GEOMETRIC QUANTIZATION OF CHERN-SIMONS THEORY VIA KAHLER POLARIZATION

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1. Requirements on resulting vector space

As a consequence of geometric quantization, we hope to get a 3-maniold invariant, or equivalently a 3-dimensional TQFT functor. There are 2 properties we might expect.

- Independent of choices : we might require that it does not depend on choices such as a metric, complex structre etc. on Σ., even though we used them to construct $Z(\Sigma)$.
- Action of mapping class group : we want $Z(\Sigma)$ to have a mapping class group action. Why? Recall our definition of TQFT functor(Sean's talk). It is a functor $Z : Bord_{23} \to Vect_{\mathbb{C}}$ which is cobordism generated, involutive, and multiplicative. Here $Bord_{23}$ is a category whose objects are smooth closed oriented surfaces and morphisms are isotopy classes of smooth oriented compact cobordism between surfaces. Now suppose we are given an orientation preserving diffeomorphism $f : \Sigma \to \Sigma$, and a 3-manifold M with boundary $\partial M = \Sigma$. Recall that $Z(M)$ is a vector in $Z(\Sigma)$. Gluing M to $\Sigma \times [0, 1]$ via f gives a new 3-manifold whose boundary is Σ, which again corresponds to a vector in $Z(\Sigma)$ udner Z. Thus f extends to a linear automorphism of $Z(\Sigma)$. This implies that we have an action of $Diff^{+}(\Sigma)$ on $Z(\Sigma)$. Since morphisms of $Bord_{23}$ are defined upto isotopy, in fact we have an action of mapping class group $MCG(\Sigma)$ on $Z(\Sigma)$.

Our goal is to construct such $Z(\Sigma)$.

2. Kahler structure on the moduli space

We have seen that the moduli space $\mathcal{M}$ of flat connections over a closed riemann surface Σ admits a symplectic structure. The sympectic form is induced from the symplectic form $\tilde{\omega}$ on the space of all connections $\mathcal{A}$, which is defined as follows.

$$\tilde{\omega}(A, B) = \int_{\Sigma} < A, B >$$

where $A, B \in \Omega^1(\Sigma, adP)$. Here we identified $\mathcal{A}$ with $\Omega^1(\Sigma, adP)$. $\mathcal{M}$ is obtained as a symplectic quotient of $\mathcal{A}$ where the moment map is given by curvature F and the gauge group is $\mathcal{G}$. Denote the induced symplectic form on $\mathcal{M}$ by ω. We will apply a procedure of geometric quantization via Kahler polarization to $(\mathcal{M}, \omega)$. In fact, we will quantize $(\mathcal{M}, k\omega)$ for any positive integer k.

To talk about a Kahler structure on $\mathcal{M}$, we need to specify a complex structure on $\mathcal{M}$. We give a complex structure on $\mathcal{M}$ as follows. Pick a complex structure J on Σ. J induces a complex structure on $\mathcal{A}$, say $J_{\mathcal{A}}$. We can check that $(\mathcal{A}, J_{\mathcal{A}}, \tilde{\omega})$ is Kahler. Note that the action of $\mathcal{G}$ on $\mathcal{A}$ extends to an action of $\mathcal{G}_{\mathbb{C}}$, the complexification of $\mathcal{G}$, on $(\mathcal{A}, J_{\mathcal{A}})$.

The complex structure $J_{\mathcal{A}}$ induces a complex structure on $\mathcal{A}$ as follows. Fix any $A \in \mathcal{M}$ and $\tilde{A} \in$orbit of A in $F^{-1}(0)$. Then $T_A\mathcal{M} \simeq T_{\tilde{A}}F^{-1}(0)/T(\mathfrak{g})$. Here $T(\mathfrak{g})$ is a subspace of $T_{\tilde{A}}F^{-1}(0)$ generated by $\mathcal{G}$. One can check that

$$T_{\tilde{A}}F^{-1}(0)/T(\mathfrak{g}) \simeq T_{\tilde{A}}\mathcal{A}/T_c(\mathfrak{g}_{\mathbb{C}}).$$

at least for finite dimensional symplectic quotient. Since the right hand side has a complex structre which is induced from $\mathcal{A}$, so is $T_A\mathcal{M}$. However it is not clear that with this complex structre, $(\mathcal{M}, \omega)$ is Kahler.

To see this, assume $G = U(n)$ for a while. Let M be a complex manifold. Recall that there is a bijection between {connection on $U(n)$ principal bunlde P over M} and {connection on rank n Hermitian vector bundle E over M which is compatible with metric}. We will think of a connection as an element of the right hand side for a while.

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Note that E is just a smooth complex vector bundle. If there is a holomorphic structre $\mathcal{E}$ on E, then it defines an operator $\bar{\partial}_{\mathcal{E}} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ satisfying Leibniz rule. Also note that by taking $(0,1)$ part, any connection A on E defines $\bar{\partial}_A : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ which satisfies the Leibniz rule.

Definition 2.1. A partial connection is a $\mathbb{C}$-linear operator $\bar{\partial} : \Omega^{0,0}(E) \to \Omega^{0,1}(E)$ which satisfy the Leibniz rule.

Any partial connection extends to a family of operators from $\Omega^{p,q}(E)$ to $\Omega^{p,q+1}(E)$. Holomorphic structres and connections induce partial connections. Question : Given a parital connection, how can we know that it is induced from a connection or a holomorphic structure?

Lemma 2.2. *For any partial connection $\bar{\partial}$ on E, there exists a unique connection A such that $\bar{\partial}_A = \partial$.*

Proof. Let α be a connection 1-form of ∂. Define connection 1-from of A by $\alpha - \alpha^*$. It actually defines a connection and unitarity imposes that this is the only possibility. □

Theorem 2.3. *A partial connection ∂ on a smooth complex vector bundle E over M comes from a holomorphic structure if and only if $\partial^2 = 0$.*

Proof. See section 2.2.2. of [2]. □

To summarize, we have a bijection between {holomorphic structure on E} and $\mathcal{A}^{1,1} :=$ { unitary connections whose curvature is of (1,1) type}. Indeed, a holomprhic structure $\mathcal{E}$ defines a partial differential, and by the lemma this partial differential defines a unitary connection. Its curvature satisfies $F^{0,2} = \partial^2 = 0$ and $F^{2,0} = -(F^{0,2})^* = 0$ hence of $(1,1)$ type. Conversely, any element in $\mathcal{A}^{1,1}$ defines a partial connection $\bar{\partial}_A$, and $\bar{\partial}_A^2 = 0$ since curvature of A is (1,1) type. By the theorem, $\bar{\partial}_A$ comes from a holomorphic structure on E.

Also note that a holomorphic structure on E correspond to a holomorphic structure on comlexified principal $G_{\mathbf{C}}$ bundle $P_{\mathbf{C}}$ over M.

If $M = \Sigma$, riemann surface with complex structure J, any connection has a curvature of (1,1) type. Thus we have $\mathcal{A}/\mathcal{G}_{\mathbf{C}}$=moduli space of holomorphic structure on $P_{\mathbf{C}} \to M$. We may write this space as $\mathcal{M}_J$.

By the theorem, we have a map $i : \mathcal{M} \to \mathcal{M}_J$ from the moduli space of flat connections to moduli space of holomorphic structures. Geometric invariant theory says following. See e.g. [4]

Theorem 2.4.
- *M_J is a complex projective variety.*
- *i is a diffeomorphism*
- *symplectic form on $\mathcal{M}$ becomes a Kahler form on $\mathcal{M}_J$ under i.*

analogy : $X = \mathbf{C}^{n+1} - \{0\}$, $G = GL_1(\mathbf{C}) = K_{\mathbf{C}}$ acts on X where $K = U(1)$. There is a momoent map $\mu : X \to Lie(K) = \mathbf{R}$. X/G is a complex projective variety $\mathbf{CP}^n$ which can be seen as a symplectic quotient $\mu^{-1}(1)/K = S^{2n+1}/U(1) = \mathbf{CP}^n$. X(resp. K, G, X/G, $\mu^{-1}(1)/K$) correspond to $\mathcal{A}$(resp. $\mathcal{G}$, $\mathcal{G}_{\mathbf{C}}$, $\mathcal{M}_J$, $\mathcal{M}$).

So far we have talked about $U(n)$, but the theorem is true for other nice Lie groups. That is, $\mathcal{A}/\mathcal{G}_{\mathbf{C}}$ is a moduli space of holomorphic structure on complexified principal bundle, i is a diffeomorphism, $\mathcal{M}_J$ is Kahler and the symplectic form on $\mathcal{M}$ is a Kahler form on $\mathcal{M}_J$. Note that we have an isomorphism between tangent spaces which mentioned before: $T_A\mathcal{M} \simeq T_{\tilde{A}}\mathcal{A}/T(\mathfrak{g}_{\mathbf{C}})$.

We have an another observation: isotopic compelx structures J and J' give the same complex structure on $\mathcal{M}$. Suppose f is an orientation preserving diffeomorphism which is isotopic to the identity such that $f^*J' = J$.

We have a following diagram. $f^* : \mathcal{M}'_J \to \mathcal{M}_J$ is a pull back of holomorphic structure. If we identify $\mathcal{M}$ with $Hom(\pi_1(\Sigma), G)/G$, then $i_J^{-1} \circ f^* \circ i_{J'}$ is just the identity because a diffeomorphism isotopic to the identity induces the identity map between fundamental groups. Thus a complex sturcture of $\mathcal{M}_J$ depend on a compelx structure J on Σ only up to isotopy.

$$\begin{tikzcd}
Hom(\pi_1(\Sigma), G)/G \arrow[r, equal] \arrow[d, "f^*=id"'] & \mathcal{M} \arrow[r, "i_{J'}"] \arrow[d, "i_J^{-1}\circ f^*\circ i_{J'}"'] & \mathcal{M}_{J'} \arrow[d, "f^*"] \\
Hom(\pi_1(\Sigma), G)/G \arrow[r, equal] & \mathcal{M} \arrow[r, "i_J"] & \mathcal{M}_J
\end{tikzcd}$$

So we have : $\mathcal{M} \times \mathcal{T} \to \mathcal{T}$ where $\mathcal{T}$ is the Teichmuller space, and whose fiber at J is a Kahler manifold $\mathcal{M}_J$. All fibers are $\mathcal{M}$ as a symplectic manifold, but with different Kahler structre. We have many choices which is parametrized by Teichmuller space to apply Kahler polarization.

3. Determinant Line bundle

In this section, we will find the prequantum line bundle to apply geometric quantization. Reference for this section is [5]. We saw that a holomorphic structure $\mathcal{E}$ on E defines an operator $\tilde{\partial} : \Omega^{0,0}(E) \to \Omega^{0,1}(E)$. Consider all of these operators. We can identify

$$\text{space of } \partial \text{ operators} \simeq \Omega^{0,1}(E) \simeq \Omega^1(EndE) \simeq \mathcal{A}$$

We define 1-dimensional $\mathbb{C}$ vector space associated to an element in $\Omega^{0,1}(EndE)$.

Definition 3.1. For any $D \in \Omega^{0,1}(EndE)$, define $\mathcal{L}_D = \Lambda(KerD)^* \otimes \Lambda(CokerD)$

Here Λ denotes the top exterior power. Note that since we are working over Riemann surfaces, $KerD$ and $CokerD$ are E-valued Dolbeualt cohomology. i.e. $KerD \simeq H^{0,0}(\Sigma, E), CokerD = H^{0,1}(\Sigma, E)$. Hence they are finite dimensional by the Hodge theorem.

We defined a 1-dimensional complex vector space correspond to each point in $\Omega^{0,1}(EndE)$. There is a way to give a trivialization so that the family of $\mathcal{L}_D$ forms a holomorphic line bundle over $\Omega^{0,1}(EndE)$, hence on $\mathcal{A}$ after choosing complex sturcture on Σ. This line bundle is called the determinant line bundle. We may write the determinant line bundle as $\mathcal{L}$.

Now we choose an inner product on E and Σ. This will induce inner products on $\Omega^{0,q}(E)$ so that we can define an adjoint D^* for each D. Also we have induced inner products on $KerD$ and $CokerD = KerD^*$. This will give an inner product on L_D. But we do further. Definte zeta function of D^*D by $\zeta(s) = \sum \lambda^{-s}$ where the sum is taken over eigenvalues of D^*D. $\zeta(s)$ is holomorphic for $Re(s) \gg 0$ and it can be continued to a meromorphic function on $\mathbb{C}$. We define an inner product on L_D as an inner prouct on L_D induced from $KerD$ and $KerD^*$ times $exp(\zeta'(0))$. We consider the connection on the line bundle $\mathcal{L} \to \mathcal{A}$ which is compatible with this metric.

Theorem 3.2. *The curvature of the connection is equal to the symplectic form on $\mathcal{A}$.*

Restict $\mathcal{L}$ to flat connections. Taking quotient by gauge group, we have a line bundle $\mathcal{L}_{\mathcal{M}}$ over $\mathcal{M}$. Abusing notation, we may write $\mathcal{L}$ for $\mathcal{L}_M$. So far we have talked about a vector bundle. We can do the same procedure to principal bundles by fixing a representation of G and considering associated vector bundle.

One might worry about that the determinant line bundle depend on a choice of complex sturcture on Σ. However we can choose canonical one. Indeed, we know that prequantum line bundles are parametrized by $H^1(\mathcal{M}, U(1))$(Nilay's talk), and for a compact semisimple G, $H^1(\mathcal{M}, U(1))$ is finite.[1]. Thus we have a map from the Teichmuller space $\mathcal{T}$ to $H^1(\mathcal{M}, U(1))$ which sends a complex sturctre J to the determinant line bundle associated to J. Since $\mathcal{T}$ is connected and $H^1(\mathcal{M}, U(1))$ is finite, image is singleton. This is the prequnatum line bundle we want. Now we have a vector bundle, which we will call prequnatum bundle, $\mathcal{H}_{pr} = \Gamma(\mathcal{M}_J, \mathcal{L}) \times \mathcal{T}$ over $\mathcal{T}$.

4. Main theorem

Now we can apply a geometric quantization via Kahler polarization to each $\mathcal{L} \to \mathcal{M}_J$. Varing over $\mathcal{T}$, we have a bundle whose fiber at at J is a space of holomorphic sections of $\mathcal{L} \to \mathcal{M}$. This is in fact a vector bundle. That is, each fiber has the same dimension.

Recall that we want to quantize $(\mathcal{M}, k\omega)$ for every positive integer k. To do this, we consider prequantum line bundle $\mathcal{L}^{\otimes K}$. We can show the following.

Proposition 4.1. *$dimH^0(\mathcal{M}_J, \mathcal{L}^{\otimes K})$ does not depend of J.*

Proof. For simplicity, we assume that k is large enough. By Riemann-Roch theorem, $\sum(-1)^i dimH^i(\mathcal{M}, \mathcal{L}^{\otimes k})$ is a topological invariant. By Serre duality, we have $H^i(\mathcal{M}, \mathcal{L}^{\otimes k}) \simeq H^{n-i}(\mathcal{M}, K \otimes \mathcal{L}^{\otimes -k})$. $c_1(K\mathcal{L}^{\otimes -k}) = c_1(K) - kc_1(L)$ and $c_1(L) = \omega$. Thus if k is large, $K \otimes \mathcal{L}^{\otimes -k}$ is negative. By Kodaira vanishing theorem, $H^{n-i}(\mathcal{M}, K \otimes \mathcal{L}^{\otimes -k}) = 0$ for $i > 0$. So $H^0(\mathcal{M}, \mathcal{L}^{\otimes k})$ is topological hence does not depend on J. ∎

[1] [1], p.819.

Mapping class group action. Fix a principal bundle P over Σ. After choosing an element of $H^4(BG,\mathbb{Z})$, one can define $\mathbf{R}/2\pi\mathbb{Z}$ valued Chern-Simons functional S for 3-manifolds with boundary. Since S is $\mathbb{R}/2\pi\mathbb{Z}$-valued, e^{iS} is well defined. Let $Aut(P)$ be the automorphism group of P, not necessarily base preserving. We define $\rho : Aut(P) \times \mathcal{A} \to U(1)$ by $\rho(\Phi, A) = e^{iS(M_0,P_0,A_0)}$. Here $M_0(resp.P_0, A_0)$ is a 3-manifold with boundary(resp. principal bundle, connection) obtained by identifying gluing two $P \times I \to \Sigma \times I$ via Φ. As a result of gluing, we have again a 3-manifold with boundary of topological type $\Sigma \times I$. Let $Aut(P)'$ be the subgroup $Aut(P)$ consisting of lifting of diffeomorphisms of Σ which are isotopic to the identity. Then ρ induces a map $Aut(P)/Aut(P)' \times \mathcal{A} \to U(1)$. Note that $\Gamma_{\Sigma,P} := Aut(P)/Aut(P)'$ is the subgroup of mapping class group, whose elements are diffeomorphisms fixing topological type of P. After taking restriction and quotient, we have a $\Gamma_{\Sigma,P} \times \mathcal{M} \to U(1)$ which gives an action of $\Gamma_{\Sigma,P}$ on the line bundle $\mathcal{L}$. Now we can state the main theorem.

Theorem 4.2. *$\mathcal{M}$: moduli space of flat connections of $G \to P \to \Sigma$. $\mathcal{L}$: prequnatum line bundle with an action of $\Gamma_{\Sigma,P}$. $\mathcal{T}$: Teichmuller space of Σ. Let quantum bundle $\mathcal{H}_Q$ be the vector bundle whose fiber at $J \in \mathcal{T}$ is $H^0(\mathcal{M}_J, \mathcal{L})$. There is a projectively flat connection on the bundle $\mathcal{H}_Q$ which is invariant under $\Gamma_{\Sigma,P}$ action. In particular, we get a projective representation of $\Gamma_{\Sigma,P}$.*

A connection is projectively flat if its curvature $R(X,Y)$ is constant multiplication at each point for all vector fields X, Y. Recall that in geometric quantization, multiplication by constant has no effect hence only projectivization is canonically defined. This theorem implies that we can identify all vector spaces obtained by geometric quantization of $\mathcal{M}_J$ for all J. Hence we can say that the space of projectively flat sections of the bundle $\mathcal{H}_Q$ is the vector space obtained by geometric quantization of Chern-Simons theory via Kahler polarization.

References

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