Theo Johnson-Freyd. Feb 17, 2016

Geometric quantization of Chern-Simons Theory via real polarizations

My goal is to describe yet another quantization of the moduli space $\mathcal{M} = \mathrm{Loc}_G \Sigma$ for Σ a surface and G ~~[illegible]~~ a compact group. Mostly I assume genus > 1 and $G = SU(2)$, but you can do the other cases as well. My talk is based on the papers

L. Jeffrey and J. Weitsman. Bohr-Sommerfeld Orbits in the moduli space of flat connections and the Verlinde dimension formula. Com. Math. Phys., 150, 593-630 (1992)

—. Half density quantization of the moduli space of flat connections and Witten's semiclassical manifold invariants. Topology. Vol 32, No 3, 509-529, (1993).

—. Geometric quantization and Witten's semiclassical manifold invariants. Low-dimensional Topology and Quantum Field Theory (H. Osborn, ed.). Plenum Press (1993).

The last is a survey of the first two. These constructions provide a bridge between the Kähler and combinatorial talks from earlier.

A. Bohr-Sommerfeld quantization:

Recall from Nilay's talk: the input data for geometric quantization consists of (i) a symplectic manifold (M,ω), (ii) a line bundle $L \to M$ with connection ∇ with curvature $F_\nabla = \omega$, (iii) a polarization, which is a Lagrangian integrable subbundle of $T_M \otimes \mathbb{C}$. We will keep (i) and (ii) verbatim but modify (iii).

Defn: An integrable system on M is a surjection (preferably with connected fibers) $M \xrightarrow{p} B$ such that $\dim M = 2n$, $\dim B = n$, and $\omega|_{p^{-1}(b)} = 0$ for all $b \in B$. Think of it as a "colagrangian"!

We will handle the singular fibers of p (i.e. where p is not a submersion) in due time, but first let's talk about the smooth part. Then the fibers $p^{-1}(b)$ are Lagrangian, so $\mathcal{L}|_{p^{-1}(b)}$ is flat.

<u>Defn:</u> $b \in B$ is <u>Bohr-Sommerfeld</u> if $\mathcal{L}|_{p^{-1}(b)}$ has a flat section.

~~We will come back to these, but the basic idea is that the~~

<u>Defn:</u> The <u>Bohr-Sommerfeld quantization</u> of $(M, \mathcal{L}, B, \ldots)$ is $\mathcal{H} = \bigoplus_{\substack{b \in B \\ \text{Bohr-Sommerfeld}}} H^0(\mathcal{L}|_{p^{-1}(b)}) \simeq \mathbb{C}^N$ where $N = \#(\text{b.s. points})$.

See, Nilay told us to look for sections of $\mathcal{L} \to M$ flat along the polarization. Here our polarization is $\ker(dp)$. There will be no smooth flat sections in general, but $\mathcal{H}$ consists of "singular" flat sections concentrated on the Bohr-Sommerfeld fibers, and a theorem of Sniatycki identifies $\mathcal{H}$ with

$\bigoplus_i H^i$ (sheaf on M whose local sections are sections of $\mathcal{L}$ that are constant along fibers).

OK, let me remind a bit more about the geometry of integrable systems. Any function on B pulls back to a function on M, and the pull-backs commute. Given $f: B \to \mathbb{R}$, the hamiltonian flow $X_f = X_{f \circ p}$ commutes with p, i.e. it acts on the fibers. Because of this, such f are called <u>action</u> functions on M. Then each fiber is acted on by group $\mathbb{R}^n$ (choose local coords around $b \in B$), and so is probably an n-torus (if M is compact). The local action functions then give elements of $H_1(p^{-1}(b) \simeq (S^1)^n) \otimes \mathbb{R}$, and you can modify them to ~~give~~ be in the lattice of integer homology classes — those are precisely the $f: B \to \mathbb{R}$ such that time-1 flow $\exp(X_f)$ is identity. See, the fibration of $(S^1)^n$s over B gives a lattice $H_1(\text{fiber}) \to B$, and the symplectic form embeds this as a sublattice of $\mathcal{O}(B)/_{\text{constants}}$. Just to emphasize, this picture breaks at the singular fibers, as there the Hamiltonian flows can vanish.

Defn: <u>Action coordinates</u> are those for which X_f has period precisely 1. (The irreducible elts of the lattice.) Or perhaps choose a basis of the lattice; those are ~~also called~~ even more accurately <u>coordinates</u>.

Remark: <u>Angle coordinates</u> are the dual coords on $(S^1)^n$s whose derivatives are the dual basis in $H^1(p^{-1}(b)) \otimes \mathbb{R}$.

Now, here's the main general result, which is proven by measuring symplectic areas of certain cylinders in M:

<u>Thm:</u> Fix a prequantum line bundle $\mathcal{L} \to M$, and suppose that $b_1, b_2 \in B$ are both Bohr–Sommerfeld (and B is connected, natch). If f is an action variable, then $f(b_1) - f(b_2) \in \mathbb{Z}$. ~~Conversely, if $f(b_1$~~

Conversely, if b_0 is Bohr–Sommerfeld and for <u>every</u> f in the lattice, $f(b) - f(b_0) \in \mathbb{Z}$, then b is also Bohr–Sommerfeld.

This means that you can count b.s. points provided you

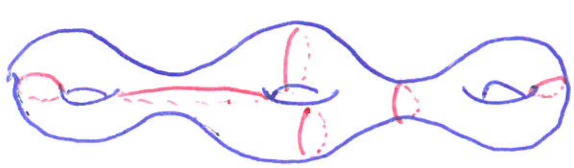
- understand the action coordinates $\{f_i\}$
- can find at least one b.s. point, and calculate $\{f_i(b_0)\}$, or at least, for each action coordinate f_i, find a b.s. pt b_i such that you can calculate $f_i(b_i)$.

Oh, I should say, remarkably a version of the theorem holds even in the singular case. Near singular points, some of the action variables f_i might be undefined (the lattice degenerates or blows up or...), but the first part of the thm holds if b_1, b_2 are both in the domain of definition of f, and the second part you can do as well.

B. Integrable systems from pants decompositions.

So this is our program to apply to $\mathcal{M} = \mathrm{Loc}_G \Sigma$. I'll recall the construction of line bundle $\mathcal{L} \to \mathcal{M}$ later; first, let's find an integrable system $\mathcal{M} \to B$.

Choose a pair-of-pants decomposition for Σ:



Set $B = \mathrm{Loc}_G(\amalg \text{ cutting circles}) \simeq \left(\frac{G}{G}\right)^{3g-3}$, where $\frac{G}{G} = H/W = \mathrm{Loc}_G(S^1)$. In $SU(2)$ case, this is the half-circle

$\frac{H}{W} = \frac{U(1)}{\mathbb{Z}/2} =$  $\subseteq \frac{\mathbb{C}}{\mathbb{Z}/2}$.

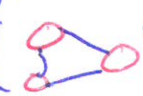
~~Pf~~ Fact: This does the job, i.e. $\dim B = \frac{1}{2} \dim M$ and fibers are Botnpiz. $\boxtimes$.

Well, almost. The map $M \xrightarrow{p} B$ might not be surjective.

To understand $p(M)$, we should look at a single pair of pants. I will do $G = SU(2)$-case.

$$\mathrm{Loc}_G\Big($$  $$\Big) \cong \mathrm{Loc}_G(8) = \frac{G \times G}{G}.$$

In $SU(2)$ case, the generic part of this B 3-dim:
Given pair of matrices (X, Y), use gauge symmetries to ~~make~~ diagonalize $Y = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \in U(1)$; then you still have $U(1)$-worth of symmetries, so you can get $X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$ into form where $b \in \mathbb{R}_{>0}$. So degrees of freedom, generically, are $a \in \mathbb{C}$, $\lambda \in U(1)/_{\mathbb{Z}/2}$, $|a| \leq 1$.

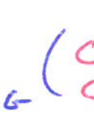
If I tell you ~~$\mathrm{tr}(X)$~~ $\mathrm{tr}(Y)$, you can get λ; if I tell you $\mathrm{tr}(X)$, you can get $\mathrm{Re}(a)$; if I tell you $\mathrm{tr}(XY)$, you can get $\mathrm{Re}(\lambda a)$; this is enough. So generically $\mathrm{Loc}_G($  $) \to \mathrm{Loc}_G($  $)$ is ~~a~~ local diffeo.

<u>Exercise</u>: In $SU(2)$, $\mathrm{tr}(A)\,\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1})$.

From this you can conclude:

<u>Proposition:</u> Use polar coordinates to identify

$$\frac{U(1)}{\mathbb{Z}/2} \simeq [0,\pi] \qquad \theta \in [0,\pi] \longmapsto \text{conj. class of } \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}.$$

Then image of $p: \mathrm{Loc}_G($  $) \to \mathrm{Loc}_G($  $) = [0,\pi]^3$

is $\{(\theta_1, \theta_2, \theta_3) : |\theta_1 - \theta_2| \leq \theta_3 \leq \min(\theta_1+\theta_2,\ 2\pi - (\theta_1+\theta_2))\}.$

<u>Exercise:</u> This is symmetric in $\{1,2,3\}$.

The non-smooth pts are precisely where you have equality.

i.e. singular fibers are above the loci:

$$\theta_1 + \theta_2 - \theta_3 = 0$$
$$\theta_2 + \theta_3 - \theta_1 = 0$$
$$\theta_3 + \theta_1 - \theta_2 = 0$$
$$\theta_1 + \theta_2 + \theta_3 = 2\pi.$$

<u>Cor:</u> Image of $\mathcal{M} \to B = \prod_{\bigcirc_s} [0,\pi]$ is

$\{$ Labels of cutting circles by angles in $[0,\pi]$: at each pair of pants, above inequality is enforced. $\}$.

moreover, it turns out that these Θ-angles are ~~[illegible]~~ essentially action coordinates:

<u>Proposition:</u> For symplectic form $k \cdot \omega$, the functions $h_i = \frac{k\Theta_i}{\pi}$ are action variables, i.e. their hamiltonian flows have period $= 1$.*

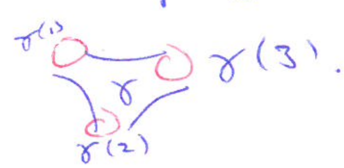
These do not $\mathbb{Z}$-span the full lattice of action variables, however.

<u>Theorem:</u> The lattice of action variables is spanned by coordinates

$$h_i, \qquad i \in \text{set of cutting circles}$$

together with

$$h_\gamma := \frac{1}{2}\left(h_{\gamma(1)} + h_{\gamma(2)} + h_{\gamma(3)}\right) \qquad \gamma \in \text{set of pairs of pants}$$



<u>Corollary:</u> Suppose

(a) For every b.s. point b at which h_i (or h_γ) is critical, (so that its flow vanishes), $h_i(b)$ (or $h_\gamma(b)$) $\in \mathbb{Z}$.

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* Actually, if i is a separating curve, then the flow has period $\frac{1}{2}$, and $h_i' = \frac{h_i}{2} = \frac{k\Theta_i}{2\pi}$ ~~[illegible]~~ is action. This won't affect the main result, so can be ignored on first pass.

(b) For every i (or γ) we can find b_i (or b_γ) b.s. s.t. h_i (or h_γ) is non-critical and $h_i(b_i)$ or $(h_\gamma(b_\gamma)) \in \mathbb{Z}$.

Then the Bohr-Sommerfeld locus is precisely the set of points in $p(\mathcal{M}) \subseteq B$ s.t. all h_i and h_γ are integral.

Remark: For us, (a) is automatic, as the bad locus for h_i is $h_i = \{0, K\}$, i.e. $\theta_i = \{0, \pi\}$.

C. Construction of some Bohr-Sommerfeld points.

Of course, the Bohr-Sommerfeld condition depends on the choice of prequantum line bundle $\mathcal{L} \to \mathcal{M}$. Recall from Pyongwon's talks: if X^3 is closed, then $\exp(i\, CS(A))$ for A a $\mathfrak{g}$-connection on X only depends on gauge class of A, but if $\partial X = \Sigma$, then

~~$\exp(i\,CS(A))$~~

a connection

a gauge transformation

$$\exp(i\, CS(A \triangleleft g)) = \exp(i\, CS(A)) \cdot \Theta(A|_\Sigma, g|_\Sigma)$$

where Θ is a $U(1)$-cocycle on $\mathcal{M}$. You can use integration by parts to write an explicit formula for Θ.

This Θ defines the line bundle $\mathcal{L}$. Namely, you take trivial bundle $\mathbb{C} \times \mathcal{A}_F \to \mathcal{A}_F$ upstairs ($\mathcal{A}_F =$ space of flat $\mathfrak{g}$-connections, say on the trivial G-bundle)

and push it to $\mathcal{M}$ by saying that $\mathcal{L}_{[A]}$, for $[A]$ a gauge class of connections, is the line of functions on {representatives of $[A]$} that transform via cocycle Θ.

Further, trivial line $\mathbb{C}\times\mathcal{A}_F$ has a connection given by 1-form $\alpha \longmapsto \int_\Sigma \mathrm{tr}\langle A\wedge\alpha\rangle$ for $\alpha \in T_A\mathcal{A}_F$.

This is the "$\frac{1}{2}(p\,dq - q\,dp)$" connection. It descends to $\mathcal{L}$, and has curvature ω.

So the point is, to construct sections of $\mathcal{L}$, you can use Chern-Simons action. For $\mathcal{L}^{\otimes k}$, you use $k\cdot CS$.

Now here's the strategy. The pants decomposition picks out a handlebody H^3 bounding Σ:

Pick a ~~knot~~ knot K in the interior of H. Let $\tilde{H}^{(k)} \to H$ denote the k-fold cover branched at K. On the boundary, it is unramified $\tilde{\Sigma} \xrightarrow{k:1} \Sigma$. It gives

$$\begin{array}{ccc} \mathcal{M}_\Sigma & \longrightarrow & \mathcal{M}_{\tilde\Sigma} = \tilde{\mathcal{M}} \\ \uparrow & & \uparrow \\ \mathcal{L}^{\otimes k} & \longrightarrow & \tilde{\mathcal{L}} \end{array}$$

by pulling back ~~connections~~ local systems

because the "k-fold" ~~map~~ raises cocycle Θ to the k^{th} power.

Now, look at ~~image~~ map $\mathrm{Loc}_G(\tilde{H}) \to \mathrm{Loc}_G(\tilde{\Sigma}) = \tilde{\mathcal{M}}_\Sigma$.

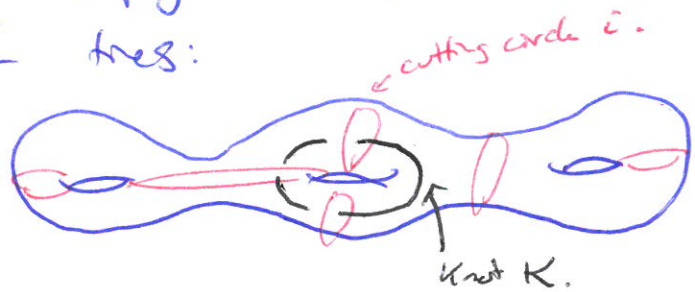
It is an inclusion, because you can map every generator of free group $\pi_1 \tilde{H}$ to a generator of $\pi_1 \tilde{\Sigma}$.

It is also Lagrangian. In fact, it is nothing but the fiber over $\tilde{B}$ in which <u>all</u> coordinates are $\tilde{\theta}_i = 0$.

for lifts of the cutting circles.

So if $[\tilde{A}] \in \tilde{\mathcal{M}}$ has all $\tilde{\theta}_i = 0$, then it extends uniquely to bulk $\tilde{H}$, and then CS(this extension to $\tilde{H}$) is a section of $\tilde{\mathcal{L}}|_{\tilde{p}^{-1}(0)}$:

Assuming circle i is non-separating we can arrange the knot simply to pass through the cutting circles either 0 or 1 times:



Consider the point b_i in B where $h_j = \frac{\theta_j}{k\pi} =$

$$= \begin{cases} 0 & \text{if } K \text{ does not link } C_i \\ 1 & \text{if } K \text{ does.} \end{cases}$$

In particular, $h_i(b_i) = 1$, so b_i is good for i.
Then $b_i \longmapsto \vec{0}$ under $B \to \tilde{B}$, where $\tilde{\mathcal{L}}|_{\tilde{p}^{-1}(\vec{0})}$ admits a section. So this pulls back to a section of $\mathcal{L}|_{p^{-1}(b_i)}$, which is what we wanted.

If the cutting circle is separating, you arrange K to link it twice, and a similar thing happens. Along with the previous theorems, and a similar construction of $h_\gamma \rightsquigarrow b_\gamma$, this proves:

<u>Thm</u>: The Bohr-Sommerfeld points are those $b \in B$ such that

(1) $h_i(b) \in [0, K]$ is integral.

(2) for each pair of pants,
$$h_\gamma(b) = \frac{1}{2}\left(h_{\gamma(1)}(b) + h_{\gamma(2)}(b) + h_{\gamma(3)}(b)\right)$$
is integral.

(3) The <u>Clebsch-Gordan inequalities</u>
$$\left|h_{\gamma(1)}(b) - h_{\gamma(2)}(b)\right| \le h_{\gamma(3)}(b) \le \min\left(h_{\gamma(1)}(b) + h_{\gamma(2)}(b),\ 2\pi - \left(h_{\gamma(1)}(b) + h_{\gamma(2)}(b)\right)\right)$$
hold for each pair of pants.

(4) h_i is even if curve i is separating,

<u>Exercise</u>: except this condition is redundant for (1,2).

But these also precisely count the Verlinde dimension. In terms of quantum groups / MTCs, think of value of h_i as indicating an $SU(2)$-rep, actually for quantum $SU(2)^{(k)}$.

Then the combinatorics counts the dimension of a certain intertwiner space. (There is a problematic shift $k \to k+2$ unaccounted for).

D. Hilbert structure and mapping class group action

All fibers (even the singular ones) of $\mathcal{M} \to B$ are (canonically!) coset spaces for compact groups. Thus you have a canonical choice of semidensity on each fiber, preserved by hamiltonian flow.

Suppose you make a different choice of cutting circles $\mathcal{M} \xrightarrow{p'} B'$. Pick $b \in B$ and $b' \in B'$ both Bohr-Sommerfeld. Then $p^{-1}(b)$ and $(p')^{-1}(b')$ are both Lagrangian and both have semidensities.*

Fact: $p^{-1}(b) \cap (p')^{-1}(b')$ is clean, meaning the tangent space to the intersection is the intersection of tangent spaces. ∎

Exercise: Suppose $L, L' \hookrightarrow (M, \omega)$ are Lagrangians with clean intersection and both equipped with semidensities. Then $L \cap L'$ carries a canonical density.

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* if b is singular, then use ~~not~~ the tangent space, but the appropriate virtual tangent space $T(\text{fiber}) \ominus T(\text{stackiness})$.

Since $\mathcal{L} \to \mathcal{M}$ is Hermitian, given $s \in \Gamma(\mathcal{L}|_{p^{-1}(b)})$ and $s' \in \Gamma(\mathcal{L}|_{p'j^{-1}(b')})$, we get a pairing

$$\langle\!\langle s, s' \rangle\!\rangle = \int_{L \cap L'} \langle s|_{L\cap L'}, s'|_{L \cap L'} \rangle_{\mathcal{L}}$$

where $\langle , \rangle_{\mathcal{L}}$ is the Hermitian inner product on $\mathcal{L}$.

Corollary: $\mathcal{H}$ constructed from polarization $\mathcal{B}$ and $\mathcal{H}'$ constructed from polarization $\mathcal{B}'$ are ~~is~~ canonically iso.

Using this gives a (projective) rep of M.C.G.

Corollary: Given closed N^3, write $N = H \cup_\Sigma H'$ where H, H' are handlebodies. Picking cutting curves ~~whose~~ that are contractible in H, H' gives $\mathcal{B}, \mathcal{B}'$ and Hilbert spaces $\mathcal{H}$, $\mathcal{H}'$ and Bohr-Sommerfeld fibers $\mathrm{Loc}_G H = p^{-1}(\vec{0})$, $\mathrm{Loc}_G H' = (p'j)^{-1}(\vec{0})$, and we can define $\langle N \rangle = \langle H, H' \rangle =$ pairing of corresponding basis vectors in $\mathcal{H}$, $\mathcal{H}'$.

<u>Thm:</u> If $Loc_G(N)$ is a finite set, then

$$\langle N \rangle = \sum_{[A] \in Loc_G(N)} e^{2\pi i\, k\, CS(A)} \sqrt{\tau^{\mathbb{Z}}(N, A)}$$

where $\tau^{\mathbb{Z}}(N, A)$ is the <u>Reidemeister Torsion</u> of the twisted de Rham complex d_A. (it depends on an integral structure lifting the integer chains for a cell decomposition of N; ~~and gen~~ Reidemeister torsion generalizes the absolute value of the determinant).

If $\dim Loc_G(N) > 0$, there is a similar integral expression.