 39.1 to 39.5, which are conveniently not erased, we see that:

$$F^{(r)}z_0 = \begin{cases} q^r x \otimes F^{(r)}y + Fx \otimes F^{(r-1)}y, & 0 < r < m+1 \\ Fx \otimes F^{(m)}y, & r = m+1 \end{cases} \tag{40.3}$$

$$F^{(r)}z_1 = [r+1]q^r x \otimes F^{(r+1)}y - q^m[m-r]Fx \otimes F^{(r)}y \tag{40.4}$$

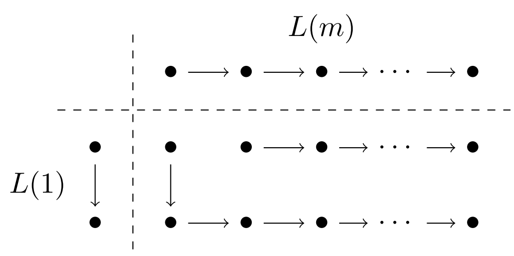
Then the point is that all the coefficients belong to A, and $[r+1]q^r = 1 \mod q$, and the terms with q^r vanish in the quotient by (q).

So we see that $F^{(i)}x \otimes F^{(j)}y$ generate and admissible lattice, and moreover we do have:

$$\tilde{F}\Big(\bar{x} \otimes \tilde{F}^i\bar{y}\Big) = \begin{cases} \tilde{F}\bar{x} \otimes \bar{y}, & i = 0 \\ \bar{x} \otimes \tilde{F}^{i+1}\bar{y}, & i > 0 \end{cases} \tag{40.5}$$

$$\tilde{F}\Big(\tilde{F}\bar{x} \otimes \tilde{F}^i\bar{y}\Big) = \tilde{F}\bar{x} \otimes \tilde{F}^{i+1}\bar{y} \tag{40.6}$$

If you like, you can think about this in terms of pictures. We have:



So this does it when $M_2 = L(m)$ and $M_1 = L(1)$.

Now we consider $M_1 \otimes (M_2 \otimes M_3)$. Then we have:

$$\tilde{F}(b_1 \otimes b_2 \otimes b_3) = \begin{cases} \tilde{F}b_1 \otimes b_2 \otimes b_3, & f(b_1) > e(b_2 \otimes b_3) \\ b_1 \otimes \tilde{F}b_2 \otimes b_3, & f(b_1) \le e(b_2 \otimes b_3) \text{ and } f(b_2) > e(b_3) \\ b_1 \otimes b_2 \otimes \tilde{F}b_3, & f(b_1) \le e(b_2 \otimes b_3) \text{ and } f(b_2) \le e(b_3) \end{cases} \tag{40.7}$$

Then we need also to consider $(M_1 \otimes M_2) \otimes M_3$. We will again have three cases: $f(b_1 \otimes b_2) > e(b_3)$ and $f(b_1) > e(b_2)$; $f(b_1 \otimes b_2) > e(b_3)$ and $f(b_2) \le e(b_3)$; $f(b_1 \otimes b_2) \le e(b_3)$. But there are simply formulas for calculating:

$$f(b_1 \otimes b_2) = \begin{cases} f(b_1) - e(b_2) + f(b_2), & f(b_1) > e(b_2) \\ f(b_2), & f(b_1) \le e(b_2) \end{cases} \tag{40.8}$$

$$e(b_1 \otimes b_2) = \begin{cases} e(b_1) - f(b_2) + e(b_2), & f(b_1) \ge e(b_2) \\ e(b_2), & f(b_1) < e(b_2) \end{cases} \tag{40.9}$$

So then you see you can proceed by induction.

But then there are a number of things you have to do ****I missed a bunch of what VS said but did not write****. You use that every module can be obtained as some tensor product of some small representations.

So in principle what we want to show is that in any irreducible representation, the canonical base is unique up to multiplication by a scalar and obtained from the highest vector.

So here there is one more notion, which is called *polarization*. Let $\sigma : \mathcal{U}_q \to \mathcal{U}_q^{\text{op}}$ be given by $\sigma(E_\alpha) = q_\alpha F_\alpha K_\alpha^{-1}$, $\sigma(F_\alpha) = q_\alpha^{-1} K_\alpha E_\alpha$, and $\sigma(K_\mu) = K_\mu$. Then a bilinear form $(,) : M \times M \to \mathbb{K}$ is σ*-invariant* or *contragradient* if $(um, m') = (m, \sigma(u)m')$. Such forms have a number of properties:

1. $\ker(,)$ is an invariant subspace.
2. $(M_\mu, M_\nu) = 0$ if $\mu \neq \nu$.
3. If $M = \bigoplus M[\lambda_i]$ are the *isotypic components* $M[\lambda_i] = L(\lambda_i)^{\oplus r}$, then $(M[\lambda_i], M[\lambda_j]) = 0$ if $i \neq j$.
4. A simple $L(\lambda)$ always admits a unique-up-to-scalar σ-invariant form.

   Because see why: define M^* a $\mathcal{U}_q$-module by $\langle u\phi, m\rangle = \langle \phi, \sigma(u)m\rangle$ — it is the dual vector space, but the module structure is defined via σ, not the antipode. Then $L(\lambda)^*$ is irreducible and has the same character as $L(\lambda)$, and so therefore $L(\lambda)^* \cong L(\lambda)$, and we can pick $(,)$ to be such an isomorphism. Uniqueness follows from the property 1.

   So what we will do is fix $v_\lambda \in L(\lambda)$, and then there is a unique form for which $(v_\lambda, v_\lambda) = 1$ — well, actually, not unique, because you can multiply by -1, but other than that it is.

5. Recall the other comultiplication from last time — this is the comultiplication with which we are defining $\otimes$. Then $\Delta' \circ \sigma = (\sigma \otimes \sigma) \circ \Delta'$.

   So, if you have a form $(,)_1$ on M_1 and $(,)_2$ on M_2, then you can construct $(,)$ on $M_1 \otimes M_2$ by $(m_1 \otimes m_2, m_1' \otimes m_2') = (m_1, m_1')_1 \otimes (m_2, m_2')_2$.

6. Let $\tilde{M}$ be an admissible lattice. Then we ask for $(,)$ such that $(,) : \tilde{M} \times \tilde{M} \to A$. Then it has a lot of nice properties:

$$(\tilde{E}_\alpha m, m') = (m, \tilde{F}_\alpha m') \mod qA \tag{40.10}$$

So if we are in such a situation, then we can define a form $(,)_0 : (\tilde{M}/q\tilde{M})^{\times 2} \to \mathbb{Q}$. Because of this, we can define a *polarization* of $(\tilde{M}, B)$ to be a σ-invariant form $(,)$ on M such that $(\tilde{M}, \tilde{M}) \subseteq A$, and such that B is an orthonormal basis for $(,)_0$, i.e. $(b, b')_0 = \delta_{b,b'}$.

Lemma 40.2 *If $(\tilde{M}, B)$ admits a polarization, then*

1. $\tilde{M} = \{x \in M \text{ s.t. } (x, x) \in A\}$.
2. $\tilde{M} = \bigoplus \tilde{M} \cap M[\lambda_i]$, *where the sum is over isotypic components.*

We will skip the proof of 1. For 2., note that if $x \in \tilde{M}$, we can write it as $\sum x_{\lambda_i}$, and so $(x_{\lambda_i}, x_{\lambda_i}) \in A$, and so $x_{\lambda_i} \in \tilde{M} \cap M[\lambda_i]$.

So we already know that the small representations admit crystal bases. And we can show that this crystal base admits a polarization:

Proposition 40.3 *Let $L(\lambda)$ be a small representation. I.e. it is spanned by $\{x_\mu, h_\alpha\}$, where $\mu \in W \cdot \lambda$ and $\alpha \in \Pi_{\rm sh}$ — you need the h_α only if $L(\lambda)$ is not minuscule. So by brute force, you construct your form, and check that it works:*

$$(x_\mu, x_\nu) = \delta_{\mu,\nu} \qquad (h_\mu, x_\nu) = 0 \tag{40.11}$$

$$(h_\beta, h_\gamma) = \begin{cases} 1 + q_\beta^2, & \beta = \gamma \\ q_\beta, & (\beta, \gamma) \neq 0 \\ 0, & (\beta, \gamma) = 0 \end{cases} \tag{40.12}$$

And then $\{\bar{x}_\mu, \bar{h}_\alpha\} = B$ is an orthonormal base.

So finally we study what happens when we tensor, and for this we use a classical lemma:

Lemma 40.4 *Let $L(\lambda_0$ be small. Then $L(\lambda) \otimes L(\lambda_0)$ has the following components:*

1. *$L(\lambda + \mu)$ with multiplicity 1 is $\mu \in W \cdot \lambda_0$ and $\lambda + \mu \in P^+$.*
2. *In the not-minuscule case: $L(\lambda)$ with multiplicity $\#\{\alpha \in \Pi_{\rm sh}$ s.t. $(\lambda, \alpha^\vee) > 0\}$.*

To prove lemma 40.4, we use the Weyl character formula:

$$\operatorname{ch} L(\lambda) \cdot \operatorname{ch} L(\lambda_0) = \frac{1}{\mathcal{D}} \sum_{\substack{w \in W \\ \mu \in W\lambda_0}} \operatorname{sign}(w)\, e^{w(\lambda+\rho)+\mu} + \frac{1}{\mathcal{D}} \left|\Pi_{\rm sh}\right| \sum_{w \in W} \operatorname{sign}(w)\, e^{w(\lambda+\rho)} \tag{40.13}$$

In the minuscule case, if $w(\lambda + \rho) + \mu$ is not dominant, then $w(\lambda + \rho) + \mu$ lies on a wall. So if ****missed**** then it cancels.

In the second case, it could happen that $\lambda + \rho + \mu$ is not dominant, and so that means that $(\lambda + \rho + \alpha) = s_\alpha(\lambda + \rho)$. Then this guy in the first sum in equation 40.13 cancels with a guy in the second sum. So you get the cancelation, and that gives the multiplicites in lemma 40.4.

So on the other hand, we also can play with the crystal graphs. Since we are out of time, we will just quote the result, which you can easily recover:

Lemma 40.5 *Assume that $\big(\tilde{L}(\lambda), B(\lambda)\big)$ is a crystal base with polarization. If $L(\lambda_0)$ is small, then we already know that $L(\lambda_0) \otimes L(\lambda)$ admits a crystal base $\big(\tilde{L}(\lambda_0) \otimes \tilde{L}(\lambda), B(\lambda) \otimes B(\lambda_0)\big)$. Write $B = B(\lambda) \otimes B(\lambda_0)$ and $\tilde{M} = \tilde{L}(\lambda_0) \otimes \tilde{L}(\lambda)$. Then $\operatorname{HW}(B_\nu) =$ the multiplicity of $L(\nu)$ in $L(\lambda_0) \otimes L(\lambda)$.*

Moreover, $\tilde{M} = \bigoplus M[\nu] \cap \tilde{M}$, where the sum is over isotypic components. Since B is orthonormal, it is also a sum of isotypic components: $B = \bigsqcup B[\nu]$. Then $\operatorname{HW}(B[\nu]) =$ multiplicity of $L(\nu)$ in $L(\lambda_0) \otimes L(\lambda)$. But then we must have $B[\nu]_\nu = \{b_1, \ldots, b_s\}$. Then we star applying $\tilde{F}_\alpha$s: we have

$\tilde{F}_{\alpha_1} \dots \tilde{F}_{\alpha_k} b_i$, and they are disjoint because we can go back, and they generate everything because if not then there is another highest vector. And we know that every vector in the canonical basis comes from some highest vector, but if it doesn't come form this one, then we must have another highest vector. And we know how many there are. So in particular $\{\tilde{F}_{\alpha_1} \dots \tilde{F}_{\alpha_k} b_i\}$ generates a copy of $L(\nu)$ in $L(\lambda_0) \otimes L(\lambda)$, and so it gives us a canonical basis of $L(\nu)$ with polarization.

That means, again, that if λ was nice, then it's still true for any ν inside the tensor product. So now we can do induction on the weights. □

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