MATH 448: RESHETIKHIN–TURAEV INVARIANTS
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1. Introduction

My goal today is to braid back together the various lectures we've had so far. I won't try to set up the results completely precisely, aiming instead for intuition and outline, and I'll stick to the SU(2) case. The best results in the literature are due to Andersen and Ueno, who prove that in the SU(n) case the RT/BHMV tqft is isomorphic to a tqft coming from geometric quantization and conformal field theory. Those papers are very recent, and depend on quite a lot of work, and no one has written up similar results away from type-A. According to a MathOverflow discussion between Jørgen Andersen and André Henriques, the result for all types (so at least covering all connected simply connected compact groups) can be extracted with some care from the existing literature — you patch together results by a whole lot of authors. Oh, and the tori and the finite groups are easier, so probably you could piece together everything. Probably there needs to be a book covering all details.

2. Geometric set-up

So, anyway, where to start? Let Σ be an oriented surface, and $G = \mathrm{SU}(2)$. There is a space $\mathrm{Loc}_G(\Sigma)$ of G-local systems on Σ, which is of mathematical interest (because G-bundles are interesting, and flat ones are...). This space is really a derived stack, but for most of its life it's been a singular manifold. It carries a symplectic structure and a prequantum line bundle — these are canonical once you pick a generator of $H^4(BG)$ and an orientation of Σ, via what we now understand as a transgression (or "AKSZ") construction. (You pull back something from BG to $\Sigma \times \mathrm{Loc}_G(\Sigma)$, and then integrate over Σ.) In physics, $\mathrm{Loc}_G(\Sigma)$ is called the "classical phase space for Chern-Simons theory". See, the Chern-Simons functional has as its critical points the flat connections, so in physics "Chern-Simons" and "flat bundle" mean the same thing (critical points = solutions to EOM, because the Euler–Lagrange equations are $dCS = 0$). The physical theory is one of a "pure gauge field" (meaning there's no "matter" fields, i.e. nothing for the gauge bundle to act on) with this "action". It's a general property of Euler–Lagrange theory that critical loci over cylinders are symplectic.

We want to quantize this theory, because that will be something about "quantum flat bundles" or "local systems for a quantum group" or something like that, and because quantization tends to be interesting. To do this, we need to break some symmetry in the system. In physical language, we "couple to gravity", meaning that we fix some background metric on Σ. As Pyongwon explained, any such choice gives a Kahler quantization. Now let the metric vary: the quantizations package together into a vector bundle over Teichmuller space $\mathcal{T}_\Sigma$, whose corresponding projective bundle is flat. (To actually put a projectively flat connection, you need to pick a Lagrangian in $H^2(\Sigma)$.) The generator of $H^4(BG)$ determined a prequantum line bundle $\mathcal{L}$; k times the generator gives the prequantum line bundle $\mathcal{L}^{\otimes k}$, and its quantization I will call $\mathcal{H}^{(k)}$, which is a bundle over Teichmuller space.

By definition, Teichmuller space is the space of complex structures on a surface Σ. I.e. complex curves with topology Σ. There are natural families of complex curves. For example, you could pick a simple closed cycle on Σ, and squeeze it down to a point. Conformally, that's the same as stretching the "neck" through that cycle to be very long. The limiting object is a nodal curve. This defines a boundary of Teichmuller space, in which you go out to these nodal curves. Andersen and Ueno showed that the vector bundle $\mathcal{H}^{(k)}$ extends out to this boundary, and the projectively-flat connection projectively-extends (by which I mean that it blows up, but not very badly, and only as a scalar, so the flat connection on $\mathbb{P}\mathcal{H}^{(k)}$ extends.)

3. Bohr–Sommerfeld description

Last week I gave an independent description of the values of $\mathcal{H}^{(k)}$ on the 0-dimensional locus in $\partial\mathcal{T}_\Sigma$, i.e. where Σ has degenerated to a bunch of thrice-punctured $\mathbb{P}^1$s linked together. The Kahler polarization used in the bulk of $\mathcal{T}_\Sigma$ degenerates to a real polarization, and I described the corresponding Hilbert space in terms of Bohr–Sommerfeld orbits.

At level k, these were indexed by assignments of numbers $0, \dots, k$ to each of the cutting circles (i.e. the nodal points). (For other groups G, the indices range over positive weights of height bounded by k.) These were subject to a condition from each pair of pants ($\mathbb{P}^1$). Suppose the indices on the three ends are i_1, i_2, i_3. Then the conditions are that $i_1 + i_2 + i_3$ is even and at most $2k$ and that i_1, i_2, i_3 solve the triangle inequalities. These are called the *Clebsch–Gordon rules*.

As a reality check, let's see that number of Bohr–Sommerfeld points is independent of the pants decomposition. (It would have to be if we are going to have a vector bundle, since this number is the dimension of Hilbert space.) How should we calculate? There is a particularly simple lattice model that calculates this. Namely, replace nodal Σ with a trivalent graph, and consider a model in which spins are assigned to edges and the spins are numbers $0, \dots, k$. Let's say that the Boltzmann weights are assigned to the vertices, and are 0 or 1 depending on whether the Clebsch–Gordon rules are satisfied. Then the partition function is the number of Bohr–Sommerfeld points.

But now note that this lattice model satisfies the following simple property *I=H*. Corollary: the partition function only depends on the genus. Exercise: write down that formula. It's the *Verlinde formula*.

4. Fusion rings

The above property tells you something more. Consider the vector space with basis $\{0, \dots, k\}$. Then the Boltzmann weights define an associative product on this vector space, which is in fact a commutative unital (via 0) Frobenius (via the obvious pairing) algebra.

A *Fusion ring* is a unital Frobenius algebra with a basis in which all structure coefficients are nonnegative integers. Fusion rings invite categorification, because the first ones arose in Frobenius's work on finite groups, namely as the representation rings of finite groups. (More generally, if you have integer Boltzman weights, you could try to find a "categorified lattice model" in which the spins are objects of a category, and the weights are vector spaces. These arise in relation to "foam" models, where the spins are on the 2-cells of a soap foam.) In our case, the answer clearly should have something to do with the representations of $SU(2)$, because those already satisfy the "infinite k" version of the Clebsh–Gordon rules (triangle + even, no bound).

I actually already told you the (essentially unique) categorification of this ring. I described it in terms of Temperley–Lieb algebra by setting quantum parameter q to $\exp(2\pi i/(k+2))$, and then killing negligible morphisms. There's a Schur–Weyl dual description: you take Jimbo's Hopf algebra $U_q\mathfrak{sl}(2)$ and look just at the completely reducible finite-dimensional modules, which is a braided monoidal subcategory, and in fact is the representation category of a certain quotient Hopf algebra called $U_q^{s.s.}\mathfrak{sl}(2)$, where "s.s." stands for "semisimplified". (In the $SU(N)$ case, you can do exactly the same thing, on the Weyl side building a version of Hecke category at $q = \exp(2\pi i/(k+N))$ and killing negligibles, and on the Schur side building a quotient of $U_q\mathfrak{sl}(n)$. In the BCD types, I think Kauffman has described the appropriate Weyl-side categories, but I don't remember the details, and I'm not sure what's been explored in exceptional types, although Kuperberg's spiders definitely do the case of rank 2, including exceptional type G_2.)

So we get a "categorified lattice model", and the associativity isomorphism still lets you change the graph to a different one with the same genus. So you get a well-defined vector space depending just on the genus — almost. A ribbon graph would give you a well-defined vector space, but the braiding is not a symmetry, so you should think of this as assigned to just the trivalent graph, but rather really to the surface. Also, there's holonomy when you go around the mapping class group, and the cutting circles are secretly picking out a Lagrangian in $H^1(\Sigma)$, and you pick up projective effects when you change that.

5. Skein theory

Since we have a skein-theoretic description of the fusion category, we can get one of this vector space. Remember that object $i \in \{0, \dots, k\}$ corresponded to the unique idempotent on i strands in TL category orthogonal to all U-generators of the i-strand TL algebra. Because we have a braiding, but a nontrivial Reidemeister-1, we should think of these strands not as living on a plane but in a 3-space, and being framed or ribbon there. Consider the solid 3-ball, which is the interior of $\mathbb{P}^1$, and three marked points on the boundary. Put i_1, i_2, i_3 strands entering the 3-ball at each of the three marked points, and a JW-idempotent near there. Then ask: what are all ways to fill in the 3-ball to connect these up? Naturally you can add and subtract these, by using the skein laws, so there's a vector space of possible ways. This vector space is precisely $\hom(i_1 \otimes i_2, i_3)$ in the category (and I'm using that $i^* = i$). The associativity then says that if you were to look at a solid ball with four marked points, you'd get a skein module that you could compute as a tensor product of the 3-marked-point skein modules in various ways.

So any filling of Σ by a handlebody, even if you didn't pick cutting circles, gives you a skein module, and this is the vector space we're after. But, actually, this is precisely the vector space assigned by BHMV construction that Sean discussed. See, he built his vector spaces by looking at fillings, and his manifolds were allowed to have these ribbons in them, and the theorem that made the construction tick was that actually every filling was equivalent to a handlebody, because of a way to remove 2-handles (nothing but a certain surgery formula).

6. Conclusion

Ok, so Jeffrey–Weitsman and BHMV constructions give the same dimensions of vector spaces, and I assert that Kahler approach does as well. This is not good enough to prove that the TQFTs are the same. To do so, you have to work hard. The zeroth step is to make sure that everyone makes sense on marked or punctured surfaces, where the special points are labeled, and that there are "lattice model" rules for connecting up at the punctures. The first hard step is to make sure that everyone satisfies the same axioms (namely, Walker's topological version of the axioms of a "Modular functor"). The second hard step is to prove that modular functors are uniquely determined by their genus-zero data: if you and I agree on genus-zero marked curves, then we agree. Finally, you need to actually build a natural isomorphism on the genus-zero parts, which really involves careful study of Kahler quantization of $\mathrm{Loc}_G(\mathbb{P}^1_n)$, where n is the number of marked points, including how to cut and glue. A remark for those who like operads: the point is that there's an intimate relationship between moduli space of curves and the framed E_2 operad.